

An Analytical Approach to Exponent-Restricted Multiple Counting Sequences

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Abstract

This study involves definitions for multiple-counting regular and summation sequences of ρ . My paper introduces and proves recurrent relationships for multiple-counting sequences and shows their association with Fermat's little theorem. I also studied matrix representations and obtained generalized Binet formulas for defined sequences. As a result of multiple-counting sequence explanation, this study leads to a better understanding for distribution of composite numbers between consecutive bases.

1 Introduction

Jacobsthal Sequence is defined as

$$j_{n+2} = j_{n+1} + 2j_n, j_1 = 1, j_0 = 0, n \geq 0 \quad (1)$$

[2] Horadam and Hoggatt has conducted many research on Jacobsthal Sequence. Jacobsthal Sequence has a lot of remarkable properties as counting microcontroller skip instructions [6] and counting number of ways to tile a $3 \times (n - 1)$ rectangle with 2×2 and 1×1 tiles [1] *et cetera*. Barry [1] also shows that Jacobsthal Sequence can be expressed in floor function notation. Fossaceca [1] mentioned at sequence A001045 that each element of Jacobsthal Sequence counts integer multiples of 3 in between corresponding exponents of 2.

Interval boundaries of exponents of 2	2^0		2^1		2^2		2^3		2^4		2^5
Numbers of integer multiples of 3 in each interval.		0		1		1		3		5	

Table 1: Number of integers which divides 3 with no remainder between exponents of 2. The sequence is identical to Jacobsthal Sequence.

It seems that there are other Jacobsthal-like recurrence relationships held in between some other exponents and bases when multiplicative numbers are counted and this study investigates various aspects of those recurrent sequences.

Another concept will be discussed in this paper is Fermat's little theorem's condition which is given in [3] as $x^{\rho-1} \equiv 1 \pmod{\rho}$. The theorem states that the given condition is satisfied when ρ and x are coprime and ρ is prime. However, vice versa is not always true. The condition is still valid for some composite ρ s called *Fermat pseudoprimes*, *Carmichael numbers* or K_1 *Knödel numbers*. Particularly for $x = 2$ case, ρ s are called *Sarrus numbers*, *Poulet numbers* or *Fermatians* if they satisfy Fermat's given condition [7].

2 Generalized Multiple-Counting Sequences between Exponents

Definition 1. Let x and ρ be elements of integer set $\{2, 3, 4, 5 \dots\}$ and n be an element of $\{0, 1, 2, 3 \dots\}$. If an integer sequence J is defined in which J_n is equal to the number of multiples of ρ which are greater than x^n and less than x^{n+1} ; then J is a *multiple-counting sequence* of ρ between exponents of x . Using floor functions, we can derive number of multiples of ρ less than x^{n+1} as $\left\lfloor \frac{x^{n+1}}{\rho} \right\rfloor$. As a natural consequence, multiples of ρ greater than x^n and less than x^{n+1} is found as follows:

$$J_n = \left\lfloor \frac{x^{n+1}}{\rho} \right\rfloor - \left\lfloor \frac{x^n}{\rho} \right\rfloor \quad (2)$$

Theorem 2. Let x and ρ be integers from set $\{1, 2, 3 \dots\}$. Let numbers x and ρ be relatively prime and satisfy Fermat's $x^{\rho-1} \equiv 1 \pmod{\rho}$ condition. For any number n from set $\{0, 1, 2 \dots\}$ with already provided starting terms $J_0, J_1 \dots J_{\rho-1}$, the sequence J_n which counts multiples of ρ between x^n and x^{n+1} , also satisfies the following recurrence relationship:

$$J_{n+\rho-1} = (x-1) \sum_{i=1}^{\rho-2} J_{n-i+\rho-1} + xJ_n \quad (3)$$

Proof. Let the expression is transformed arithmetically to the following:

$$J_{n+\rho-1} = (x-1) \sum_{i=1}^{\rho-1} J_{n-i+\rho-1} + J_n \quad (4)$$

Assume that x^n is formulated parametrically by $\lambda\rho + \varphi$, then x^{n+w} is also formulated as $x^w(\lambda\rho + \varphi)$. It is easily found that there are $(x-1)(\lambda\rho + \varphi) - 1$ integers between x^n and x^{n+1} .

x^n		x^{n+1}	...	$x^{n+\rho-1}$		$x^{n+\rho}$
	$(x-1)(\lambda\rho + \varphi) - 1$...		$x^{\rho-1}(x-1)(\lambda\rho + \varphi) - 1$	

Table 2: Numbers of integers between exponents of x.

