Enumeration of m-Endomorphisms

Louis Rubin and Brian Rushton

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Abstract

An m-endomorphism on a free semigroup is an endomorphism that sends every generator to a word of length $\leq m$. Two m-endomorphisms are combinatorially equivalent if they are conjugate under an automorphism of the semigroup. In this paper, we specialize an argument of N. G. de Bruijn to produce a formula for the number of combinatorial equivalence classes of m-endomorphisms on a rank-n semigroup. From this formula, we derive several little-known integer sequences.

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2 Introduction

Let D be a nonempty set of symbols, and let D^+ be the set of all finite strings of one or more elements of D. That is, $D^+ = \{d_1 \dots d_k | k \in \mathbb{N}, d_i \in D\}$. Paired with the operation of string concatenation, D^+ forms the **free semigroup** on D. If $d_1, \dots, d_k \in D$, then we refer to the natural number k as the **length** of the string $d_1 \dots d_k$. Denote the length of $W \in D^+$ by $|W|$.

By a **semigroup endomorphism** (or, simply, an **endomorphism**) on D^+, we mean a mapping $\phi : D^+ \to D^+$ satisfying $\phi(W_1 W_2) = \phi(W_1)\phi(W_2)$ for all $W_1, W_2 \in D^+$. Note that if ϕ is an endomorphism on D^+ and $d_1, \dots, d_k \in D$, then $\phi(d_1 \dots d_k) = \phi(d_1) \dots \phi(d_k)$; this shows that an endomorphism on D^+ is determined by its action on the elements of D. On the other hand, any mapping $f : D \to D^+$ extends uniquely to the endomorphism $\phi_f : D^+ \to D^+$ defined by $\phi_f(d_1 \dots d_k) = f(d_1) \dots f(d_k)$, and it is straightforward to verify that ϕ_f is an automorphism (that is, a bijective endomorphism) precisely when f is a bijection on D.

Example 1. Let $D = \{a, b\}$, and let $f : D \to D^+$ be defined by $f(a) = ab$ and $f(b) = a$. Then, for example,

$$\phi_f(ababa) = f(a)f(b)f(a)f(b)f(a) = abaabaab.$$

Let $End(D^+)$ be the collection of all endomorphisms on D^+, and let $m \in \mathbb{N}$. Then $\phi \in End(D^+)$ is called an ***m*-endomorphism** if and only if $|\phi(d)| \leq m$ for all $d \in D$. Note that the mapping ϕ_f from Example 1 is an m-endomorphism for all $m \geq 2$. Now let Γ be the set of all m-endomorphisms on D^+. That is,

$$\Gamma = \{\phi \in End(D^+) : \phi(D) \subseteq R\},$$

where $R = \{W \in D^+ : |W| \leq m\}$. Consider the set Ω consisting of all mappings $f : D \to R$. Then we may write

$$\Gamma = \{\phi_f : f \in \Omega\}.$$

We can put Γ into one-to-one correspondence with Ω by sending each m-endomorphism to its restriction to D. Moreover, if $|D| = n \in \mathbb{N}$, then the size of these sets is easily evaluated in view of the fact that $|R| = \sum_{i=1}^{m} n^i$. In particular, if $n > 1$, then $|R| = \frac{n^{m+1}-n}{n-1}$, and

$$|\Gamma| = |\Omega| = \left(\frac{n^{m+1} - n}{n - 1}\right)^n.$$

However, in this paper we shall be interested in counting the number of *classes* of m-endomorphisms under a particular equivalence relation. To motivate our definition of equivalence on Γ, we define a relation $\sim$ on Ω as follows:

$$f_1 \sim f_2 \iff \text{there exists a bijection } g : D \to D \text{ such that } f_2 \circ g = \phi_g \circ f_1.$$

As an exercise, the reader may wish to verify that $\sim$ satisfies the reflexive, symmetric, and transitive properties required of any equivalence relation. In §2.1, however, it will be shown that $\sim$ is a specific instance of a well-known equivalence relation induced by a group acting on a nonempty set.

Example 2. Let f be as in Example 1 (with $D = \{a, b\}$). Consider the bijection $g : D \to D$ defined by $g(a) = b$ and $g(b) = a$. Now let $f_1 : D \to D^+$ be given by $f_1(a) = b$ and $f_1(b) = ba$. Then

$$(f_1 \circ g)(a) = f_1(g(a)) = f_1(b) = ba = g(a)g(b) = \phi_g(ab) = \phi_g(f(a)) = (\phi_g \circ f)(a)$$

and

$$(f_1 \circ g)(b) = f_1(g(b)) = f_1(a) = b = g(a) = \phi_g(a) = \phi_g(f(b)) = (\phi_g \circ f)(b),$$

which shows that $f \sim f_1$.

