

A BIJECTION BETWEEN APERIODIC PALINDROMES AND CONNECTED CIRCULANT GRAPHS

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ABSTRACT. In this paper, we show that there is a one-to-one correspondence between the set of compositions (resp. prime compositions) of n and the set of circulant digraphs (resp. connected circulant digraphs) of order n . We also show that there is a one-to-one correspondence between the set of palindromes (resp. aperiodic palindromes) of n and the set of circulant graphs (resp. connected circulant graphs) of order n . As a corollary of this correspondence, we enumerate the number of connected circulant (di)graphs of order n .

1. INTRODUCTION

A *composition* $\sigma = \sigma_1\sigma_2\dots\sigma_m$ of n is an ordered word of one or more positive integers whose sum is n . The number of summands m is called the *number of parts* of the composition σ . A composition of n without order gives a partition of n . A composition of n with k parts is *aperiodic* if its period is k . In other words, a composition is aperiodic if it is not the concatenation of a proper part of given composition. A composition $\sigma = \sigma_1\sigma_2\dots\sigma_m$ is called *prime* if $\gcd(\sigma) = \gcd\{\sigma_1, \dots, \sigma_m\} = 1$. A *numeral palindrome* (or simply, *palindrome*) is a composition in which the summands in given order are the same with those in the reverse order (*i.e.*, $\sigma = \sigma^{-1}$ where $\sigma^{-1} = \sigma_m\dots\sigma_2\sigma_1$ for $\sigma = \sigma_1\sigma_2\dots\sigma_m$). It is known in [4] that the number of palindromes of $n \geq 2$ is $2^{\lfloor \frac{n}{2} \rfloor}$. The first 30 palindromes in decimal can be found as the sequence A002113 in OEIS [7]. Several types of palindromes are studied (see [7], [8]). The number of aperiodic palindromes of n with k parts ($1 \leq k \leq n$) is studied but the numbers are known only for $n \leq 55$ (see OEIS [7]). However it is still unknown whether there exist infinitely many palindromic primes or not, where a palindromic prime is a positive integer which is prime and also a palindrome. Although palindromes are often considered in the decimal system, the concept of palindromicity can be generalized to the natural numbers in any numeral system. An integer $m > 0$ is called *palindromic* in base $b \geq 2$ if it is written in standard notation

$$m = \sum_{i=0}^k a_i b^i \quad (a_k \neq 0)$$

with $k + 1$ digits a_i ($0 \leq i \leq k$) satisfying $a_i = a_{k-i}$ and $0 \leq a_i < b$ for all $0 \leq i \leq k$.

A *circulant graph* is a graph whose automorphism group includes a cyclic subgroup which acts transitively on the vertex set of the graph. For a subset $S \subseteq \mathbb{Z}_n$ satisfying

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$S = -S \pmod n$, a circulant graph of order n denoted by $G(n, S)$ is a graph with vertex set $\{0, 1, \dots, n-1\}$ and edge set E , where $\{i, j\} \in E$ if and only if $i \neq j$ and $j - i \in S \pmod n$. A *circulant digraph* is also defined without the condition $S = -S$. That is, for a subset $S \subseteq \mathbb{Z}_n$, a circulant digraph $G(n, S)$ is a digraph with vertex set $\{0, 1, \dots, n-1\}$ and arc set A , where $(i, j) \in A$ if and only if $i \neq j$ and $j - i \in S \pmod n$.

Isomorphism problem of circulant graphs had been studied by several authors (see [1, 3]) and it is completely solved by Muzychuk [6]. In [5], Kim, Kwon and Lee found degree distribution polynomials for the equivalence classes of circulant graphs of several types of order. They also found an enumeration formula for the number of equivalence classes of circulant graphs and they listed in [5, Table 1] the degree distribution polynomials $\Psi_{\mathbb{Z}_n}(x)$ and the number of equivalence classes of circulant graphs $\mathcal{E}(\mathbb{Z}_n)$ for $1 \leq n \leq 20$. We observe from [5, Table 1] that the number of equivalence classes of circulant graphs $\mathcal{E}(\mathbb{Z}_n)$ is equal to the number of aperiodic palindromes of n for $1 \leq n \leq 20$. This leads us to study a coincidence between the set of circulant graphs of order n and the set of aperiodic palindromes of n . In this paper, we study the coincidence and we extend this to circulant digraphs of order n and compositions of n .

