

CANONICAL BASES OF A COIDEAL SUBALGEBRA IN $U_q(\mathfrak{sl}_2)$

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ABSTRACT. We consider tensor products of finite-dimensional representations of a coideal subalgebra in $U_q(\mathfrak{sl}_2)$. We present an explicit expression for the dual of the canonical bases through a diagrammatic presentation. We show that the decomposition of tensor products of dual canonical bases and the action of the coideal subalgebra have integral and positive properties. As an application, we consider the eigensystem of the generator of the coideal subalgebra on the dual canonical bases. We provide all the eigenvalues and obtain an explicit expression of the eigenfunction for the largest eigenvalue. The sum of the components of this eigenfunction is conjectured to be equal to the total number of arrangements of bishops with a certain symmetry.

1. INTRODUCTION

In [18], Lusztig introduced the notion of the canonical basis of the q -analogue of enveloping algebras $U_q(\mathfrak{g})$ associated with a simple finite-dimensional Lie algebra \mathfrak{g} . This basis is characterized by three conditions: the basis is integral, bar-invariant and spans a $\mathbb{Z}[q^{-1}]$ -lattice \mathcal{L} with a certain image in the quotient $\mathcal{L}/q^{-1}\mathcal{L}$. In [10, 11, 12], Kashiwara introduced the notion of (global) crystal bases and showed its existence and uniqueness. The coincidence of two concepts, the canonical basis and the global crystal basis, was shown in [19, 9]. Lusztig constructed a canonical basis in the tensor product and proved its associativity in [20]. In the case of $\mathfrak{g} = \mathfrak{sl}_2$, Frenkel and Khovanov provided a diagrammatic depiction of the dual of the canonical basis in the tensor products of finite-dimensional irreducible representations and gave the action of the quantum group on these bases [8]. This diagrammatic method together with the positive integral property led to the categorification of $U_q(\mathfrak{sl}_2)$ [3, 7].

Let θ be an involution of \mathfrak{g} and \mathfrak{g}^θ be the fixed Lie subalgebra. The symmetric pair $(\mathfrak{g}, \mathfrak{g}^\theta)$ forms symmetric spaces in the classical case. In [22, 23, 24], Noumi, Sugitani and Dijkhuizen constructed quantum symmetric spaces by using the solution of the reflection equation. In [15, 16, 17], Letzter constructed quantum symmetric spaces by using the involution θ and described the generators of the q -analogue of $U(\mathfrak{g}^\theta)$ which is a coideal subalgebra of $U_q(\mathfrak{g})$. These two approaches produce the same coideal subalgebras [15]. A general theory for quantum symmetric spaces in the case of symmetrizable Kac–Moody algebras was developed in [14]. In the study of Kazhdan–Lusztig theory of type B, Bao and Wang introduced the notion of the quasi- R -matrix and canonical bases (ι -canonical bases in [1]) for the quantum symmetric pair in the case of $\mathfrak{g} = \mathfrak{sl}_n$ [1].

In this paper, we consider tensor products of finite-dimensional representations of a coideal subalgebra U in $U_q(\mathfrak{sl}_2)$. We present an explicit expression for the dual of the canonical bases and provide the action of the coideal subalgebra on these bases. The diagrammatic presentation of the dual bases is also provided. For this presentation, we make use of a diagrammatic presentation for Kazhdan–Lusztig bases of Hecke algebra of type B studied in [26]. We show that the expansion coefficients of a (dual) canonical basis in terms of standard bases are written in terms of Kazhdan–Lusztig polynomials. We also show that the decomposition of tensor products of dual canonical bases and the action of the coideal subalgebra have integral and positive properties. As an application, we consider the eigensystem of the generator of the coideal subalgebra on the dual canonical bases. In quantum integrable systems, a coideal subalgebra is the

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symmetry of the system with a boundary, that is, the generators of a coideal subalgebra commute with the Hamiltonian. Therefore, a knowledge of the eigensystem of the generators of U is important to study the eigensystem of the Hamiltonian. We provide all the eigenvalues of the generator of the coideal subalgebra on the dual canonical bases. An explicit expression of the eigenfunction Ψ for the largest (at $q = 1$) eigenvalue is obtained (see Theorem 6.21). We show that this eigenfunction Ψ has a positive integral property, *i.e.*, $\Psi \in \mathbb{N}[q, q^{-1}]$. From this observation, we have a conjecture that the sum of components of Ψ at $q = 1$ is equal to the total number of arrangements of bishops with a symmetry.

The paper is organized as follows. In section 2, we briefly recall the parabolic Kazhdan–Lusztig polynomials in the case of the Hecke algebra of type B. In section 3, we review the definitions and results about the quantum group $U_q(\mathfrak{sl}_2)$ and its coideal subalgebra U . In Section 4, we introduce the notion of canonical bases for both $U_q(\mathfrak{sl}_2)$ and U . The graphical depiction of the dual canonical basis is presented. In Section 5, we introduce another graphical method to connect canonical bases with Kazhdan–Lusztig polynomials. We extend the diagrammatic rules in [26, Section 3] and provide a new inversion formula regarding Kazhdan–Lusztig polynomials. Section 6 is devoted to an analysis of integral and positive properties of dual canonical bases. We study the eigensystem of the generator of U in details and obtain an explicit formula for the eigenvector Ψ . A conjecture on this eigenvector is presented. In Appendix A, we collect two technical lemmas used in Section 6.

