

A New Lower Bound for the Distinct Distance Constant

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Abstract

The reciprocal sum of Zhang sequence is not equal to the Distinct Distance Constant. This note introduces a B_2 -sequence with larger reciprocal sum, and provides a more precise estimate of the reciprocal sums of Mian-Chowla sequence and Zhang sequence.

1 Introduction

A *Sidon sequence*, also called a B_2 -sequence, is a sequence of positive integers

$$a_1 < a_2 < a_3 < \dots$$

such that all the sums $a_i + a_j$ ($i \leq j$) are distinct.

The *distinct distance constant* (DDC) is the supremum of the set of the reciprocal sums of Sidon sequences. Levine [1] observed that

$$\text{DDC} \leq \sum_{n=0}^{\infty} \frac{1}{1 + \frac{n(n+1)}{2}} = \frac{2\pi}{\sqrt{7}} \tanh\left(\frac{\sqrt{7}}{2}\pi\right) < 2.37366. \quad (1)$$

Let S_A be the reciprocal sum of the sequence A ; that is, $S_A = \sum_{i=1}^{\infty} \frac{1}{a_i}$. It is an open problem to find, if it exists, a Sidon sequence U whose reciprocal sum is equal to the DDC [2, p. 351]. We only know from Taylor and Yovanof [3] that, if some Sidon sequence achieves the DDC, then it must begin with the values 1, 2, 4.

The B_2 -sequence with the largest known reciprocal sum had been, for a long time, the one produced by the greedy algorithm, $G = \{1, 2, 4, 8, 13, 21, 31, 45, 66, 81, \dots\}$. The sequence G is named the *Mian-Chowla sequence*. Lewis found that

$$2.158435 \leq S_G \leq 2.158677, \quad (2)$$

where S_G is the reciprocal sum of G [4, pp. 164–165].

In 1991, Zhang [5] found a Sidon sequence with a reciprocal sum greater than S_A . Zhang's sequence Z is obtained by running the greedy algorithm for the first 14 terms, setting $z_{15} = 229$, and then continuing with the greedy algorithm. Zhang proved that

$$S_Z > 2.1597. \quad (3)$$

The aim of this note is to exhibit a Sidon sequence H such that $S_H > S_Z$.

2 Computation of reciprocal sums

2.1 Preliminary considerations

To estimate the reciprocal sum, we will use two basic properties of Sidon sequences: they are growing sequences, and their differences $a_i - a_j$ ($i \geq j$) are all distinct. That is,

$$a_i \geq a_j + (i - j), \quad i \geq j \quad (4)$$

$$a_i > \frac{i(i-1)}{2}. \quad (5)$$

Therefore, if we know the values of a_n for $1 \leq n \leq k$, we also know that

$$\sum_{n=1}^k \frac{1}{a_n} < S_A < \sum_{n=1}^k \frac{1}{a_n} + \sum_{n=k}^{\infty} \frac{1}{\max\left(a_k + n - k, \frac{n(n-1)}{2}\right)}. \quad (6)$$

Further assumptions could be made on a_i , but for large values of k they would not yield a significant improvement of the bounds.

2.2 The reciprocal sum of the Mian-Chowla sequence

Let G be the B_2 -sequence constructed by the greedy algorithm. The values of g_n for $1 \leq n \leq 25000$, computed on my notebook, are listed in the accompanying file `MianChowla.txt`. We get the following bounds for S_G :

$$\sum_{n=1}^{25000} \frac{1}{g_n} < S_G < \sum_{n=1}^{25000} \frac{1}{g_n} + \sum_{n=25001}^{510096} \frac{1}{g_{25000} + n - 25000} + \sum_{n=510097}^{\infty} \frac{2}{n(n-1)}, \quad (7)$$

i.e.,

$$2.15845268 < S_G < 2.15846062. \quad (8)$$

2.3 The reciprocal sum of the Zhang sequence

The Zhang sequence Z [5] is, at the moment, the known Sidon sequence with the largest reciprocal sum. In the accompanying file `Zhang.txt` there are the first 25000 terms of Z . Their values allow us to compute the following bounds for S_Z :

$$\sum_{n=1}^{25000} \frac{1}{z_n} < S_Z < \sum_{n=1}^{25000} \frac{1}{z_n} + \sum_{n=25001}^{510290} \frac{1}{z_{25000} + n - 25000} + \sum_{n=510291}^{\infty} \frac{2}{n(n-1)} \quad (9)$$

$$S_G < 2.16007769 < S_Z < 2.16008532. \quad (10)$$

2.4 The sequence H and its reciprocal sum

Definition 1.

$$h_n = \begin{cases} 1, & \text{if } n = 1; \\ 229, & \text{if } n = 15; \\ 962, & \text{if } n = 27; \\ \min\{x \mid \forall i, j, k \leq n, a_i + a_j \neq a_k + x\}, & \text{otherwise.} \end{cases}$$

The first 26 terms of the sequence H are the same of the Zhang sequence, the 27th term is 962, and from there on the values are provided by the greedy algorithm.

We will now prove that the sequence H satisfies our aim, i.e., $S_H > S_Z$.

We can find the first values of h_n ($1 \leq n \leq 25000$) in the accompanying file `H.txt`. Hence we get

$$\sum_{n=1}^{25000} \frac{1}{h_n} < S_H < \sum_{n=1}^{25000} \frac{1}{h_n} + \sum_{n=25001}^{510140} \frac{1}{h_{25000} + n - 25000} + \sum_{n=510141}^{\infty} \frac{2}{n(n-1)} \quad (11)$$

$$2.16027651 < S_H < 2.16028417. \quad (12)$$

In conclusion,

$$S_Z < 2.16027651 < S_H \leq \text{DDC}, \quad (13)$$

which was what we wanted to prove.

I found the sequence H using an algorithm which proceeds as follows: given a finite Sidon sequence b_1, b_2, \dots, b_n , choose as b_{n+1} the value that would yield the largest reciprocal sum if the sequence were continued with the greedy algorithm; then repeat with the sequence $b_1, b_2, \dots, b_n, b_{n+1}$. The program considered 20 candidate values in each step, and estimated the reciprocal sums with the terms up to 64000. Giving as input the first 13 terms of the Mian-Chowla sequence, I obtained the first 30 terms of H . I have experimented with a number of other starting sequences, without finding any other sequence with a reciprocal sum greater than S_Z .

The most likely place to find a B_2 -sequence X achieving a reciprocal sum $S_X > S_H$, if it exists, is between the sequences having the first 27 terms in common with H , or with the Zhang sequence.

References

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(Concerned with sequence [A005282](#))