Remark 1. Perhaps a more intuitive illustration of $\sim$ is as follows. If we let f and f_1 be as in the preceding example, then the respective graphs of f and f_1 are $\{(a, ab), (b, a)\}$ and $\{(a, b), (b, ba)\}$. But the graph of f_1 can be obtained by applying the bijection g to each element of D that appears in the graph of f. In other words,

$$\{(g(a), g(a)g(b)), (g(b), g(a))\} = \{(a, b), (b, ba)\}.$$

Since the graphs of f and f_1 are "the same" up to a permutation of a and b, we wish to consider these mappings equivalent, and $\sim$ provides the desired equivalence relation.

Extending $\sim$ to an equivalence relation on Γ leads to the following definition. If $f, h \in \Omega$, then ϕ_f is **combinatorially equivalent** to ϕ_h if and only if there exists a bijection $g : D \to D$ such that $\phi_h \circ \phi_g = \phi_g \circ \phi_f$. To state precisely the aim of this paper: Given a set of symbols D with $|D| = n$, we wish to produce a formula for the number of equivalence classes in Γ under the relation of combinatorial equivalence. To this end, we shall specialize an argument of N. G. de Bruijn (namely, that for Theorem 1 in [1]) to produce a formula for the number of classes in Ω under the relation $\sim$. But it is easy to check that for all $f, h \in \Omega$, $f \sim h$ if and only if ϕ_f is combinatorially equivalent to ϕ_h. Hence, there is a well-defined correspondence given by

$$[f] \leftrightarrow [\phi_f]$$

between the equivalence classes in Ω and those in Γ, and it follows that our formula will also provide the number of m-endomorphisms on D^+ up to combinatorial equivalence. Moreover, once this formula is obtained, we can fix one of the variables n, m and let the other run through the natural numbers in order to derive integer sequences, many of which appear to be little-known.

2.1 Group Actions

For the reader's convenience, we review group actions. The following material (through Proposition 1) is paraphrased from [5]. Let G be a group and S a nonempty set. A **left action** of G on S is a function

$$\cdot : G \times S \to S,$$
$$\cdot(g, s) \to g \cdot s$$

such that for all $g_1, g_2 \in G$ and for all $s \in S$,

1. $(g_1 g_2) \cdot s = g_1 \cdot (g_2 \cdot s)$ (where $g_1 g_2$ denotes the product of g_1, g_2 in G), and
2. $e \cdot s = s$ (where e is the identity element of G).

A left action induces the well-known equivalence relation E on the set S given by

$$(a,b) \in E \iff g \cdot a = b \text{ for some } g \in G,$$

for all $a, b \in S$. We refer to the equivalence classes under this relation as the **orbits** of G on S. The following result (known as "Burnside's Lemma") gives an expression for the number of these, provided that G and S are finite.

Proposition 1. [5] Let S be a finite, nonempty set, and suppose there is a left action of a finite group G on S. Then the number of orbits of G on S is

$$\tfrac{1}{|G|} \sum_{g \in G} |\{s \in S : g \cdot s = s\}|.$$

Thus, the number of orbits of G on S equals the average number of elements of S that are "fixed" by an element of G. We now show that the relation $\sim$ from §2 is a specific instance of the relation E described above. To see this, let D be a finite nonempty set, and let $Sym(D)$ denote the symmetric group on D (i.e., the group of all bijections on D). Then $Sym(D)$ acts on the set Ω according to the rule

$$g \cdot f = \phi_g \circ f \circ g^{-1},$$

for all $g \in Sym(D), f \in \Omega$. (One can easily verify that $\cdot$ defined in this way is indeed a left action.) Now, for any $f_1, f_2 \in \Omega$, we have

$$\begin{aligned} f_1 \sim f_2 &\iff f_2 \circ g = \phi_g \circ f_1 \text{ for some } g \in Sym(D) \\ &\iff f_2 = \phi_g \circ f_1 \circ g^{-1} \text{ for some } g \in Sym(D) \\ &\iff g \cdot f_1 = f_2 \text{ for some } g \in Sym(D) \\ &\iff (f_1, f_2) \in E. \end{aligned}$$

It follows that the equivalence classes in Ω under the relation $\sim$ are just the orbits of $Sym(D)$ on Ω. Enumerating the elements of $Sym(D)$ by $g_1, \ldots, g_{n!}$, we find the number of orbits to be

$$\frac{1}{n!} \sum_{r=1}^{n!} |\{f \in \Omega : f \circ g_r = \phi_{g_r} \circ f\}| . \tag{1}$$

For any permutation g of a finite set, and for each natural number j, let $c(g,j)$ denote the number of cycles of length [1] j occurring in the cycle decomposition of g. (This notation comes from [1].) The quantities $c(g,j)$ will play a role in the evaluation of $|\{f \in \Omega : f \circ g_r = \phi_{g_r} \circ f\}|$, which occurs in the next section. Our evaluation is a modification of de Bruijn's counting argument in §5.12 of [2].

[1] There should be no confusion between the notions of 'string length' and 'cycle length'.