The outline of this paper is as follows. In Section 2, we show that there is a one-to-one correspondence between the set of compositions of n and the set of circulant digraphs of order n (see Theorem 2.1). In particular, we also show that this bijection in Theorem 2.1 guarantees a one-to-one correspondence between the set of prime compositions of n and the set of connected circulant digraphs of order n (see Theorem 2.3). As an application, we enumerate the number of connected circulant digraphs (i.e., the number of prime compositions), disconnected circulant digraphs and circulant digraphs of outdegree k (see Corollary 2.4). In Section 3, we first show that there is a one-to-one correspondence between the set of palindromes of n and the set of circulant graphs of order n (see Theorem 3.1). In particular, we also show that this bijection in Theorem 3.1 guarantees a one-to-one correspondence between the set of aperiodic palindromes of n and the set of connected circulant graphs of order n (see Theorem 3.5). As a corollary, we give an enumeration formula of the number of aperiodic palindromes of n (i.e., the number of connected circulant graphs of order n) in Corollary 3.6.

2. ONE-TO-ONE CORRESPONDENCE BETWEEN THE CIRCULANT DIGRAPHS OF ORDER n AND THE COMPOSITIONS OF n

In this section, we first show that there is a bijection between the set of compositions of n and the set of circulant digraphs of order n in Theorem 2.1. In particular, we also show that this bijection in Theorem 2.1 guarantees a bijection between the set of prime compositions of n and the set of connected circulant digraphs of order n (see Theorem 2.3). Moreover, we will enumerate the number of connected circulant digraphs of order n (i.e., the number of prime compositions of n) by using the Möbius inversion formula (see Corollary 2.4).

For each integer $n \geq 1$, we define

$$C(n) = \{\sigma \mid \sigma \text{ is a composition of } n\} \text{ and } \mathcal{G}_C(n) = \{\Omega \mid \Omega \subseteq \mathbb{Z}_n, 0 \in \Omega\}.$$

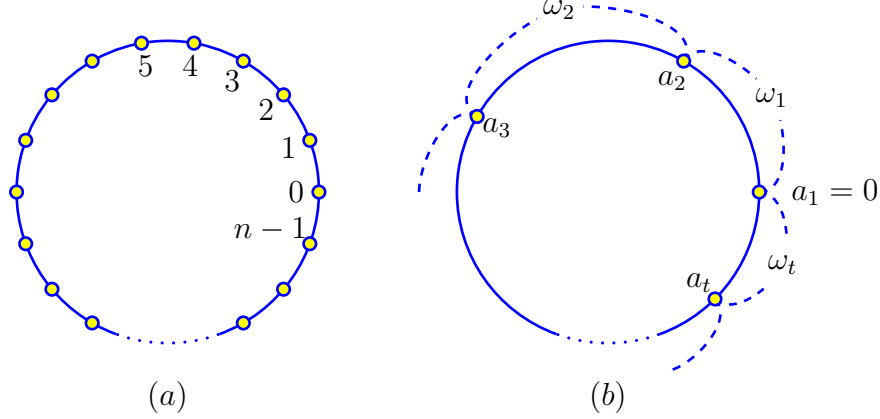


FIGURE 1. (a) C_n (b) $\Omega = \{a_1, a_2, \dots, a_t\}$ and $\sigma_\Omega = \omega_1\omega_2 \cdots \omega_t$

Without loss of generality, each element $\Omega \in \mathcal{G}_C(n)$ with $|\Omega| = t \geq 1$ is denoted by $\Omega = \{a_1, a_2, \dots, a_t\}$ with $0 = a_1 < a_2 < \cdots < a_t$.

Note here that in view of [5, Theorem 2.2], there is a bijection between the set $\mathcal{G}_C(n)$ and the set of equivalence classes of circulant digraphs of order n . In Table 1, we consider the elements Ω of $\mathcal{G}_C(5)$, the corresponding circulant digraphs $G(5, \Omega)$ of order 5 and the corresponding compositions σ_Ω of 5 which will be defined in (1).

In the following, we define two notations which will be used to consider the relationship between $C(n)$ and $\mathcal{G}_C(n)$. Let C_n be the cycle of length n with vertex set $\mathbb{Z}_n = \{0, 1, 2, \dots, n-1\}$ (see Figure 1 (a)). We consider each element $\Omega \in \mathcal{G}_C(n)$ as a subset of the vertex set of C_n . For each element $\Omega = \{a_1, a_2, \dots, a_t\}$ of $\mathcal{G}_C(n)$, we define a composition

$$(1) \quad \sigma_\Omega = \omega_1\omega_2 \cdots \omega_t \in C(n)$$

where $\omega_i = a_{i+1} - a_i$ for each $i = 1, 2, \dots, t-1$ and $\omega_t = n - a_t$ (see Figure 1 (b)). Notice that ω_i is the number of encountered edges when we move anticlockwise from a_i to a_{i+1} for each $1 \leq i \leq t$ where $a_{t+1} := a_1$.

