# On the structure and the behaviour of Collatz 3n + 1 sequences

Finite subsequences and the role of the Fibonacci sequence

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**Abstract**

The number theoretic function $T(n) = \frac{n}{2}$ if $n$ is even, $T(n) = \frac{3n+1}{2}$ if $n$ is odd, generates for each starting number $s \in \mathbb{N}$ a Collatz sequence $C(s) = \left(T^k(s)_{k=0}^{\infty}\right), T^0(s) = s, T^k(s) = T\left(T^{k-1}(s)\right)$. A $C(s)$ can only assume two possible forms. Either it falls into a cycle or it grows to infinity. The unproved conjecture to this problem is that each $C(s)$ enters the cycle $(1, 2)$.

It is shown that every $C(s)$ consists only of same structured finite subsequences $C^h(s) = \left(T^k(s)_{k=0}^{h}\right)$ for $s \equiv 9 \ (mod\ 12)$ or $C^t(s) = \left(T^k(s)_{k=0}^{t}\right)$ for $s \equiv 3, 7 \ (mod\ 12)$. For starting numbers of specific residue classes $(mod\ 12 \cdot 2^h)$ or $(mod\ 12 \cdot 2^{t+1})$ the finite subsequences have the same length $h, t$. It is conjectured that for each $h, t \geq 2$ the number of all admissible residue classes is given exactly by the Fibonacci sequence. This has been proved for $2 \leq h, t \leq 50$.

Collatz's conjecture is equivalent to the conjecture that for each $s \in \mathbb{N}, s > 1$, there exists $k \in \mathbb{N}$ such that $T^k(s) < s$. The least $k \in \mathbb{N}$ such that $T^k(s) < s$ is called the stopping time of $s$, which we will denote by $\sigma(s)$. It is shown that Collatz's conjecture is true, if every starting number $s \equiv 3, 7 \ (mod\ 12)$ have finite stopping time.

We denote $\tau(s)$ as the number of $C^t(s)$ until $\sigma(s)$ is reached for a starting number $s \equiv 3, 7 \ (mod\ 12)$. Starting numbers of specific residue classes $(mod\ 3 \cdot 2^{\sigma(s)})$ have the same stopping times $\sigma(s)$ and $\tau(s)$. By using $\tau(s)$ it is shown that almost all $s \equiv 3, 7 \ (mod\ 12)$ have finite stopping time and statistically two out of three $s \equiv 3, 7 \ (mod\ 12)$ have $\tau(s) = 1$. Further it is shown what consequences it entails, if a $C(s)$ grows to infinity.

# Contents

# 1 Introduction

Looking at the behaviour of different Collatz sequences, we quickly gain the impression that here is pure chaos at work. This impression is not deceptive. Guenther Wirsching, an international expert on the subject, writes: "The mathematical difficulties in studying the dynamics of 3n + 1 iterations seem to be associated with the fact that we are dealing with a deterministic process that simulates stochastic behaviour. This connects the subject with the mathematical approach of the chaos."[9]

By dividing a Collatz sequence into finite subsequences, we can bring a little order in their dynamic behaviour.

A Collatz sequence $C^{\infty}(s)$ with an infinite growth must reach after a finite number of iterations a term of the residue class $[3]_4$. From there $C^{\infty}(s)$ consists only of finite subsequences $C^t(s)$ for $s \equiv 3, 7\ (mod\ 12)$. If we take a sorted endless list of all $C^t(s)$, as shown in appendix 7.2, then $C^{\infty}(s)$ must go through the subsequences in this list - not all, but each only once. This usually happens chaotic, jumping back and forth, but with the tendency ever upward through the list.

Now it can be shown that the distribution of the $C^t(s)$ in this list, according to their length (number of terms), is not chaotic, but follows a strict pattern based on the Fibonacci sequence. This has consequences for the growth behaviour of a $C^{\infty}(s)$.

# 2 The Collatz $3n + 1$ function

The Collatz $3n + 1$ function is defined as a function $T : \mathbb{N} \to \mathbb{N}$ on the set of positive integers by

$$T(n) := \begin{cases} T_0 := \dfrac{n}{2} & \text{if } n \text{ is even,} \\ \\ T_1 := \dfrac{3n+1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Let $T^0(s) = s$ and $T^k(s) = T\left(T^{k-1}(s)\right)$ for $k \in \mathbb{N}$. Then the Collatz sequence for $s \in \mathbb{N}$ is $C(s) = \left(T^k(s) \mid k = 0, 1, 2, 3, \dots\right)$.

For example, the starting number $s = 11$ generates the Collatz sequence

$$C(11) = (11, 17, 26, 13, 20, 10, 5, 8, 4, 2, 1, 2, 1, 2, 1, \dots).$$

A Collatz sequence can only assume two possible forms. Either it falls into a cycle or it grows to infinity. The unproved conjecture to this problem is that each Collatz sequence enters the cycle $(1, 2)$.

# 3 Term Lemmata

## 3.1 The residue classes modulo $2^n$

**Lemma 1**: For every $T^k(s) \equiv 0\ (mod\ 2)$ there is $T^{k+1}(s) < T^k(s)$.

*Proof*: For $T^k(s) = 2x$ with $x \in \mathbb{N}$, $x \geq 1$, there is

$$T^{k+1}(s) = \frac{2x}{2} = x < T^k(s) = 2x.$$

□

**Lemma 2**: For every $T^k(s) \equiv 1\ (mod\ 4)$ there is $T^{k+2}(s) < T^k(s)$ with $T^{k+2}(s) \equiv 1\ (mod\ 3)$.

*Proof*: For $T^k(s) = 4x + 1$ with $x \in \mathbb{N}$, $x \geq 1$, there is

$$T^{k+1}(s) = \frac{3(4x+1)+1}{2} = \frac{12x+4}{2} = 6x + 2.$$

$$T^{k+2}(s) = \frac{6x+2}{2} = 3x + 1 < T^k(s) = 4x + 1.$$

□

**Lemma 3**: For every $T^k(s) \equiv 6\ (mod\ 8)$ there is $T^{k+1}(s) < T^k(s)$ with $T^{k+1}(s) \equiv 3\ (mod\ 4)$.

*Proof*: For $T^k(s) = 8x + 6$ with $x \in \mathbb{N}$, $x \geq 0$, there is

$$T^{k+1}(s) = \frac{8x+6}{2} = 4x + 3 < T^k(s) = 8x + 6.$$

□

**Lemma 4**: For every $T^k(s) = 2^n$ in $C(s)$ the term "1" is reached after exactly $n$ iterations.

*Proof*: For $T^k(s) = 2^n$ with $n \in \mathbb{N}$ there is

$$T^{k+1}(s) = \frac{2n}{2} = 2^{n-1}.$$

Each further iteration generates an even term. Because of this, after $n$ iterations there is

$$T^{k+n}(s) = 2^{n-n} = 1.$$

□

**Lemma 5**: For every $T^k(s) \equiv (2^n - 1)\ (mod\ 2^{n+1})$, $n \in \mathbb{N}$, $n \geq 2$, there is $T^{k+1}(s) \equiv (2^{n-1} - 1)\ (mod\ 2^n)$.

*Proof*: For $T^k(s) = 2^{n+1}x + 2^n - 1$ with $x, n \in \mathbb{N}$, $x \geq 0$, $n \geq 2$, there is

$$\begin{aligned} T^{k+1}(s) &= \frac{3(2^{n+1}x + 2^n - 1) + 1}{2} &&= \frac{3 \cdot 2^{n+1}x + 3(2^n - 1) + 1}{2} \\ &= \frac{3 \cdot 2^{n+1}x + 3 \cdot 2^n - 2}{2} &&= \frac{3 \cdot 2^{n+1}x + (2+1) \cdot 2^n - 2}{2} \\ &= \frac{3 \cdot 2^{n+1}x + 2^{n+1} + 2^n - 2}{2} &&= 3 \cdot 2^n x + 2^n + 2^{n-1} - 1 \\ &= 2^n(3x + 1) + 2^{n-1} - 1. \end{aligned}$$

□

**Lemma 6**: For every $T^k(s) \equiv (2^n - 1)\ (mod\ 2^{n+1})$, $n \in \mathbb{N}, n \geq 2$, $C(s)$ contains a term $T^{n-1}(s) \equiv 1\ (mod\ 4)$.

*Proof*: According to Lemma 5, a number $s \in \mathbb{N}$ of the residue class $[2^n - 1]_{2^{n+1}}$ generates in $C(s)$ after the first iteration a term of the residue class $[2^{n-1} - 1]_{2^n}$. By induction over $n$ follows that after $n-1$ iterations a term of the residue class $[2^{n-(n-1)} - 1]_{2^{n+1-(n-1)}} = [1]_4$ is reached. □

**Lemma 7**: For each $s \in \mathbb{N}$ the union of the sets $\{s \equiv (2^n - 1)\ (mod\ 2^{n+1})\}$ for each $n \in \mathbb{N}$, $n \geq 2$, is equal to the set $\{s \equiv 3\ (mod\ 4)\}$. □

**Lemma 8**: Of every $T^k(s) \equiv 3\ (mod\ 4)$ only the terms of the residue class $[11]_{12}$ generates in $C(s)$ a term $T^{k-1}(s) \equiv 3\ (mod\ 4) < T^k(s) \equiv 3\ (mod\ 4)$.

*Proof*: For $T^k(s) = 4x + 3$ with $x \in \mathbb{N}$, $x \geq 0$, there is

$$T^{k-1}(s)\frac{2(4x + 3) - 1}{3} = \frac{8x + 5}{3} < 4x + 3.$$

□

## 3.2 The residue classes modulo 12

**Lemma 9**: For every $T^k(s) \equiv 5\ (mod\ 12)$ there is $T^{k-1}(s) \equiv 3\ (mod\ 4)$.

*Proof*: For $T^k(s) = 12x+5$ with $x \in \mathbb{N}$, $x \geq 0$, there is

$$T^{k-1}(s) = \frac{2(12x+5)-1}{3} = \frac{24x+9}{3} = 8x+3.$$

It is $[3]_8 \subseteq [3]_4$. □

**Lemma 10**: For $T^k(s) \equiv 1 \ (mod\ 12)$ there is $T^{k-2}(s) \equiv 5,9 \ (mod\ 12)$ or $T^{k-4}(s) \equiv 5,9 \ (mod\ 12)$.

*Proof*: For every $T^k(s) = 12x+1$ with $x \in \mathbb{N}$, $x \geq 1$, there is

$$T^{k-1}(s) = 2(12x+1) = 24x+2.$$

$$T^{k-2}(s) = \frac{2(24x+2)-1}{3} = \frac{48x+3}{3} = 16x+1.$$

We distinguish three different cases for $x$. For $x \equiv 1 \ (mod\ 3)$ there is $T^{k-2}(s) \equiv 17 \ (mod\ 48)$. For $x \equiv 2 \ (mod\ 3)$ there is $T^{k-2}(s) \equiv 33 \ (mod\ 48)$. For $x \equiv 0 \ (mod\ 3)$ there is $T^{k-2}(s) \equiv 1 \ (mod\ 48)$.

It is $[17]_{48} \subseteq [5]_{12}$, $[33]_{48} \subseteq [9]_{12}$ and $[1]_{48} \subseteq [1]_{12}$, whereby $T^{k-2}(s) \equiv 1 \ (mod\ 48)$ is a number of the residue class $[17]_{48}$ or $[33]_{48}$. □

# 4 Finite subsequences

Let $C^a(s) = \big(T^k(s) \mid k = 0, \dots, a\big)$ with $a \geq 1$ be a finite subsequence of $C(s)$. Then we define for the global extrema in $C^a(s)$ the following terms. $T^{min_o}(s)$ is an odd minimum, $T^{min_e}(s)$ is an even minimum, $T^{max_o}(s)$ is an odd maximum and $T^{max_e}(s)$ is an even maximum.

## 4.1 Finite subsequences $C^t(s)$ for $s \equiv 3,7 \ (mod\ 12)$

Let $C^t(s) = \big(T^k(s) \mid k = 0, \dots, t\big)$ with $t \geq 2$ be a finite subsequence of $C(s)$ for each $s \equiv 3,7 \ (mod\ 12)$. Then a $C^t(s)$ has the structure

$$C^t(s) = \left(T^0(s), \dots, T^{max_o}(s), T^{max_e}(s), \dots, T^t(s)\right),$$

where $T^0(s), \dots, T^{max_o}(s), T^{max_e}(s)$ with $T^0(s) \equiv 3,7 \ (mod\ 12)$, $T^1(s), \dots, T^{max_o - 1}(s) \equiv 3 \ (mod\ 4)$, $T^{max_o}(s) \equiv 1 \ (mod\ 4)$. The further course, dependent on $s$, is different for each subsequence. In the shortest case is $T^t(s) = T^{max_e}(s) \equiv 6 \ (mod\ 8)$. In the longest case is $T^{max_e}(s) \equiv 2 \ (mod\ 6)$, $T^{max_e+1}(s)$, $\dots$, $T^{t-1}(s) \equiv 1 \ (mod\ 4)$ or $0,2,4 \ (mod\ 8)$ and $T^t(s) = T^{min_e}(s) \equiv 6 \ (mod\ 8)$ or $T^t(s) = T^{min_o}(s) = 1$. Therefore the further course is bounded by a term of the residue class $[6]_8$ or the term "1".

The following Theorem 1 discusses only the longest case of a $C^t(s)$, in which all possible shorter cases are included.

**Theorem 1**: $C^t(s)$ is strictly increasing until a maximum $T^{max_o}(s), T^{max_e}(s)$ is reached. Then strictly decreasing (separately for even and odd terms) until a minimum $T^t(s)$ is reached. Let $C^{t_A}(s)$ be the subsequence with $T^{min_e}(s) \equiv 6 \ (mod\ 8)$, and let $C^{t_B}(s)$ be the subsequence with $T^{min_o}(s) = 1$. Then the residue class structure of the subsequences are

$$C^{t_A}(s) = ([3,7]_{12}, \dots, [3]_8, [1]_4, [2]_6, \dots, [6]_8)\,,$$
$$C^{t_B}(s) = ([3,7]_{12}, \dots, [3]_8, [1]_4, [2]_6, \dots, [0]_{2^n}, \dots, 1)\,.$$

*Proof*: It is $[3]_4 = [3]_{12} \sqcup [7]_{12} \sqcup [11]_{12}$. According to Lemma 8, the numbers $s$ of the residue classes $[3]_{12}$ and $[7]_{12}$ are exactly the only numbers of the residue class $[3]_4$ which have in $C(s)$ no smaller predecessor term of the residue class $[3]_4$. From Lemma 5 and 6, in use of Lemma 7, it follows that $C(s)$ is strictly increasing for numbers of the residue class $[3]_4$, passing only through terms of the residue class $[3]_4$, until an odd maximum of the residue class $[1]_4$ is reached. According to Lemma 2, the next term is an even maximum of the residue class $[2]_6$. According to Lemma 1 and 2, the next terms are only of the residue classes $[1]_4$ and $[0,2,4]_8$, even and odd terms separately strictly decreasing, until an even minimum of the residue class $[6]_8$ or the odd minimum "1" is reached. According to Lemma 4, the term "1" is exactly reached, if a term of the residue classes $[0,2,4]_8$ is a power of two. According to Lemma 3, the next term after a term of the residue class $[6]_8$ is a smaller odd term of the residue class $[3]_4$. □

The first subsequences $C^t(s)$ are

$$\begin{aligned}
C^{5_B}(3) &= (3, 5, 8, 4, 2, 1),\\
C^{11_B}(7) &= (7, 11, 17, 26, 13, 20, 10, 5, 8, 4, 2, 1),\\
C^{12_B}(15) &= (15, 23, 35, 53, 80, 40, 20, 10, 5, 8, 4, 2, 1),\\
C^{3_A}(19) &= (19, 29, 44, 22),\\
C^{2_A}(27) &= (27, 41, 62),\\
C^{7_A}(31) &= (31, 47, 71, 107, 161, 242, 121, 182),\\
C^{3_A}(39) &= (39, 59, 89, 134),\\
C^{8_A}(43) &= (43, 65, 98, 49, 74, 37, 56, 28, 14),\\
&\ and\ so\ forth.
\end{aligned}$$

Appendix 7.2 shows a list of the first subsequences $C^t(s)$ up to $s = 1047$.

## 4.2 Finite subsequences $C^h(s)$ for $s \equiv 9\ (mod\ 12)$

Let $C^h(s) = \big(T^k(s) \mid k = 0, \dots, h\big)$ with $h \geq 2$ be a finite subsequence of $C(s)$ for each $s \equiv 9\ (mod\ 12)$. Then a $C^h(s)$ has the structure

$$C^h(s) = \left(T^{max_o}(s), T^{max_e}(s), \dots, T^{min_o}(s)\right),$$

where $T^{max_o}(s) = T^0(s) \equiv 9\ (mod\ 12)$, $T^1(s), \dots, T^{min_o-1}(s) \equiv 1\ (mod\ 4)$ or $0\ (mod\ 2)$ and $T^{min_o}(s) = T^h(s) \equiv 3\ (mod\ 4)$ or 1.

**Theorem 2**: $C^h(s)$ is strictly decreasing (separately for even and odd terms) and has an odd minimum $T^{min_o}(s)$. Let $C^{h_A}(s)$ be the subsequence with $T^{min_o}(s) \equiv 3\ (mod\ 4)$, and $C^{h_B}(s)$ be the subsequence with $T^{min_o}(s) = 1$. Then the residue class structure of the subsequences are

$$C^{h_A}(s) = ([1]_4, [2]_6, \dots, [6]_8, [3]_4)\,,$$
$$C^{h_B}(s) = ([1]_4, [2]_6, \dots, [0]_{2^n}, \dots, 1)\,.$$

*Proof*: Because $[9]_{12} \subseteq [1]_4$, the residue class structure of a $C^h(s)$ is equal to the residue class structure of a $C^t(s)$, beginning with the term $T^{max_o}(s) \equiv 1\ (mod\ 4)$ of a $C^t(s)$. Therefore the proof of Theorem 2 is identically equal to the proof of Theorem 1. □

The first subsequences $C^h(s)$ are

$$C^{2_A}(9) = (9, 14, 7),$$
$$C^{6_B}(21) = (21, 32, 16, 8, 4, 2, 1),$$
$$C^{4_A}(33) = (33, 50, 25, 38, 19, )$$
$$C^{12_B}(45) = (45, 68, 34, 17, 26, 13, 20, 10, 5, 8, 4, 2, 1),$$
$$C^{2_A}(57) = (57, 86, 43),$$
$$C^{11_B}(69) = (69, 104, 52, 26, 13, 20, 10, 5, 8, 4, 2, 1),$$
$$C^{5_A}(81) = (81, 122, 61, 92, 46, 23),$$
$$C^{2_A}(93) = (93, 140, 70, 35),$$

*and so forth.*

Appendix 7.1 shows a list of the first subsequences $C^h(s)$ up to $s = 2073$.

## 4.3 How the subsequences works

**Theorem 3**: Every Collatz's sequence $C(s)$ consists only of the subsequences $C^t(s)$ and $C^h(s)$.

*Proof*: It is $\mathbb{N} \subseteq [0]_2 \sqcup [1]_2 \sqcup [3]_4 \sqcup [1]_{12} \sqcup [5]_{12} \sqcup [9]_{12}$. According to the definition of a $C(s)$, every even number generates an odd number. According to Lemma 9, every $T^k(s) \equiv 5\ (mod\ 12)$ has a predecessor $T^{k-1}(s) \equiv 3\ (mod\ 4)$. According to Lemma 10, every $T^k(s) \equiv 1\ (mod\ 12)$ has a predecessor $T^{k-2}(s) \equiv 5, 9\ (mod\ 12)$ or $T^{k-4}(s) \equiv 5, 9\ (mod\ 12)$. Therefore, with the use of Lemma 9, every $T^k(s) \equiv 1\ (mod\ 12)$ has a predecessor $T^{k-3}(s) \equiv 3\ (mod\ 4)$ or $T^{k-5}(s) \equiv 3\ (mod\ 4)$. According to Theorem 1, the union of all $C^t(s)$ contains the set of all positive numbers of the residue class $[3]_4$, whereby each positive number of the residue class $[3]_4$ is *uniquely* assigned to one $C^t(s)$. Every $C^t(s)$ contains no term of the residue class $[9]_{12}$. Because of this a $C^h(s)$ can only be the beginning of a $C(s)$ or leads to a $C(s)$. According to Theorem 2, after a finite number of iterations a $C^h(s)$ contains a term of the residue class $[3]_4$. Therefore it leads a $C^t(s)$ or it leads directly to the term "1". This shows that every natural number is included in at least one $C^t(s)$ or $C^h(s)$. □

*Example*: For a better understanding of how a Collatz sequence consists out of their subsequences let us take a look on the sequence with the starting number $s = 27$.