3 Main Results

We now produce a formula for the number of equivalence classes in Ω under the relation $\sim$. Let D be a finite set, and suppose that $g \in Sym(D)$ is the product of disjoint cycles of lengths $k_1, k_2, \ldots, k_\ell$, where $k_1 \le k_2 \le \ldots \le k_\ell$. Then the sequence $k_1, k_2, \ldots, k_\ell$ is called the **cycle type** of g. For example, if $D = \{a, b, c, d, e\}$, then the permutation $g = (a)(b, c)(d, e)$ has cycle type 1,2,2. The following lemma will be useful.

Lemma 1. *Let D be a finite set, and let $g \in Sym(D)$ have cycle type $k_1, k_2, \ldots, k_\ell$. For each $1 \le i \le \ell$, select a single $d_i \in D$ from the cycle corresponding to k_i. (Thus, k_i is the smallest natural number such that $g^{k_i}(d_i) = d_i$.) Now suppose that $f \in \Omega$. Then $f \circ g = \phi_g \circ f$ if and only if for each $1 \le i \le \ell$, the following holds:*

1. *$(f \circ g^j)(d_i) = (\phi_g^j \circ f)(d_i)$ for all $j \in \mathbb{N}$.*
2. *$f(d_i)$ is of the form $d'_1 \ldots d'_{k \le m}$, where $d'_1, \ldots, d'_k \in D$ each belong to a cycle in g whose length divides k_i.*

Proof. First assume that $f \circ g = \phi_g \circ f$. Then condition 1 follows from an inductive argument. But $f(d_i) = f(g^{k_i}(d_i)) = \phi_g^{k_i}(f(d_i))$. Write $f(d_i) = d'_1 \ldots d'_k$, where $d'_1, \ldots, d'_k \in D$ and $k \le m$. Then

$$d'_1 \ldots d'_k = \phi_g^{k_i}(d'_1 \ldots d'_k) = g^{k_i}(d'_1) \ldots g^{k_i}(d'_k).$$

In particular, for each $1 \le t \le k$, we have $d'_t = g^{k_i}(d'_t)$. This implies that

$$\left(d'_t, g(d'_t), g^2(d'_t), \ldots, g^{k_i - 1}(d'_t)\right)$$

is a cycle whose length divides k_i. The conclusion follows.

Conversely, suppose that condition 1 holds. (Condition 2 is superfluous here.) Let $d \in D$. Then there exist $i, j \in \mathbb{N}$ such that $d = g^j(d_i)$. Now,

$$\begin{aligned} f(g(d)) &= f(g(g^j(d_i))) \\ &= f(g^{1+j}(d_i)) \\ &= \phi_g^{1+j}(f(d_i)) \\ &= \phi_g(\phi_g^j(f(d_i))) \\ &= \phi_g(f(g^j(d_i))) \\ &= \phi_g(f(d)). \end{aligned}$$

Therefore, $f \circ g = \phi_g \circ f$, so the proof is complete. □

Once again, suppose that $|D| = n$, and label the elements of $Sym(D)$ by $g_1, \ldots, g_{n!}$. For each $1 \le r \le n!$, we can find the number of $f \in \Omega$ satisfying

$$f \circ g_r = \phi_{g_r} \circ f. \tag{2}$$

Suppose that g_r has cycle type $k_{r1}, k_{r2}, \ldots, k_{r\ell_r}$. For each $1 \leq i \leq \ell_r$, select a single element $d_{ri} \in D$ from the cycle corresponding to k_{ri}. Then Lemma 1 implies that any $f \in \Omega$ satisfying (2) is determined by its values on each d_{ri}. Hence, to find the number of f satisfying (2), we need only count the number of possible images of d_{ri} under such an f, and then take the product over all i. But the m or fewer elements of D comprising the string $f(d_{ri})$ must each belong to a cycle in the decomposition of g_r whose length divides k_{ri}. For each $1 \leq k \leq m$, there are

$$\Big(\sum_{j|k_{ri}} jc(g_r, j)\Big)^k$$

choices of $f(d_{ri})$ such that $|f(d_{ri})| = k$. Hence, there are

$$\sum_{k=1}^{m} \Big(\sum_{j|k_{ri}} jc(g_r, j)\Big)^k$$

total choices of $f(d_{ri})$. Taking the product over all i, it follows that the number of f satisfying (2) is

$$\prod_{i=1}^{\ell_r} \Big(\sum_{k=1}^{m} \Big(\sum_{j|k_{ri}} jc(g_r, j)\Big)^k\Big). \tag{3}$$

Thus, we've evaluated $|\{f \in \Omega : f \circ g_r = \phi_{g_r} \circ f\}|$, and putting together (1) and (3) gives an expression for the number of equivalence classes in Ω under the relation $\sim$. Recalling that these classes are in one-to-one correspondence with the classes in Γ under the relation of combinatorial equivalence, we obtain our main result:

Theorem 1. *If $|D| = n$, then the number of m-endomorphisms on D^+, up to combinatorial equivalence, is the value of*

$$\frac{1}{n!} \sum_{r=1}^{n!} \Bigg(\prod_{i=1}^{\ell_r} \Big(\sum_{k=1}^{m} \Big(\sum_{j|k_{ri}} jc(g_r, j)\Big)^k\Big)\Bigg), \tag{4}$$

where $g_1, \ldots, g_{n!}$ are the elements of $Sym(D)$, and $k_{r1}, \ldots, k_{r\ell_r}$ is the cycle type of g_r.