2. HECKE ALGEBRA OF TYPE B AND KAZHDAN–LUSZTIG POLYNOMIALS

Let \mathcal{S}_N be the symmetric group and \mathcal{S}_N^C be the Weyl group associated with the Dynkin diagram of type C. We use a partial order in \mathcal{S}_N^C , the (strong) *Bruhat order*. We write $w' \leq w$ if and only if w' can be obtained as a subexpression of a reduced expression of w . The *Hecke algebra* \mathcal{H}_N of type B is the unital, associative algebra over the ring $R := \mathbb{Z}[t, t^{-1}]$ with generators T_i , $i = 1, \dots, N$, and relations

$$\begin{aligned} (T_i - t)(T_i + t^{-1}) &= 0 & 1 \leq i \leq N, \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} & 1 \leq i < N - 1, \\ T_{N-1} T_N T_{N-1} T_N &= T_N T_{N-1} T_N T_{N-1}, \\ T_i T_j &= T_j T_i & |i - j| > 1. \end{aligned}$$

The Hecke algebra \mathcal{H}_N has standard basis $(T_w)_{w \in \mathcal{S}_N^C}$ where $T_w = T_{i_1} T_{i_2} \cdots T_{i_r}$ for a reduced word $w = s_{i_1} s_{i_2} \cdots s_{i_r}$ written in terms of elementary transpositions s_i . The involutive ring automorphism of \mathcal{H}_N , $a \rightarrow \bar{a}$, is defined by $T_i \rightarrow T_i^{-1}$ and $t \rightarrow t^{-1}$. Then

Theorem 2.1 (Kazhdan and Lusztig [13]). *There exists a unique basis C_w such that $\overline{C_w} = C_w$ and*

$$C_w = \sum_{v \leq w} P_{v,w}(t^{-1}) T_v,$$

where $P_{v,w}(t^{-1}) \in t^{-1}\mathbb{Z}[t^{-1}]$ and $P_{v,v} = 1$.

Let $\epsilon \in \{+, -\}$. We have simple bijections among the following three sets [26, Section 2].

- (1) An element of $\mathcal{S}_N^C / \mathcal{S}_N$.
- (2) A binary string in $\{+, -\}^N$.
- (3) A path from $(0, 0)$ to (N, n) with $|n| \leq N$ and $N - n \in 2\mathbb{Z}$ where each step is in the direction $(1, \pm 1)$.

Bijections are realized by the natural action of \mathcal{S}_N^C on $\{+, -\}^N$ with representative $(+\dots+)$ for $\epsilon = +$ and $(-\dots-)$ for $\epsilon = -$. A sequence $\mathbf{v} = (v_1, \dots, v_N) \in \{+, -\}^N$ is identified with the path with the i -th step $(1, v_i)$. We denote by \mathcal{P}_N the set of paths from $(0, 0)$ to (N, n) .

Definition 2.2. Let α, β be two paths in \mathcal{P}_N . Then, $\alpha \leq \beta$ if and only if α is below β for $\epsilon = -$, above β for $\epsilon = +$.

Note that this definition is compatible with the induced Bruhat order in $\mathcal{S}_N^C/\mathcal{S}_N$.

We define a free R -module \mathcal{M}_N with a basis indexed by \mathcal{P}_N , that is, $\mathcal{M}_N := \langle m_\nu : \nu \in \mathcal{P}_N \rangle$. We have two modules \mathcal{M}_N^ϵ , $\epsilon = \pm$, corresponding to two natural projection maps from $\mathbb{C}[\mathcal{S}_N^C]$ to $\mathbb{C}[\mathcal{S}_N^C/\mathcal{S}_N]$ [6]. The action of \mathcal{H}_N on the modules \mathcal{M}_N^ϵ is as follows:

$$\begin{aligned}
\epsilon = + : \quad & T_i m_{\dots \alpha \alpha \dots} = t m_{\dots \alpha \alpha \dots}, \quad \alpha = \pm, \quad 1 \leq i \leq N-1 \\
& T_i m_{\dots + \dots} = m_{\dots + \dots}, \quad 1 \leq i \leq N-1 \\
& T_i m_{\dots - \dots} = (t - t^{-1}) m_{\dots - \dots} + m_{\dots + \dots}, \quad 1 \leq i \leq N-1 \\
& T_N m_{\dots +} = m_{\dots -}, \\
& T_N m_{\dots -} = m_{\dots +} + (t - t^{-1}) m_{\dots -}, \\
\epsilon = - : \quad & T_i m_{\dots \alpha \alpha \dots} = -t^{-1} m_{\dots \alpha \alpha \dots}, \quad \alpha = \pm, \quad 1 \leq i \leq N-1 \\
& T_i m_{\dots - \dots} = m_{\dots + \dots}, \quad 1 \leq i \leq N-1 \\
& T_i m_{\dots + \dots} = (t - t^{-1}) m_{\dots + \dots} + m_{\dots - \dots}, \quad 1 \leq i \leq N-1 \\
& T_N m_{\dots -} = m_{\dots +}, \\
& T_N m_{\dots +} = m_{\dots -} + (t - t^{-1}) m_{\dots +},
\end{aligned}$$

Note that the module \mathcal{M}_N^+ (resp. \mathcal{M}_N^-) has a generating vector $m_{+\dots+}$ (resp. $m_{-\dots-}$).

We introduce the parabolic analogue of the Kazhdan–Lusztig bases and polynomials. We are interested in the maximal parabolically induced module \mathcal{M}_N^ϵ .

Theorem 2.3 (Deodhar [6]). *There exists a unique basis $(C_\alpha^\pm)_{\alpha \in \mathcal{P}_N}$ of \mathcal{M}_N^\pm such that $\overline{C_\alpha^\pm} = C_\alpha^\pm$ and*

$$C_\beta^\pm = \sum_{\alpha \leq \beta} P_{\alpha, \beta}^\pm(t^{-1}) m_\alpha$$

where $\alpha \leq \beta$ is in the order of paths associated with the sign $\epsilon = \pm$ and the polynomials $P_{\alpha, \beta}^\pm(t^{-1}) \in t^{-1}\mathbb{Z}[t^{-1}]$ if $\alpha < \beta$ and $P_{\alpha, \alpha}^\pm(t^{-1}) = 1$.