$C(27) = (27, 41, 62, 31, 47, 71, 107, 161, 242, 121, 182, 91, 137, 206, 103, 155, 233, 350, 175, 263, 395, 593, 890, 445, 668, 334, 167, 251, 377, 566, 283, 425, 638, 319, 479, 719, 1079, 1619, 2429, 3644, 1822, 911, 1367, 2051, 3077, 4616, 2308, 1154, 577, 866, 433, 650, 325, 488, 244, 122, 61, 92, 46, 23, 35, 53, 80, 40, 20, 10, 5, 8, 4, 2, 1)$.

This sequence consists of 59 terms, which can be subdivided into 10 subsequences $C^t(s)$.

(27, 41, 62),
(31, 47, 71, 107, 161, 242, 121, 182),
(91, 137, 206),
(103, 155, 233, 350),
(175, 263, 395, 593, 890, 445, 668, 334),
(111, 167, 251, 377, 566),
(283, 425, 638),
(319, 479, 719, 1079, 1619, 2429, 3644, 1822),
(607, 911, 1367, 2051, 3077, 4616, 2308, 1154, 577, 866, 433, 650, 325, 488, 244, 122, 61, 92, 46),
(15, 23, 35, 53, 80, 40, 20, 10, 5, 8, 4, 2, 1).

## 4.4 Finite subsequences of the same length

It is not hard to verify that for starting numbers of specific residue classes $(mod\ 12 \cdot 2^h)$ or $(mod\ 12 \cdot 2^{t+1})$ the subsequences have the same length $h$ or $t$.

For the $C^h(s)$ there is

$$h = 2 \quad \text{if} \quad s \equiv 9\ (mod\ 48),$$

$h = 3$ if $s \equiv 93$ $(mod\ 96)$,
$h = 4$ if $s \equiv 33, 165$ $(mod\ 192)$,
$h = 5$ if $s \equiv 81, 117, 237$ $(mod\ 384)$,
$h = 6$ if $s \equiv 129, 333, 405, 561, 645$ $(mod\ 768)$,
$h = 7$ if $s \equiv 429, 657, 837, 981, 1293, 1461, 1473, 1521$ $(mod\ 1536)$,
$h = 8$ if $s \equiv 177, 309, 513, 597, 1089, 1221, 1557, 1581, \ldots$ $(mod\ 3072)$,
and so forth.

For the $C^t(s)$ there is

$t = 2$ if $s \equiv 27, 91$ $(mod\ 96)$,
$t = 3$ if $s \equiv 19, 39, 103, 147$ $(mod\ 192)$,
$t = 4$ if $s \equiv 55, 67, 111, 183, 195, 235, 363, 367$ $(mod\ 384)$,
$t = 5$ if $s \equiv 139, 159, 163, 207, 243, 327, 415, \ldots$ $(mod\ 768)$,
$t = 6$ if $s \equiv 51, 99, 259, 279, 427, 447, 559, 655, \ldots$ $(mod\ 1536)$,
$t = 7$ if $s \equiv 31, 135, 175, 291, 319, 331, 375, 627, \ldots$ $(mod\ 3072)$,
$t = 8$ if $s \equiv 43, 63, 199, 223, 271, 351, 355, 435, 519, \ldots$ $(mod\ 6144)$,
and so forth.

Appendix 7.3 and 7.4 show lists of all admissible residue classes for the $C^h(s)$ and the $C^t(s)$ up to $h = 16$ and $t = 14$.

## 4.5 The role of the Fibonacci sequence

When counting the residue classes of the subsequences with the same length, we find that for each $h, t \leq 50$ the number of residue classes for the $C^h(s)$ is given exactly by the Fibonacci sequence (OEIS A000045) and for the $C^t(s)$ exactly by a sequence based on the Fibonacci sequence (OEIS A019274). Therefore, we can formulate the following conjectures.

**Conjecture 1**: Let $A(h)$ be the number of residue classes $(mod\ 12 \cdot 2^h)$, then for each $h \in \mathbb{N}$, $h \geq 2$, it is

$$A(h) = \frac{1}{\sqrt{5}} \left[ \left( \frac{1+\sqrt{5}}{2} \right)^{h-1} - \left( \frac{1-\sqrt{5}}{2} \right)^{h-1} \right].$$

**Conjecture 2**: Let $A(t)$ be the number of residue classes $(mod\ 12 \cdot 2^{t+1})$, then for each $t \in \mathbb{N}$, $t \geq 2$, it is

$$A(t) = \frac{2}{\sqrt{5}} \left[ \left( \frac{1+\sqrt{5}}{2} \right)^{t+1} - \left( \frac{1-\sqrt{5}}{2} \right)^{t+1} \right] - 2.$$

Chapter 4.4 and 4.5 instantly raises a question to a connection to the number of residue classes for Collatz sequences with finite stopping time.

# 5 Stopping time

## 5.1 The stopping time $\sigma(s)$

Collatz's conjecture is equivalent to the conjecture that for each $s \in \mathbb{N}, s > 1$, there exists $k \in \mathbb{N}$ such that $T^k(s) < s$. The least $k \in \mathbb{N}$ such that $T^k(s) < s$ is called the stopping time of $s$, which we will denote by $\sigma(s)$. It is not hard to verify that

$\sigma(s) = 1$ if $s \equiv 0$ ($mod$ 2),
$\sigma(s) = 2$ if $s \equiv 1$ ($mod$ 4),
$\sigma(s) = 4$ if $s \equiv 3$ ($mod$ 16),
$\sigma(s) = 5$ if $s \equiv 11, 23$ ($mod$ 32),
$\sigma(s) = 7$ if $s \equiv 7, 15, 59$ ($mod$ 128),
$\sigma(s) = 8$ if $s \equiv 39, 79, 95, 123, 175, 199, 219$ ($mod$ 256),
$\sigma(s) = 10$ if $s \equiv 287, 347, 367, 423, 507, 575, 583, \ldots$ ($mod$ 1024),
and so forth.

As a general rule: For each $n \in \mathbb{N}$, $n \geq 0$, there is

$$\sigma(s) = \lfloor 1 + n \cdot \log_2 3 \rfloor \quad \text{if} \quad s \equiv s_1, s_2, s_3, \ldots, s_z \ (mod\ 2^{\sigma(s)}).$$

Let $z(n)$ be the number of residue classes ($mod\ 2^{\sigma(s)}$) for each $n \geq 0$.

Everett[1] proves that almost all $k \in \mathbb{N}$ have finite stopping time, and Terras[3] gives a probability distribution function for stopping times.

The possible stopping times $\sigma(s)$ are listed in OEIS A020914. The associated residue classes ($mod\ 2^{\sigma(s)}$) are listed in OEIS A177789. The number of residue classes $z(n)$ for $n \geq 1$ are listed in OEIS A100982.

Appendix 7.5 shows a list of the first residue classes ($mod\ 2^{\sigma(s)}$) up to $\sigma(s) = 16$.

**Theorem 4**: Collatz's conjecture is true, if every starting number $s \equiv 3, 7$ ($mod$ 12) have finite stopping time.

*Proof*: According to Lemma 1, there is $\sigma(s) = 1$ for every $s \equiv 0$ ($mod$ 2), and according to Lemma 2, there is $\sigma(s) = 2$ for every $s \equiv 1$ ($mod$ 4). Because of $\mathbb{N} \subseteq [0]_2 \sqcup [1]_4 \sqcup [3]_4$ it remains only to clarify the stopping time behaviour of every starting number $s \equiv 3$ ($mod$ 4). According to Theorem 1, the union of all $C^t(s)$ contains the set of all positive numbers of the residue class $[3]_4$, whereby each positive number of the residue class $[3]_4$ is *uniquely* assigned to one $C^t(s)$. If a $C^t(s)$ contains a term $T^{max_g+1}(s), \ldots, T^t(s) < T^0(s)$, then all terms $T^0(s), \ldots,\ T^{max_u-1}(s) \equiv 3\ (mod\ 4)$ have finite stopping time. According to Lemma 2, every $C^h(s)$ have finite stopping time $\sigma(s) = 2$, because $[9]_{12} \subseteq [1]_4$. □

## 5.2 The stopping time $\tau(s)$

Let $\tau(s)$ be the number of subsequences $C^t(s)$ until $\sigma(s)$ is reached for a starting number $s \equiv 3, 7\ (mod\ 12)$.

*Note*: For the special case $T^t(s) > T^0(s)$ with $T^{t+1}(s) = \frac{T^t(s)}{2} < T^0(s)$, let $\sigma(s) = t+1$ for the *least* subsequence where $\sigma(s)$ is reached . Although $T^{t+1}(s)$ is per definition not a term of a $C^t(s)$.

*Example 1*: There is $\tau(19) = 1$, because $\sigma(19)$ is reached in the *first* subsequence. There is $C^3(19) = (19, 29, 44, 22)$ and $\sigma(19) = 4$ with $T^4(19) = 11$.

*Example 2*: There is $\tau(187) = 2$, because $C^2(187) = (187, 281, 422)$ and $C^3(211) = (211, 317, 476, 238)$. With $T^4(211) = 119$, $\sigma(187)$ is reached in the *second* subsequence. There is $\sigma(187) = 2 + 1 + 4 = 7$, because $T^0 = 211$ is also a counted term.

*Example 3*: According to chapter 5, there is $\tau(27) = 9$.

It is not hard to verify that starting numbers of specific residue classes $(mod\ 3 \cdot 2^{\sigma(s)})$ have the same stopping times $\sigma(s)$ and $\tau(s)$.

For $\tau(s) = 1$ there is

$\sigma(s) = 4$ if $s \equiv 3, 19\ (mod\ 48)$,
$\sigma(s) = 5$ if $s \equiv 43, 55, 75, 87\ (mod\ 96)$,
$\sigma(s) = 7$ if $s \equiv 7, 15, 135, 271\ (mod\ 384)$,
$\sigma(s) = 8$ if $s \equiv 79, 175, 199, 351, 591, 607, 687, 711\ (mod\ 768)$,
$\sigma(s) = 10$ if $s \equiv 735, 1311, 1599, 1759, 1839, \ldots\ (mod\ 3072)$,
$\sigma(s) = 12$ if $s \equiv 1087, 1855, 2239, 3295, 4479, \ldots\ (mod\ 12288)$,
$\sigma(s) = 13$ if $s \equiv 255, 303, 543, 1215, 1567, 2431, \ldots\ (mod\ 24576)$,
and so forth.

For $\tau(s) = 2$ there is

$\sigma(s) = 7$ if $s \equiv 187, 315\ (mod\ 384)$,
$\sigma(s) = 8$ if $s \equiv 39, 123, 219, 295, 379, 475\ (mod\ 768)$,
$\sigma(s) = 10$ if $s \equiv 367, 423, 583, 975, 999, 1371, \ldots\ (mod\ 3072)$,
$\sigma(s) = 12$ if $s \equiv 231, 463, 615, 879, 1231, 1435, \ldots\ (mod\ 12288)$,

$\sigma(s) = 13$ if $s \equiv 207, 799, 1071, 1327, 1563, 1983, \ldots$ $(mod\ 24576)$,
$\sigma(s) = 15$ if $s \equiv 415, 2719, 2767, 2799, 2847, \ldots$ $(mod\ 98304)$,
$\sigma(s) = 16$ if $s \equiv 1183, 1351, 2367, 3103, 4335, \ldots$ $(mod\ 196608)$,
and so forth.

For $\tau(s) = 3$ there is

$\sigma(s) = 10$ if $s \equiv 507, 1531$ $(mod\ 3072)$,
$\sigma(s) = 12$ if $s \equiv 3675, 5115, 5799, 5883, 7771, \ldots$ $(mod\ 12288)$,
$\sigma(s) = 13$ if $s \equiv 679, 1135, 1191, 3067, 3835, \ldots$ $(mod\ 24576)$,
$\sigma(s) = 15$ if $s \equiv 411, 1095, 1275, 1903, 2119, \ldots$ $(mod\ 98304)$,
$\sigma(s) = 16$ if $s \equiv 559, 859, 1179, 1519, 2407, \ldots$ $(mod\ 196608)$,
$\sigma(s) = 18$ if $s \equiv 4543, 5167, 6055, 6079, 6367, \ldots$ $(mod\ 786432)$,
$\sigma(s) = 20$ if $s \equiv 2175, 3279, 3871, 4167, 4351, \ldots$ $(mod\ 3145728)$,
and so forth.

Appendix 7.6 shows a list of the first residue classes $(mod\ 3 \cdot 2^{\sigma(s)})$ up to $\tau(s) = 6$.

The next table shows the number of residue classes for the possible stopping times $(mod\ 3 \cdot 2^{\sigma(s)})$ for $\tau(s) = 1, \ldots, 7$.

| | n | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $\sigma(s)$ | 4 | 5 | 7 | 8 | 10 | 12 | 13 | 15 | 16 | 18 | 20 | 21 | 23 |
| $\tau(s)$ | | | | | | | | | | | | | | |
| 1 | | 2 | 4 | 4 | 8 | 8 | 16 | 32 | 32 | 64 | 64 | 128 | 256 | 256 |
| 2 | | | | 2 | 6 | 14 | 36 | 96 | 160 | 384 | 544 | 1248 | 2880 | 3776 |
| 3 | | | | | | 2 | 8 | 40 | 136 | 416 | 912 | 2480 | 6976 | 12736 |
| 4 | | | | | | | | 2 | 18 | 86 | 372 | 1290 | 4924 | 13508 |
| 5 | | | | | | | | | | 2 | 30 | 156 | 1008 | 4584 |
| 6 | | | | | | | | | | | | 2 | 46 | 410 |
| 7 | | | | | | | | | | | | | | 2 |

The algorithmic correlation between $\sigma(s)$ and $\tau(s)$ becomes clearly, if we calculate the sum of the $\tau(s)$ for each column $n$. If the result is halved, we get the same values for $z(n)$ as in chapter 5.1 (OEIS A100982).

**Conjecture 3**: Let $A_{\tau(s)}(n)$ be the number of residue classes $(mod\ 3 \cdot 2^{\sigma(s)})$ for $\tau(s) \geq 1$ with $\sigma(s) = \lfloor 1 + n \cdot \log_2 3 \rfloor$, $n \geq 2$, there is

$$z(n) = \frac{1}{2} \cdot \sum_{\tau(s)=1}^{\lfloor \frac{n}{2} \rfloor} A_{\tau(s)}(n).$$

*Example*: For $n = 8$ there is $z(8) = 85$, because

$$z(8) = \frac{1}{2} \cdot \sum_{\tau(s)=1}^{4} A_{\tau(s)}(8) = \frac{1}{2}\left(A_1(8) + A_2(8) + A_3(8) + A_4(8)\right)$$

$$= \frac{32 + 96 + 40 + 2}{2} = 85.$$

For $\tau(s) = 1$ we conjecture that for every $n \geq 2$, $z(n)$ is always a power of two.

**Conjecture 4**: Let $A_{\tau(s)}(n)$ be the number of residue classes $(mod\ 3 \cdot 2^{\sigma(s)})$ for $\tau(s) \geq 1$ with $\sigma(s) = \lfloor 1 + n \cdot \log_2 3 \rfloor$. Than for $\tau(s) = 1$ with $n \geq 2$ there is

$$A_1(n) = 2^m \quad \text{with} \quad m = \lfloor 1 + (n-1) \cdot \log_2 3 \rfloor - (n-1).$$

The sequence of the exponents $m$ is listed in OEIS A098294.

# 6 Limiting values and asymptotic densities

## 6.1 The subsequences $C^t(s)$ and $\tau(s) = 1$

**Theorem 5**: Let $A(s)$ be the number of $C^t(s)$ for all $s \equiv 3, 7\ (mod\ 12)$ until an upper limit $s = 2^G$, and let $A_\sigma^G(s)$ be the number of these $C^t(s)$ with $\tau(s) = 1$, there is

$$\lim_{G \to \infty} \frac{A(s)}{A_\sigma^G(s)} \approx 1,5.$$

*Proof*: In the set of the first $s = 2^G$ positive integers, there are exactly $\left\lfloor \frac{s+5}{6} \right\rfloor$ numbers of the residue classes $3, 7\ (mod\ 12)$. Therefore there is

$$\left\lfloor \frac{s+5}{6} \right\rfloor = \left\lfloor \frac{2^G+5}{6} \right\rfloor, \quad \text{and} \quad A(s) \approx \frac{2^{G-1}}{3}.$$

Let $A_\sigma(s)$ be the number of subsequences with finite stopping time $\sigma(s) = \lfloor 1 + n \cdot \log_2 3 \rfloor$ for a *specific* $n$ from the set of the $A(s)$ subsequences, there is

$$A_\sigma(s) = \frac{s \cdot A_{\tau(s)}(n)}{3 \cdot 2^{\sigma(s)}} = \frac{2^G \cdot A_{\tau(s)}(n)}{3 \cdot 2^{\sigma(s)}}.$$

According to Conjecture 4, for $\tau(s) = 1$, there is

$$A_\sigma(s) = \frac{2^G \cdot A_1(n)}{3 \cdot 2^{\sigma(s)}} = \frac{2^G \cdot 2^m}{3 \cdot 2^{\sigma(s)}}.$$

Because $\sigma(s) > m$ for every $n \geq 0$, there is

$$A_\sigma(s) = \frac{2^G}{3 \cdot 2^{\sigma(s)-m}}.$$

For the sum $A_\sigma^G(s)$ of all $A_\sigma(s)$ for $n = 2, \ldots, G$, there is

$$A_\sigma^G(s) = \sum_{n=2}^{G} A_\sigma(s) = \sum_{n=2}^{G} \frac{2^G}{3 \cdot 2^{\sigma(s)-m}}.$$

For the term $\frac{A(s)}{A_\sigma^G(s)}$, there is

$$\frac{A(s)}{A_\sigma^G(s)} = \frac{\frac{2^{G-1}}{3}}{\sum_{n=2}^{G} \frac{2^G}{3 \cdot 2^{\sigma(s)-m}}} = \frac{2^{G-1}}{\sum_{n=2}^{G} \frac{2^G}{3 \cdot 2^{\sigma(s)-m}}}.$$

With $\sigma(s) = \lfloor 1 + n \cdot \log_2 3 \rfloor$ and $m = \lfloor 1 + (n-1) \cdot \log_2 3 \rfloor - (n-1)$, there is

$$\frac{A(s)}{A_\sigma^G(s)} = \frac{2^{G-1}}{\sum_{n=2}^{G} \frac{2^G}{2^{\lfloor n \cdot \log_2 3 \rfloor - \lfloor (n-1) \cdot \log_2 3 \rfloor + n - 1}}}.$$

For every $n \geq 2$, there is

$$\lfloor n \cdot \log_2 3 \rfloor - \lfloor (n-1) \cdot \log_2 3 \rfloor + n - 1 = n + \beta_n,$$

with the binary sequence

$$\beta_n = 1, 0, 1, 0, 1, 1, 0, 1, 0, 1, 1, 0, 1, 0, 1, 0, 1, 1, 0, 1, 0, \ldots$$

Probably the binary sequence $\beta_n$ is equal to the *sturmian word* of slope of the square root of two, listed in OEIS A144611. For the calculation of the limiting value, however, this does not matter. There is

$$\frac{A(s)}{A_\sigma^G(s)} = \frac{2^{G-1}}{\sum_{n=2}^{G} \frac{2^G}{2^{n+\beta_n}}} = \frac{2^{G-1}}{\sum_{n=2}^{G} 2^{G-n-\beta_n}}$$

$$= \frac{2^{G-1}}{2^{G-3} + 2^{G-3} + 2^{G-5} + 2^{G-5} + 2^{G-7} + \cdots + 2^{2-\beta_{G-2}} + 2^{1-\beta_{G-1}} + 2^{-\beta_G}}.$$

If the value of $G$ tends to infinity, there is

$$\lim_{G \to \infty} \frac{2^{G-1}}{\sum_{n=2}^{G} 2^{G-n-\beta_n}} = 1,5121861\ldots.$$

□

The sequence of quotients converges quite early. Therefore, the limiting value applies even for small $s$.