Conversely, for each composition $\sigma = \sigma_1\sigma_2 \cdots \sigma_\ell \in C(n)$, we define a set

$$(2) \quad \Omega_\sigma = \{s_1, s_2, \dots, s_\ell\} \in \mathcal{G}_C(n)$$

where $s_1 = 0$, $s_i = \sum_{j=1}^{i-1} \sigma_j$ for each $i = 2, \dots, \ell$. Note that if $\sigma = n \in C(n)$ and $\sigma' = \overbrace{11 \cdots 1}^{n \text{ times of } 1} \in C(n)$ then $\Omega_\sigma = \{0\}$ and $\Omega_{\sigma'} = \{0, 1, 2, \dots, n-1\}$, respectively.

In the following theorem, we will show that these constructions σ_Ω and Ω_σ in (1) and (2) give a one-to-one correspondence between the sets $\mathcal{G}_C(n)$ and $C(n)$.

Theorem 2.1. *For each integer $n \geq 1$, the map $\psi : \mathcal{G}_C(n) \rightarrow C(n)$ defined by $\psi(\Omega) = \sigma_\Omega$ is a bijection (i.e., there is a one-to-one correspondence between the set of circulant digraphs of order n and the set of compositions of n).*

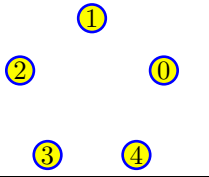
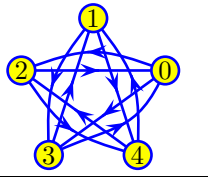
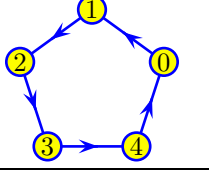
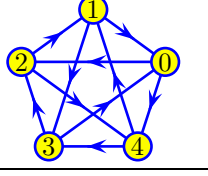
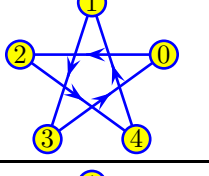
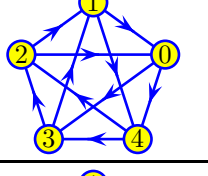
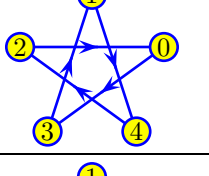
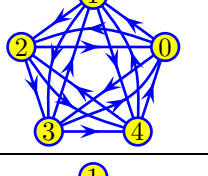
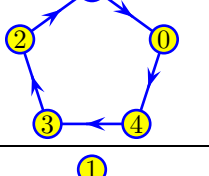
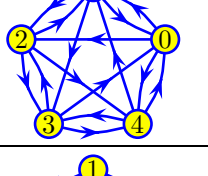
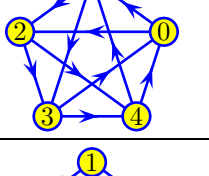
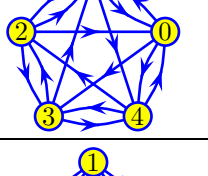
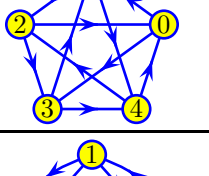
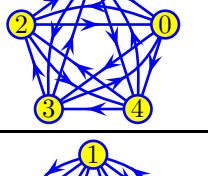
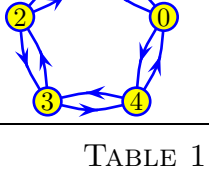
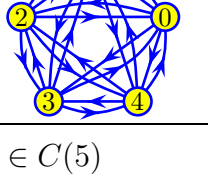
$\Omega \in \mathcal{G}_C(5)$	$G(5, \Omega)$	$\sigma_\Omega \in C(5)$	$\Omega \in \mathcal{G}_C(5)$	$G(5, \Omega)$	$\sigma_\Omega \in C(5)$
$\{0\}$		5	$\{0, 2, 3\}$		212
$\{0, 1\}$		14	$\{0, 2, 4\}$		221
$\{0, 2\}$		23	$\{0, 3, 4\}$		311
$\{0, 3\}$		32	$\{0, 1, 2, 3\}$		1112
$\{0, 4\}$		41	$\{0, 1, 2, 4\}$		1121
$\{0, 1, 2\}$		113	$\{0, 1, 3, 4\}$		1211
$\{0, 1, 3\}$		122	$\{0, 2, 3, 4\}$		2111
$\{0, 1, 4\}$		131	$\{0, 1, 2, 3, 4\}$		11111

TABLE 1. $\Omega \in \mathcal{G}_C(5)$, $G(5, \Omega)$ and $\sigma_\Omega \in C(5)$