Example 3. Let $D = \{a, b\}$. We find the number of classes of 1-endomorphisms on D^+. The elements of $Sym(D)$ (in cycle notation) are $g_1 = (a)(b)$ and $g_2 = (a, b)$. Evidently, $c(g_1, 1) = 2$, $c(g_2, 1) = 0$, and $c(g_2, 2) = 1$. Using Theorem 1, there are

$$\begin{aligned} \frac{1}{2}\Big(c(g_1, 1)^2 + 2c(g_2, 2)\Big) &= \frac{1}{2}(2^2 + 2) \\ &= 3 \end{aligned}$$

classes of 1-endomorphisms on D^+. These are given by

$$\left\{\begin{array}{ccc} a & \to & a \\ b & \to & b \end{array}\right\}, \left\{\begin{array}{ccc} a & \to & b \\ b & \to & a \end{array}\right\}, \text{ and } \left\{\begin{array}{ccccccc} a & \to & a & \equiv & a & \to & b \\ b & \to & a & & b & \to & b \end{array}\right\}.$$

We can extend the result of Example 3 by fixing $n = 2$ and letting m be arbitrary. From (4), we find that the number of classes m-endomorphisms on D^+, where $|D| = 2$, is

$$\frac{1}{2}\Big((2^{m+1}-2)^2 + (2^{m+1}-2)\Big).$$

Running m through the natural numbers, we obtain values $3, 21, 105, 465, 1953, \ldots$. This is the sequence A134057 in the On-line Encyclopedia of Integers. (See [3].) However, for $n = 3$, the number of classes of m-endomorphisms becomes

$$\frac{1}{6}\left(\left(\frac{3^{m+1}-3}{2}\right)^3 + 3m\left(\frac{3^{m+1}-3}{2}\right) + 2\left(\frac{3^{m+1}-3}{2}\right)\right).$$

Letting $m = 1, 2, 3, 4, \ldots$ gives values $7, 304, 9958, 288280, \ldots$. This sequence appears to be little-known, and has been submitted by the authors to the OEIS.

3.1 An Alternative Formulation of Theorem 1

We now present a slight rewording of Theorem 1. In order to compute the number of equivalence classes of m-endomorphisms (where $|D| = n$), we need not, in practice, consider each element of $Sym(D)$ individually. Rather, we need only consider the cycle types of these permutations. The following well-known result gives the number of permutations in $Sym(D)$ of a given cycle type.

Proposition 2. [4] Let $|D| = n$, and let $g \in Sym(D)$. Suppose that $m_1, m_2, \ldots, m_s$ are the *distinct* integers appearing in the cycle type of g. For each $j \in \{1, 2, \ldots, s\}$, abbreviate $c_j = c(g, m_j)$. Let C_g be the set of all permutations in $Sym(D)$ whose cycle type is that of g. Then

$$|C_g| = \frac{n!}{\prod_{j=1}^{s} c_j!\, m_j^{c_j}}. \tag{5}$$

For convenience, we shall say that $g \in Sym(D)$ **fixes** the mapping $f \in \Omega$ if and only if $f \circ g = \phi_g \circ f$. Now, two bijections in $Sym(D)$ with the same cycle type must fix the same number of $f \in \Omega$. Therefore, in order to derive an expression for the number of classes of m-endomorphisms on D^+, we can select a single representative in $Sym(D)$ of each possible cycle type, then determine the number of $f \in \Omega$ fixed by each representative using expression (3), multiply this number by the corresponding value of (5), and then sum up over all of our representatives and divide by $n!$. But the cycle types in $Sym(D)$ are precisely the **integer partitions** of n, namely, the nondecreasing sequences of natural numbers whose sum is n. If $p(n)$ denotes the number of integer partitions of n, then we may restate Theorem 1 as follows.

Corollary 1. *Let $|D| = n$, and suppose that $g_1, \dots, g_{p(n)} \in Sym(D)$ have distinct cycle types. Then the number of m-endomorphisms on D^+, up to combinatorial equivalence, is the value of*

$$\frac{1}{n!}\sum_{r=1}^{p(n)}\left(|C_{g_r}|\prod_{i=1}^{\ell_r}\left(\sum_{k=1}^{m}\Big(\sum_{j|k_{ri}} jc(g_r,j)\Big)^k\right)\right), \tag{6}$$

where $k_{r1}, \dots, k_{r\ell_r}$ is the cycle type of g_r, and C_{g_r} is as in Proposition 2.