Our definition of $P_{\alpha, \beta}^\pm$ differs from the original parabolic Kazhdan–Lusztig polynomials by the factor t^{-d} for some $d \in \mathbb{N}$. The polynomial $P^-(t^{-1})$ is a monomial of t^{-1} [5]. The algorithm to compute $P_{\alpha, \beta}^+(t^{-1})$ was found by Boe [4]. See [26] for a unified treatment of $P_{\alpha, \beta}^\pm$ in terms of paths.

3. QUANTUM GROUP $U_q(\mathfrak{sl}_2)$

In this section, we briefly summarize the quantum group $U_q(\mathfrak{sl}_2)$ and a coideal subalgebra in $U_q(\mathfrak{sl}_2)$. We follow the notation used in [1, 8].

3.1. Quantum Group $U_q(\mathfrak{sl}_2)$. Let $\mathbb{C}(q)$ be the field of rational functions in an indeterminate q . We denote by $\bar{\cdot} : \mathbb{C}(q) \rightarrow \mathbb{C}(q)$ the \mathbb{C} -algebra involution such that $q^n \mapsto q^{-n}$ for all n . The quantum group $U_q(\mathfrak{sl}_2)$ is an

associative algebra over $\mathbb{C}(q)$ with generators $K^{\pm 1}, E, F$ and relations

$$\begin{aligned} KK^{-1} &= K^{-1}K = 1, \\ KEK^{-1} &= q^2E, \\ KFK^{-1} &= q^{-2}F, \\ [E, F] &= \frac{K - K^{-1}}{q - q^{-1}}. \end{aligned}$$

We introduce the quantum integer $[n] := (q^n - q^{-n})/(q - q^{-1})$, the quantum factorial $[n]! := \prod_{k=1}^n [k]$ and the q -analogue for the binomial coefficient

$$\begin{bmatrix} n \\ m \end{bmatrix} := \frac{[n]!}{[m]![n-m]}.$$

We set $E^{(n)} := E^n/[n]!$ and $F^{(n)} := F^n/[n]!$.

We define the two involutions. One is *the Cartan involution* denoted by ω :

$$\begin{aligned} \omega(E) &= F, & \omega(F) &= E, & \omega(K^{\pm 1}) &= K^{\pm 1}, & \omega(q^{\pm 1}) &= q^{\pm 1}, \\ \omega(xy) &= \omega(y)\omega(x), & x, y &\in U_q(\mathfrak{sl}_2). \end{aligned}$$

The other involution is *the bar involution* ψ and defined by

$$\begin{aligned} \psi(E) &= E, & \psi(F) &= F, & \psi(K^{\pm 1}) &= K^{\mp 1}, & \psi(q^{\pm 1}) &= q^{\mp 1}, \\ \psi(xy) &= \psi(x)\psi(y), & x, y &\in U_q(\mathfrak{sl}_2). \end{aligned}$$

The irreducible $(n+1)$ -dimensional representations V_n , $n \geq 1$, has a basis $\{v_m \mid -n \leq m \leq n, m \equiv n \pmod{2}\}$. The action of $U_q(\mathfrak{sl}_2)$ is

$$\begin{aligned} K^{\pm 1}v_m &= q^{\pm m}v_m, \\ Ev_m &= \left[\frac{n+m}{2} + 1 \right] v_{m+2}, \\ Fv_m &= \left[\frac{n-m}{2} + 1 \right] v_{m-2}. \end{aligned}$$

Note that the bases $\{v_m\}$ are canonical bases in the sense of [18]. All $U_q(\mathfrak{sl}_2)$ -modules in this paper will be finite-dimensional representations of type I.

We define a bilinear symmetric pairing in V_n by $\langle xu, v \rangle = \langle u, \omega(x)v \rangle$ and $\langle v_n, v_n \rangle = 1$ where $u, v \in V_n$ and $x \in U_q(\mathfrak{sl}_2)$. Let $\{v^m \mid -n \leq m \leq n, n \equiv m \pmod{2}\}$ be the dual bases of $\{v_m\}$ with respect to \langle, \rangle . The action of $U_q(\mathfrak{sl}_2)$ on the dual basis is given explicitly in [8].

For $\kappa = (\kappa_1, \dots, \kappa_n)$ with $\kappa_i = \pm 1, 1 \leq i \leq n$, we define

$$|\kappa| := \sum_{i=1}^n \kappa_i, \quad \|\kappa\|_- = \sum_{i < j} \theta(\kappa_i < \kappa_j),$$

where $\theta(P) = 1$ if P is true and zero otherwise. We define the projection $\pi_n : V_1^{\otimes n} \mapsto V_n$ by

$$(1) \quad \pi_n(v^{\kappa_1} \otimes \dots \otimes v^{\kappa_n}) = q^{-\|\kappa\|_-} v^{|\kappa|}.$$

The quantum group $U_q(\mathfrak{sl}_2)$ has a Hopf algebra structure with comultiplication. We have two different comultiplications Δ_{\pm} :

$$\begin{aligned} \Delta_+(K^{\pm 1}) &= K^{\pm 1} \otimes K^{\pm 1}, \\ \Delta_+(E) &= E \otimes 1 + K \otimes E, \\ \Delta_+(F) &= F \otimes K^{-1} + 1 \otimes F. \end{aligned}$$

and

$$\begin{aligned}\Delta_-(K^{\pm 1}) &= K^{\pm 1} \otimes K^{\pm 1}, \\ \Delta_-(E) &= E \otimes K^{-1} + 1 \otimes E, \\ \Delta_-(F) &= F \otimes 1 + K \otimes F.\end{aligned}$$

We define another comultiplications as $\bar{\Delta}_{\pm}(x) := (\psi \otimes \psi)\Delta_{\pm}(\psi(x))$, $x \in U_q(\mathfrak{sl}_2)$. We also have counit and antipode, but we do not need them in this paper.