*Example*: For $G = 11$, there is

$$\frac{A(s)}{A_\sigma^{11}(s)} = \frac{2^{11-1}}{\sum_{n=2}^{11} 2^{11-n-\beta_n}}$$

$$= \frac{2^{10}}{2^8 + 2^8 + 2^6 + 2^6 + 2^4 + 2^3 + 2^3 + 2^1 + 2^1 + 2^{-1}} = \frac{1024}{676,5} \approx 1,51367.$$

## 6.2 The subsequences $C^t(s)$ and $\tau(s) \geq 1$

**Theorem 6**:Let $A(s)$ be the number of $C^t(s)$ for all $s \equiv 3,7$ $(mod\ 12)$ until an upper limit $s = 2^G$, and let $A_\sigma^G(s)$ be the number of these $C^t(s)$ with $\tau(s) \geq 1$, there is

$$\lim_{G\to\infty} \frac{A(s)}{A_\sigma^G(s)} = 1.$$

*Proof*: In the set of the first $s = 2^G$ positive integers, there are exactly $\lfloor \frac{s+5}{6} \rfloor$ numbers of the residue classes $3,7$ $(mod\ 12)$. Therefore there is

$$\left\lfloor \frac{s+5}{6} \right\rfloor = \left\lfloor \frac{2^G+5}{6} \right\rfloor, \quad \text{and} \quad A(s) \approx \frac{2^{G-1}}{3}.$$

Let $A_\sigma(s)$ be the number of subsequences with finite stopping time $\sigma(s) = \lfloor 1+n\cdot\log_2 3 \rfloor$ for $\tau(s) \geq 1$ for a *specific* $n$ from the set of the $A(s)$ subsequences, there is, according to Conjecture 3

$$A_\sigma(s) = \frac{s \cdot \sum_{\tau(s)=1}^{\lfloor \frac{n}{2} \rfloor} A_{\tau(s)}(n)}{3 \cdot 2^{\sigma(s)}} = \frac{s \cdot 2 \cdot z(n)}{3 \cdot 2^{\sigma(s)}} = \frac{2^G \cdot 2 \cdot z(n)}{3 \cdot 2^{\sigma(s)}} = \frac{2^{G+1} \cdot z(n)}{3 \cdot 2^{\sigma(s)}}.$$

For the sum $A_\sigma^G(s)$ of all $A_\sigma(s)$ for $n = 2, \ldots, G$, there is

$$A_\sigma^G(s) = \sum_{n=2}^{G} A_\sigma(s) = \sum_{n=2}^{G} \frac{2^{G+1} \cdot z(n)}{3 \cdot 2^{\sigma(s)}}.$$

For the term $\frac{A(s)}{A_\sigma^G(s)}$ with $\sigma(s) = \lfloor 1 + n \cdot \log_2 3 \rfloor$, there is

$$\frac{A(s)}{A_\sigma^G(s)} = \frac{\frac{2^{G-1}}{3}}{\sum_{n=2}^{G} \frac{2^{G+1}\cdot z(n)}{3\cdot 2^{\sigma(s)}}} = \frac{2^{G-1}}{\sum_{n=2}^{G} \frac{2^{G+1}\cdot z(n)}{2^{\sigma(s)}}} = \frac{2^{G-1}}{\sum_{n=2}^{G} 2^{G+1-\sigma(s)} \cdot z(n)}$$

$$= \frac{2^{G-1}}{\sum_{n=2}^{G} 2^{G+1-\lfloor 1+n\cdot\log_2 3\rfloor}\cdot z(n)} = \frac{2^{G-1}}{\sum_{n=2}^{G} 2^{G-\lfloor n\cdot\log_2 3\rfloor}\cdot z(n)}$$

$$= \frac{2^{G-1}}{2^{G-3}z(2) + 2^{G-4}z(3) + \cdots + 2^{G-\lfloor (G-1)\cdot\log_2 3\rfloor}z(G-1) + 2^{G-\lfloor G\cdot\log_2 3\rfloor}z(G)}$$

with $z(n) = 1, 2, 3, 7, 12, 30, 85, 173, 476, 961, 2652, 8045, 17637, \ldots$ .

If the value of $G$ tends to infinity, there is

$$\lim_{G\to\infty} \frac{2^{G-1}}{\sum_{n=2}^{G} 2^{G-\lfloor n\cdot\log_2 3\rfloor}\cdot z(n)} = 1.$$

□

*Examples*: For $G = 100$, there is $\frac{A(s)}{A_\sigma^{100}(s)} \approx 1,00007$. For $G = 200$, there is $\frac{A(s)}{A_\sigma^{200}(s)} \approx 1,0000001$.

## 6.3 Interpretation of the limiting values

The limiting value $\lim_{G\to\infty} \frac{A(s)}{A_\sigma^G(s)} = 1$ of Theorem 6 shows that from a sufficient big $s$ almost all subsequences $C^t(s)$, and also almost all $s \equiv 3, 7$ ($mod$ 12), have finite stopping time. This is the same result as proved by Everett and Terras in 1979.

The limiting value $\lim_{G\to\infty} \frac{A(s)}{A_\sigma^G(s)} \approx 1,5$ of Theorem 5 shows that statistically two out of three subsequences of all $C^t(s)$ or two out of three immediately consecutive subsequences $C^t(s)$ have finite stopping time.

## 6.4 Consequences for infinite growth

Let a $C^t(s)$ with $\tau(s) = 1$ be named a *stopping-sequence*, and with $\tau(s) > 1$ a *growing-sequence*.

Let $C^\infty(s)$ be a Collatz sequence with an infinite growth. Then, according to Theorem 1 and 2, $C^\infty(s)$ must reach after a finite number of iterations a term of the residue class $[3]_4$. From there $C^\infty(s)$ consists only of finite subsequences $C^{t_A}(s)$. Also $C^\infty(s)$ must have an odd minimum of the residue class $[3]_4$. This minimum would be a starting number *without* finite stopping time. We can assume that $C^\infty(s)$ go through the $C^{t_A}(s)$ in a chaotic way, jumping back and forth, but with the tendency ever upward through the list, passing bigger and bigger numbers of the residue class $[3]_4$.

But despite this dynamic behaviour, $C^\infty(s)$ must follow a very regular pattern to stay in growth. For an constantly growth process, $C^\infty(s)$ must run through more growing-sequences as stopping-sequences on average. To disturb the growth process not to much, the stopping-sequences permitted to disturb this rhythm only short. Very long stopping-sequences should never be run through, certainly not in close succession. This would lead to an immediate collapse of the growth process.

The Chapters 4 and 5 show that the distribution of the $C^t(s)$ among each other, according to their length (number of terms), follows a strict pattern based on the Fibonacci sequence. This regularity of the statistical length distribution must also be reflected in the expansion of every $C^t(s)$ with longer stopping time. This means that $C^\infty(s)$ must run through $C^t(s)$ with bigger $t$ periodically. But such $C^t(s)$ usually leading to very small terms.

## 6.5 Growth and collapse of a $C(s)$ - two final examples

Two final examples are intended to illustrate the necessarily limited growth process of a Collatz sequence. Stopping-sequences are written in red.

*Example 1*: For $s = 27$ with $\sigma(s) = 59$ and $\tau(s) = 9$, we get

(27, 41, 62),
(31, 47, 71, 107, 161, 242, 121, 182),
(91, 137, 206),
(103, 155, 233, 350),
(175, 263, 395, 593, 890, 445, 668, 334),
(111, 167, 251, 377, 566),
(283, 425, 638),
(319, 479, 719, 1079, 1619, 2429, 3644, 1822),
(607, 911, 1367, 2051, 3077, 4616, 2308, 1154, 577, 866, 433, 650, 325, 488, 244, 122, 61, 92, 46),
(15, 23, 35, 53, 80, 40, 20, 10, 5, 8, 4, 2, 1).

Although $C(27)$ contains of only ten $C^t(s)$, we can already see the growing process and its collapse. The chaotic jumping back and forth between the $C^t(s)$ will be even clearer with the use of the list in Appendix 7.2.

Until the eighth $C^t(s)$, the statistically necessary and expected growth behaviour is seen. The Collatz sequence could grows to infinity in this way. But with the ninth $C^t(s)$, which is more than twice as long as the average and accordingly has a small last term, the collapse of the growth process begins. Here begins the great shrinkage of the terms, picked up by the next $C^t(s)$, again a stopping-sequence, and continued to the term "1". Of two successively stopping-sequences, one of which also has many terms, the growth process at this early phase can not recover and is stopped.

*Example 2*: $s = 2602714556700227743$ with $\sigma(s) = 1005$ and $\tau(s) = 165$.

This example deals with very big numbers. For clear display of the $C^t(s)$, we write for each term just the symbol $\circ$. The expansion of the $C^t(s)$ reads in columns from top left to bottom right.

Until the blue line, there is a steady growth process, here and there briefly interrupted by shorter stopping-sequences (red). Because of the size of the numbers, the $C^t(s)$ are on average longer than in the first example. But here, too, we see the statistically necessary growth behaviour. The growing-sequences occur more frequently, also in direct succession, and are on average longer than the stopping-sequences. But from the blue line this process is reversed. From here begins a steady shrinkage process. The stopping-sequences occur more frequently, also in direct succession, and are substantially longer than the growing-sequences.

# 7 Appendix

## 7.1 The first subsequences $C^h(s)$

9, 14, 7
21, 32, 16, 8, 4, 2, 1
33, 50, 25, 38, 19
45, 68, 34, 17, 26, 13, 20, 10, 5, 8, 4, 2, 1
57, 86, 43
69, 104, 52, 26, 13, 20, 10, 5, 8, 4, 2, 1
81, 122, 61, 92, 46, 23
93, 140, 70, 35
105, 158, 79
117, 176, 88, 44, 22, 11
129, 194, 97, 146, 73, 110, 55
141, 212, 106, 53, 80, 40, 20, 10, 5, 8, 4, 2, 1
153, 230, 115
165, 248, 124, 62, 31
177, 266, 133, 200, 100, 50, 25, 38, 19
189, 284, 142, 71
201, 302, 151
213, 320, 160, 80, 40, 20, 10, 5, 8, 4, 2, 1
225, 338, 169, 254, 127
237, 356, 178, 89, 134, 67
249, 374, 187
261, 392, 196, 98, 49, 74, 37, 56, 28, 14, 7
273, 410, 205, 308, 154, 77, 116, 58, 29, 44, 22, 11
285, 428, 214, 107
297, 446, 223
309, 464, 232, 116, 58, 29, 44, 22, 11
321, 482, 241, 362, 181, 272, 136, 68, 34, 17, 26, 13, 20, 10, 5, 8, 4, 2, 1
333, 500, 250, 125, 188, 94, 47
345, 518, 259
357, 536, 268, 134, 67
369, 554, 277, 416, 208, 104, 52, 26, 13, 20, 10, 5, 8, 4, 2, 1
381, 572, 286, 143
393, 590, 295
405, 608, 304, 152, 76, 38, 19
417, 626, 313, 470, 235
429, 644, 322, 161, 242, 121, 182, 91
441, 662, 331
453, 680, 340, 170, 85, 128, 64, 32, 16, 8, 4, 2, 1
465, 698, 349, 524, 262, 131
477, 716, 358, 179
489, 734, 367
501, 752, 376, 188, 94, 47
513, 770, 385, 578, 289, 434, 217, 326, 163
525, 788, 394, 197, 296, 148, 74, 37, 56, 28, 14, 7
537, 806, 403
549, 824, 412, 206, 103
561, 842, 421, 632, 316, 158, 79
573, 860, 430, 215
585, 878, 439
597, 896, 448, 224, 112, 56, 28, 14, 7
609, 914, 457, 686, 343
621, 932, 466, 233, 350, 175
633, 950, 475
645, 968, 484, 242, 121, 182, 91
657, 986, 493, 740, 370, 185, 278, 139
669, 1004, 502, 251
681, 1022, 511
693, 1040, 520, 260, 130, 65, 98, 49, 74, 37, 56, 28, 14, 7
705, 1058, 529, 794, 397, 596, 298, 149, 224, 112, 56, 28, 14, 7
717, 1076, 538, 269, 404, 202, 101, 152, 76, 38, 19
729, 1094, 547
741, 1112, 556, 278, 139
753, 1130, 565, 848, 424, 212, 106, 53, 80, 40, 20, 10, 5, 8, 4, 2, 1
765, 1148, 574, 287
777, 1166, 583
789, 1184, 592, 296, 148, 74, 37, 56, 28, 14, 7
801, 1202, 601, 902, 451
813, 1220, 610, 305, 458, 229, 344, 172, 86, 43
825, 1238, 619
837, 1256, 628, 314, 157, 236, 118, 59
849, 1274, 637, 956, 478, 239
861, 1292, 646, 323
873, 1310, 655
885, 1328, 664, 332, 166, 83
897, 1346, 673, 1010, 505, 758, 379
909, 1364, 682, 341, 512, 256, 128, 64, 32, 16, 8, 4, 2, 1
921, 1382, 691
933, 1400, 700, 350, 175
945, 1418, 709, 1064, 532, 266, 133, 200, 100, 50, 25, 38, 19
957, 1436, 718, 359
969, 1454, 727
981, 1472, 736, 368, 184, 92, 46, 23
993, 1490, 745, 1118, 559
1005, 1508, 754, 377, 566, 283

1017, 1526, 763
1029, 1544, 772, 386, 193, 290, 145, 218, 109, 164, 82, 41, 62, 31
1041, 1562, 781, 1172, 586, 293, 440, 220, 110, 55
1053, 1580, 790, 395
1065, 1598, 799
1077, 1616, 808, 404, 202, 101, 152, 76, 38, 19
1089, 1634, 817, 1226, 613, 920, 460, 230, 115
1101, 1652, 826, 413, 620, 310, 155
1113, 1670, 835
1125, 1688, 844, 422, 211
1137, 1706, 853, 1280, 640, 320, 160, 80, 40, 20, 10, 5, 8, 4, 2, 1
1149, 1724, 862, 431
1161, 1742, 871
1173, 1760, 880, 440, 220, 110, 55
1185, 1778, 889, 1334, 667
1197, 1796, 898, 449, 674, 337, 506, 253, 380, 190, 95
1209, 1814, 907
1221, 1832, 916, 458, 229, 344, 172, 86, 43
1233, 1850, 925, 1388, 694, 347
1245, 1868, 934, 467
1257, 1886, 943
1269, 1904, 952, 476, 238, 119
1281, 1922, 961, 1442, 721, 1082, 541, 812, 406, 203
1293, 1940, 970, 485, 728, 364, 182, 91
1305, 1958, 979
1317, 1976, 988, 494, 247
1329, 1994, 997, 1496, 748, 374, 187
1341, 2012, 1006, 503
1353, 2030, 1015
1365, 2048, 1024, 512, 256, 128, 64, 32, 16, 8, 4, 2, 1
1377, 2066, 1033, 1550, 775
1389, 2084, 1042, 521, 782, 391
1401, 2102, 1051
1413, 2120, 1060, 530, 265, 398, 199
1425, 2138, 1069, 1604, 802, 401, 602, 301, 452, 226, 113, 170, 85, 128, 64, 32, 16, 8, 4, 2, 1
1437, 2156, 1078, 539
1449, 2174, 1087
1461, 2192, 1096, 548, 274, 137, 206, 103
1473, 2210, 1105, 1658, 829, 1244, 622, 311
1485, 2228, 1114, 557, 836, 418, 209, 314, 157, 236, 118, 59
1497, 2246, 1123
1509, 2264, 1132, 566, 283
1521, 2282, 1141, 1712, 856, 428, 214, 107
1533, 2300, 1150, 575
1545, 2318, 1159
1557, 2336, 1168, 584, 292, 146, 73, 110, 55
1569, 2354, 1177, 1766, 883
1581, 2372, 1186, 593, 890, 445, 668, 334, 167
1593, 2390, 1195
1605, 2408, 1204, 602, 301, 452, 226, 113, 170, 85, 128, 64, 32, 16, 8, 4, 2, 1
1617, 2426, 1213, 1820, 910, 455
1629, 2444, 1222, 611
1641, 2462, 1231
1653, 2480, 1240, 620, 310, 155
1665, 2498, 1249, 1874, 937, 1406, 703
1677, 2516, 1258, 629, 944, 472, 236, 118, 59
1689, 2534, 1267
1701, 2552, 1276, 638, 319
1713, 2570, 1285, 1928, 964, 482, 241, 362, 181, 272, 136, 68, 34, 17, 26, 13, 20, 10, 5, 8 4, 2, 1
1725, 2588, 1294, 647
1737, 2606, 1303
1749, 2624, 1312, 656, 328, 164, 82, 41, 62, 31
1761, 2642, 1321, 1982, 991
1773, 2660, 1330, 665, 998, 499
1785, 2678, 1339
1797, 2696, 1348, 674, 337, 506, 253, 380, 190, 95
1809, 2714, 1357, 2036, 1018, 509, 764, 382, 19
1821, 2732, 1366, 683
1833, 2750, 1375
1845, 2768, 1384, 692, 346, 173, 260, 130, 65, 98, 49, 74, 37, 56, 28, 14, 7
1857, 2786, 1393, 2090, 1045, 1568, 784, 392, 196, 98, 49, 74, 37, 56, 28, 14, 7
1869, 2804, 1402, 701, 1052, 526, 263
1881, 2822, 1411
1893, 2840, 1420, 710, 355
1905, 2858, 1429, 2144, 1072, 536, 268, 134, 67
1917, 2876, 1438, 719
1929, 2894, 1447
1941, 2912, 1456, 728, 364, 182, 91
1953, 2930, 1465, 2198, 1099
1965, 2948, 1474, 737, 1106, 553, 830, 415
1977, 2966, 1483
1989, 2984, 1492, 746, 373, 560, 280, 140, 70, 35
2001, 3002, 1501, 2252, 1126, 563
2013, 3020, 1510, 755
2025, 3038, 1519
2037, 3056, 1528, 764, 382, 191
2049, 3074, 1537, 2306, 1153, 1730, 865, 1298, 649, 974, 487
2061, 3092, 1546, 773, 1160, 580, 290, 145, 218, 109, 164, 82, 41, 62, 31
2073, 3110, 1555

and so forth.