Example 4. To illustrate Corollary 1, we find the number of classes of m-endomorphisms when $|D| = 4$. Let $D = \{a, b, c, d\}$. As previously mentioned, the cycle types in $Sym(D)$ are the integer partitions of 4. There are five such partitions:

$$\begin{aligned} 4 &= 1+1+1+1 \\ &= 1+1+2 \\ &= 2+2 \\ &= 1+3 \\ &= 4. \end{aligned}$$

Hence, the bijections

$$g_1 = (a)(b)(c)(d),\ g_2 = (a)(b)(c,d),\ g_3 = (a,b)(c,d),\ g_4 = (a)(b,c,d), \text{ and } g_5 = (a,b,c,d)$$

encompass all possible cycle types in $Sym(D)$. Direct calculation using (5) yields

$$|C_{g_1}| = 1,\ |C_{g_2}| = 6,\ |C_{g_3}| = 3,\ |C_{g_4}| = 8, \text{ and } |C_{g_5}| = 6.$$

Thus, by Corollary 1, the number of classes of m-endomorphisms when $n = 4$ is

$$\tfrac{1}{24}\left(\left(\tfrac{4^{m+1}-4}{3}\right)^4 + 6\left(2^{m+1}-2\right)^2\left(\tfrac{4^{m+1}-4}{3}\right) + 3\left(\tfrac{4^{m+1}-4}{3}\right)^2 \right.$$
$$\left. +8m\left(\tfrac{4^{m+1}-4}{3}\right) + 6\left(\tfrac{4^{m+1}-4}{3}\right)\right).$$

Proceeding along the lines of Example 4, we find that there are

$$\frac{1}{120}\Bigg(\left(\frac{5^{m+1}-5}{4}\right)^5+10\left(\frac{3^{m+1}-3}{2}\right)^3\left(\frac{5^{m+1}-5}{4}\right)+15m\left(\frac{5^{m+1}-5}{4}\right)^2$$

$$+20(2^{m+1}-2)^2\left(\frac{5^{m+1}-5}{4}\right)+20(2^{m+1}-2)\left(\frac{3^{m+1}-3}{2}\right)$$

$$+30m\left(\frac{5^{m+1}-5}{4}\right)+24\left(\frac{5^{m+1}-5}{4}\right)\Bigg)$$

classes of m-endomorphisms when $n = 5$, and

$$\frac{1}{720}\Bigg(\left(\frac{6^{m+1}-6}{5}\right)^6+15\left(\frac{4^{m+1}-4}{3}\right)^4\left(\frac{6^{m+1}-6}{5}\right)+45(2^{m+1}-2)^2\left(\frac{6^{m+1}-6}{5}\right)^2$$

$$+15\left(\frac{6^{m+1}-6}{5}\right)^3+40\left(\frac{3^{m+1}-3}{2}\right)^3\left(\frac{6^{m+1}-6}{5}\right)+120m\left(\frac{3^{m+1}-3}{2}\right)\left(\frac{4^{m+1}-4}{3}\right)$$

$$+40\left(\frac{6^{m+1}-6}{5}\right)^2+90(2^{m+1}-2)^2\left(\frac{6^{m+1}-6}{5}\right)+90(2^{m+1}-2)\left(\frac{6^{m+1}-6}{5}\right)$$

$$+144m\left(\frac{6^{m+1}-6}{5}\right)+120\left(\frac{6^{m+1}-6}{5}\right)\Bigg)$$

classes of m-endomorphisms when $n = 6$. Letting m run through $\mathbb{N}$ in these cases, we again obtain sequences that are not well-known. The following tables display the values of (6) for $n, m \le 6$.

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
$m=1$	1	3	7	19	47
$m=2$	2	21	304	6,915	207,258
$m=3$	3	105	9,958	2,079,567	746,331,322
$m=4$	4	465	288,280	556,898,155	2,406,091,382,736
$m=5$	5	1,953	7,973,053	144,228,436,231	7,567,019,254,708,782
$m=6$	6	8,001	217,032,088	37,030,504,349,475	23,677,181,825,841,420,408

	$n=6$
$m=1$	130
$m=2$	7,773,622
$m=3$	409,893,967,167
$m=4$	19,560,646,482,079,624
$m=5$	916,131,223,607,107,471,135
$m=6$	42,770,482,829,102,570,213,645,988

Remark 2. The sequence $1, 3, 7, 19, 47, 130, \ldots$ is sequence A001372 in the OEIS.

4 Two Natural Variations

In this section, we highlight two natural variations of Corollary 1. First, we restrict our attention to endomorphisms on D^+ that send each element of D to a string of length exactly m. We then consider m-endomorphisms of the so-called free monoid, which contains the empty string. Expressions analogous to those in §3 are derived in each case.