Following [20, 21], we define the quasi- R -matrix Θ_{\pm} associated with Δ_{\pm} :

$$\Theta_+ := \sum_{k \geq 0} (-1)^k q^{-k(k-1)/2} \frac{(q - q^{-1})^k}{[k]!} F^k \otimes E^k,$$

and $\Theta_- := \sigma \bar{\Theta}_+$ where σ is the permutation of the tensor factors. For finite-dimensional representations M and N , all but finitely many terms of Θ_{\pm} act as zero on any given vector $m \otimes n \in M \otimes N$. The quasi- R -matrix has the property such that $\Theta_{\pm} \bar{\Delta}_{\pm}(x) = \Delta_{\pm}(x) \Theta_{\pm}$ and $\Theta_{\pm} \bar{\Theta}_{\pm} = \bar{\Theta}_{\pm} \Theta_{\pm} = 1$. We also define $\Theta^{(3)} = (1 \otimes \Delta) \Theta \cdot \Theta_{23}$ and in general

$$\Theta^{(n)} := (1 \otimes \Delta^{n-2}) \Theta \cdot \Theta_{2, \dots, n}^{(n-1)},$$

where $\Theta = \Theta_{\pm}$ and $\Delta = \Delta_{\pm}$.

Let M be a finite-dimensional $U_q(\mathfrak{sl}_2)$ -module of type I, B be a $\mathbb{C}(q)$ -basis of M and the pair (M, B) be a based module as in [21, Section 27]. We define an involution $\psi : M \rightarrow M$ by $\psi(ab) = \bar{a}b$ for all $a \in \mathbb{C}(q)$ and $b \in B$. This involution is compatible with the involution ψ on $U_q(\mathfrak{sl}_2)$ in the sense that $\psi(um) = \psi(u)\psi(m)$ for all $u \in U_q(\mathfrak{sl}_2)$ and $m \in M$. Suppose M and N are finite-dimensional $U_q(\mathfrak{sl}_2)$ -modules of type I with the involution ψ . Following [21, Section 27.3], we define an involution ψ_{\pm} on the tensor product $M \otimes N$:

$$\psi_{\pm}(m \otimes n) := \Theta_{\pm}(\psi(m) \otimes \psi(n)), \quad m \in M, n \in N.$$

In general, let M_i , $1 \leq i \leq r$, be involutive $U_q(\mathfrak{sl}_2)$ -modules. Then the involution $\psi_{\pm} : M_1 \otimes \dots \otimes M_r \rightarrow M_1 \otimes \dots \otimes M_r$ is recursively given by

$$\psi_{\pm}(m_1 \otimes \dots \otimes m_r) = \Theta_{\pm}(\psi_{\pm}(m_1 \otimes \dots \otimes m_p) \otimes \psi_{\pm}(m_{p+1} \otimes \dots \otimes m_r))$$

for $1 \leq p \leq r-1$ and $m_i \in M_i$.

3.2. Coideal subalgebra. We consider the Dynkin diagram of type A_1 and the identity involution. By a general theory of quantum symmetric pairs [14, 15], we have coideal subalgebras of $U_q(\mathfrak{sl}_2)$. A coideal subalgebra U is defined as a polynomial algebra in X , namely, $U := \mathbb{C}(q)[X]$. The pair $(U_q(\mathfrak{sl}_2), U)$ is a quantum symmetric pair. The coideal subalgebra U has an antilinear bar involution ψ' such that $\psi'(X) = X$ and $\psi'(q) = q^{-1}$. There exists an injective $\mathbb{C}(q)$ -algebra homomorphism $\iota : U \rightarrow U_q(\mathfrak{sl}_2)$, $X \mapsto E + qFK^{-1} + K^{-1}$. In the dual picture, we consider the generator $Y := \psi(\omega(X)) = F + q^{-1}KE + K$. The comultiplication $\Delta : U \rightarrow U_q(\mathfrak{sl}_2) \times U$ is given by

$$(2) \quad \Delta(X) = K^{-1} \otimes X + qFK^{-1} \otimes 1 + E \otimes 1,$$

$$(3) \quad \Delta(Y) = K \otimes Y + q^{-1}KE \otimes 1 + F \otimes 1.$$

Note that U is left coideal since $\Delta(U) \subset U_q(\mathfrak{sl}_2) \times U$.

A general theory of constructing the quasi- R -matrix for a quantum symmetric pair was developed in [1]. We collect the facts about the quasi- R -matrix for the quantum symmetric pair $(U_q(\mathfrak{sl}_2), U)$ in this subsection. See [1] for a detailed exposition.

The *intertwiner* Υ_{\pm} for the quantum symmetric pair $(U_q(\mathfrak{sl}_2), U)$ satisfy

$$(4) \quad \iota(\psi'(u))\Upsilon_+ = \Upsilon_+\psi(\iota(u)), \quad u \in U.$$

The solution of Eqn.(4) is explicitly given in [1, Section 4]. We have $\Upsilon_+ = \sum_{n \geq 0} \Upsilon_{+,n}$ with $\Upsilon_{+,n} = c_n F^{(n)}$. The coefficients c_n satisfy the recurrence relation

$$c_n = -q^{-(n-1)}(q - q^{-1})(q[n-1]c_{n-2} + c_{n-1}).$$

with $c_0 = 0$ and $c_1 = 1$. Similarly, we define $\Upsilon_- = \sum_{n \geq 0} \Upsilon_{-,n}$ with $\Upsilon_{-,n} = \overline{c_n} E^{(n)}$. We have normalized Υ_{\pm} such that $\Upsilon_{\pm,0} = 1$.