Proof. It follows by (1) that the map ψ is well-defined and injective. For any composition $\sigma \in C(n)$, a set Ω_σ in $\mathcal{G}_C(n)$ satisfies $\psi(\Omega_\sigma) = \sigma$, see (2). Thus the map ψ is surjective and this completes the proof. \square

Let $n \geq 2$ be an integer. For each integer $1 \leq k \leq n$, let

$$C(n, k) = \{\sigma \in C(n) \mid \sigma \text{ has } k \text{ parts}\} \text{ and } \mathcal{G}_C(n, k) = \{\Omega \in \mathcal{G}_C(n) \mid |\Omega| = k\}.$$

For each $\Omega \in \mathcal{G}_C(n, k)$, the corresponding circulant digraph $G(\mathbb{Z}_n, \Omega)$ is a regular graph with outdegree $k - 1$. Since the bijection ψ in Theorem 2.1 naturally matches $\mathcal{G}_C(n, k)$ and $C(n, k)$, and by $|\mathcal{G}_C(n, k)| = \binom{n-1}{k-1}$, we can find immediately the number of compositions of n with k parts as follows.

Corollary 2.2. *Let $n \geq 2$ be an integer. For each integer $1 \leq k \leq n$, the number of compositions of n with k parts is $\binom{n-1}{k-1}$.*

For each $\sigma = \sigma_1 \sigma_2 \cdots \sigma_\ell \in C(n)$ and for each $\Omega = \{a_1, a_2, \dots, a_t\} \in \mathcal{G}_C(n)$, define

$$\gcd(\sigma) = \gcd\{\sigma_1, \sigma_2, \dots, \sigma_\ell\} \text{ and } \gcd(\Omega) = \gcd\{a_1, a_2, \dots, a_t\}.$$

For each integer $n \geq 2$, we define

$$C(n)^* = \{\sigma \in C(n) \mid \gcd(\sigma) = 1\} \text{ and } \mathcal{G}_C(n)^* = \{\Omega \in \mathcal{G}_C(n) \mid \gcd(\Omega) = 1\}.$$

In the following theorem, we will show that the bijection ψ in Theorem 2.1 guarantees a one-to-one correspondence between $\mathcal{G}_C(n)^*$ and $C(n)^*$.

Theorem 2.3. *For each integer $n \geq 2$, there is a one-to-one correspondence between $\mathcal{G}_C(n)^*$ and $C(n)^*$.*

Proof. To prove the result, we will show that the map $\psi|_{\mathcal{G}_C(n)^*} : \mathcal{G}_C(n)^* \rightarrow C(n)^*$ defined by $\psi|_{\mathcal{G}_C(n)^*}(\Omega) = \psi(\Omega)$ is a bijection, where $\psi : \mathcal{G}_C(n) \rightarrow C(n)$ is the bijection defined in Theorem 2.1. It is enough to show that $\psi|_{\mathcal{G}_C(n)^*}$ is well-defined and surjective since it is injective by Theorem 2.1. For each $\Omega = \{a_1, a_2, \dots, a_t\} \in \mathcal{G}_C(n)^*$, $\gcd\{a_1, a_2, \dots, a_t\} = 1$ and thus it follows by the Euclidean algorithm that

$$1 = \gcd\{a_1, a_2, \dots, a_t\} = \gcd\{a_2 - a_1, a_3 - a_2, \dots, n - a_t\} = \gcd(\sigma_\Omega) = \gcd(\psi|_{\mathcal{G}_C(n)^*}(\Omega)).$$

Hence $\psi|_{\mathcal{G}_C(n)^*}(\Omega) \in C(n)^*$ and thus $\psi|_{\mathcal{G}_C(n)^*}$ is well-defined. Similarly, one can check that $\psi|_{\mathcal{G}_C(n)^*}$ is surjective since for each $\sigma \in C(n)^*$, $\psi|_{\mathcal{G}_C(n)^*}(\Omega_\sigma) = \sigma$ holds for $\Omega_\sigma \in \mathcal{G}_C(n)^*$. \square

As an application, we now enumerate the number of connected circulant digraphs. It follows by Theorem 2.1 that the cardinality of $C(n)$ is equal to the number of subsets of \mathbb{Z}_n that contain 0, i.e., $|C(n)| = 2^{n-1}$. By Theorem 2.3, $|C(n)^*|$ is equal to the number of generating set of \mathbb{Z}_n that contains 0. Hence $|C(n)^*|$ is the number of connected circulant digraphs of order n , and thus the cardinality of $C(n) \setminus C(n)^* = \{\sigma \in C(n) \mid \gcd(\sigma) \neq 1\}$ is the number of disconnected circulant digraphs of order n . In the following corollary, we enumerate these two numbers $|C(n)^*|$ and $|C(n) \setminus C(n)^*|$.