4.1 *m*-Uniform Endomorphisms

Fix $n, m \in \mathbb{N}$, and suppose that $|D| = n$. Then $\phi \in End(D^+)$ is called an **m-uniform endomorphism** if and only if $|\phi(d)| = m$ for each $d \in D$. In this section, we produce a formula for the number of m-uniform endomorphisms on D^+ up to combinatorial equivalence. To begin, let $g_1, \ldots, g_{p(n)} \in Sym(D)$ have distinct cycle types. We now put $R = \{W \in D^+ : |W| = m\}$ and take Ω to be the set of all mappings of D into R. For each $1 \leq r \leq p(n)$, we ask for the number of $f \in \Omega$ satisfying

$$f \circ g_r = \phi_{g_r} \circ f.$$

Once again, if g_r has cycle type $k_{r1}, \ldots, k_{r\ell_r}$, then for each $1 \leq i \leq \ell_r$ we select an element d_{ri} from the cycle corresponding to k_{ri}, and count the number of possible values of $f(d_{ri})$. In this case, we must have $|f(d_{ri})| = m$, where the elements of D comprising the string $f(d_{ri})$ each belong to a cycle whose length divides k_{ri}. Hence, there are

$$\Big(\sum_{j|k_{ri}} jc(g_r, j)\Big)^m$$

choices of $f(d_{ri})$, and multiplying over all i yields

$$\prod_{i=1}^{\ell_r} \Big(\sum_{j|k_{ri}} jc(g_r, j)\Big)^m$$

as the value of $|\{f \in \Omega : f \circ g_r = \phi_{g_r} \circ f\}|$. Noting that permutations in $Sym(D)$ of the same cycle type fix the same number of $f \in \Omega$, we multiply by $|C_{g_r}|$, sum with respect to r, and divide by $n!$ to obtain the following.

Corollary 2. *If $|D| = n$ and $g_1, \ldots, g_{p(n)} \in Sym(D)$ have distinct cycle types, then the number of m-uniform endomorphisms on D^+, up to combinatorial equivalence, is the value of*

$$\frac{1}{n!} \sum_{r=1}^{p(n)} \left(|C_{g_r}| \prod_{i=1}^{\ell_r} \Big(\sum_{j|k_{ri}} jc(g_r, j) \Big)^m \right), \tag{7}$$

where $k_{r1}, \ldots, k_{r\ell_r}$ is the cycle type of g_r, and C_{g_r} is as in Proposition 2.

When $n = 2$, the number of m-uniform endomorphisms on D^+, up to combinatorial equivalence, is

$$\frac{1}{2}(2^{2m} + 2^m).$$

Letting $m = 1, 2, 3, 4, \ldots$ gives values $3, 10, 36, 136, \ldots$. This is the sequence A007582 from the OEIS. Moreover, when $n = 3$ there are

$$\frac{1}{6}(3^{3m} + 3 \cdot 3^m + 2 \cdot 3^m)$$

classes of m-uniform endomorphisms, and letting m run through $\mathbb{N}$ gives the sequence $7, 129, 3303, 88641, \ldots$, which is not well-known. Continuing, the expressions when $n = 4, 5, 6$ are

$$\frac{1}{24}(4^{4m} + 6 \cdot 2^{2m} \cdot 4^m + 3 \cdot 4^{2m} + 8 \cdot 4^m + 6 \cdot 4^m),$$

$$\frac{1}{120}(5^{5m} + 10 \cdot 3^{3m} \cdot 5^m + 15 \cdot 5^{2m} + 20 \cdot 2^{2m} \cdot 5^m + 20 \cdot 2^m \cdot 3^m + 30 \cdot 5^m + 24 \cdot 5^m),$$

and

$$\frac{1}{720}(6^{6m} + 15 \cdot 4^{4m} \cdot 6^m + 45 \cdot 2^{2m} \cdot 6^{2m} + 15 \cdot 6^{3m} + 40 \cdot 3^{3m} \cdot 6^m$$

$$+120 \cdot 3^m \cdot 4^m + 40 \cdot 6^{2m} + 90 \cdot 2^{2m} \cdot 6^m + 90 \cdot 2^m \cdot 6^m + 144 \cdot 6^m + 120 \cdot 6^m),$$

respectively. The following tables display the values of (7) for $n, m \leq 6$.