The quasi- R -matrix Θ_{\pm}^t is defined by (see [1, Section 3])

$$\Theta_{\pm}^t := \Delta_{\pm}(\Upsilon_{\pm})\Theta_{\pm}(1 \otimes \Upsilon_{\pm}^{-1}).$$

The quasi- R -matrix satisfies $\Theta_{\pm}^t \overline{\Theta_{\pm}^t} = 1$ and $\Delta_{\pm}(u)\Theta_{\pm}^t = \Theta_{\pm}^t \overline{\Delta_{\pm}(u)}$ for $u \in U$.

Let (M, B) be a based module of $U_q(\mathfrak{sl}_2)$ as in Section 3.1. We regard M as a U -module. We define an involution $\psi'_{\pm} : M \rightarrow M$ by $\psi'_{\pm} := \Upsilon_{\pm} \circ \psi$. The involution ψ'_{\pm} is compatible with ψ^t on U in the sense that $\psi'_{\pm}(um) = \psi^t(u)\psi'_{\pm}(m)$ for all $u \in U$ and $m \in M$. Suppose M is a $U_q(\mathfrak{sl}_2)$ -module equipped with ψ and N is a U -module equipped with ψ'_{\pm} . We regard $M \otimes N$ as a U -module. Following [1, Section 3.4], we define the involution ψ'_{\pm} on the tensor product $M \otimes N$:

$$\psi'_{\pm}(m \otimes n) := \Theta_{\pm}^t(\psi(m) \otimes \psi'_{\pm}(n)), \quad m \in M, n \in N.$$

In general, let $M_i, 1 \leq i \leq r$, be $U_q(\mathfrak{sl}_2)$ -modules and N be a U -module. The involution ψ'_{\pm} is recursively given by

$$\psi'_{\pm}(m_1 \otimes \dots \otimes m_r \otimes n) = \Theta_{\pm}^t(\psi_{\pm}(m_1 \otimes \dots \otimes m_p) \otimes \psi'_{\pm}(m_{p+1} \otimes \dots \otimes m_r \otimes n)),$$

where $m_i \in M_i, 1 \leq p \leq r$, and $n \in N$.

4. CANONICAL BASES OF $U_q(\mathfrak{sl}_2)$ AND U

Let $\mathbf{k} = (k_1, \dots, k_n)$ and $\mathbf{l} = (l_1, \dots, l_n)$. When $\sum_{i=1}^m k_i \leq \sum_{i=1}^m l_i$ for all $1 \leq m \leq n$, we denote it by $\mathbf{k} \leq_+ \mathbf{l}$ or $\mathbf{l} \leq_- \mathbf{k}$. For $\mathbf{m} := (m_1, \dots, m_n) \in \mathbb{N}_+^n$, we define

$$I_{\mathbf{m}} := \{k_i, 1 \leq i \leq n \mid -m_i \leq k_i \leq m_i, k_i \equiv m_i \pmod{2}\}.$$

In this section, we follow the notation and the convention used in [2, 21].

4.1. Canonical bases. Let (M, B) and (M', B') be based modules. The tensor product of two based modules $M \otimes M'$ has a basis $B \otimes B'$. This basis is not compatible with the involution ψ in general. We introduce a modified basis $B \diamond B'$ in the tensor product following [20]. We call the basis $B \diamond B'$ a *canonical basis*. More in general, we obtain canonical bases $\{v_{k_1} \diamond \dots \diamond v_{k_n}\}_{\mathbf{k} \in I_{\mathbf{m}}}$ in the tensor product $V_{m_1} \otimes \dots \otimes V_{m_n}$. Note that we have associativity of tensor products. The canonical basis is characterized as follows.

Theorem 4.1 (Lusztig [20]).

(1) *There exists a unique element $v_{k_1} \diamond \dots \diamond v_{k_n} \in V_{m_1} \otimes \dots \otimes V_{m_n}$ such that*

$$\begin{aligned} \psi_+(v_{k_1} \diamond \dots \diamond v_{k_n}) &= v_{k_1} \diamond \dots \diamond v_{k_n}, \\ v_{k_1} \diamond \dots \diamond v_{k_n} - v_{k_1} \otimes \dots \otimes v_{k_n} &\in q^{-1} \cdot {}_{\mathbb{Z}[q^{-1}]} V_{m_1} \otimes \dots \otimes V_{m_n}. \end{aligned}$$

(2) *The vector $v_{k_1} \diamond \dots \diamond v_{k_n} - v_{k_1} \otimes \dots \otimes v_{k_n}$ is a linear combination of $v_{l_1} \otimes \dots \otimes v_{l_n}$ with $\mathbf{l} \neq \mathbf{k}$, $\sum_{i=1}^n k_i = \sum_{i=1}^n l_i$ and $\mathbf{l} \leq_+ \mathbf{k}$. The coefficients are in $q^{-1} \mathbb{Z}[q^{-1}]$.*

(3) *The elements $v_{k_1} \diamond \dots \diamond v_{k_n}$ form a $\mathbb{C}(q)$ -basis of $V_{m_1} \otimes \dots \otimes V_{m_n}$, a $\mathbb{Z}[q, q^{-1}]$ -basis of ${}_{\mathbb{Z}[q, q^{-1}]} V_{m_1} \otimes \dots \otimes V_{m_n}$, and a $\mathbb{Z}[q^{-1}]$ -basis of ${}_{\mathbb{Z}[q^{-1}]} V_{m_1} \otimes \dots \otimes V_{m_n}$.*

We define the bilinear pairing of $V_{m_1} \otimes \dots \otimes V_{m_n}$ and $V_{m_1} \otimes \dots \otimes V_{m_n}$ by

$$\langle v_{k_1} \otimes \dots \otimes v_{k_n}, v^{k'_1} \otimes \dots \otimes v^{k'_n} \rangle = \delta_{k_1}^{k'_1} \dots \delta_{k_n}^{k'_n}.$$

We define the dual canonical basis $v^{k_1} \heartsuit \dots \heartsuit v^{k_n}$ with respect to the bilinear pairing:

$$\langle v_{k_1} \diamond \dots \diamond v_{k_n}, v^{l_1} \heartsuit \dots \heartsuit v^{l_n} \rangle = \delta_{k_1}^{l_1} \dots \delta_{k_n}^{l_n}.$$

The dual statement of Theorem 4.1 is

Theorem 4.2 (Frenkel and Khovanov [8, Theorem 1.8]).