Corollary 2.4. *Let $n \geq 2$ be an integer. Then the following hold:*

- (i) $|C(n)| = 2^{n-1}$,
- (ii) $|C(n)^*| = \sum_{d|n} \mu\left(\frac{n}{d}\right) 2^{d-1}$ and
- (iii) $|C(n) \setminus C(n)^*| = \sum_{d|n, d \neq n} |C(d)^*| = \sum_{d|n, d \neq n} -\mu\left(\frac{n}{d}\right) 2^{d-1}$,

n	1	2	3	4	5	6	7	8	9	10
$ C(n)^* $	1	1	3	6	15	27	63	120	252	495
$ C(n) \setminus C(n)^* $	0	1	1	2	1	5	1	8	4	17
n	11	12	13	14	15	16	17	18	19	20
$ C(n)^* $	1,023	2,010	4,095	8,127	16,365	32,640	65,535	130,788	262,143	523,770
$ C(n) \setminus C(n)^* $	1	38	1	65	19	128	1	284	1	518
n	21	22	23	24	25	26	27	28	29	30
$ C(n) \setminus C(n)^* $	67	1,025	1	2,168	16	4,097	256	9,198	1	16,905
n	31	32	33	34	35	36	37	38	39	40
$ C(n) \setminus C(n)^* $	1	32,768	1,027	65,537	79	133,090	1	262,145	4,099	524,282

TABLE 2. $|C(n)^*|$ and $|C(n) \setminus C(n)^*|$ up to $n = 40$.

where μ is the Möbius function.

Proof. (i): It follows by Theorem 2.1.

(ii)-(iii): Note that for each $\sigma = \sigma_1 \cdots \sigma_\ell \in C(n)$ with $\gcd(\sigma) = d \geq 1$, composition $\frac{1}{d}\sigma := (\frac{\sigma_1}{d}) \cdots (\frac{\sigma_\ell}{d})$ satisfies $\frac{1}{d}\sigma \in C(\frac{n}{d})^*$. We now observe $|C(n)| = |\cup_{d|n} C(d)^*| = \sum_{d|n} |C(d)^*|$. Now, by applying the Möbius inversion formula with $|C(d)| = 2^{d-1}$ by (i), the result (ii) follows. Now (iii) follows as $|C(n) \setminus C(n)^*| = |C(n)| - |C(n)^*|$ and $\mu(1) = 1$. \square

Using Corollary 2.4, we enumerate $|C(n)^*|$ and $|C(n) \setminus C(n)^*|$ up to $n = 40$ in Table 2.

Example 2.5. *The number of connected circulant digraphs and the number of disconnected circulant digraphs of order 72 are 23, 611, 832, 414, 004, 545, 432, 040 and 34, 368, 074, 808, respectively.*

Proof. The prime factorization of 72 is $2^3 \cdot 3^2$. There are 11 proper divisors, 1, 2, 3, 4, 6, 8, 9, 12, 18, 24 and 36. In Table 2, we find their $|C(k)^*|$ as 1, 1, 3, 6, 25, 120, 252, 2012, 130790, 8336440 and 34359605278. By summing all, we have $|C(72) \setminus C(72)^*| = 34, 368, 074, 808$ and $|C(72)^*| = 23, 611, 832, 414, 004, 545, 432, 040$. \square

3. ONE-TO-ONE CORRESPONDENCE BETWEEN THE PALINDROMES OF n AND THE CIRCULANT GRAPHS OF ORDER n

In this section, we first show that there is a bijection between the set of palindromes of n and the set of circulant graphs of order n in Theorem 3.1. In particular, we also show that this bijection in Theorem 3.1 guarantees a one-to-one correspondence between the set of aperiodic palindromes of n and the set of connected circulant graphs of order n (see Theorem 3.5).

For each integer $n \geq 2$, we define

$$\mathcal{G}_P(n) = \{\Omega \in \mathcal{G}_C(n) \mid \Omega = \Omega^{-1}\} \text{ and } P(n) = \{\sigma \in C(n) \mid \sigma = \sigma^{-1}\}.$$

Note that the set $P(n)$ is the set of palindromes of n . Without loss of generality, each element $\Omega \in \mathcal{G}_P(n)$ with $|\Omega| = t \geq 2$ is denoted by a set $\Omega = \{a_1, a_2, \dots, a_t\}$ satisfying

$$(3) \quad 0 = a_1 < a_2 < \dots < a_t \text{ and } a_i + a_{t+2-i} = n, \quad i = 2, \dots, t.$$

In the following theorem, we will show that the construction σ_Ω in (1) for an element Ω of $\mathcal{G}_P(n)$ guarantees one-to-one correspondence between $\mathcal{G}_P(n)$ and $P(n)$.