	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$
$m = 1$	1	3	7	19	47
$m = 2$	1	10	129	2,836	83,061
$m = 3$	1	36	3,303	700,624	254,521,561
$m = 4$	1	136	88,641	178,981,696	794,756,352,216
$m = 5$	1	528	7,973,053	45,813,378,304	2,483,530,604,092,546
$m = 6$	1	2,080	64,570,689	11,728,130,323,456	7,761,021,959,623,948,401

	$n = 6$
$m = 1$	130
$m = 2$	3,076,386
$m = 3$	141,131,630,530
$m = 4$	6,581,201,266,858,896
$m = 5$	307,047,288,863,992,988,160
$m = 6$	14,325,590,271,500,876,382,987,456

4.2 The Free Monoid

If we adjoin the unique string of length 0 (denoted by ϵ) to the set D^+, then we form the set D^*. Paired with the operation of string concatenation, D^* forms the **free monoid** on D. We refer to ϵ as the **empty string**, and it serves as the identity element in D^*. That is, for any $W \in D^*$,

$$W\epsilon = W = \epsilon W.$$

We define an endomorphism on D^* to be a mapping $\phi : D^* \to D^*$ such that $\phi(W_1 W_2) = \phi(W_1)\phi(W_2)$ for all $W_1, W_2 \in D^*$.

Remark 3. Note that if ϕ is an endomorphism on D^*, then $\phi(\epsilon) = \epsilon$. This follows since for any $W \in D^*$, we have

$$\phi(W) = \phi(\epsilon W) = \phi(\epsilon)\phi(W),$$

which implies that $\phi(\epsilon)$ has length 0.

Now, an m-endomorphism on D^* is an endomorphism such that $|\phi(d)| \leq m$ for all $d \in D$. Thus, an m-endomorphism on D^* can map elements of D to ϵ. To determine the number of m-endomorphisms on D^* up to combinatorial equivalence, we put $R = \{W \in D^* : |W| \leq m\}$, and for each $g \in Sym(D)$, we ask for the number of $f : D \to R$ that are fixed by g. Again, it suffices to count the number of possible images under such an f of a single $d \in D$ from each cycle in the decomposition of g, and then multiply over all the cycles. But there is now one additional possible value of $f(d)$: the empty string. Hence, if d belongs to a cycle of length k_i, then we have

$$1 + \sum_{k=1}^{m} \Big(\sum_{j|k_i} jc(g_r, j) \Big)^k = \sum_{k=0}^{m} \Big(\sum_{j|k_i} jc(g_r, j) \Big)^k$$

choices of $f(d)$. From this observation, we deduce the following.

Corollary 3. *Let $|D| = n$, and suppose that $g_1, \ldots, g_{p(n)} \in Sym(D)$ have distinct cycle types. Then the number of m-endomorphisms on D^*, up to combinatorial equivalence, is the value of*

$$\frac{1}{n!} \sum_{r=1}^{p(n)} \left(|C_{g_r}| \prod_{i=1}^{\ell_r} \left(\sum_{k=0}^{m} \Big(\sum_{j|k_{ri}} jc(g_r, j) \Big)^k \right) \right), \tag{8}$$

where $k_{r1}, \ldots, k_{r\ell_r}$ is the cycle type of g_r, and C_{g_r} is as in Proposition 2.

When $n = 2$, the number of m-endomorphisms on D^*, up to combinatorial equivalence, is

$$\frac{1}{2}\Big((2^{m+1} - 1)^2 + (2^{m+1} - 1) \Big).$$

This is sequence A006516 from the OEIS. The corresponding expressions for $n = 3, 4, 5, 6$ are

$$\frac{1}{6}\left(\left(\frac{3^{m+1} - 1}{2} \right)^3 + 3(m+1)\left(\frac{3^{m+1} - 1}{2} \right) + 2\left(\frac{3^{m+1} - 1}{2} \right) \right),$$

$$\frac{1}{24}\left(\left(\frac{4^{m+1}-1}{3}\right)^4+6\left(2^{m+1}-1\right)^2\left(\frac{4^{m+1}-1}{3}\right)+3\left(\frac{4^{m+1}-1}{3}\right)^2\right.$$

$$\left.+8(m+1)\left(\frac{4^{m+1}-1}{3}\right)+6\left(\frac{4^{m+1}-1}{3}\right)\right),$$

$$\frac{1}{120}\left(\left(\frac{5^{m+1}-1}{4}\right)^5+10\left(\frac{3^{m+1}-1}{2}\right)^3\left(\frac{5^{m+1}-1}{4}\right)+15(m+1)\left(\frac{5^{m+1}-1}{4}\right)^2\right.$$

$$+20(2^{m+1}-1)^2\left(\frac{5^{m+1}-1}{4}\right)+20(2^{m+1}-1)\left(\frac{3^{m+1}-1}{2}\right)$$

$$\left.+30(m+1)\left(\frac{5^{m+1}-1}{4}\right)+24\left(\frac{5^{m+1}-1}{4}\right)\right),$$

and

$$\frac{1}{720}\left(\left(\frac{6^{m+1}-1}{5}\right)^6+15\left(\frac{4^{m+1}-1}{3}\right)^4\left(\frac{6^{m+1}-1}{5}\right)\right.$$

$$+45(2^{m+1}-1)^2\left(\frac{6^{m+1}-1}{5}\right)^2+15\left(\frac{6^{m+1}-1}{5}\right)^3$$

$$+40\left(\frac{3^{m+1}-1}{2}\right)^3\left(\frac{6^{m+1}-1}{5}\right)+120(m+1)\left(\frac{3^{m+1}-1}{2}\right)\left(\frac{4^{m+1}-1}{3}\right)$$

$$+40\left(\frac{6^{m+1}-1}{5}\right)^2+90(2^{m+1}-1)^2\left(\frac{6^{m+1}-1}{5}\right)$$

$$\left.+90(2^{m+1}-1)\left(\frac{6^{m+1}-1}{5}\right)+144(m+1)\left(\frac{6^{m+1}-1}{5}\right)+120\left(\frac{6^{m+1}-1}{5}\right)\right).$$