(1) There exists a unique element $v^{k_1} \heartsuit \dots \heartsuit v^{k_n} \in V_{m_1} \otimes \dots \otimes V_{m_n}$ such that

$$\begin{aligned} \psi_-(v^{k_1} \heartsuit \dots \heartsuit v^{k_n}) &= v^{k_1} \heartsuit \dots \heartsuit v^{k_n}, \\ v^{k_1} \heartsuit \dots \heartsuit v^{k_n} - v^{k_1} \otimes \dots \otimes v^{k_n} &\in q^{-1} \cdot_{\mathbb{Z}[q^{-1}]} V_{m_1} \otimes \dots \otimes V_{m_n} \end{aligned}$$

(2) The vector $v^{k_1} \heartsuit \dots \heartsuit v^{k_n} - v^{k_1} \otimes \dots \otimes v^{k_n}$ is a linear combination of $v^{l_1} \otimes \dots \otimes v^{l_n}$ with $\mathbf{l} \neq \mathbf{k}$, $\sum_{i=1}^n k_i = \sum_{i=1}^n l_i$ and $\mathbf{k} <_+ \mathbf{l}$. The coefficients are in $q^{-1}\mathbb{Z}[q^{-1}]$.

(3) The elements $v_{k_1} \heartsuit \dots \heartsuit v_{k_n}$ form a $\mathbb{C}(q)$ -basis of $V_{m_1} \otimes \dots \otimes V_{m_n}$, a $\mathbb{Z}[q, q^{-1}]$ -basis of ${}_{\mathbb{Z}[q, q^{-1}]} V_{m_1} \otimes \dots \otimes V_{m_n}$, and a $\mathbb{Z}[q^{-1}]$ -basis of ${}_{\mathbb{Z}[q^{-1}]} V_{m_1} \otimes \dots \otimes V_{m_n}$.

We regard a based module (M, B) as a finite-dimensional U -module. When $b \in B$ is a lowest weight vector, $\psi_+(b) = b$. Similarly, when $b \in B$ is a highest weight vector, $\psi_-(b) = b$. For any other $b \in B$, we have $\psi_{\pm}(b) \neq b$, that is, the basis B is not compatible with ψ_{\pm} . We can introduce a modified basis B' with the property $\psi'(b') = b'$ for $b' \in B'$. By abuse of notation, we call the pair (M, B') a based module of U and the basis B' a canonical basis of U .

Let (M, B) and (M', B') be based modules of $U_q(\mathfrak{sl}_2)$ and U respectively. By a similar argument to canonical bases of $U_q(\mathfrak{sl}_2)$, we can introduce a modified basis $B \bowtie B'$ in the tensor product of two based modules $M \otimes M'$. In general, we obtain bases $\{v^{k_1} \heartsuit \dots \heartsuit v^{k_n}\}_{\mathbf{k} \in I_m}$ in the tensor product $V_{m_1} \otimes \dots \otimes V_{m_n}$. We call $v^{k_1} \heartsuit \dots \heartsuit v^{k_n}$, $\mathbf{k} \in I_m$, a canonical basis of U . The canonical bases of U are characterized as follows. The proofs are similar to the one for Theorem 4.1.

Theorem 4.3.

(1) There exists a unique element $v_{k_1} \heartsuit \dots \heartsuit v_{k_n} \in V_{m_1} \otimes \dots \otimes V_{m_n}$ such that

$$\begin{aligned} \psi_+(v_{k_1} \heartsuit \dots \heartsuit v_{k_n}) &= v_{k_1} \heartsuit \dots \heartsuit v_{k_n}, \\ v_{k_1} \heartsuit \dots \heartsuit v_{k_n} - v_{k_1} \otimes \dots \otimes v_{k_n} &\in q^{-1} \cdot_{\mathbb{Z}[q^{-1}]} V_{m_1} \otimes \dots \otimes V_{m_n}. \end{aligned}$$

(2) The vector $v_{k_1} \heartsuit \dots \heartsuit v_{k_n} - v_{k_1} \otimes \dots \otimes v_{k_n}$ is a linear combination of $v_{l_1} \otimes \dots \otimes v_{l_n}$ with $\mathbf{l} \leq_+ \mathbf{k}$. The coefficients are in $q^{-1}\mathbb{Z}[q^{-1}]$.

(3) The elements $v_{k_1} \heartsuit \dots \heartsuit v_{k_n}$ form a $\mathbb{C}(q)$ -basis of $V_{m_1} \otimes \dots \otimes V_{m_n}$, a $\mathbb{Z}[q, q^{-1}]$ -basis of ${}_{\mathbb{Z}[q, q^{-1}]} V_{m_1} \otimes \dots \otimes V_{m_n}$, and a $\mathbb{Z}[q^{-1}]$ -basis of ${}_{\mathbb{Z}[q^{-1}]} V_{m_1} \otimes \dots \otimes V_{m_n}$.

We define the dual of a canonical basis, $v^{l_1} \heartsuit \dots \heartsuit v^{l_n}$, with respect to the bilinear pairing:

$$(5) \quad \langle v_{l_1} \heartsuit \dots \heartsuit v_{l_n}, v^{l'_1} \heartsuit \dots \heartsuit v^{l'_n} \rangle = \delta_{l_1}^{l'_1} \dots \delta_{l_n}^{l'_n}.$$

The dual statement of Theorem 4.3 is

Theorem 4.4.