## 7.2 The first subsequences $C^t(s)$

3, 5, 8, 4, 2, 1
7, 11, 17, 26, 13, 20, 10, 5, 8, 4, 2, 1
15, 23, 35, 53, 80, 40, 20, 10, 5, 8, 4, 2, 1
19, 29, 44, 22
27, 41, 62
31, 47, 71, 107, 161, 242, 121, 182
39, 59, 89, 134
43, 65, 98, 49, 74, 37, 56, 28, 14
51, 77, 116, 58, 29, 44, 22
55, 83, 125, 188, 94
63, 95, 143, 215, 323, 485, 728, 364, 182
67, 101, 152, 76, 38
75, 113, 170, 85, 128, 64, 32, 16, 8, 4, 2, 1
79, 119, 179, 269, 404, 202, 101, 152, 76, 38
87, 131, 197, 296, 148, 74, 37, 56, 28, 14
91, 137, 206
99, 149, 224, 112, 56, 28, 14
103, 155, 233, 350
111, 167, 251, 377, 566
115, 173, 260, 130, 65, 98, 49, 74, 37, 56, 28, 14
123, 185, 278,
127, 191, 287, 431, 647, 971, 1457, 2186, 1093, 1640, 820, 410, 205, 308, 154, 77, 116, 58, 29, 44, 22
135, 203, 305, 458, 229, 344, 172, 86
139, 209, 314, 157, 236, 118
147, 221, 332, 166
151, 227, 341, 512, 256, 128, 64, 32, 16, 8, 4, 2, 1
159, 239, 359, 539, 809, 1214
163, 245, 368, 184, 92, 46
171, 257, 386, 193, 290, 145, 218, 109, 164, 82, 41, 62
175, 263, 395, 593, 890, 445, 668, 334
183, 275, 413, 620, 310
187, 281, 422
195, 293, 440, 220, 110
199, 299, 449, 674, 337, 506, 253, 380, 190
207, 311, 467, 701, 1052, 526
211, 317, 476, 238
219, 329, 494
223, 335, 503, 755, 1133, 1700, 850, 425, 638
231, 347, 521, 782
235, 353, 530, 265, 398
243, 365, 548, 274, 137, 206
247, 371, 557, 836, 418, 209, 314, 157, 236, 118
255, 383, 575, 863, 1295, 1943, 2915, 4373, 6560, 3280, 1640, 820, 410, 205, 308, 154, 77, 116, 58, 29, 44, 22
259, 389, 584, 292, 146, 73, 110
267, 401, 602, 301, 452, 226, 113, 170, 85, 128, 64, 32, 16, 8, 4, 2, 1
271, 407, 611, 917, 1376, 688, 344, 172, 86
279, 419, 629, 944, 472, 236, 118
283, 425, 638
291, 437, 656, 328, 164, 82, 41, 62
295, 443, 665, 998
303, 455, 683, 1025, 1538, 769, 1154, 577, 866, 433, 650, 325, 488, 244, 122, 61, 92, 46
307, 461, 692, 346, 173, 260, 130, 65, 98, 49, 74, 37, 56, 28, 14
315, 473, 710
319, 479, 719, 1079, 1619, 2429, 3644, 1822
327, 491, 737, 1106, 553, 830
331, 497, 746, 373, 560, 280, 140, 70
339, 509, 764, 382
343, 515, 773, 1160, 580, 290, 145, 218, 109, 164, 82, 41, 62
351, 527, 791, 1187, 1781, 2672, 1336, 668, 334
355, 533, 800, 400, 200, 100, 50, 25, 38
363, 545, 818, 409, 614
367, 551, 827, 1241, 1862
375, 563, 845, 1268, 634, 317, 476, 238
379, 569, 854
387, 581, 872, 436, 218, 109, 164, 82, 41, 62
391, 587, 881, 1322, 661, 992, 496, 248, 124, 62
399, 599, 899, 1349, 2024, 1012, 506, 253, 380, 190
403, 605, 908, 454
411, 617, 926
415, 623, 935, 1403, 2105, 3158
423, 635, 953, 1430
427, 641, 962, 481, 722, 361, 542
435, 653, 980, 490, 245, 368, 184, 92, 46
439, 659, 989, 1484, 742
447, 671, 1007, 1511, 2267, 3401, 5102
451, 677, 1016, 508, 254
459, 689, 1034, 517, 776, 388, 194, 97, 146, 73, 110
463, 695, 1043, 1565, 2348, 1174
471, 707, 1061, 1592, 796, 398
475, 713, 1070
483, 725, 1088, 544, 272, 136, 68, 34, 17, 26, 13, 20, 10, 5, 8, 4, 2, 1
487, 731, 1097, 1646
495, 743, 1115, 1673, 2510
499, 749, 1124, 562, 281, 422
507, 761, 1142
511, 767, 1151, 1727, 2591, 3887, 5831, 8747, 13121, 19682, 9841, 14762, 7381, 11072, 5536, 2768, 1384, 692, 346,
173, 260, 130, 65, 98, 49, 74, 37, 56, 28, 14
519, 779, 1169, 1754, 877, 1316, 658, 329, 494
523, 785, 1178, 589, 884, 442, 221, 332, 166

531, 797, 1196, 598
535, 803, 1205, 1808, 904, 452, 226, 113, 170, 85, 128, 64, 32, 16, 8, 4, 2, 1
543, 815, 1223, 1835, 2753, 4130, 2065, 3098, 1549, 2324, 1162, 581, 872, 436, 218, 109, 164, 82, 41, 62
547, 821, 1232, 616, 308, 154, 77, 116, 58, 29, 44, 22
555, 833, 1250, 625, 938, 469, 704, 352, 176, 88, 44, 22
559, 839, 1259, 1889, 2834, 1417, 2126
567, 851, 1277, 1916, 958
571, 857, 1286
579, 869, 1304, 652, 326
583, 875, 1313, 1970, 985, 1478
591, 887, 1331, 1997, 2996, 1498, 749, 1124, 562, 281, 422
595, 893, 1340, 670
603, 905, 1358
607, 911, 1367, 2051, 3077, 4616, 2308, 1154, 577, 866, 433, 650, 325, 488, 244, 122, 61, 92, 46
615, 923, 1385, 2078
619, 929, 1394, 697, 1046
627, 941, 1412, 706, 353, 530, 265, 398
631, 947, 1421, 2132, 1066, 533, 800, 400, 200, 100, 50, 25, 38
639, 959, 1439, 2159, 3239, 4859, 7289, 10934
643, 965, 1448, 724, 362, 181, 272, 136, 68, 34, 17, 26, 13, 20, 10, 5, 8, 4, 2, 1
651, 977, 1466, 733, 1100, 550
655, 983, 1475, 2213, 3320, 1660, 830
663, 995, 1493, 2240, 1120, 560, 280, 140, 70
667, 1001, 1502
675, 1013, 1520, 760, 380, 190
679, 1019, 1529, 2294
687, 1031, 1547, 2321, 3482, 1741, 2612, 1306, 653, 980, 490, 245, 368, 184, 92, 46
691, 1037, 1556, 778, 389, 584, 292, 146, 73, 110
699, 1049, 1574
703, 1055, 1583, 2375, 3563, 5345, 8018, 4009, 6014
711, 1067, 1601, 2402, 1201, 1802, 901, 1352, 676, 338, 169, 254
715, 1073, 1610, 805, 1208, 604, 302
723, 1085, 1628, 814
727, 1091, 1637, 2456, 1228, 614
735, 1103, 1655, 2483, 3725, 5588, 2794, 1397, 2096, 1048, 524, 262
739, 1109, 1664, 832, 416, 208, 104, 52, 26, 13, 20, 10, 5, 8, 4, 2, 1
747, 1121, 1682, 841, 1262
751, 1127, 1691, 2537, 3806
759, 1139, 1709, 2564, 1282, 641, 962, 481, 722, 361, 542
763, 1145, 1718
771, 1157, 1736, 868, 434, 217, 326
775, 1163, 1745, 2618, 1309, 1964, 982
783, 1175, 1763, 2645, 3968, 1984, 992, 496, 248, 124, 62
787, 1181, 1772, 886
795, 1193, 1790
799, 1199, 1799, 2699, 4049, 6074, 3037, 4556, 2278
807, 1211, 1817, 2726
811, 1217, 1826, 913, 1370, 685, 1028, 514, 257, 386, 193, 290, 145, 218, 109, 164, 82, 41, 62
819, 1229, 1844, 922, 461, 692, 346, 173, 260, 130, 65, 98, 49, 74, 37, 56, 28, 14
823, 1235, 1853, 2780, 1390
831, 1247, 1871, 2807, 4211, 6317, 9476, 4738, 2369, 3554, 1777, 2666, 1333, 2000, 1000, 500, 250, 125, 188, 94
835, 1253, 1880, 940, 470
843, 1265, 1898, 949, 1424, 712, 356, 178, 89, 134
847, 1271, 1907, 2861, 4292, 2146, 1073, 1610, 805, 1208, 604, 302
855, 1283, 1925, 2888, 1444, 722, 361, 542
859, 1289, 1934
867, 1301, 1952, 976, 488, 244, 122, 61, 92, 46
871, 1307, 1961, 2942
879, 1319, 1979, 2969, 4454
883, 1325, 1988, 994, 497, 746, 373, 560, 280, 140, 70
891, 1337, 2006
895, 1343, 2015, 3023, 4535, 6803, 10205, 15308, 7654
903, 1355, 2033, 3050, 1525, 2288, 1144, 572, 286
907, 1361, 2042, 1021, 1532, 766
915, 1373, 2060, 1030
919, 1379, 2069, 3104, 1552, 776, 388, 194, 97, 146, 73, 110
927, 1391, 2087, 3131, 4697, 7046
931, 1397, 2096, 1048, 524, 26
939, 1409, 2114, 1057, 1586, 793, 1190
943, 1415, 2123, 3185, 4778, 2389, 3584, 1792, 896, 448, 224, 112, 56, 28, 14
951, 1427, 2141, 3212, 1606
955, 1433, 2150
963, 1445, 2168, 1084, 542
967, 1451, 2177, 3266, 1633, 2450, 1225, 1838
975, 1463, 2195, 3293, 4940, 2470
979, 1469, 2204, 1102
987, 1481, 2222
991, 1487, 2231, 3347, 5021, 7532, 3766
999, 1499, 2249, 3374
1003, 1505, 2258, 1129, 1694
1011, 1517, 2276, 1138, 569, 854
1015, 1523, 2285, 3428, 1714, 857, 1286
1023, 1535, 2303, 3455, 5183, 7775, 11663, 17495, 26243, 39365, 59048, 29524, 14762, 7381, 11072, 5536, 2768, 1384,
692, 346, 173, 260, 130, 65, 98, 49, 74, 37, 56, 28, 14
1027, 1541, 2312, 1156, 578, 289, 434, 217, 326
1035, 1553, 2330, 1165, 1748, 874, 437, 656, 328, 164, 82, 41, 62
1039, 1559, 2339, 3509, 5264, 2632, 1316, 658, 329, 494
1047, 1571, 2357, 3536, 1768, 884, 442, 221, 332, 166

and so forth.

## 7.3 Residue classes for $C^h(s)$

$h = 2$
if $s \equiv$ 9 $(mod\ 48)$

$h = 3$
if $s \equiv$ 93 $(mod\ 96)$

$h = 4$
if $s \equiv$ 33, 165 $(mod\ 192)$

$h = 5$
if $s \equiv$ 81, 117, 237 $(mod\ 384)$

$h = 6$
if $s \equiv$ 129, 333, 405, 561, 645 $(mod\ 768)$

$h = 7$
if $s \equiv$ 429, 657, 837, 981, 1293, 1461, 1473, 1521 $(mod\ 1536)$

$h = 8$
if $s \equiv$ 177, 309, 513, 597, 1089, 1221, 1557, 1581, 1677, 1809, 1905, 2565, 3021 $(mod\ 3072)$

$h = 9$
if $s \equiv$ 813, 1041, 1077, 1281, 1749, 1797, 1989, 2241, 2253, 2445, 2481, 2673, 3393, 3765, 4677, 5133, 5361, 5397, 5805, 5973, 6033 $(mod\ 6144)$

$h = 105$
if $s \equiv$ 261, 717, 789, 1197, 2049, 2133, 2229, 2613, 2817, 3117, 3141, 3441, 3525, 3597, 3777, 3981, 4785, 5421, 5697, 6165, 7089, 7569, 7989, 8001, 8901, 9357, 9429, 9489, 9969, 10245, 10353, 10581, 10701, 11793 $(mod\ 12288)$

$h = 11$
if $s \equiv$ 273, 525, 1485, 2289, 2325, 2349, 4113, 4149, 5061, 5205, 5313, 5325, 6285, 6837, 6849, 6897, 7053, 7509, 7749, 7857, 8721, 8769, 9933, 11445, 11697, 12357, 12501, 12609, 12657, 13317, 13425, 15021, 15189, 15381, 16149, 16305, 16557, 16641, 17157, 17205, 17217, 17805, 17973, 18117, 18177, 18477, 18885, 20493, 21393, 21765, 22317, 22929, 23253, 23553, 24177 $(mod\ 24576)$

$h = 12$
if $s \equiv$ 945, 2757, 3345, 4557, 5301, 6405, 6417, 6513, 7629, 8193, 8277, 8433, 8877, 10125, 10161, 10257, 10497, 11469, 11841, 11973, 13893, 14913, 15501, 15573, 15621, 16113, 17937, 18033, 18753, 19149, 19221, 19569, 21045, 21333, 21549, 21573, 23493, 24597, 24621, 24717, 25029, 25329, 25941, 26001, 26421, 26433, 26637, 27153, 28365, 28461, 29073, 30789, 33813, 34317, 34497, 35505, 35637, 35649, 35925, 36141, 36237, 37557, 37569, 37773, 38577, 39825, 40965, 41073, 41301, 41685, 41985, 42417, 43221, 43533, 44037, 44205, 45237, 45333, 45357, 45825, 46449, 46533, 46869, 47157, 47277, 47877, 48321, 48693, 48897 $(mod\ 49152)$

$h = 135$
if $s \equiv$ 693, 705, 1029, 3093, 3285, 3333, 4269, 5517, 5553, 5685, 5745, 5889, 6357, 6933, 7173, 8373, 9105, 9261, 9585, 10005, 10413, 11265, 12033, 12813, 13041, 13233, 13713, 16401, 16437, 17601, 17613, 20037, 20481, 21525, 22209, 22545, 23733, 23757, 23949, 24897, 24945, 26181, 27309, 27477, 27789, 28593, 28677, 28785, 28929, 30225, 31041, 31173, 31245, 31437, 31857, 31917, 33537, 33861, 34605, 34869, 38709, 38721, 40749, 41925, 42069, 42165, 43149, 43377, 44721, 45585, 46797, 46989, 47025, 48429, 49221, 49365, 50061, 50865, 53013, 53589, 54021, 54069, 54081, 57909, 60117, 60165, 60357, 60981, 61581, 61713, 62805, 62925, 63729, 63765, 63789, 64017, 64197, 65229, 67653, 67845, 67857, 69069, 69873, 70209, 72369, 72501, 72513, 74865, 76353, 76689, 77553, 78549, 78849, 79557, 79917, 81933, 82197, 82221, 82773, 83205, 83397, 84369, 85185, 85617, 86037, 86061, 86469, 88077, 89361, 90513, 90573, 92853, 92865, 92913, 93069, 93525, 94293, 94893, 95757, 96513, 97365, 97713, 97857, 97989 $(mod\ 98304)$

$h = 14$
if $s \equiv$ 2061, 3789, 3825, 4209, 4497, 6213, 6597, 6669, 9477, 10929, 11061, 11073, 11349, 11889, 12741, 14349, 15249, 16785, 17409, 18645, 18801, 19185, 19797, 20145, 20661, 20757, 20781, 25521, 27921, 28437, 32769, 32853, 32949, 33333, 33681, 34581, 35781, 35841, 36045, 37137, 37809, 38229, 38469, 40077, 40149, 40197, 40653, 41013, 42513, 42609, 43077, 43329, 47925, 47937, 48309, 48333, 49905, 50289, 50577, 50757, 52365, 53037, 53169, 54801, 55617, 56433, 57357, 57621, 57645, 59793, 60609, 62349, 65325, 65997, 67725, 69717, 70161, 70317, 71733, 71937, 72453, 73281, 76485, 77013, 77061, 78165, 79029, 79413, 79473, 81357, 82161, 82989, 83853, 83889, 84693, 84741, 85557, 86769, 88641, 88773, 91341, 93645, 93765, 94449, 95937, 98325, 98349, 98445, 98625, 98757, 99669, 100881, 100929, 101205, 102405, 104133, 105645, 107265, 107349, 108045, 108333, 108597, 110613, 110637, 111285, 111297, 115413, 115797, 115893, 116145, 117765, 120177, 120261, 120333, 120597, 120717, 120753, 121005, 121941, 122049, 122625, 123093, 123573, 123585, 125973, 127437, 127749, 128397, 128433, 129237, 130053, 131757, 132465, 133137, 133293, 133377, 134721, 134853, 134913, 135309, 135693, 136653, 137457, 137745, 140481, 142029, 142101, 142449, 143361, 143937, 144429, 144453, 145425, 146097, 147825, 149301, 149313, 150189, 150417, 151809, 152277, 152577, 153645, 154317, 156693, 156741, 157125, 157377, 159021, 159117, 161457, 161589, 161601, 163089, 163845, 163953, 164181, 164805, 166413, 166581, 166593, 166797, 167085, 168705, 169677, 171309, 171441, 171573, 171717, 172101, 172941, 173061, 173745, 176301, 176949, 176961, 177921, 178437, 178449, 178545, 178965, 185745, 186645, 186669, 188145, 188433, 190725, 190737, 193077, 193365, 193557, 195525, 195981 $(mod\ 196608)$

$h = 15$
if $s \equiv$ 8469, 8493, 9237, 9285, 10641, 12429, 13197, 13773, 14133, 14145, 17349, 18693, 18705, 18957, 22221, 23853, 24129, 25485, 27333, 27861, 27909, 28401, 32205, 33837, 36465, 37317, 37617, 40209, 41361, 43269, 43281, 44613, 46785, 48069, 49473, 52053, 52941, 52977, 53253, 56085, 56493, 57525, 58113, 59181, 61041, 62385, 62865, 65553, 65589, 65937, 66645, 66741, 68949, 70677, 71025, 71445, 71565, 71601, 75333, 76941, 77073, 78597, 80193, 80325, 80397, 81009, 82005, 82101, 82833, 83733, 83985, 84993, 86157, 86961, 88305, 89901, 90165, 91761, 94737, 94785, 95301, 96945, 99729, 99909, 100149, 100161, 101265, 101517, 103425, 104493, 104769, 109269, 109317, 109509, 109869, 110133, 110865, 111957, 114381, 114477, 115029, 117429, 117441, 119025, 119313, 120885, 122289, 123909, 124017, 125505, 127149, 127317, 128709, 128769, 129285, 129297, 129813, 133521, 133845, 133893, 135189, 135213, 138993, 143445, 143601, 144213, 144405, 144909, 147009, 148149, 148161, 150033, 150081, 151953, 152973, 153009, 153141, 153201, 153285, 153813, 154053, 154125, 154629, 156501, 156717, 157869, 158805, 159489, 159765, 159789, 164565, 167493, 167601, 168117, 169413, 169485, 169665, 171093, 172353, 172401, 172725, 172737, 172977, 176133, 177549, 177585, 178893, 179205, 179373, 180993, 182061, 182325, 182445, 183237, 183873, 184005, 184065, 185925, 187533, 188109, 189633, 190785, 192513, 193581, 196677, 196821, 196977, 197517, 197745, 198321, 199341, 200493, 200961, 201477, 201525, 201537, 203469, 204813, 206529, 208065, 208269, 211185, 211221, 211245, 211473, 212997, 213105, 213453, 216237, 217173, 217665, 217773, 217857, 219393, 219825, 219909, 221253, 222897, 224145, 226005, 226101, 226113, 226305, 226869, 226929, 227373, 227697, 229617, 230661, 230853, 235533, 235797, 235821, 236097,

238797, 240309, 240321, 240369, 240525, 240981, 242229, 245169, 245445, 245781, 245805, 246213, 246789, 250029, 251649, 251973, 254865, 255237, 255501, 255765, 256053, 256689, 256821, 256833, 257025, 258501, 258741, 258753, 260109, 261009, 263169, 263601, 264945, 265221, 266517, 266541, 268461, 269493, 269517, 269709, 270081, 270549, 274353, 274545, 275985, 284109, 285201, 285909, 285957, 288909, 289137, 289485, 289905, 290481, 294069, 294093, 295665, 298773, 298929, 299349, 299733, 300561, 302193, 303669, 304581, 308913, 309957, 310545, 313413, 313485, 314253, 314829, 318261, 318273, 319029, 319173, 324789, 327693, 327957, 327981, 328533, 329613, 329649, 330945, 331317, 333201, 334533, 335889, 336333, 339405, 340653, 342273, 343125, 343437, 344205, 345429, 346125, 347157, 347349, 347397, 348273, 353649, 356877, 357105, 361665, 361677, 362709, 362865, 364545, 366609, 367809, 371373, 371541, 371733, 372501, 372993, 373197, 374997, 376833, 377397, 377517, 377925, 378225, 379137, 380109, 381201, 381453, 382293, 382773, 382785, 385989, 386133, 386229, 386577, 387141, 387861, 390861, 391053, 391089, 391185, 391989, 392001, 392493 $(mod\ 393216)$