Theorem 3.1. *For each integer $n \geq 2$, there is a one-to-one correspondence between the set of palindromes of n and the set of circulant graphs of order n .*

Proof. To prove the result, we will show that the map $\psi|_{\mathcal{G}_P(n)} : \mathcal{G}_P(n) \rightarrow P(n)$ defined by $\psi|_{\mathcal{G}_P(n)}(\Omega) = \psi(\Omega) = \sigma_\Omega$ is a bijection, where $\psi : \mathcal{G}_C(n) \rightarrow C(n)$ is the bijection defined in Theorem 2.1. To show that the map $\psi|_{\mathcal{G}_P(n)}$ is well-defined and injective, it is enough to show $\sigma_\Omega = \sigma_\Omega^{-1}$ for each $\Omega = \{a_1, \dots, a_t\} \in \mathcal{G}_P(n)$ as ψ is bijective. If $t = 1$ then $\Omega = \{a_1 = 0\}$ and $\sigma_\Omega = n = \sigma_\Omega^{-1}$. Now let $t \geq 2$ and $\sigma_\Omega = \omega_1 \cdots \omega_t$ in (1). As $\omega_t = n - a_t = a_2 = \omega_1$ and $\omega_{t+1-i} = a_{t+2-i} - a_{t+1-i} = (n - a_i) - (n - a_{i+1}) = a_{i+1} - a_i = \omega_i$ ($i = 2, \dots, t-1$) follow by (1) and (3), we find $\sigma_\Omega = \sigma_\Omega^{-1}$. Let $\sigma = \sigma_1 \cdots \sigma_m$ be an element in $P(n)$ and let $\Omega_\sigma = \{a_1, \dots, a_m\}$ (see (2)). To show $\psi|_{\mathcal{G}_P(n)}$ is surjective, we will show $\Omega_\sigma = \Omega_\sigma^{-1}$ as $\Omega_\sigma \in \mathcal{G}_C(n)$ and $\psi(\Omega_\sigma) = \sigma$ in Theorem 2.1. If $m = 1$ then $\sigma = n$ and thus we find $\Omega_\sigma = \{0\} = \Omega_\sigma^{-1}$. Let $m \geq 2$. As $\sigma \in P(n)$, $\sigma_i = \sigma_{m+1-i}$ for all $i = 1, \dots, m$ and $\sum_{j=1}^m \sigma_j = n$. Thus $a_1 = 0$, $a_2 + a_m = \sigma_1 + \sum_{j=1}^{m-1} \sigma_j = \sum_{j=1}^m \sigma_j = n$ by (2), and

$$a_i + a_{m+2-i} = \sum_{j=1}^{i-1} \sigma_j + \sum_{k=1}^{m+1-i} \sigma_k = \sum_{j=1}^{i-1} \sigma_j + \sum_{k=1}^{m+1-i} \sigma_{m+1-k} = \sum_{j=1}^m \sigma_j = n, \quad i = 2, \dots, m.$$

This shows $\Omega_\sigma = \Omega_\sigma^{-1}$, and thus the map $\psi|_{\mathcal{G}_P(n)}$ is surjective. This completes the proof. \square

As we mentioned earlier, it is already known that the number of palindromes of $n \geq 2$ is $2^{\lfloor \frac{n}{2} \rfloor}$ (see [4]). By using Theorem 3.1, we have very short and elementary proof for the same statement.

Corollary 3.2. *For each integer $n \geq 2$, the number of palindromes of n is $2^{\lfloor \frac{n}{2} \rfloor}$.*

Proof. Let $n \geq 2$ be an integer and consider $\lfloor \frac{n}{2} \rfloor$ sets $\{i, n-i\}$, $i = 1, \dots, \lfloor \frac{n}{2} \rfloor$. As $\Omega^{-1} = \{n-i \mid i \in \Omega\}$, the cardinality of the set $\mathcal{G}_P(n) = \{\Omega \in \mathcal{G}_C(n) \mid \Omega = \Omega^{-1}\}$ is $2^{\lfloor \frac{n}{2} \rfloor}$. As the number of palindromes of n is equal to $|\mathcal{G}_P(n)|$ by Theorem 3.1, the result follows. \square

Example 3.3. *In Table 3, the circulant graphs of order 8 and the corresponding palindromes in Theorem 3.1 are presented.*

There are four disconnected circulant graphs out of sixteen circulant graphs of order 8, where their generating sets are $\{0\}$, $\{0, 4\}$, $\{0, 2, 6\}$ and $\{0, 2, 4, 6\}$, see Table 3. On the other hand, there are four periodic palindromes $(4)^2$, $(2)^4$, $(121)^2$ and $(1)^8$. One can see that the bijection in Theorem 3.1 does not preserve the connectedness of the circulant graph $G(n, \Omega)$ to the aperiodicity of the corresponding palindromes σ_Ω as we have seen in Example 3.3.