(1) There exists a unique element $v^{k_1} \heartsuit \dots \heartsuit v^{k_n} \in V_{m_1} \otimes \dots \otimes V_{m_n}$ such that

$$\begin{aligned} \psi_-(v^{k_1} \heartsuit \dots \heartsuit v^{k_n}) &= v^{k_1} \heartsuit \dots \heartsuit v^{k_n}, \\ v^{k_1} \heartsuit \dots \heartsuit v^{k_n} - v^{k_1} \otimes \dots \otimes v^{k_n} &\in q^{-1} \cdot_{\mathbb{Z}[q^{-1}]} V_{m_1} \otimes \dots \otimes V_{m_n} \end{aligned}$$

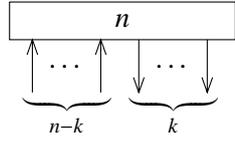
- (2) The vector $v^{k_1} \diamond \dots \diamond v^{k_n} - v^{k_1} \otimes \dots \otimes v^{k_n}$ is a linear combination of $v^{l_1} \otimes \dots \otimes v^{l_n}$ with $\mathbf{l} \leq \mathbf{k}$. The coefficients are in $q^{-1}\mathbb{Z}[q^{-1}]$.
- (3) The elements $v_{k_1} \diamond \dots \diamond v_{k_n}$ form a $\mathbb{C}(q)$ -basis of $V_{m_1} \otimes \dots \otimes V_{m_n}$, a $\mathbb{Z}[q, q^{-1}]$ -basis of ${}_{\mathbb{Z}[q, q^{-1}]}V_{m_1} \otimes \dots \otimes V_{m_n}$, and a $\mathbb{Z}[q^{-1}]$ -basis of ${}_{\mathbb{Z}[q^{-1}]}V_{m_1} \otimes \dots \otimes V_{m_n}$.

4.2. Graphical depiction of the standard bases. The graphical calculus for $U_q(\mathfrak{sl}_2)$ was developed in [8, Section 2]. Since the comultiplication in the dual space is different from [8], we briefly summarize the graphical calculus for standard bases in this subsection.

The dual basis v^{n-2k} is written as

$$v^{n-2k} = \pi_n((v^1)^{\otimes(n-k)} \otimes (v^{-1})^{\otimes k}).$$

The diagram for v^{n-2k} is depicted as



where the box marked by n with n lines corresponds to the projector π_n . The graph for a tensor product $v^{n_1-2k_1} \otimes \dots \otimes v^{n_r-2k_r}$ is depicted by placing the diagram for $v^{n_i-2k_i}$, $1 \leq i \leq r$, from left to right in parallel.

4.3. Graphical depiction of dual canonical bases. The diagram for $v^{n_1-2k_1} \diamond \dots \diamond v^{n_r-2k_r}$ is obtained from the diagram for $v^{n_1-2k_1} \otimes \dots \otimes v^{n_r-2k_r}$ by the following rules. The rules are essentially the same as the ones for the Kazhdan–Lusztig basis studied in [26] ($m = 1$ of Case B in [26]).

- (A) Make a pair between adjacent down arrow and up arrow (in this order). Connect this pair of two arrows into a simple unoriented arc. The arc does not intersect with anything.
- (B) Repeat the above procedure (making pairs) until all the up arrows are to the left of all down arrows.
- (C) Put a star (★) on the rightmost down arrow if it exists.
- (D) For the remaining down arrows, we make a pair of two adjacent down arrows from right. Connect this pair of two arrows into a simple unoriented dashed arc.

After applying the rules (A)-(D), there may be a down arrow which does not form a dashed arc. We call this down arrow an *unpaired down arrow*. The diagram for $v^{m_1-2k_1} \heartsuit \dots \heartsuit v^{m_n-2k_n}$ is obtained by the rules (A) and (B) [8, Section 2.3].

Each building block (a simple oriented arc, a dashed arc and an arrow with a star) is a vector in V_1 or $V_1 \otimes V_1$:

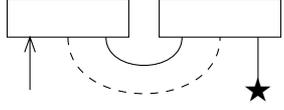
$$(6) \quad \text{---} \cup \text{---} = v^{-1} \otimes v^1 - q^{-1} v^1 \otimes v^{-1},$$

$$(7) \quad \text{---} \text{---} \text{---} = v^{-1} \otimes v^{-1} - q^{-1} v^1 \otimes v^1,$$

$$(8) \quad \downarrow \star = v^{-1} - q^{-1} v^1,$$

and an up arrow (resp. an unpaired down arrow) corresponds to v^1 (resp. v^{-1}). A vector corresponding to $v^{k_1} \diamond \dots \diamond v^{2k_r}$ is obtained by acting the projection on a tensor product of vectors for building blocks.

Example 4.5. The graph and the vector corresponding to $v^{-1} \diamond v^{-1} \in V_3 \otimes V_3$ are as follows:



$$= [-1, -1] - q^{-2}[1, -3] - q^{-1}[1, 1] - q^{-2}[-1, 1] + q^{-3}[3, -1] \\ + q^{-5}[1, -1] + q^{-2}[1, 3] - q^{-5}[3, 1].$$

where $[i, j] := v^i \otimes v^j$.

First, we describe the dual canonical basis of $V_1^{\otimes n}$. Set $t = -q$ in the notation of Section 2 and let $\kappa_i = \pm 1$ for $1 \leq i \leq n$.

Lemma 4.6. The diagram for $v^{\kappa_1} \diamond \dots \diamond v^{\kappa_n}$ provides the dual canonical basis of $V_1^{\otimes n}$.