Table 2 shows that there are $x^{\rho-1}(x-1)(\lambda\rho + \varphi) - 1$ numbers in between $x^{n+\rho-1}$ and $x^{n+\rho}$. Let δ be a unique arithmetical minimum plus or maximum minus process which makes $(x-1)(\lambda\rho + \varphi) - 1$ a certain multiple of ρ . After δ process is applied, number of integers which divide ρ in chosen interval is easily found by dividing the number of elements in chosen interval by ρ .

$$J_n = \frac{(x-1)(\lambda\rho + \varphi) - 1 + \delta}{\rho} \quad (5)$$

Because of $x^{\rho-1} \equiv 1 \pmod{\rho}$ initial condition, the expression $x^{\rho-1}(x-1)(\lambda\rho + \varphi) - 1$ also needs the identical δ process to be made a multiple of ρ .

$$J_{n+\rho-1} = \frac{x^{\rho-1}(x-1)(\lambda\rho + \varphi) - 1 + \delta}{\rho} \quad (6)$$

$(Mod\rho) \equiv$	1	2	3	...	φ	$\varphi + 1$...	$\rho - 1$	ρ
					$\lambda\rho + \varphi$	*	*	*	*
	*	*	*	*	*	*	*	*	*

	*	*	*	*	*	*	*	*	*
	*	*	*	*	$x^{\rho-1}(\lambda\rho + \varphi)$				

Table 3: Distribution for ε . Asterisk signs refer to the consecutive integers between boundaries $\lambda\rho + \varphi$ and $x^{\rho-1}(\lambda\rho + \varphi)$. ρ rows include terms which are multiple of ρ .

On the other hand, let ε be another unique process which makes the number of integers at given interval divides ρ . We see at Table 3 that ε process must be equal to +1 if we want to find the number of integers which divides ρ . Then the sigma notation part of equation is found as follows:

$$(x-1) \sum_{i=1}^{\rho-2} J_{n-i+\rho-1} = \frac{(x-1)(x^{\rho-1}(\lambda\rho + \varphi) - \lambda\rho - \varphi - 1 + \varepsilon)}{\rho} \quad (7)$$

Rewrite expression (4) by combining (5), (6) and (7) and consider that $\varepsilon = +1$, then proof is provided after simplification. \square

Lemma 3. *An alternative simpler proof for the previous theorem can be derived from the floor function notation of J_n in (2).*

Proof. For a multiple-counting sequence which satisfies the recurrence relationship given in (4), it is also known that $x^{\rho-1} \equiv 1 \pmod{\rho}$. Replace J_n with its floor function notation (2). The following expression is obtained when intermediate elements of the sigma notation are eliminated by themselves.

$$\left\lfloor \frac{x^{n+\rho}}{\rho} \right\rfloor - \left\lfloor \frac{x^{n+\rho-1}}{\rho} \right\rfloor = (x-1) \left(\left\lfloor \frac{x^{n+\rho-1}}{\rho} \right\rfloor - \left\lfloor \frac{x^n}{\rho} \right\rfloor \right) + \left\lfloor \frac{x^n + 1}{\rho} \right\rfloor - \left\lfloor \frac{x^n}{\rho} \right\rfloor \quad (8)$$

Let a be an arbitrary constant. Assume $x^{\rho-1}$ is equal to $a\rho + 1$ as a result of the $x^{\rho-1} \equiv 1 \pmod{\rho}$ condition. When $a\rho + 1$ is substituted to the (8), floor function takes a -terms out because of its integer exclusion property. Right and left hand sides are simplified and proof is completed. \square