Once again, the sequences given by these expressions appear to be little-known. The following tables give the values of (8) for $n, m \leq 6$.

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
$m=1$	2	6	16	45	121
$m=2$	3	28	390	8,442	244,910
$m=3$	4	120	10,760	2,180,845	770,763,470
$m=4$	5	496	295,603	563,483,404	2,421,556,983,901
$m=5$	6	2,016	8,039,304	144,651,898,755	2,370,422,688,990,078
$m=6$	7	8,128	217,629,416	37,057,640,711,850	23,683,244,198,577,149,289

	$n=6$
$m=1$	338
$m=2$	8,967,034
$m=3$	419,527,164,799
$m=4$	19,636,295,549,860,505
$m=5$	916,720,535,022,517,503,173
$m=6$	42,775,066,732,111,188,868,070,978

5 (χ,ζ)-Patterns

In closing, we briefly place the relation $\sim$ from §2 into a more general context. Let G be a finite group, and let N and M be finite nonempty sets. Suppose that $\chi : G \to Sym(N)$ and $\zeta : G \to Sym(M)$ are group homomorphisms. Denote the set of all functions from N into M by M^N. (This notation comes from [1].) De Bruijn introduced the equivalence relation $E_{\chi,\zeta}$ on M^N defined by

$$(f_1, f_2) \in E_{\chi,\zeta} \iff f_2 \circ \chi(\gamma) = \zeta(\gamma) \circ f_1 \text{ for some } \gamma \in G.$$

Example 5. [1] Suppose that N is a set of size $n \in \mathbb{N}$, and define an equivalence relation S on the set of all mappings of N into itself by

$$(f_1, f_2) \in S \iff f_2 \circ \gamma = \gamma \circ f_1 \text{ for some } \gamma \in Sym(N).$$

Letting $G = Sym(N)$, $M = N$, and $\chi = \zeta$ be the identity homomorphism on $Sym(N)$ shows that S is a special case of the relation $E_{\chi,\zeta}$. Moreover, the sequence in Remark 2 gives the number of equivalence classes under S for $n = 1, 2, 3\ldots$. (See §3 of [1].)

The relation $E_{\chi,\zeta}$ stems from the left action of G on M^N given by

$$\gamma \cdot f = \zeta(\gamma) \circ f \circ \chi(\gamma^{-1}),$$

for all $\gamma \in G$, $f \in M^N$. De Bruijn referred to the orbits of G on M^N as **(χ,ζ)-patterns**, and provided a formula for the number of these by applying Burnside's Lemma, and then evaluating $\left|\{f \in M^N : \gamma \cdot f = f\}\right|$ for each $\gamma \in G$. (See [1].) But the relation $\sim$ on the set $\Omega = \{\text{mappings of } D \text{ into } R\}$, where $0 < |D| < \infty$ and $R = \{W \in D^+ : |W| \leq m\}$, is a special instance of the relation $E_{\chi,\zeta}$. To see this, take $N = D$, $M = R$, and $G = Sym(D)$. Let χ be the identity homomorphism on $Sym(D)$, and define $\zeta : G \to Sym(R)$ by

$$\zeta(g) = \phi_g|_R,$$

for all $g \in Sym(D)$. Then for any $g, g' \in Sym(D)$,

$$\zeta(g \circ g') = \phi_{g\circ g'}|_R = (\phi_g \circ \phi_{g'})|_R = \phi_g|_R \circ \phi_{g'}|_R = \zeta(g) \circ \zeta(g'),$$

so ζ is a group homomophism. Now, for any $f_1, f_2 \in \Omega$, we have

$$\begin{aligned} f_1 \sim f_2 &\iff f_2 \circ g = \phi_g \circ f_1 = \phi_g|_R \circ f_1 \text{ for some } g \in Sym(D) \\ &\iff f_2 \circ \chi(g) = \zeta(g) \circ f_1 \text{ for some } g \in Sym(D) \\ &\iff (f_1, f_2) \in E_{\chi,\zeta}. \end{aligned}$$

It follows that the equivalence classes in Ω under the relation $\sim$ are (χ, ζ)-patterns, for χ,ζ chosen as above. In particular, our Theorem 1 is a special case of de Bruijn's formula.

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