$h = 16$
if $s \equiv$ 1845, 1857, 2961, 5121, 5649, 6189, 8757, 11565, 16725, 17841, 18501, 18573, 19125, 19137, 24321, 24789, 25605, 27669, 28845, 30981, 30993, 34161, 34701, 34737, 35073, 37389, 38349, 40689, 43725, 43797, 44145, 44493, 45909, 46101, 47121, 47361, 50517, 53649, 53973, 54669, 54705, 55509, 55749, 55821, 61185, 63153, 63285, 63297, 66501, 66765, 69813, 72897, 73005, 73269, 73413, 74097, 74637, 76629, 76821, 78285, 80085, 80589, 82605, 83313, 84933, 86541, 89229, 90129, 92421, 92433, 92481, 92949, 96141, 96177, 96273, 100017, 102093, 102129, 102189, 102513, 103221, 103233, 106509, 109761, 112077, 112437, 112449, 112917, 112941, 115089, 115149, 115653, 119469, 121605, 122157, 131073, 131157, 131253, 131313, 131637, 132357, 132885, 134349, 135441, 136533, 137793, 140817, 140913, 141381, 141573, 141585, 142065, 142677, 147477, 147501, 150669, 151245, 151281, 153921, 154389, 157197, 158097, 160437, 160449, 160653, 163629, 163857, 164241, 166917, 167253, 170037, 170157, 171189, 171213, 172245, 175377, 177681, 179661, 180309, 180405, 181137, 182997, 183045, 183297, 190065, 190605, 191601, 192753, 196929, 198033, 199185, 199233, 199509, 199821, 201045, 201429, 202437, 203073, 205569, 206637, 208917, 208941, 209169, 212241, 212685, 212781, 216525, 218481, 218565, 218637, 218901, 219021, 219057, 219189, 220245, 220725, 220869, 221877, 221889, 222321, 225621, 226053, 228357, 230229, 231441, 231597, 231825, 232149, 232197, 234897, 235761, 241749, 241905, 242241, 244401, 247605, 247617, 247821, 248337, 248385, 248721, 248853, 249969, 250113, 250881, 251445, 251505, 251589, 251949, 254805, 255345, 257325, 257421, 258069, 258093, 258573, 264885, 264897, 267009, 267717, 267789, 268305, 270405, 270657, 271029, 271041, 271365, 274605, 274689, 276741, 276753, 276849, 277269, 277509, 278529, 279093, 279297, 280365, 280629, 280749, 282177, 284469, 284481, 286449, 287685, 287937, 288837, 290817, 291885, 294189, 297645, 299409, 299781, 304833, 305841, 305973, 305985, 306261, 306573, 307653, 308109, 309261, 309489, 309777, 310161, 311301, 312321, 313605, 313617, 314541, 315573, 315669, 315693, 315969, 318129, 319041, 319557, 322245, 322449, 323313, 324609, 325677, 326001, 328749, 329157, 334989, 335061, 335109, 335565, 336273, 338241, 338613, 338625, 338829, 339525, 341697, 344817, 345093, 345201, 346965, 347949, 348081, 348165, 348333, 351405, 352269, 352437, 355125, 355137, 355521, 358413, 360909, 364629, 364821, 364845, 365229, 366357, 366477, 366513, 367365, 368013, 371853, 373941, 374325, 374385, 375105, 375921, 377073, 383553, 383685, 384213, 384261, 384813, 387441, 390213, 393237, 393261, 393357, 393669, 393969, 394581, 397077, 397233, 400497, 402957, 403509, 404181, 404229, 405045, 406197, 406209, 412677, 413937, 415917, 416565, 416577, 416961, 417201, 420417, 422349, 423093, 423621, 423681, 424149, 424725, 425997, 426261, 426285, 426669, 429249, 429621, 430101, 430125, 432657, 432837, 434637, 438957, 439821, 442509, 443061, 443073, 445653, 445701, 449541, 451605, 451653, 452037, 452781, 453717, 455409, 458001, 461325, 461709, 462513, 464589, 467889, 477441, 480657, 480837, 483021, 484437, 484533, 490437, 490893, 491589, 492429, 492657, 493581, 500757, 500805, 503409, 503949, 510165, 510321, 510477, 511317, 512085, 513741, 514305, 517005, 519441, 519957, 521781, 521841, 525201, 527361, 529329, 530445, 531729, 532533, 533709, 539445, 539457, 539589, 541125, 542097, 542277, 549141, 549165, 549777, 550677, 550965, 551937, 552561, 554385, 558513, 561681, 562197, 564801, 564993, 568005, 568533, 568581, 569265, 569685, 571845, 571917, 574509, 575253, 578289, 578481, 579021, 580113, 581685, 584397, 585285, 587457, 591429, 593925, 597165, 598869, 599493, 601029, 603477, 603825, 604869, 606933, 607317, 607413, 609165, 610833, 619917, 619953, 626241, 626373, 626433, 626829, 630801, 632001, 634881, 635949, 635973, 638037, 638349, 638529, 639345, 641709, 645837, 648237, 648897, 655365, 655473, 655701, 656085, 656577, 657621, 657777, 658605, 659013, 659457, 661185, 662613, 662961, 665265, 666285, 667413, 667653, 668469, 668481, 669069, 669105, 669441, 670893, 672837, 675021, 675525, 675585, 676113, 677205, 678165, 678189, 681489, 684597, 684885, 685077, 685773, 686901, 686913, 688341, 688497, 692481, 694341, 694725, 694797, 694989, 696597, 696621, 697605, 698769, 704625, 707313, 708273, 709377, 713649, 714417, 715989, 716037, 717525, 717621, 717633, 720333, 723909, 724593, 725445, 725745, 726597, 727317, 727341, 733749, 736437, 736461, 736689, 736965, 737601, 742929, 743169, 743493, 744561, 746241, 746757, 747309, 748209, 750021, 750513, 752529, 753717, 754689, 754773, 754869, 755853, 756465, 759153, 760065, 763461, 766065, 766725, 771981, 772017, 772113, 773685, 774285, 776433, 779469, 781773, 782001, 782865, 782913, 785073, 785589, 785613 $(mod\ 786432)$

and so forth.

## 7.4 Residue classes for $C^t(s)$

$t = 2$
if $s \equiv$ 27, 91 $(mod\ 96)$

$t = 3$
if $s \equiv$ 19, 39, 147, 103 $(mod\ 192)$

$t = 4$
if $s \equiv$ 55, 67, 111, 183, 195, 235, 363, 367 $(mod\ 384)$

$t = 5$
if $s \equiv$ 139, 159, 163, 207, 243, 327, 415, 463, 471, 499, 583, 651, 675, 727 $(mod\ 768)$

$t = 6$
if $s \equiv$ 51, 99, 259, 279, 427, 447, 559, 655, 715, 771, 775, 939, 991, 1015, 1071, 1075, 1123, 1167, 1227, 1287, 1303, 1471, 1503, 1527 $(mod\ 1536)$

$t = 7$
if $s \equiv$ 31, 135, 175, 291, 319, 331, 375, 627, 639, 855, 967, 1119, 1159, 1203, 1315, 1323, 1359, 1399, 1431, 1551, 1651, 1663, 1803, 1879, 1923, 2019, 2079, 2143, 2223, 2227, 2347, 2367, 2379, 2383, 2455, 2575, 2827, 2947, 3015, 3043 $(mod\ 3072)$

$t = 8$
if $s \equiv$ 43, 63, 199, 223, 271, 351, 355, 435, 519, 523, 663, 703, 799, 895, 903, 1027, 1327, 1455, 1483, 1615, 1707, 1783, 1791, 1935, 2091, 2167, 2179, 2247, 2271, 2319, 2403, 2419, 2571, 2635, 2647, 2751, 2847, 2943, 3075, 3375, 3379, 3531, 3607, 3619, 3663, 3811, 3831, 4159, 4215, 4227, 4447, 4467, 4531, 4615, 4683, 4695, 4759, 4999, 5427,

5551, 5655, 5667, 5803, 5859, 5887, 6031 $(mod\ 6144)$

$t = 9$
if $s \equiv$ 79, 87, 247, 387, 391, 399, 691, 843, 867, 1039, 1047, 1251, 1291, 1407, 1599, 1711, 1807, 2059, 2083, 2335, 2503, 2559, 2563, 2655, 2691, 2859, 2863, 2911, 2967, 2995, 3019, 3079, 3147, 3199, 3327, 3415, 3615, 3627, 3807, 3919, 4183, 4483, 4495, 4939, 4963, 4983, 5143, 5235, 5271, 5347, 5503, 5695, 5751, 6003, 6063, 6315, 6627, 6655, 6751, 6787, 6855, 6955, 6963, 6975, 7063, 7243, 7359, 7423, 7711, 7723, 7815, 7903, 7971, 8271, 8439, 8583, 8883, 9079, 9231, 9331, 9367, 9483, 9847, 9903, 9999, 10099, 10159, 10251, 10275, 10411, 10527, 10695, 10723, 10755, 10951, 11055, 11059, 11071, 11103, 11187, 11211, 11271, 11391, 11455, 11607, 11911, 12067, 12111 $(mod\ 12288)$

$t = 10$
if $s \equiv$ 459, 591, 759, 783, 883, 1311, 1479, 1687, 1839, 1911, 1995, 2175, 2479, 2583, 2595, 2895, 2991, 3423, 3507, 3555, 3591, 3595, 3871, 3915, 4099, 4119, 4171, 4287, 4323, 4399, 4639, 4671, 4743, 4783, 4863, 5007, 5131, 5343, 5635, 5719, 5727, 6039, 6159, 6259, 6295, 6399, 6883, 7167, 7203, 7231, 7255, 7339, 7539, 7555, 7755, 7879, 7999, 8115, 9075, 9259, 9295, 9463, 9571, 9879, 10111, 10507, 10671, 10795, 10975, 11383, 11395, 11787, 12063, 12127, 12291, 12295, 12363, 12591, 12595, 12631, 12831, 12919, 12975, 13323, 13711, 13827, 13911, 14131, 14179, 14215, 14451, 14487, 14719, 15019, 15075, 15139, 15423, 15447, 15531, 15559, 15631, 15747, 15751, 15871, 16003, 16051, 16063, 16071, 16171, 16191, 16843, 16975, 17143, 17167, 17451, 17487, 17655, 17695, 17763, 17863, 18223, 18295, 18303, 18379, 18559, 18699, 18967, 18979, 18987, 19167, 19279, 19375, 19575, 19587, 19807, 19891, 19939, 19975, 20299, 20319, 20487, 20503, 20671, 20707, 20787, 20823, 21055, 21111, 21127, 21247, 21391, 21727, 21903, 22111, 22323, 22371, 22407, 22423, 22543, 22783, 22911, 23211, 23331, 23551, 23587, 23751, 23823, 23923, 23943, 24063, 24139, 24195, 24243, 24255, 24363, 24499 $(mod\ 24576)$

$t = 11$
if $s \equiv$ 115, 171, 547, 555, 711, 735, 847, 919, 1059, 1155, 1195, 1279, 1395, 1543, 1735, 1855, 2127, 2623, 2739, 2935, 2959, 3151, 3339, 3679, 3955, 3991, 4015, 4351, 4363, 4551, 4579, 4779, 4911, 4959, 5119, 5239, 5247, 5319, 5643, 5767, 5811, 5967, 5983, 6447, 6451, 6687, 6735, 6987, 7011, 7027, 7183, 7191, 7455, 7743, 7831, 7855, 8071, 8203, 8227, 8319, 8343, 8575, 8623, 8703, 8799, 8835, 8931, 9003, 9279, 9487, 9567, 9651, 9727, 10027, 10059, 10239, 10263, 10467, 10627, 10699, 11275, 11307, 11511, 13503, 13699, 13795, 13855, 13867, 13899, 14343, 14679, 15475, 15511, 15715, 15759, 16227, 16255, 16555, 16899, 16939, 17095, 17119, 17331, 17443, 17539, 17779, 18051, 18511, 18519, 19119, 19123, 19723, 19971, 20343, 20935, 21015, 21163, 21295, 21343, 21423, 21579, 21591, 21631, 21703, 22027, 22195, 22239, 22351, 22719, 22831, 23071, 23119, 23175, 23295, 23371, 23395, 23575, 23775, 23839, 24127, 24183, 24591, 24703, 24727, 25087, 25183, 25219, 25315, 25387, 25395, 25407, 25663, 25951, 26035, 26403, 26443, 26547, 26623, 26647, 26851, 26871, 27015, 27255, 27691, 27895, 28431, 28467, 28959, 29475, 29643, 29703, 29887, 30039, 30087, 30283, 30399, 30495, 30723, 30727, 30795, 31063, 31119, 31479, 31503, 31971, 32127, 32143, 32199, 32343, 32559, 32611, 32715, 32883, 33283, 33315, 33615, 33687, 33715, 33963, 34047, 34311, 34435, 34503, 34623, 34903, 35391, 35503, 35703, 35727, 35919, 36355, 36447, 36723, 36727, 36759, 36783, 37119, 37131, 37347, 37399, 37807, 37887, 37963, 37975, 38007, 38535, 38623, 38751, 39103, 39219, 39559, 39679, 39795, 39951, 40159, 40567, 40599, 40623, 40839, 40971, 40975, 40995, 41343, 41391, 41779, 41791, 42255, 42495, 42787, 42795, 42931, 43255, 43395, 43399, 43467, 43639, 44043, 44815, 44851, 45343, 45859, 46027, 46087, 46423, 46467, 46471, 46563, 46623, 46635, 46783, 46879, 47107, 47179, 47503, 47863, 47887, 48243, 48279, 48355, 48483, 48511, 48583, 48727, 48943, 49023, 49099 $(mod\ 49152)$

$t = 12$
if $s \equiv$ 343, 631, 1035, 1099, 1111, 1423, 1579, 1843, 1891, 2391, 3087, 3459, 3463, 3471, 3903, 4527, 4927, 5199, 5475, 5491, 5763, 6007, 6015, 6411, 6679, 6691, 6699, 6775, 6835, 6991, 7087, 7179, 7987, 8031, 8055, 8383, 8727, 8839, 9103, 9135, 9343, 9559, 9603, 9607, 9823, 10063, 10495, 10543, 10639, 10783, 10887, 10999, 11287, 11487, 11535, 11619, 11719, 11775, 11955, 12075, 12079, 12159, 12747, 12835, 12843, 12879, 12895, 13023, 13027, 13171, 13375, 13599, 13831, 13975, 14115, 14259, 14335, 14415, 14767, 14911, 15223, 15607, 15627, 15711, 16143, 16159, 16303, 16387, 16407, 16459, 16611, 16639, 16671, 16867, 17067, 17247, 17355, 17407, 17607, 18007, 18055, 18099, 18435, 18439, 18547, 19299, 19315, 19627, 20043, 20055, 20119, 20167, 20323, 20515, 20595, 20995, 21291, 21399, 21675, 21759, 22215, 22335, 22551, 22615, 23263, 23671, 23683, 24075, 24879, 24883, 25119, 25719, 26115, 26143, 26187, 26335, 26503, 26815, 26931, 27007, 27307, 27363, 27391, 27711, 27735, 27763, 27799, 27847, 28279, 28551, 28671, 28683, 28687, 28843, 29383, 29491, 29739, 29827, 30847, 31107, 31111, 31407, 32179, 32227, 32259, 32587, 33111, 33399, 33867, 33879, 34191, 34347, 34591, 34611, 34659, 34891, 35407, 35875, 36067, 36127, 36211, 36231, 36991, 37375, 37507, 37695, 38239, 38259, 38323, 38731, 38775, 39139, 39447, 39459, 39543, 39603, 39759, 39855, 40015, 40755, 40819, 41151, 41227, 41607, 41647, 41871, 42111, 42175, 42327, 42375, 42591, 42831, 42847, 43159, 43263, 43311, 43407, 43551, 43767, 44055, 44487, 44719, 44847, 45439, 45603, 45663, 45795, 45939, 46003, 46143, 46327, 46351, 46591, 46599, 46743, 46891, 47103, 47535, 47563, 47679, 47991, 48375, 48927, 49071, 49155, 49159, 49227, 49407, 49635, 50175, 50659, 50775, 50823, 51207, 51315, 52003, 52083, 52395, 52867, 52887, 52927, 52935, 53091, 53283, 53763, 54007, 54031, 54307, 54727, 55087, 55243, 55383, 55543, 56031, 56439, 56451, 56839, 57651, 57799, 57919, 58111, 58147, 58159, 58375, 58591, 58891, 58911, 59071, 59103, 59271, 59287, 59407, 59583, 59775, 60075, 60159, 60175, 60235, 60415, 60531, 60567, 60615, 60799, 60991, 61047, 61363, 61387, 61455, 61591, 61611, 62151, 62259, 62287, 62595, 63615, 63879, 64399, 64555, 64947, 64995, 65119, 65355, 65431, 66571, 67359, 67659, 67927, 68175, 68623, 68643, 68835, 68895, 68979, 68995, 69007, 69439, 69759, 70063, 70143, 70275, 70735, 71007, 71011, 71091, 71299, 71499, 71551, 71907, 71947, 72235, 72715, 72783, 73567, 73587, 73591, 73995, 74263, 74415, 74671, 74943, 75139, 75615, 75927, 76423, 77023, 77071, 77155, 77311, 77487, 77491, 77611, 77695, 78207, 78283, 78379, 78415, 78559, 78771, 79095, 79119, 79135, 79359, 79651, 79659, 79795, 79951, 80331, 81163, 81247, 81679, 81927, 81943, 82147, 82207, 82603, 82783, 82891, 83143, 83427, 83635, 83971, 84771, 84835, 85579, 85591, 85635, 85695, 86131, 86775, 86799, 86827, 86935, 87075, 87211, 87295, 87495, 87751, 87855, 87871, 88011, 88087, 88311, 89607, 89611, 90415, 90567, 90655, 90687, 90879, 90915, 90927, 91143, 91255, 91359, 91651, 91659, 91723, 91839, 92055, 92175, 92467, 92899, 92943, 93003, 93183, 93247, 93271, 93567, 93759, 94087, 94131, 94155, 94207, 94219, 94359, 95055, 95275, 96643, 96943, 97167, 97323, 97795, 97887, 98199 $(mod\ 98304)$

$t = 13$
if $s \equiv$ 1215, 1411, 1539, 1759, 2071, 2199, 2239, 2295, 2355, 2815, 3115, 3159, 3703, 3975, 4555, 4651, 4831, 5367, 5919, 6219, 6271, 6927, 7435, 8011, 8823, 8875, 9039, 9055, 9415, 10927, 11107, 11491, 11779, 11907, 12555, 12823, 13099, 13387, 13399, 13683, 14175, 14359, 14983, 15027, 15183, 15883, 16047, 16651, 16687, 16767, 16927, 17071, 17215, 17415, 17535, 17679, 18211, 18219, 18271, 18355, 18687, 19063, 19171, 19215, 21087, 21219, 21451, 21547, 21567, 21775, 22315, 22531, 23439, 23907, 24067, 24151, 24159, 24447, 24691, 25311, 25495, 25635, 25675, 25687, 25771, 25855, 26083, 26119, 26311, 26467, 26631, 27199, 27427, 27663, 27915, 28515, 29103, 29355, 29503, 29535, 29695, 29815, 29895, 30343, 30387, 30667, 30987, 31255, 31351, 31455, 32263, 32319, 32607, 32779, 32803, 32919, 32959, 33223, 33343, 33415, 33579, 33583, 34135, 34711, 35203, 35215, 35499, 35575, 36039, 36111, 36195, 36295, 36351, 36651, 36655, 36735, 36879, 37683, 38431, 38443, 38475, 38991, 39255, 39303, 39799, 40051, 40087, 40335, 40371, 40879, 40959, 40963, 41035, 41131, 41671, 41995, 42583, 42627, 42675, 42783, 43083, 43123, 43351, 44203, 44419, 44431, 44743, 44919, 45091, 45999, 46975, 47659, 47695, 48151, 48207, 48247, 48351, 48415, 49011, 49015, 49279, 49759, 49795, 49891, 50095, 50239, 50611, 50719, 50763, 52287, 52375, 52471, 52911, 53007, 53043, 53247, 53535, 53631, 53839, 54463, 54559, 54615, 54663, 54783, 55219, 55695, 55755, 55983, 56055, 56775, 57103, 57135, 57187, 57367, 57571, 57687, 57859, 57891, 58767, 59199, 59479, 60279, 60451, 61119, 61359, 61567, 61695, 61923, 61951, 62083, 62199, 62223, 62499, 62815, 62899, 62919, 63279, 63351, 63715, 63795, 64255, 64371, 64431, 65143, 65175, 65331, 65415, 66751, 66951, 67075, 67083, 67735, 67831, 67891, 68427, 68607, 68695, 69183, 69511, 69783,