Example 4. Table 4 contains several examples of multiple-counting sequences for different ρ s and x s

ρ	x	Summary of Sequence
3	2	Identical to Jacobsthal Sequence: $J_n = J_{n-1} + 2J_{n-2}, n \geq 2, J_0 = 0, J_1 = 1$
5	3	$J_n = 2J_{n-1} + 2J_{n-2} + 2J_{n-3} + 3J_{n-4}, n \geq 4, J_0 = 0, J_1 = 1, J_2 = 4, J_3 = 11$
3	10	$J_n = 9J_{n-1} + 10J_{n-2}, n \geq 2, J_0 = 3, J_1 = 30$ J_n counts (n+1)-digit numbers which divides 3

Table 4: Some multiple-counting sequences

Additionally, Cooka and Bacon [8] defines sequences J_n s for various ρ s called ‘‘Higher Order Jacobsthal Sequences’’ when $x = 2$.

Remark 5. For any element of sequence J_n , Binet expression

$$J_n = \sum_{i=1}^n \lambda_i^{n-i+2} v_{1,i} \left(\sum_{j=1}^n v_{i,j}^{-1} u_j^{\rho-2} \right) \quad (9)$$

is always satisfied when key matrix

$$K_{(\rho-1) \times (\rho-1)} = \begin{pmatrix} x-1 & x-1 & \dots & x-1 & x-1 & x \\ 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & \ddots & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & 1 & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix}$$

satisfies equation $Ku^n = u^{n+1}$ for vector

$$u^n = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{p-1} \end{pmatrix} = \begin{pmatrix} J_n \\ J_{n-1} \\ \vdots \\ J_{n-p+2} \end{pmatrix}$$

if K is diagonalizable and K 's eigenvector matrix V is non-singular. Note that elements u_i, v_{ij} belong to u, V .

Proof. Using Kalman's method [5], explicit Binet form of our sequence is easily obtained. After $K = V\Lambda V^{-1}$ decomposition; equation $Ku^n = u^{n+1}$ is transformed to $V\Lambda V^{-1}u^n = u^{n+1}$ where Λ is diagonal eigenvalue matrix of K . We only need top elements of vectors, thus multiply the expression by

$$\begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix}$$

$(\rho - 1) \times 1$ matrix from the left. Any particular term of sequence J_n satisfies equation:

$$J_n = \begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix} (V\Lambda^{n-\rho+2}(V^{-1}u^{\rho-2})) \quad (10)$$

Since

$$(V^{-1}u^{\rho-2}) = \begin{pmatrix} v_{1,1}^{-1} & v_{1,2}^{-1} & \cdots & v_{1,\rho-1}^{-1} \\ v_{2,1}^{-1} & v_{2,2}^{-1} & \cdots & v_{2,\rho-1}^{-1} \\ \cdots & \cdots & \ddots & \cdots \\ v_{\rho-1,1}^{-1} & v_{\rho-1,2}^{-1} & \cdots & v_{\rho-1,\rho-1}^{-1} \end{pmatrix} \begin{pmatrix} J_{\rho-2} \\ J_{\rho-3} \\ \vdots \\ J_0 \end{pmatrix}$$

and

$$V = \begin{pmatrix} v_{1,1} & v_{1,2} & \cdots & v_{1,\rho-1} \\ v_{2,1} & v_{2,2} & \cdots & v_{2,\rho-1} \\ \cdots & \cdots & \ddots & \cdots \\ v_{\rho-1,1} & v_{\rho-1,2} & \cdots & v_{\rho-1,\rho-1} \end{pmatrix}$$

and

$$\Lambda^{n-\rho+2} = \begin{pmatrix} \lambda_1^{n-\rho+2} & 0 & \cdots & 0 \\ 0 & \lambda_2^{n-\rho+2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_{\rho-1}^{n-\rho+2} \end{pmatrix}$$

are known, we can compute (10) after required substitution. Remember that $\lambda_1, \lambda_2, \dots, \lambda_n$ denote eigenvalues of K . After the outcome of the computation is simplified, we obtain the initial expression (9). \square