For each integer $n \geq 2$, we now define

$$P_A(n) = \{\sigma \in P(n) \mid \sigma \text{ is aperiodic}\} \text{ and } \mathcal{G}(n) = \{\Omega \in \mathcal{G}_P(n) \mid \langle \Omega \rangle = \mathbb{Z}_n\}.$$

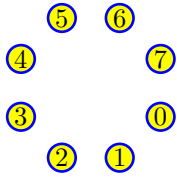
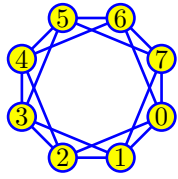
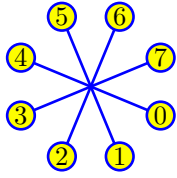
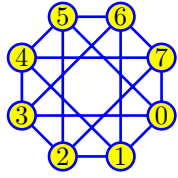
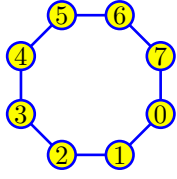
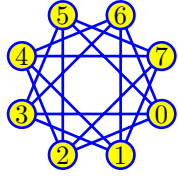
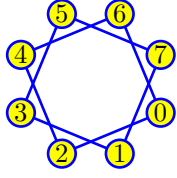
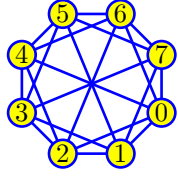
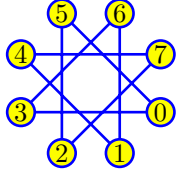
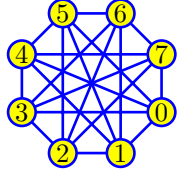
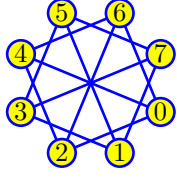
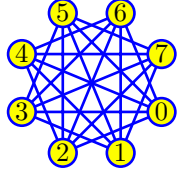
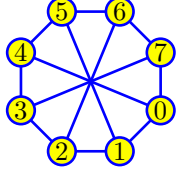
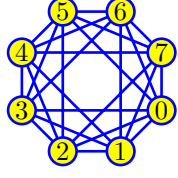
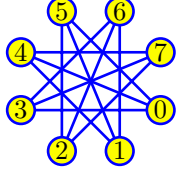
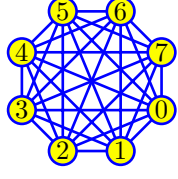
$\Omega \in \mathcal{G}_P(8)$	$G(8, \Omega)$	$\sigma_\Omega \in P(8)$	$\Omega \in \mathcal{G}_P(8)$	$G(8, \Omega)$	$\sigma_\Omega \in P(8)$
$\{0\}$		8	$\{0, 1, 2, 6, 7\}$		11411
$\{0, 4\}$		$(4)^2$	$\{0, 1, 3, 5, 7\}$		12221
$\{0, 1, 7\}$		161	$\{0, 2, 3, 5, 6\}$		21212
$\{0, 2, 6\}$		242	$\{0, 1, 2, 4, 6, 7\}$		112211
$\{0, 3, 5\}$		323	$\{0, 1, 3, 4, 5, 7\}$		$(121)^2$
$\{0, 2, 4, 6\}$		$(2)^4$	$\{0, 2, 3, 4, 5, 6\}$		211112
$\{0, 1, 4, 7\}$		1331	$\{0, 1, 2, 3, 5, 6, 7\}$		1112111
$\{0, 3, 4, 5\}$		3113	$\{0, 1, 2, 3, 4, 5, 6, 7\}$		$(1)^8$

TABLE 3. The circulant graphs of order 8 and the corresponding palindromes

In the following lemma, we will show how $\gcd(\sigma_\Omega)$ classifies the connectedness of the corresponding circulant graphs $G(n, \Omega)$, and the result will be used to prove Theorem 3.5.

Lemma 3.4. *For each integer $n \geq 2$, the following are equivalent.*

- (i) *The circulant graph $G(n, \Omega)$ is connected.*
- (ii) *$\Omega \in \mathcal{G}(n)$.*
- (iii) *$\gcd(\sigma_\Omega) = 1$.*

Proof. Note that for each $\Omega \in \mathcal{G}_P(n)$, Ω is a generating set for group \mathbb{Z}_n if and only if $\gcd(\Omega) = 1$. By $\gcd(\Omega) = \gcd(\sigma_\Omega)$, the result now follows immediately. \square

To define a map between $P_A(n)$ and $\mathcal{G}(n)$ in Theorem 3.5, we need to define the following notation.