70179, 70903, 71175, 71455, 71755, 72255, 72463, 72747, 73503, 73983, 74211, 74359, 74575, 74751, 74883, 75123, 75399, 76467, 76659, 77443, 77463, 78091, 78975, 79203, 79219, 79491, 79695, 79711, 80175, 80415, 80563, 80719, 81583, 82227, 82303, 82431, 82455, 82951, 83071, 83215, 83295, 83755, 83787, 83847, 83967, 84195, 84223, 84351, 84615, 84751, 85107, 85215, 85683, 86187, 86623, 86727, 86755, 87103, 87171, 87843, 88071, 88975, 89439, 89443, 89571, 89695, 89983, 90399, 90847, 91059, 91083, 91171, 92163, 92167, 92751, 93199, 93451, 93471, 93555, 93771, 93783, 94051, 94323, 94639, 94891, 95071, 95127, 95403, 95431, 95487, 95923, 95943, 95967, 96063, 96075, 96447, 96523, 96991, 97855, 98143, 98319, 98455, 99115, 99123, 99447, 100743, 101035, 101575, 101647, 101731, 101887, 102187, 102271, 102411, 102415, 103203, 103219, 103347, 103431, 104011, 104127, 104527, 104791, 104835, 104839, 105699, 105855, 105871, 105907, 106443, 106495, 106503, 108075, 108163, 108211, 108319, 108339, 108619, 109455, 109647, 109959, 110175, 110451, 110455, 110487, 111535, 112887, 113187, 113679, 113743, 113887, 114547, 115119, 115455, 115491, 115599, 115935, 116223, 116299, 116319, 116415, 116559, 116751, 117039, 117195, 117279, 117771, 117783, 117823, 118143, 118447, 118543, 118579, 118707, 118731, 118783, 119071, 119167, 119631, 119667, 120151, 120195, 120199, 120291, 120319, 120471, 120831, 121231, 121263, 121291, 121519, 121591, 122103, 122311, 122671, 122775, 123223, 123427, 123435, 124303, 124735, 125007, 125815, 126655, 126783, 126895, 127231, 127431, 127459, 127491, 127735, 127759, 127791, 128035, 128079, 128455, 128523, 128815, 128887, 129111, 129331, 129867, 129891, 129907, 129967, 130059, 130179, 130711, 130867, 130951, 132483, 132487, 132619, 132831, 133143, 133311, 133887, 133963, 134143, 134187, 134719, 134775, 135319, 135627, 135715, 135723, 135903, 136711, 137343, 137791, 138283, 138507, 139039, 139083, 139519, 139747, 139947, 140127, 140287, 140419, 140487, 140659, 140935, 141999, 142003, 142179, 142195, 142563, 142851, 142999, 143895, 144171, 144459, 144471, 144511, 144739, 145027, 145231, 145431, 145711, 145951, 146055, 146955, 147723, 147759, 147763, 147967, 147991, 147999, 148143, 148287, 148831, 149283, 149323, 149343, 149383, 149427, 149503, 149731, 149887, 150135, 150151, 150243, 150643, 150751, 151219, 151723, 152263, 152523, 152619, 152707, 152847, 153379, 153387, 153603, 153607, 154975, 155107, 155139, 155223, 155763, 155935, 156567, 156595, 156619, 156747, 156759, 156843, 156927, 157155, 157191, 157383, 157539, 157699, 158271, 158287, 158499, 159007, 159091, 159307, 159319, 159859, 160575, 160663, 160767, 160887, 160939, 161023, 161415, 161479, 161503, 161599, 161611, 161739, 161983, 162327, 162423, 163335, 163851, 163855, 163875, 164031, 164295, 164415, 164487, 164655, 164659, 164983, 165207, 165783, 166275, 166279, 166287, 166647, 167367, 167727, 167947, 168739, 168883, 168967, 169503, 169515, 169663, 170371, 170871, 171123, 171159, 171235, 171391, 171951, 171979, 172035, 172039, 172107, 172203, 172743, 173067, 173611, 173655, 173875, 174195, 174423, 174991, 175183, 175275, 175491, 175495, 175503, 175711, 175815, 175987, 176023, 176163, 178047, 178423, 178723, 178731, 178767, 179215, 179223, 179319, 179487, 180087, 180351, 180655, 180831, 180867, 180963, 180991, 181027, 181135, 181167, 181311, 181471, 181683, 181759, 181791, 181855, 181951, 182095, 182287, 182575, 182731, 182815, 183307, 183319, 183447, 183543, 183679, 184243, 184267, 184911, 185167, 185203, 185535, 185631, 185731, 185827, 186007, 186291, 186367, 186799, 187639, 188175, 188259, 188311, 188439, 188643, 188931, 188971, 190543, 190551, 191523, 192319, 192639, 192967, 193023, 193027, 193155, 193327, 193615, 193887, 193971, 194059, 194647, 194787, 195327, 195403, 195427, 195595, 195715, 196215 $(mod\ 196608)$

$t = 14$

if $s \equiv$ 307, 943, 1459, 1927, 2047, 2431, 2787, 3187, 3759, 4131, 4267, 4479, 4611, 4807, 5163, 5311, 5631, 6231, 6391, 7287, 7299, 7651, 7683, 7951, 8419, 8959, 8995, 9111, 9291, 9303, 9439, 9759, 9919, 11083, 11127, 11263, 11635, 11839, 12207, 12211, 12235, 12439, 12543, 12771, 13047, 13119, 13347, 13431, 13767, 14127, 14155, 14871, 15403, 16023, 16575, 17031, 17203, 17751, 18679, 18739, 18819, 18823, 18831, 19191, 19275, 19911, 20271, 20287, 20359, 20607, 21855, 21859, 21939, 22059, 22147, 22755, 23563, 23595, 23779, 24579, 24583, 25111, 25207, 25263, 25731, 26199, 26739, 26775, 27271, 27819, 28339, 28359, 28531, 28707, 29407, 29943, 29967, 30051, 30067, 30499, 30507, 31567, 31767, 32095, 33055, 33279, 33303, 33375, 33507, 33571, 33739, 33855, 33919, 34015, 34143, 34335, 34495, 34819, 34831, 35071, 35463, 36063, 36087, 36427, 36439, 36483, 36531, 36787, 36811, 36979, 37603, 37675, 37711, 38059, 38371, 38599, 38691, 40291, 40831, 41263, 41475, 41503, 41695, 41907, 41931, 41991, 42903, 43011, 43015, 43095, 44619, 44631, 44863, 45067, 45171, 45183, 45567, 46123, 46159, 46251, 46515, 46771, 46791, 46911, 47871, 47947, 48139, 49963, 50995, 51043, 51351, 51543, 51883, 52239, 52267, 52423, 52611, 52615, 52623, 53259, 53263, 53679, 53707, 54279, 54859, 54975, 55071, 55167, 55831, 55843, 55851, 55927, 56587, 57991, 58027, 58207, 58567, 58711, 59187, 59791, 60151, 60687, 60771, 60807, 60871, 60927, 61059, 61227, 61231, 61335, 61975, 62323, 63487, 63567, 63919, 64035, 64135, 64527, 64591, 65035, 65199, 65539, 65559, 65611, 65763, 65839, 65919, 65967, 66079, 66447, 66831, 67159, 67167, 67251, 67363, 67371, 67407, 67507, 67699, 67887, 68127, 68779, 69319, 69427, 70515, 70603, 70999, 71043, 71047, 71683, 72079, 72139, 72439, 72591, 73159, 73303, 73311, 73519, 73843, 74275, 74283, 74923, 75007, 75267, 75339, 75463, 75487, 75619, 75855, 75967, 76543, 76863, 76887, 76951, 77431, 77503, 78607, 79999, 80139, 80503, 80739, 80755, 81739, 81759, 81931, 82071, 82263, 83343, 83467, 83991, 84735, 84991, 85027, 85219, 85347, 85567, 85623, 86167, 86527, 86659, 88191, 88407, 88755, 88911, 88951, 89487, 89907, 89931, 90031, 90187, 90379, 91327, 91527, 91779, 91827, 91999, 93699, 94755, 94947, 95307, 95319, 95751, 95875, 96831, 97399, 97503, 98079, 98559, 98571, 98991, 99811, 99915, 100171, 100191, 100579, 101155, 101235, 101439, 101527, 104395, 105135, 105823, 105991, 106339, 106783, 106951, 107071, 107311, 108039, 108351, 108927, 109119, 109227, 109327, 109347, 109603, 109767, 111235, 111411, 111511, 111615, 111871, 111967, 112503, 112587, 112867, 113031, 113551, 113583, 114099, 114147, 114271, 114295, 114483, 114687, 114723, 115723, 115831, 115903, 116103, 117327, 118047, 119331, 119587, 120055, 120651, 120907, 121099, 121219, 122239, 122655, 122719, 122739, 122743, 122827, 123135, 123823, 123903, 123915, 124459, 124551, 125619, 125811, 125839, 126559, 126847, 127243, 127567, 128127, 128287, 128511, 128847, 128863, 128947, 129327, 129567, 129615, 130335, 130831, 130935, 131379, 132015, 132103, 132531, 132999, 133119, 133503, 133903, 134155, 134167, 134259, 135339, 135879, 136383, 136855, 137215, 137463, 137647, 138487, 138723, 139023, 139491, 140031, 140067, 140323, 140511, 140991, 141619, 142155, 142335, 142351, 142707, 142911, 143239, 143283, 143307, 143359, 143511, 143791, 143875, 145227, 145495, 146095, 146475, 146563, 147007, 148087, 148275, 149215, 149695, 149751, 149811, 149895, 150799, 151039, 151339, 151359, 151423, 151431, 152931, 152947, 153219, 153463, 153679, 154543, 154635, 154851, 155655, 156183, 156279, 156799, 157951, 158343, 158383, 158947, 159411, 159603, 159607, 160351, 160479, 160687, 160831, 161139, 161571, 162639, 163167, 164127, 164323, 164643, 164671, 164811, 164863, 164991, 165087, 165511, 165567, 165891, 165895, 165903, 166143, 166519, 167499, 167511, 167575, 167695, 167779, 167859, 167883, 167935, 168051, 168223, 168675, 168747, 168783, 169131, 169231, 169443, 169671, 169771, 170719, 171363, 171903, 172335, 172339, 172375, 172575, 172767, 173455, 173539, 173959, 174087, 175219, 175935, 176139, 176143, 176299, 176839, 177195, 177231, 177283, 177799, 177843, 178483, 179019, 179211, 179719, 180103, 180679, 180799, 181035, 181039, 182047, 182067, 182115, 182167, 182347, 182955, 183339, 183495, 183667, 183687, 184335, 184779, 185863, 185887, 185931, 186903, 186915, 186943, 186999, 187471, 187507, 187543, 187659, 188587, 188899, 189063, 189099, 189127, 189279, 189639, 189783, 189811, 190807, 190863, 191223, 191875, 191887, 191943, 192151, 192175, 192303, 192895, 193047, 193395, 193459, 194431, 194559, 194991, 195019, 195115, 195207, 195663, 196107, 196611, 196615, 196683, 196735, 196911, 197151, 197251, 198231, 198435, 198579, 198771, 199851, 200383, 200391, 200499, 201295, 201463, 201487, 201675, 201763, 201859, 202015, 202071, 202119, 202183, 202543, 202675, 202755, 202759, 203151, 203211, 203511, 204231, 204375, 204591, 204823, 204915, 205347, 205995, 206079, 206347, 206535, 206559, 206691, 206863, 207039, 207439, 207615, 208023, 208159, 208255, 208503, 208575, 209679, 209743, 210655, 211071, 211135, 211575, 211827, 212811, 212887, 213003, 213007, 214531, 214539, 216063, 216099, 216151, 216291, 216639, 217239, 217599, 217731, 218035, 218191, 219919, 220023, 221103, 221259, 221451, 222031, 222399, 222595, 223071, 224335, 224479, 225739, 225835, 226947, 227575, 228019, 228471, 228619, 230059, 230143, 230239, 230599, 230883, 230911, 231243, 231651, 231883, 232227, 232291, 232599, 232831, 234079, 234391, 234559, 234751, 235327, 235467, 235543, 236895, 237063, 237067, 237411, 237463, 237855, 238023, 238143, 238383, 238711, 240355, 240399, 240675, 240907, 241507, 242119, 242307, 242347, 242479, 242527, 242583, 242887, 242943, 243039, 243211, 243939, 244099, 244447, 244623, 245251, 245343, 245367, 246795, 246859, 246871, 246903, 246975, 247171, 247339, 250411, 250591, 250659, 250675, 250687, 251127, 251979, 252171, 252291, 252295, 253311, 253363, 253791, 253815, 253899, 253951, 254143, 254863, 254895, 255531, 255583, 255775, 255823, 256075, 256303, 256543, 256867, 256911, 257047, 257631, 257919, 258315, 258639, 258859, 259359, 259735, 259935, 260019, 260119, 260983, 261367, 261903, 262003, 262063, 263175, 264931, 264975, 265227, 265239, 265903, 266275, 266623, 266755, 267307, 267775, 267927, 268287, 268375, 268719, 269431, 269443, 269559, 269827, 271255, 271395, 271435, 271447, 271903, 272691, 273271, 273423, 274311, 274351, 274431, 274687, 274863, 274915, 274947, 275191, 275263, 275491, 275575, 275911, 276271, 276567, 277015, 277167, 277635, 278079, 278167, 278719, 279159, 279175, 279895, 280287, 280767, 280963, 280975, 281335, 281419, 281871, 282055, 282111, 282411, 282415, 282495, 282751, 283999, 284019, 284083, 284203, 284535, 284751, 284899, 285615, 285739, 286723, 287407, 287871, 287875, 288343, 288883, 288919, 289023, 289455, 289963, 290019, 290503, 290679, 290851, 291423, 291759, 291903, 292087, 292111, 292195, 292651, 293911, 295395, 295423, 295447, 295519, 295651, 295743, 295935, 295999, 296287, 296479, 296583, 296967, 297591, 297607, 298207, 298231, 298627, 298647, 298675, 298767, 298851, 299007, 299295, 300303, 300835, 300843, 301791, 303411, 303447, 303619, 304051, 304075, 304135, 304527, 304611, 305031, 305047, 305155, 305239, 306291, 306763, 306775, 307215, 307315, 307327, 307371, 307711, 307911, 308355, 308395, 308659, 308871, 308935, 309055, 309555, 310015, 310791, 311175, 311751, 311871, 312111, 313119, 313239, 313419, 313495, 313687, 314383, 314739, 314755, 314767, 315403, 315823, 316423, 316935, 316959, 317119, 317215, 317311, 317995, 318015, 318543, 318579, 318615, 319659, 319971, 320199, 320883, 321331, 321879, 322831, 322915, 322947, 322951, 322959, 323071, 323203, 323223, 323247, 323371, 323479, 323967, 324531, 325503, 325711, 326091, 326179, 326187, 326671, 327343, 327687, 327703, 327807, 327907, 328063, 328111, 328323, 328591, 328975, 329311, 329395, 329515, 329551, 330031, 330271, 331455, 332367, 332535, 332559, 332659, 332835, 332931, 333087, 333187, 333255, 333615, 333747, 333831, 334735, 335455, 335895, 336427, 337411, 337419, 337483, 337935, 337999, 338511, 339007, 339031, 339231, 339327, 340815, 341727, 342207, 342283, 342883, 343903, 343959, 344079, 344215, 344407, 345487, 345603, 346135, 346879, 347223, 347491, 347767, 349107, 349263, 350335, 350551, 350899, 350991, 351055, 351631, 352051, 352075, 353103, 353667, 353671, 353923, 353971, 355407, 355551, 355843, 356811, 356899, 356907, 357091, 357451, 357463, 357895, 358647, 358975, 359091, 359647, 359691, 360223, 360703, 360715, 361131, 361135, 361215, 361311, 361671, 361983, 362059, 362335, 362955, 363363, 363379, 363583, 363903, 365151, 365463, 365631, 365823, 366399, 366615, 367279, 368139, 368535, 369783, 370183, 370495, 371071, 371263, 371371, 371427, 371491, 371911, 371979, 372579, 373191, 373419, 373551, 373555, 373599, 373759, 373959, 374283, 374647, 374731, 375171, 375175, 375519, 375727, 376243, 376291, 376323, 376627, 376831, 376867, 377931, 377943, 378243, 378247, 378411, 379471, 380191, 381475, 381483, 381663, 381747, 381759, 382795, 383367, 384435, 384799, 384883, 385023, 385215, 385279, 385935, 386047, 386059, 386655, 386695, 386847, 386895, 387147, 387375, 387615, 387763, 387939, 387955, 388119, 389931, 390271, 390655, 390807, 390991, 391191, 391471, 391711, 391759, 392055, 392439, 392479, 393075, 393079, 393135 $(mod\ 393216)$

and so forth.