3 Generalized Multiple-Counting Summation Sequences until Exponents

Definition 6. Let n, x and ρ be elements of integer set $A = \{2, 3, 4, 5, \dots\}$ and n be element of $\{0, 1, 2, 3, \dots\}$. If an integer sequence S is defined when S_n is equal to the number of integer multiples of ρ , which are greater than 0 and less than x^{n+1} ; then J is a *multiple-counting summation sequence of ρ* . *Summation sequence has a more compact form as follows:*

$$S_n = \left\lfloor \frac{x^{n+1}}{\rho} \right\rfloor \quad (11)$$

The strong interrelation between regular multiple-counting sequences and multiple-counting summation sequences is expressed as

$$S_n = \sum_{i=0}^n J_i \quad (12)$$

The summation idea of a multiple-counting sequence is very similar to the *Jacobsthal Representation Sequence* which Horadam [4] mentions. The summation sequence satisfies the same recurrence relationship which regular sequence also does, except for a particularly determined constant.

Theorem 7. Let x and ρ be integers from set $\{1, 2, 3, \dots\}$ and let numbers x and ρ satisfy Fermat's $x^{\rho-1} \equiv 1 \pmod{\rho}$ condition. For any number n from set $R = \{0, 1, 2, \dots\}$, with already provided starting terms $S_0, S_1 \dots S_{\rho-1}$, the sequence S_n which counts multiples of ρ between 0 and x^{n+1} , satisfies the following recurrence relationship:

$$S_{n+\rho-1} = (x-1) \sum_{i=1}^{\rho-2} S_{n-i+\rho-1} + xS_n + \pi \quad (13)$$

or

$$S_{n+\rho-1} = (x-1) \sum_{i=1}^{\rho-1} S_{n-i+\rho-1} + S_n + \pi \quad (14)$$

, where π is a unique constant for a particular combination of ρ, x and $J_0, J_1 \dots J_{\rho-2}$ initial conditions. Formula for π is:

$$\pi = \sum_{i=0}^{\rho-1} J_i - (x-1) \sum_{i=1}^{\rho-1} \sum_{j=0}^{i-2} J_j \quad (15)$$

Proof. Rewrite (14) using property (12):

$$\sum_{i=0}^{n+\rho-1} J_i = (x-1) \sum_{i=1}^{\rho-1} \sum_{j=0}^{n-i+\rho-1} J_j + \sum_{i=0}^n J_i + \pi \quad (16)$$

Separate it into n -dependent and n -independent parts:

$$\sum_{i=0}^{\rho-2} J_i + \sum_{i=\rho-1}^{n+\rho-1} J_i = (x-1) \sum_{i=1}^{\rho-1} \left(\sum_{j=0}^{i-2} J_j + \sum_{j=i-1}^{n-i+\rho-1} J_j \right) + \sum_{i=0}^n J_i + \pi \quad (17)$$

$$\sum_{i=0}^{\rho-2} J_i + \underbrace{\sum_{i=\rho-1}^{n+\rho-1} J_i}_{\text{n-dependent}} = (x-1) \sum_{i=1}^{\rho-1} \sum_{j=0}^{i-2} J_j + \underbrace{(x-1) \sum_{i=1}^{\rho-1} \sum_{j=i-1}^{n-i+\rho-1} J_j}_{\text{n-dependent}} + \underbrace{\sum_{i=0}^n J_i + \pi}_{\text{n-independent}} \quad (18)$$

n -dependent underbraced parts of equation (18) are eliminated by induction on (4). Since n -independent parts of expression above are constant for whole sequence, π also depends on a constant value and required proof is provided. \square

Remark 8. To obtain a matrix-based Binet representation for sequence S , we use a similar approach with the sequence J . Since the only difference between definitions of two sequences J and S is a constant π , we can easily modify the key matrix to insert π dependency with notational changes to get a simple formula.

$$S_n = \sum_{i=1}^n \mu_i^{n-i+2} w_{1,i} \left(\sum_{j=1}^n w_{i,j}^{-1} \tau_j^{\rho-1} \right) \quad (19)$$