For any periodic palindrome of the form $\sigma = \overbrace{(c_1 \cdots c_k) \cdots (c_1 \cdots c_k)}^{r \text{ times of } (c_1 \cdots c_k)}$ (i.e., r times repetition of aperiodic palindrome $c_1 \cdots c_k$), we denote it by $\sigma = (c_1 \cdots c_k)^r$. For any aperiodic palindromes of the form $\sigma' = \sigma_1 \sigma_2 \cdots \sigma_m$, if $\gcd(\sigma') = d \neq 1$ then we define a periodic composition $\nu(\sigma') := (\frac{\sigma_1}{d} \frac{\sigma_2}{d} \cdots \frac{\sigma_m}{d})^d$.

In Example 3.3, among twelve aperiodic palindromes of 8, two decompositions 242, 8 have $\gcd(242) = 2$ and $\gcd(8) = 8$ and thus $\nu(242) = (121)^2$ and $\nu(8) = (1)^8$.

Theorem 3.5. *For each integer $n \geq 2$, there is a one-to-one correspondence between the set of aperiodic palindromes of n and the set of connected circulant graphs of order n .*

Proof. Define a map $\tau : P_A(n) \rightarrow \mathcal{G}(n)$ by

$$\tau(\sigma) = \begin{cases} (\psi|_{\mathcal{G}_P(n)})^{-1}(\sigma) = \Omega_\sigma & \text{if } \gcd(\sigma) = 1 \\ (\psi|_{\mathcal{G}_P(n)})^{-1}(\nu(\sigma)) = \Omega_{\nu(\sigma)} & \text{if } \gcd(\sigma) = d \neq 1 \end{cases}$$

where $\nu(\sigma) = (\frac{\sigma_1}{d} \frac{\sigma_2}{d} \cdots \frac{\sigma_m}{d})^d$ and $\psi|_{\mathcal{G}_P(n)}$ is the bijective map defined in Theorem 3.1. Then the map τ is well-defined as $\tau(\sigma) \in \mathcal{G}_P(n)$ by Theorem 3.1 and $\langle \Omega \rangle = \mathbb{Z}_n$ by Lemma 3.4 for $\gcd(\nu(\sigma)) = 1$. As the map τ is injective by Theorem 3.1, to prove the result, it is enough to show that τ is surjective. Take an element $\Omega = \{a_1, a_2, \dots, a_t\} \in \mathcal{G}(n)$. If $\psi|_{\mathcal{G}_P(n)}(\Omega) \in P_A(n)$ then it is easy to see that $\tau(\psi|_{\mathcal{G}_P(n)}(\Omega)) = \Omega$. Now suppose $\psi|_{\mathcal{G}_P(n)}(\Omega) \notin P_A(n)$. Then $\psi|_{\mathcal{G}_P(n)}(\Omega)$ is a periodic palindrome and $\gcd(\psi|_{\mathcal{G}_P(n)}(\Omega)) = 1$ by Lemma 3.4. Thus put $\psi|_{\mathcal{G}_P(n)}(\Omega) = (\sigma_1 \sigma_2 \cdots \sigma_m)^r$ for some $m \geq 1$ and $r \geq 2$, where palindrome $\sigma_1 \sigma_2 \cdots \sigma_m$ is aperiodic and $\gcd(\psi|_{\mathcal{G}_P(n)}(\Omega)) = \gcd\{\sigma_1, \sigma_2, \dots, \sigma_m\} = 1$. Then palindrome $(r\sigma_1)(r\sigma_2) \cdots (r\sigma_m)$ satisfies $(r\sigma_1)(r\sigma_2) \cdots (r\sigma_m) \in P_A(n)$ and $\tau((r\sigma_1)(r\sigma_2) \cdots (r\sigma_m)) = \Omega_{\nu((r\sigma_1)(r\sigma_2) \cdots (r\sigma_m))} = \Omega_{(\sigma_1 \sigma_2 \cdots \sigma_m)^r} = \Omega$, and thus the map τ is surjective. This completes the proof. \square

By combining Theorem 3.5 and [5, Corollary 3.2], the following corollary follows immediately.

Corollary 3.6. *For each integer $n \geq 2$, the number of aperiodic palindromes of n is*

$$\sum_{d|n} \mu\left(\frac{n}{d}\right) \left(2^{\lfloor \frac{d}{2} \rfloor} - 1\right)$$

where μ is the Möbius function.

Remark 3.7. One may find the number of aperiodic palindromes of n with $n \leq 127$ in [2]. This result improves the previously known result $n \leq 55$ in [7].

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