## 7.5 Stopping time residue classes

$\sigma(s) = 1$
if $s \equiv$ 0 $(mod\ 2)$

$\sigma(s) = 2$
if $s \equiv$ 1 $(mod\ 4)$

$\sigma(s) = 4$
if $s \equiv$ 3 $(mod\ 16)$

$\sigma(s) = 5$
if $s \equiv$ 11, 23 $(mod\ 32)$

$\sigma(s) = 7$
if $s \equiv$ 7, 15, 59 $(mod\ 128)$

$\sigma(s) = 8$
if $s \equiv$ 39, 79, 95, 123, 175, 199, 219 $(mod\ 256)$

$\sigma(s) = 10$
if $s \equiv$ 287, 347, 367, 423, 507, 575, 583, 735, 815, 923, 975, 999 $(mod\ 1024)$

$\sigma(s) = 12$
if $s \equiv$ 231, 383, 463, 615, 879, 935, 1019, 1087, 1231, 1435, 1647, 1703, 1787, 1823, 1855, 2031, 2203, 2239, 2351, 2587, 2591, 2907, 2975, 3119, 3143, 3295, 3559, 3675, 3911, 4063 $(mod\ 4096)$

$\sigma(s) = 13$
if $s \equiv$ 191, 207, 255, 303, 539, 543, 623, 679, 719, 799, 1071, 1135, 1191, 1215, 1247, 1327, 1563, 1567, 1727, 1983, 2015, 2075, 2079, 2095, 2271, 2331, 2431, 2607, 2663, 3039, 3067, 3135, 3455, 3483, 3551, 3687, 3835, 3903, 3967, 4079, 4091, 4159, 4199, 4223, 4251, 4455, 4507, 4859, 4927, 4955, 5023, 5103, 5191, 5275, 5371, 5439, 5607, 5615, 5723, 5787, 5871, 5959, 5979, 6047, 6215, 6375, 6559, 6607, 6631, 6747, 6815, 6983, 7023, 7079, 7259, 7375, 7399, 7495, 7631, 7791, 7847, 7911, 7967, 8047, 8103 $(mod\ 8192)$

$\sigma(s) = 15$
if $s \equiv$ 127, 411, 415, 831, 839, 1095, 1151, 1275, 1775, 1903, 2119, 2279, 2299, 2303, 2719, 2727, 2767, 2799, 2847, 2983, 3163, 3303, 3611, 3743, 4007, 4031, 4187, 4287, 4655, 5231, 5311, 5599, 5631, 6175, 6255, 6503, 6759, 6783, 6907, 7163, 7199, 7487, 7783, 8063, 8187, 8347, 8431, 8795, 9051, 9087, 9371, 9375, 9679, 9711, 9959, 10055, 10075, 10655, 10735, 10863, 11079, 11119, 11567, 11679, 11807, 11943, 11967, 12063, 12143, 12511, 12543, 12571, 12827, 12967, 13007, 13087, 13567, 13695, 13851, 14031, 14271, 14399, 14439, 14895, 15295, 15343, 15839, 15919, 16027, 16123, 16287, 16743, 16863, 16871, 17147, 17727, 17735, 17767, 18011, 18639, 18751, 18895, 19035, 19199, 19623, 19919, 20079, 20199, 20507, 20527, 20783, 20927, 21023, 21103, 21223, 21471, 21727, 21807, 22047, 22207, 22655, 22751, 22811, 22911, 22939, 23231, 23359, 23399, 23615, 23803, 23835, 23935, 24303, 24559, 24639, 24647, 24679, 25247, 25503, 25583, 25691, 25703, 25831, 26087, 26267, 26527, 26535, 27111, 27291, 27759, 27839, 27855, 27975, 28703, 28879, 28999, 29467, 29743, 29863, 30311, 30591, 30687, 30715, 30747, 30767, 30887, 31711, 31771, 31899, 32155, 32239, 32575, 32603 $(mod\ 32768)$

$\sigma(s) = 16$
if $s \equiv$ 359, 479, 559, 603, 767, 859, 1179, 1183, 1351, 1519, 1535, 1627, 2367, 2407, 2495, 2671, 2687, 2791, 2887, 2927, 3103, 3239, 3487, 3535, 3695, 3815, 4319, 4335, 4379, 4635, 4775, 4799, 4815, 4895, 4991, 5087, 5343, 5375, 5423, 5583, 5663, 5823, 5863, 6207, 6247, 6555, 6639, 6703, 6975, 7015, 7103, 7231, 7451, 7471, 7551, 7711, 7835, 7871, 7931, 8095, 8263, 8551, 8671, 8863, 9119, 9199, 9319, 9543, 9599, 9819, 9935, 10151, 10559, 10727, 10907, 11035, 11247, 11431, 11727, 11823, 11887, 12007, 12319, 12495, 12615, 12775, 12799, 13279, 13339, 13535, 13615, 13671, 13855, 13927, 13951, 14015, 14207, 14303, 14363, 14383, 14503, 14543, 14747, 15103, 15167, 15207, 15423,

15487, 15515, 15599, 15643, 15743, 15771, 15855, 16191, 16411, 16431, 16455, 16511, 16635, 16831, 17055, 17127, 17135, 17223, 17311, 17391, 17479, 17511, 17659, 18159, 18343, 18523, 18559, 18919, 19099, 19111, 19135, 19151, 19231, 19367, 19547, 19687, 19707, 20127, 20207, 20511, 20591, 20687, 20807, 21039, 21595, 21615, 21695, 21735, 22015, 22119, 22399, 22495, 22555, 22575, 22695, 22887, 23143, 23167, 23583, 23663, 23707, 23711, 23743, 23963, 24047, 24383, 24571, 24703, 24731, 24815, 25371, 25415, 25471, 25599, 25671, 25851, 26015, 26063, 26343, 26351, 26367, 26439, 26459, 26619, 27039, 27119, 27303, 27343, 27423, 27559, 27675, 27739, 27879, 27903, 27951, 28095, 28191, 28319, 28327, 28351, 28447, 28507, 28527, 28927, 29087, 29231, 29631, 29807, 29823, 29887, 30079, 30207, 30235, 30415, 30575, 30655, 30971, 30975, 31079, 31199, 31335, 31359, 31471, 31727, 31775, 32223, 32283, 32303, 32703, 32763, 32859, 32923, 33007, 33087, 33255, 33531, 33663, 34111, 34151, 34255, 34271, 34535, 34631, 34651, 34927, 35023, 35231, 35279, 35311, 35419, 35579, 35583, 36143, 36159, 36383, 36519, 36543, 36635, 36639, 36719, 36891, 36911, 37119, 37167, 37311, 37407, 37467, 37487, 37607, 37735, 38047, 38171, 38271, 38427, 38607, 38847, 39039, 39135, 39195, 39295, 39535, 39615, 39919, 40039, 40187, 40351, 40415, 40495, 40687, 40943, 41023, 41063, 41183, 41243, 41447, 41627, 41723, 42075, 42215, 42239, 42303, 42343, 42471, 42651, 42911, 43071, 43111, 43215, 43335, 43471, 43611, 43775, 43967, 44143, 44223, 44239, 44359, 44699, 44959, 45083, 45103, 45223, 45359, 45503, 45535, 45599, 45679, 45799, 45851, 46127, 46247, 46407, 47099, 47231, 47327, 47387, 47423, 47487, 47807, 48095, 48155, 48295, 48379, 48879, 48987, 49135, 49215, 49255, 49311, 49563, 49567, 49983, 50143, 50267, 50303, 50407, 50663, 50843, 50847, 51055, 51103, 51271, 51431, 51451, 51455, 51611, 51871, 51951, 52031, 52071, 52335, 52415, 52431, 52507, 52551, 52735, 52763, 53159, 53183, 53319, 53339, 53439, 53887, 53919, 54043, 54303, 54319, 54375, 54439, 54751, 55207, 55291, 55327, 55407, 55535, 55963, 56059, 56191, 56287, 56315, 56347, 56639, 56935, 57179, 57215, 57375, 57671, 57755, 57759, 57839, 57947, 58175, 58203, 58495, 58523, 58527, 58863, 58983, 59247, 59263, 59463, 59559, 59623, 59643, 59647, 60015, 60063, 60143, 60231, 60271, 60571, 60831, 60911, 60955, 61135, 61351, 61375, 61531, 61631, 61663, 61723, 61979, 62119, 62159, 62239, 62279, 62719, 62943, 63023, 63335, 63519, 63551, 63591, 63599, 64047, 64167, 64207, 64251, 64287, 64447, 64507, 64831, 64871, 65127, 65179, 65183, 65275, 65407, 65439 $(mod\ 65536)$

and so forth.

## 7.6 The first residue classes $(mod\ 3 \cdot 2^{\sigma(s)})$ for $\tau(s) = 1...6$

### The first residue classes $(mod\ 3 \cdot 2^{\sigma(s)})$ for $\tau(s) = 1$

$n = 2, \sigma(s) = 4, A_1(n) = 2$
3, 19

$n = 3, \sigma(s) = 5, A_1(n) = 4$
43, 55, 75, 87

$n = 4, \sigma(s) = 7, A_1(n) = 4$
7, 15, 135, 271

$n = 5, \sigma(s) = 8, A_1(n) = 8$
79, 175, 199, 351, 591, 607, 687, 711

$n = 6, \sigma(s) = 10, A_1(n) = 8$
735, 1311, 1599, 1759, 1839, 2335, 2623, 2863

$n = 7, \sigma(s) = 12, A_1(n) = 16$
1087, 1855, 2239, 3295, 4479, 5919, 6447, 6687, 8575, 9279, 10015, 10047, 10431, 10543, 10783, 11487

$n = 8, \sigma(s) = 13, A_1(n) = 32$
255, 303, 543, 1215, 1567, 2431, 3135, 3903, 3967, 4927, 8383, 9439, 9759, 9919, 10623, 11647, 12159, 12415, 13119, 16159, 16575, 16639, 16687, 16927, 17599, 17631, 18111, 19519, 19839, 20287, 20607, 24351

$n = 9, \sigma(s) = 15, A_1(n) = 32$
127, 831, 5311, 5631, 11967, 12543, 13567, 22047, 22207, 23935, 30591, 32895, 33919, 35071, 38079, 40831, 46335, 51967, 54975, 55423, 56703, 66367, 66687, 67839, 71167, 73599, 77503, 78079, 84735, 87583, 88191, 96127

$n = 10, \sigma(s) = 16, A_1(n) = 64$
5823, 7551, 12799, 15103, 22015, 22399, 25599, 28351, 28927, 29823, 30207, 35583, 36543, 39039, 49983, 52735, 58495, 59647, 62719, 66303, 67071, 70911, 71199, 71359, 73087, 79743, 82047, 87231, 91135, 95359, 95743, 101119, 102079, 104575, 107775, 109311, 112767, 115519, 115839, 116991, 122751, 131839, 132607, 136447, 136735, 143871, 145279, 146175, 147583, 152767, 153087, 153471, 159423, 159999, 173311, 174847, 178303, 181375, 182527, 183807, 188287, 189567, 190719, 193791

$n = 11, \sigma(s) = 18, A_1(n) = 64$
511, 1023, 3583, 3775, 9471, 11007, 58623, 91263, 107263, 111103, 113407, 159231, 160255, 162559, 164607, 165375, 169215, 193663, 209919, 213759, 214527, 242815, 251775, 263167, 271615, 273151, 314367, 315903, 320511, 320767, 325119, 353407, 366591, 369663, 407679, 421375, 426751, 427519, 431359, 472063, 475903, 476671, 478719, 485631, 513919, 524799, 527871, 528063, 576511, 578047, 582655, 587263, 628735, 631551, 631807, 635391, 637695, 669823, 684543, 686847, 717951, 740863, 747775, 767103

$n = 12, \sigma(s) = 20, A_1(n) = 128$
14463, 33535, 69631, 71679, 78847, 97023, 184831, 194047, 238335, 244479, 246271, 281599, 303871, 354559, 393727, 396799, 489471, 492543, 560127, 562431, 563199, 566271, 607743, 636031, 666367, 673279, 699391, 701439, 707583, 713727, 773119, 807423, 814591, 820735, 869119, 881151, 921279, 984063, 1016319, 1028607, 1041663, 1063039, 1080063, 1120255, 1145599, 1189119, 1192959, 1270911, 1286911, 1293055, 1298943, 1398783, 1431423, 1437183, 1507839, 1510911, 1538047, 1541119, 1546239, 1549311, 1607679, 1608703, 1611007, 1611775, 1614847, 1656319, 1658367, 1750015, 1750143, 1756159, 1762303, 1855999, 1864191, 1897599, 1909503, 1928703, 1929727, 1967103, 1969855, 1983231, 2024703, 2028543, 2032639, 2064895, 2077183, 2090239, 2128639, 2130687, 2166783, 2175999, 2237695, 2241535, 2281983, 2291199, 2319487, 2343423, 2347519, 2378751, 2401023, 2447359, 2451711, 2479999, 2485759, 2490879, 2493951, 2556415, 2559487, 2594815, 2597887, 2656255, 2706943, 2733183, 2763519, 2770431, 2796543, 2798719, 2870271, 2911743, 2912767, 2917887, 2946175, 2958079, 2966271, 2977279, 3015679, 3031807, 3073279, 3077119

$n = 13, \sigma(s) = 21, A_1(n) = 256$
20991, 35839, 38143, 41983, 96255, 166911, 177151, 183295, 189439, 255231, 259071, 306175, 308223, 333823, 336639, 356863, 372735, 396991, 406527, 451839, 455679, 459775, 492031, 504319, 573951, 603135, 612351, 623103, 650751,

653311, 718335, 721407, 745471, 751615, 770559, 774655, 822271, 828415, 886783, 907135, 918015, 963583, 986623, 1025023, 1083391, 1123071, 1134079, 1143295, 1147903, 1165311, 1190655, 1214463, 1225855, 1242111, 1246207, 1278463, 1297407, 1339903, 1373311, 1404415, 1442815, 1445887, 1458943, 1462783, 1556991, 1561087, 1587327, 1606399, 1642495, 1644543, 1724415, 1757695, 1771519, 1773055, 1773567, 1854463, 1865727, 1869823, 1876735, 1927423, 1936383, 1969663, 1985535, 2012287, 2054143, 2062335, 2075647, 2083839, 2092543, 2132991, 2135295, 2139135, 2190847, 2208895, 2245375, 2246143, 2272255, 2274303, 2280447, 2286591, 2330623, 2387455, 2393599, 2403327, 2428927, 2430975, 2441983, 2454015, 2494143, 2556927, 2564863, 2589183, 2601471, 2640895, 2663167, 2683903, 2718463, 2750463, 2842623, 2848767, 2859775, 2865919, 2871807, 2919423, 2925567, 2960383, 2983935, 2998783, 3004287, 3032575, 3058687, 3060735, 3083775, 3113983, 3122175, 3180543, 3184639, 3229183, 3231231, 3240447, 3245055, 3323007, 3343359, 3375615, 3428863, 3437055, 3470463, 3501567, 3504895, 3539967, 3543039, 3556095, 3559935, 3658239, 3663103, 3701503, 3703551, 3739647, 3810559, 3814399, 3823615, 3854847, 3868671, 3870207, 3872767, 3892351, 3900415, 3951615, 3966975, 3971071, 3973887, 4020223, 4024575, 4058623, 4066815, 4109439, 4129279, 4151295, 4167679, 4172799, 4189695, 4215295, 4287999, 4290559, 4306047, 4342527, 4343295, 4361215, 4369407, 4427775, 4449535, 4453375, 4484607, 4490751, 4502527, 4526079, 4530943, 4539135, 4567039, 4600831, 4646143, 4649983, 4662015, 4738047, 4760319, 4768255, 4781055, 4797439, 4806655, 4815615, 4817407, 4845055, 4912639, 4915711, 4956927, 4963071, 4964863, 5057535, 5095935, 5112319, 5129727, 5155839, 5211135, 5281791, 5317375, 5326335, 5359615, 5384959, 5408767, 5436415, 5491711, 5526015, 5602047, 5751295, 5760255, 5781631, 5798655, 5838847, 5907711, 5911551, 5918719, 5920767, 5967871, 5969919, 5989503, 5997567, 6060031, 6068223, 6117375, 6130687, 6155775, 6179839, 6226431, 6256639, 6264831, 6278143

and so forth.

## The first residue classes $(mod\ 3 \cdot 2^{\sigma(s)})$ for $\tau(s) = 2$

$n = 4, \sigma(s) = 7, A_2(n) = 2$
187, 315

$n = 5, \sigma(s) = 8, A_2(n) = 6$
39, 123, 219, 295, 379, 475

$n = 4, \sigma(s) = 10, A_2(n) = 14$
367, 423, 583, 975, 999, 1371, 1447, 1947, 1999, 2023, 2395, 2415, 2631, 2971

$n = 6, \sigma(s) = 12, A_2(n) = 36$
231, 463, 615, 879, 1231, 1435, 1647, 2031, 2203, 2587, 2907, 3559, 4063, 4327, 4711, 4975, 5031, 5743, 6127, 7003, 7071, 7215, 7239, 8007, 8655, 9127, 9423, 9627, 10395, 10779, 11167, 11311, 11335, 11751, 12103, 12255

$n = 4, \sigma(s) = 13, A_2(n) = 96$
207, 799, 1071, 1327, 1563, 1983, 2079, 2095, 2271, 2331, 2607, 3039, 3483, 3687, 4159, 4251, 4455, 5023, 5103, 5191, 5275, 5439, 5607, 5871, 5959, 6375, 6559, 6607, 7023, 7375, 7399, 7495, 7791, 8731, 8815, 8911, 8991, 9519, 10207, 10267, 10287, 10855, 11743, 12271, 12351, 12391, 13147, 13215, 13383, 13467, 13807, 14151, 14239, 14751, 14799, 15007, 15175, 15271, 15567, 15591, 15687, 16591, 16923, 17007, 17103, 17455, 17947, 18367, 18399, 18459, 18463, 18655, 18715, 18991, 19047, 19423, 19867, 19935, 20071, 20463, 20583, 20635, 20839, 21339, 21487, 21823, 21991, 21999, 22255, 22431, 22759, 23199, 23367, 23407, 23463, 24175

$n = 7, \sigma(s) = 15, A_2(n) = 160$
415, 2719, 2767, 2799, 2847, 3303, 4287, 5599, 6175, 6783, 7783, 8431, 9087, 9375, 9711, 11079, 12063, 12511, 12571, 13087, 13695, 13851, 14031, 14271, 14439, 14895, 15295, 15919, 16743, 16863, 17727, 17767, 18751, 21103, 21727, 21807, 22911, 23359, 23835, 24303, 24639, 24679, 26527, 27975, 28879, 28999, 29467, 30747, 31711, 32239, 32575, 33183, 34543, 35487, 35535, 36511, 36799, 37423, 38367, 38943, 39271, 39967, 40255, 40551, 41199, 42139, 43423, 44335, 44575, 45279, 45339, 45775, 45855, 47167, 48063, 48607, 48687, 50503, 50535, 51519, 53275, 53551, 53695, 53791, 53871, 54495, 55519, 55579, 55999, 56127, 56167, 56383, 57447, 58015, 58351, 59295, 60607, 61471, 61647, 61767, 62235, 63535, 64479, 65007, 65343, 67311, 68335, 68383, 68839, 69279, 69567, 69823, 70191, 72039, 72319, 72735, 73023, 74623, 74907, 74911, 75247, 76191, 76615, 77103, 77343, 77599, 78543, 79231, 79387, 79567, 79807, 79935, 79975, 80431, 81375, 82279, 82399, 83263, 83271, 86043, 86319, 86463, 86559, 87343, 88287, 88347, 88447, 88767, 88935, 89151, 89371, 89839, 90175, 90783, 91119, 93375, 93511, 94239, 96283, 96303

$n = 8, \sigma(s) = 16, A_2(n) = 384$
1183, 1351, 2367, 3103, 4335, 5343, 6207, 6247, 6975, 7015, 7231, 7471, 7711, 8671, 8863, 9199, 9543, 11035, 11823, 12319, 12495, 12615, 13671, 13855, 13951, 14383, 15207, 15423, 15487, 15643, 15855, 16191, 16831, 17055, 18159, 18559, 19135, 19231, 19687, 20127, 20511, 21039, 22887, 23167, 23583, 23743, 24703, 25371, 25471, 26367, 27039, 27343, 27423, 27903, 27951, 28095, 28191, 28447, 29887, 30079, 30235, 30415, 30655, 30975, 31359, 31471, 32223, 33007, 33087, 33663, 34111, 36639, 36891, 37119, 37167, 37311, 37407, 37735, 38047, 38271, 38607, 39135, 39195, 39295, 39615, 40495, 40687, 41023, 42343, 43071, 43335, 44223, 44359, 45535, 47487, 48879, 49311, 49567, 51871, 51951, 52071, 52507, 53439, 53887, 54043, 54303, 54751, 55327, 56191, 56935, 57759, 58527, 58863, 59263, 60231, 61135, 61375, 61663, 61723, 62239, 62943, 63591, 64047, 64287, 64447, 64831, 65407, 65895, 66015, 67903, 68031, 68223, 69855, 69871, 69915, 70335, 70527, 70879, 70959, 71743, 72511, 72639, 72987, 73407, 75079, 75135, 75471, 76095, 77359, 78031, 78151, 79071, 79207, 79551, 79899, 80703, 80743, 80959, 81135, 81279, 81391, 81727, 82591, 83695, 84687, 85663, 86047, 86343, 86575, 88423, 89119, 89247, 89583, 89919, 90351, 90907, 91903, 92575, 92959, 93439, 93487, 93631, 93727, 93855, 94767, 96511, 96615, 96735, 96895, 97311, 97759, 97839, 98623, 99199, 99687, 101679, 101919, 102171, 102175, 102427, 102655, 102703, 102847, 102943, 103023, 103807, 104143, 104671, 104731, 105151, 106599, 106719, 108447, 108607, 108871, 109503, 109759, 110619, 110895, 111039, 111135, 111387, 112959, 113023, 113343, 113631, 114415, 114847, 117487, 117567, 117607, 117951, 118719, 118975, 119839, 121071, 122175, 123207, 123295, 123711, 124059, 124063, 124399, 125679, 125767, 127167, 127695, 128479, 128559, 128871, 129087, 129127, 129583, 129823, 130719, 131431, 131551, 132255, 132423, 133567, 133759, 134175, 135391, 135451, 135871, 136063, 136495, 137319, 138087, 138175, 138303, 138523, 138543, 138783, 138943, 139743, 139935, 140271, 140671, 141007, 141631, 142107, 143391, 144607, 144927, 145023, 145087, 145435, 145455, 146239, 146559, 146671, 146715, 146815, 147903, 149631, 150207, 150223, 150303, 150759, 151879, 154239, 154783, 154815, 155119, 155455, 155775, 155887, 156543, 158415, 159391, 159519, 160303, 160959, 161151, 161307, 161487, 161727, 162151, 162271, 162543, 162847, 163375, 164079, 165183, 165223, 167215, 167455, 167707, 168559, 168807, 169119, 170367, 171567, 171759, 172095, 172135, 172255, 173415, 173983, 175039, 175431, 176155, 176431, 176575, 176607, 176671, 176923, 178495, 178879, 179167, 180639, 182943, 183103, 183487, 183579, 184255, 184959, 185115, 185823, 186399, 186607, 187263, 187711, 188007, 188743, 189247, 189595, 190335, 191215, 192207, 192447, 192703, 192735, 192795, 193231, 193311, 194095, 194407, 194623, 195519, 195903, 196255, 196479

and so forth.