Assuming L is the such a key matrix

$$L_{\rho \times \rho} = \begin{pmatrix} x-1 & x-1 & \dots & x-1 & x-1 & x & 1 \\ 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 1 & \ddots & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \\ 0 & 0 & \ddots & 1 & 0 & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

of $L\tau^n = \tau^{n+1}$ where τ holds equation:

$$\tau^n = \begin{pmatrix} \tau_1 \\ \tau_2 \\ \vdots \\ \tau_{p-1} \\ \pi \end{pmatrix} = \begin{pmatrix} S_n \\ S_{n-1} \\ \vdots \\ S_{n-p+2} \\ \pi \end{pmatrix}$$

if L is diagonalizable and L 's eigenvector matrix W is non-singular. Note that elements w_{ij} are belonged to W and $\mu_1, \mu_2, \dots, \mu_n$ denote eigenvalues of L .

Proof. After $L = W\Lambda W^{-1}$ decomposition; equation $L\tau^n = \tau^{n+1}$ is converted to $W\Lambda W^{-1}\tau^n = \tau^{n+1}$ where Λ is diagonal eigenvalue matrix of L . We only need top elements of vectors, thus each side is multiplied by

$$\begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix}$$

$(\rho \times 1)$ matrix from right. Any particular term of sequence S_n satisfies

$$S_n = \begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix} (W\Lambda^{n-\rho+2}(W^{-1}\tau^{\rho-2})) \quad (20)$$

equation. Since

$$W^{-1}\tau^{\rho-2} = \begin{pmatrix} w_{1,1}^{-1} & w_{1,2}^{-1} & \dots & w_{1,\rho}^{-1} \\ w_{2,1}^{-1} & w_{2,2}^{-1} & \dots & w_{2,\rho}^{-1} \\ \dots & \dots & \ddots & \dots \\ w_{\rho,1}^{-1} & w_{\rho,2}^{-1} & \dots & w_{\rho,\rho}^{-1} \end{pmatrix} \begin{pmatrix} S_{\rho-2} \\ S_{\rho-3} \\ \vdots \\ S_0 \\ \pi \end{pmatrix}$$

and

$$W = \begin{pmatrix} w_{1,1} & w_{1,2} & \dots & w_{1,\rho} \\ w_{2,1} & w_{2,2} & \dots & w_{2,\rho} \\ \dots & \dots & \ddots & \dots \\ w_{\rho,1} & w_{\rho,2} & \dots & w_{\rho,\rho} \end{pmatrix}$$

and

$$\Lambda^{n-\rho+2} = \begin{pmatrix} \mu_1^{n-\rho+2} & 0 & \dots & 0 \\ 0 & \mu_2^{n-\rho+2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \mu_\rho^{n-\rho+2} \end{pmatrix}$$

are known, (20) can be computed. After (20) is simplified, initial expression (19) is obtained. \square

4 Concluding Remarks

Several sequences in OEIS [1] seem identical to some multiple counting sequences. A007910 is a multiple counting sequence when $(x = 2, \rho = 5)$ and A077947 when $(x = 2, \rho = 7)$ which are also called Fourth and Sixth Order Jacobsthal Sequences [8]. Additionally A000302 and A093138 are multiple counting sequences for $(x = 4, \rho = 3)$ and $(x = 10, \rho = 3)$ definitions. Alternative examples goes on.

Unlike prime-counting function $\Pi(x)$ which is A006880 of OEIS [1], sequences J_n and S_n count composites of given prime ρ . There is only one exception for both sequences in their initial terms. If ρ is prime, J_n and S_n counts all composites of prime ρ .

As this study has generalized relationships between defined sequences and numbers of multiples between defined boundaries, outcomes of our research can be used in various electronics, telecommunications and computer science applications. Another considerable consequence of our study might be an implementation of a new prime prediction and factorization method due to the strong connection with Fermat's little theorem. Distribution of prime numbers is a key concept in present number theory and as a measure of this phenomenon, the role of multiple counting sequences is undeniable.

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(Concerned with sequences [A001045](#), [A007910](#), [A093138](#), [A077947](#), and [A000302](#).)

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