## The first residue classes $(mod\ 3 \cdot 2^{\sigma(s)})$ for $\tau(s) = 3$

$n = 6, \sigma(s) = 10, A_3(n) = 2$
507, 1531

$n = 7, \sigma(s) = 12, A_3(n) = 8$
3675, 5115, 5799, 5883, 7771, 9211, 9895, 9979

$n = 8, \sigma(s) = 13, A_3(n) = 40$
679, 1135, 1191, 3067, 3835, 4507, 5371, 5787, 5979, 6631, 6747, 7911, 8047, 8103, 8871, 9327, 11259, 12027, 12699,
13051, 13563, 13915, 14407, 14823, 15451, 15823, 16039, 16239, 17575, 21243, 22107, 22171, 22363, 22599, 23131,
23643, 24015, 24231, 24295, 24487

$n = 9, \sigma(s) = 15, A_3(n) = 136$
411, 1095, 1275, 1903, 2119, 2299, 2983, 3163, 6255, 6759, 8347, 9051, 9679, 10075, 10735, 10863, 11119, 11679,
12967, 15343, 16027, 16287, 18639, 18895, 19035, 19623, 20079, 20199, 20527, 21223, 21471, 22939, 23803, 24559,
25503, 25831, 26535, 27111, 27291, 27759, 27855, 29743, 29863, 30687, 31771, 31899, 32155, 33607, 34671, 34887,
35047, 35067, 35751, 35931, 36379, 36775, 36955, 37999, 41115, 42447, 42727, 42823, 42843, 43503, 43887, 44911,
45595, 45735, 48111, 48795, 49639, 49915, 51663, 52687, 53295, 53991, 55707, 56571, 57327, 57415, 58459, 58471,
58599, 58855, 59035, 62511, 62631, 63079, 63655, 64539, 64923, 65371, 65947, 66375, 66631, 66811, 67815, 69147,
69543, 69723, 70767, 71791, 72295, 74587, 75495, 75591, 76399, 77215, 77679, 78363, 81823, 82407, 82683, 84175,
84571, 85159, 85455, 85615, 85735, 87007, 90183, 91039, 91227, 91239, 91623, 91803, 92071, 92647, 92827, 93295,
93391, 95847, 96223, 96423, 97435, 98139

$n = 10, \sigma(s) = 16, A_3(n) = 416$
559, 859, 1179, 1519, 2407, 2671, 2791, 2887, 3487, 3535, 4635, 4815, 5583, 6555, 6639, 6703, 8095, 8263, 8551,
9319, 11247, 11431, 11887, 12775, 13279, 13339, 13615, 13927, 14503, 15771, 16411, 16431, 16455, 16635, 17127,
17223, 17311, 17391, 17479, 17511, 17659, 18343, 18523, 19099, 21615, 21735, 22119, 22495, 22555, 22575, 22695,
23143, 23707, 25671, 26343, 26439, 27559, 27675, 27879, 28507, 28527, 29631, 31335, 32283, 32703, 32859, 32923,
33255, 33531, 34255, 34927, 35023, 35311, 35419, 36159, 38427, 38847, 39535, 39919, 40039, 40351, 42075, 42303,
42471, 42651, 43111, 43215, 43471, 44143, 44239, 44959, 45103, 45223, 45679, 45799, 46407, 48379, 48987, 49135,
49215, 49255, 49563, 50143, 50407, 50847, 51055, 51103, 51271, 51451, 52335, 52431, 52551, 53319, 53919, 54319,
54375, 55207, 55407, 55963, 56287, 56347, 57375, 58203, 58983, 59247, 59559, 59623, 59643, 60015, 60063, 60271,
60571, 60831, 60955, 61531, 62119, 63519, 64207, 65127, 65179, 65439, 66715, 68463, 68775, 69231, 69351, 70171,
70351, 70431, 70623, 71119, 72091, 72175, 73371, 74655, 75687, 76263, 76443, 76783, 79839, 80079, 80283, 81051,
81307, 81967, 81991, 82171, 82663, 82671, 82759, 82927, 83047, 84903, 85083, 85743, 86127, 86223, 87151, 87271,
87655, 88111, 88231, 89499, 90267, 90951, 91207, 91551, 91599, 91879, 91887, 91975, 91995, 92655, 93211, 93415,
94063, 94623, 95167, 96111, 96507, 96871, 97263, 97819, 98239, 98395, 98791, 99067, 99807, 100071, 100167, 100767,
100815, 101695, 102447, 103143, 103707, 103963, 104383, 105723, 105951, 106479, 106779, 106983, 107163, 107611,
107751, 107839, 108007, 108187, 108751, 111663, 111783, 111943, 112863, 112923, 113691, 114523, 114751, 115099,
115803, 116199, 116383, 116967, 117147, 117871, 117967, 118087, 118299, 118695, 118855, 118875, 119455, 119911,
120943, 122715, 122911, 123375, 123739, 124519, 124783, 125095, 125179, 125551, 125599, 126367, 126447, 127515,
127815, 129055, 129135, 130407, 130663, 130975, 131631, 131931, 132591, 133479, 133743, 133863, 133959, 133999,
134311, 134559, 134607, 134767, 134887, 135967, 136159, 137775, 138907, 139167, 139335, 139623, 140191, 140391,
141223, 141799, 141979, 142503, 142959, 143847, 144351, 144411, 144687, 144999, 145375, 145575, 145615, 145819,
146587, 147483, 148207, 148383, 148551, 148731, 149415, 149595, 150171, 150439, 150619, 151279, 151663, 151759,
153567, 153627, 154215, 154779, 155035, 155803, 156487, 157087, 157135, 157423, 157531, 158191, 158631, 159579,
160159, 161647, 162043, 162799, 163995, 165327, 165343, 165607, 165703, 165999, 166095, 166303, 166351, 166383,
166491, 167983, 168679, 169243, 170607, 170991, 171111, 171259, 171423, 171487, 172015, 172315, 172519, 172699,
173287, 174183, 174543, 175215, 175311, 176031, 176175, 176295, 176751, 176871, 177199, 177319, 178399, 178459,
179227, 179451, 180207, 180327, 181215, 181339, 181479, 181735, 182127, 182175, 182343, 182503, 182523, 182683,
183835, 184231, 184411, 185391, 186279, 187035, 187359, 187419, 188251, 188911, 190695, 191343, 191643, 191983,
192027, 192603, 193051, 193191, 193351, 194671, 195279, 195943, 196251

and so forth.

## The first residue classes ($mod\ 3 \cdot 2^{\sigma(s)}$) for $\tau(s) = 4$

$n = 8, \sigma(s) = 13, A_4(n) = 2$
12283, 20475

$n = 9, \sigma(s) = 15, A_4(n) = 18$
2727, 6907, 8187, 11943, 16123, 30715, 39675, 39931, 41563, 48891, 50779, 63483, 68263, 72699, 73723, 74331, 77479,
83547, 9 15 18

$n = 10, \sigma(s) = 16, A_4(n) = 86$
603, 1627, 5863, 9819, 11727, 12007, 18919, 19111, 19707, 21595, 24571, 25851, 26619, 27303, 27739, 28327, 34651,
36519, 37467, 43611, 48295, 54439, 55291, 56059, 59463, 61351, 64167, 64251, 64507, 65275, 66139, 70311, 73467,
75355, 77263, 85243, 89199, 91387, 92155, 92839, 95343, 101115, 102055, 102255, 103003, 107259, 109147, 110235,
112635, 116379, 121851, 123291, 123483, 124999, 129703, 129787, 132699, 135847, 136935, 139003, 143079, 149991,
150183, 152667, 154735, 155643, 158811, 159399, 160879, 165723, 166651, 167791, 172795, 175771, 178171, 179367,
181915, 185511, 186363, 187131, 187387, 188827, 189019, 192423, 195579, 196347

$n = 11, \sigma(s) = 18, A_4(n) = 372$
2043, 2811, 3183, 4143, 5287, 7419, 8955, 9883, 10095, 12199, 13359, 13479, 15355, 15775, 16795, 20391, 26695,
28063, 29799, 32671, 33895, 38503, 41887, 42087, 46695, 51963, 52315, 55911, 56431, 58087, 58107, 59227, 59599,
62575, 67791, 68635, 69487, 70375, 74983, 75847, 77007, 80923, 84039, 84199, 85531, 94747, 95199, 98907, 100519,
101019, 105127, 105583, 105627, 108123, 111727, 112359, 112807, 115963, 116175, 116455, 118639, 119919, 123367,
124647, 126831, 131559, 140775, 143451, 151195, 154791, 157275, 158887, 162471, 162555, 163483, 165799, 167079,
168615, 170395, 171771, 173991, 175567, 178671, 179611, 181063, 182767, 183207, 187375, 190279, 190959, 193627,
195567, 201307, 204783, 205915, 206959, 207099, 207451, 208539, 211623, 212827, 214683, 217767, 219247, 220923,
221595, 222043, 223855, 224719, 229447, 232911, 233071, 233191, 237639, 240103, 246855, 250459, 250971, 256603,
257115, 257275, 257691, 261231, 263835, 264027, 264187, 264795, 264955, 265327, 266287, 269563, 270747, 271099,
272239, 273519, 275503, 275623, 280431, 282015, 282535, 286623, 289647, 291943, 300123, 304231, 306267, 308839,
313179, 314107, 314619, 318055, 320251, 320763, 321531, 322215, 324327, 328443, 328935, 329935, 331431, 334875,
339151, 339483, 346183, 357343, 359067, 361051, 363163, 363771, 365535, 366759, 367771, 369915, 370267, 370683,
371355, 374503, 374751, 375963, 378319, 380583, 382063, 385179, 386511, 386791, 388975, 393703, 395727, 401499,
402919, 405595, 406107, 413787, 416935, 417435, 419419, 424347, 424615, 424699, 429223, 430759, 431847, 433915,
435015, 436135, 437991, 440815, 444903, 445351, 453103, 457711, 459867, 465147, 466779, 466927, 469243, 469755,
470683, 473199, 473767, 474279, 476827, 477435, 477807, 479911, 480423, 480999, 481275, 483067, 483739, 487143,
487335, 488103, 490491, 494055, 495055, 499783, 508999, 513115, 519259, 519835, 520239, 523375, 523431, 523515,
525979, 526171, 526939, 529575, 532891, 534171, 535663, 536487, 539643, 540063, 541083, 542575, 544159, 548767,

550983, 551791, 552351, 556959, 558183, 562267, 562791, 566175, 568411, 575323, 576603, 576763, 580719, 582375, 582907, 583515, 583675, 583887, 584359, 586471, 586863, 590587, 591079, 592923, 593575, 593775, 594663, 597019, 599271, 600135, 601627, 605211, 608487, 609819, 619035, 621211, 624807, 625915, 627679, 628903, 629415, 629871, 632059, 632827, 633499, 636015, 636895, 637095, 638107, 640251, 640743, 642727, 642927, 647323, 647655, 648655, 657871, 663643, 668251, 675483, 675931, 679579, 683175, 686491, 687771, 690087, 693991, 694683, 697159, 699855, 700135, 703899, 705351, 707047, 707055, 711663, 714567, 717915, 722011, 725595, 727291, 728923, 730203, 731247, 731739, 731899, 735343, 736423, 737115, 739579, 739951, 742567, 743143, 743419, 743535, 746331, 748143, 749007, 749287, 749479, 750247, 752635, 753735, 756199, 757359, 757479, 764391, 774747, 780891, 781563, 782383, 785575, 785659

and so forth.

## The first residue classes $(mod\ 3 \cdot 2^{\sigma(s)})$ for $\tau(s) = 5$

$n = 10, \sigma(s) = 16, A_5(n) = 2$
32763, 98299

$n = 11, \sigma(s) = 18, A_5(n) = 30$
6139, 14331, 23547, 117415, 122875, 126631, 148059, 211707, 276475, 285691, 327675, 371367, 376827, 410203, 418395, 427611, 473851, 482043, 491259, 530427, 589819, 633511, 638971, 641703, 647163, 650919, 680539, 689755, 744187, 753403

$n = 12, \sigma(s) = 20, A_5(n) = 156$
5211, 49147, 57339, 76455, 104167, 110311, 115291, 196603, 199419, 205563, 223303, 350971, 359079, 360187, 365223, 397915, 399355, 404059, 407547, 408315, 416763, 448231, 460635, 473083, 482299, 510631, 511227, 517371, 519847, 541275, 543483, 567975, 606811, 615003, 621307, 624219, 630523, 670887, 677031, 677883, 687099, 703143, 709723, 715867, 741979, 777327, 780967, 783471, 790183, 855291, 896463, 903655, 965371, 1014951, 1021947, 1032187, 1040379, 1053787, 1105915, 1121391, 1125031, 1191579, 1197723, 1247995, 1254139, 1304571, 1391355, 1407655, 1413799, 1456123, 1456891, 1465083, 1465339, 1470375, 1474299, 1509211, 1513467, 1535643, 1551015, 1559803, 1565947, 1576815, 1589851, 1592059, 1598043, 1607259, 1616551, 1624743, 1633959, 1648635, 1654779, 1663579, 1672795, 1719463, 1725607, 1726459, 1735675, 1751719, 1825903, 1832047, 1868379, 1877595, 1903867, 1945039, 1991067, 2063527, 2070523, 2088955, 2146299, 2169967, 2201319, 2207463, 2212443, 2240155, 2246299, 2293755, 2320455, 2353147, 2439931, 2448123, 2457339, 2495067, 2496507, 2501211, 2513659, 2518951, 2522875, 2545383, 2562043, 2570235, 2579451, 2584219, 2599591, 2607783, 2616999, 2625391, 2646619, 2655835, 2673319, 2682535, 2697211, 2703355, 2703963, 2718459, 2727675, 2806875, 2813019, 2839131, 2878119, 2887335, 2916955, 2926171, 3000807, 3039643, 3062523, 3129339

and so forth.

## The first residue classes $(mod\ 3 \cdot 2^{\sigma(s)})$ for $\tau(s) = 6$

$n = 12, \sigma(s) = 20, A_6(n) = 2$
1310715, 2359291

$n = 13, \sigma(s) = 21, A_6(n) = 46$
174759, 245755, 253947, 524283, 604923, 669691, 803419, 811611, 1063675, 1120251, 1507323, 1670907, 1813159, 1821351, 1869403, 1966075, 2342907, 2710267, 2758651, 2766843, 2801659, 2879143, 2900571, 3160827, 3359323, 3824635, 3910311, 3932155, 3966555, 4063227, 4369063, 4448251, 4718587, 4799227, 4807419, 4855803, 4898811, 4976295, 5005915, 5314555, 5456475, 5701627, 5865211, 5921787, 6015655, 6029307

$n = 14, \sigma(s) = 23, A_6(n) = 410$
19195, 34395, 90715, 359163, 398427, 497659, 597103, 639579, 764583, 780283, 867067, 875259, 917499, 922203, 936699, 983035, 1000027, 1006587, 1011355, 1302523, 1415163, 1416955, 1427943, 1514607, 1528411, 1567399, 1682895, 1747623, 1796167, 1830567, 1928859, 1933051, 1938087, 1981179, 2083495, 2090235, 2144935, 2168827, 2242555, 2282587, 2297083, 2310747, 2376283, 2384475, 2538235, 2587815, 2605051, 2613243, 2820859, 3047163, 3189415, 3197607, 3298471, 3299323, 3305383, 3348475, 3356667, 3398767, 3441243, 3442267, 3455739, 3490395, 3599451, 3719163, 3813019, 4047451, 4106919, 4118247, 4137723, 4209403, 4255399, 4276219, 4283131, 4330075, 4414459, 4499035, 4556379, 4635303, 4653051, 4663975, 4715631, 4787451, 4828903, 4835815, 5129883, 5179227, 5193723, 5339899, 5345959, 5389051, 5389479, 5397243, 5483175, 5498107, 5636091, 5904379, 5912571, 5947387, 5995687, 6021115, 6046299, 6295407, 6367227, 6455035, 6549159, 6605479, 6663631, 6709659, 6849115, 6887419, 6919911, 6970363, 7029339, 7112283, 7127035, 7135227, 7154343, 7215099, 7219803, 7365211, 7426651, 7436967, 7526119, 7589115, 7605927, 7764987, 7869531, 7944955, 7953147, 7995387, 8281083, 8337403, 8407803, 8471131, 8479323, 8580187, 8587099, 8797351, 8886267, 8927995, 8985711, 9010939, 9118459, 9161383, 9168891, 9246715, 9255675, 9371643, 9388635, 9399963, 9537115, 9616039, 9682939, 9691131, 9696367, 9703279, 9768187, 9775099, 9805563, 9917019, 9945691, 9956007, 10110619, 10117531, 10184775, 10321659, 10327783, 10377979, 10463911, 10472103, 10533543, 10557435, 10622971, 10627675, 10631163, 10665979, 10671195, 10685691, 10764891, 10926843, 10993659, 11013799, 11209467, 11277403, 11287291, 11529895, 11578023, 11687079, 11687931, 11688955, 11693991, 11737083, 11787375, 11815675, 11830875, 11887195, 11893927, 12115963, 12135079, 12201627, 12294139, 12393583, 12417703, 12436059, 12569851, 12576763, 12598011, 12644007, 12663547, 12664827, 12671739, 12718683, 12745723, 12803067, 12807835, 12887643, 13025275, 13052583, 13217511, 13224423, 13424359, 13592647, 13728507, 13729531, 13734567, 13777659, 13806247, 13879975, 13886715, 14079067, 14155771, 14292987, 14334715, 14335995, 14352379, 14384295, 14401531, 14409723, 14443099, 14617339, 14786299, 14843643, 14897755, 14936743, 14985895, 14994087, 15052239, 15094951, 15095803, 15195247, 15237723, 15276027, 15358971, 15515643, 15609499, 15611899, 15673339, 15745627, 15753819, 15815259, 15914727, 16051879, 16072699, 16295515, 16333563, 16625383, 16674727, 16717819, 16726011, 16811611, 16859739, 16968795, 16975707, 17136379, 17175643, 17185959, 17316603, 17399547, 17416795, 17507067, 17541799, 17549991, 17635323, 17652475, 17694715, 17699419, 17713915, 17783803, 17925723, 18004647, 18071547, 18084975, 18091887, 18156795, 18163707, 18192379, 18205159, 18291823, 18334299, 18460111, 18499227, 18506139, 18524839, 18607783, 18706075, 18715303, 18716391, 18758395, 18766587, 18852519, 18867451, 19011579, 19016283, 19054587, 19087963, 19161691, 19365031, 19390459, 19402407, 19666011, 19675899, 19824379, 19918503, 19974823, 20077563, 20133883, 20204283, 20218459, 20232955, 20267611, 20275803, 20282535, 20376667, 20496379, 20504571, 20523687, 20682747, 20782191, 20806311, 20884135, 20895463, 20914939, 20958459, 20965371, 21052155, 21134331, 21196443, 21333595, 21412519, 21413883, 21430267, 21492847, 21564667, 21812967, 21907099, 21956443, 21970939, 21981255, 22118139, 22166695, 22174459, 22194855, 22260391, 22268583, 22413307, 22467675, 22544379, 22689787, 22723323, 22740987, 22790139, 22823515, 22831707, 23005947, 23072623, 23144443, 23174907, 23286363, 23325351, 23326375, 23374503, 23483559, 23484411, 23486875, 23583855, 23697127, 23806555, 23889499, 23912443, 23931559, 23992315, 23997019, 23998107, 24000507, 24061947, 24134235, 24214183, 24366331, 24383143, 24440487, 24461307, 24542203, 24646747, 24684123, 24730363, 24772603, 25013991, 25058299, 25063335, 25106427

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