Proof. The diagram for $v^{\kappa_1} \diamond \dots \diamond v^{\kappa_n}$ is the same as the diagram for the Kazhdan–Lusztig basis in [26]. From [1, Theorem 5.8], there exists an involution which compatible with both the bar involution on \mathcal{H}_N and the bar involution ψ_-^t . Thus, the Kazhdan–Lusztig basis on \mathcal{M}_N^- coincides with the canonical basis of $V_1^{\otimes N}$. \square

For $\mathbf{k} \in I_{\mathbf{m}}$, let $\kappa := (\kappa_1, \dots, \kappa_N)$, $N = \sum_i^n m_i$, be a sequence of $+1$ and -1 such that it starts with $(m_1 + k_1)/2$ copies of $+1$, followed by $(m_1 - k_1)/2$ copies of -1 , followed by $(m_2 + k_2)/2$ copies of $+1$, followed by $(m_2 - k_2)/2$ copies of -1 , \dots and ends with $(m_n - k_n)/2$ copies of -1 . We have the following theorem about the dual canonical basis:

Theorem 4.7. The diagram for $v^{\kappa_1} \diamond \dots \diamond v^{\kappa_n}$ provides the dual canonical basis of U in $V_{m_1} \otimes \dots \otimes V_{m_n}$, i.e.,

$$(9) \quad v^{\kappa_1} \diamond \dots \diamond v^{\kappa_n} = (\pi_{m_1} \otimes \dots \otimes \pi_{m_n}) v^{\kappa_1} \diamond \dots \diamond v^{\kappa_n}.$$

Proof. We denote $\pi := \pi_{m_1} \otimes \dots \otimes \pi_{m_n}$. We will first show that $\pi(v^{\kappa_1} \diamond \dots \diamond v^{\kappa_n})$ is invariant under the action of ψ_-^t . The involution ψ_-^t is written as

$$\psi_-^t = \Delta^{n-1}(\Upsilon_-) \Theta_-^{(n)} \psi \otimes \dots \otimes \psi.$$

We expand $v^{\kappa_1} \diamond \dots \diamond v^{\kappa_n}$ in terms of the dual canonical basis $v^{k'_1} \heartsuit \dots \heartsuit v^{k'_n}$:

$$v^{\kappa_1} \diamond \dots \diamond v^{\kappa_n} = \sum_{\mathbf{k}'} c_{\mathbf{k}, \mathbf{k}'} v^{k'_1} \heartsuit \dots \heartsuit v^{k'_n},$$

where $c_{\mathbf{k}, \mathbf{k}'} \in q^{-1}\mathbb{Z}[q^{-1}]$ for $\mathbf{k} \neq \mathbf{k}'$ and $c_{\mathbf{k}, \mathbf{k}} = 1$.

From Lemma 4.6, $v^{\kappa_1} \diamond \dots \diamond v^{\kappa_n}$ is invariant under the action of ψ_-^t . Together with the fact that $v^{k'_1} \heartsuit \dots \heartsuit v^{k'_n}$ is invariant under the action of $\Theta_-^{(N)} \psi \otimes \dots \otimes \psi$, we have

$$\Delta^{n-1}(\Upsilon_-) \sum_{\mathbf{k}'} \overline{c_{\mathbf{k}, \mathbf{k}'}} v^{k'_1} \heartsuit \dots \heartsuit v^{k'_n} = \sum_{\mathbf{k}'} c_{\mathbf{k}, \mathbf{k}'} v^{k'_1} \heartsuit \dots \heartsuit v^{k'_n}.$$

We act with π on both sides. Since π is a projection, the action of π commutes with the one of $\Delta^{n-1}(\Upsilon_-)$. Further, the action of π on $v^{k'_1} \heartsuit \dots \heartsuit v^{k'_n}$ may vanish by using

$$\boxed{n} = 0.$$

Note that this vanishing condition is compatible with

$$\begin{array}{c} \boxed{n} \\ \downarrow \quad \uparrow \\ \dots \quad \dots \end{array} = q^{-1} \begin{array}{c} \boxed{n} \\ \uparrow \quad \downarrow \\ \dots \quad \dots \end{array}.$$

Therefore, $\pi(v^{k'_1} \heartsuit \dots \heartsuit v^{k'_n})$ is equal to $v^{k'_1} \heartsuit \dots \heartsuit v^{k'_n}$ or zero. For non-zero $\pi(v^{k'_1} \heartsuit \dots \heartsuit v^{k'_n})$, we can associate an sequence of integers \mathbf{k}' with κ' by the inverse of the map from \mathbf{k} to κ described just above Theorem 4.7.

For such \mathbf{k}' and κ' , we abbreviate $c_{\kappa, \kappa'}$ as $c'_{\mathbf{k}, \mathbf{k}'}$. Note that the coefficients $c'_{\mathbf{k}, \mathbf{k}'}$ is nothing but the expansion coefficients $c_{\mathbf{k}, \mathbf{k}'}$. Thus, We have

$$\Delta^{n-1}(\Upsilon_-) \sum_{\mathbf{k}'} \overline{c_{\mathbf{k}, \mathbf{k}'}} v^{k'_1} \heartsuit \dots \heartsuit v^{k'_n} = \sum_{\mathbf{k}'} c_{\mathbf{k}, \mathbf{k}'} v^{k'_1} \heartsuit \dots \heartsuit v^{k'_n}.$$

This shows that $v^{k_1} \heartsuit \dots \heartsuit v^{k_n}$ is invariant under the action of ψ'_- .

The expansion coefficients of $v^{k_1} \heartsuit \dots \heartsuit v^{k_n}$ in terms of the standard bases $v^{k'_1} \otimes \dots \otimes v^{k'_n}$ are in $q^{-1}\mathbb{Z}[q^{-1}]$ for $\mathbf{k} \neq \mathbf{k}'$ and one for $\mathbf{k} = \mathbf{k}'$ by construction. From Eqns.(6) to (8), a building block of a diagram has the form $v^{k_1} \otimes v^{k_2} + q^{-1}\mathbb{Z}[q^{-1}]v^{k'_1} \otimes v^{k'_2}$ with $(\kappa'_1, \kappa'_2) \prec_- (\kappa_1, \kappa_2)$ or $v^{k_1} + q^{-1}\mathbb{Z}[q^{-1}]v^{k'_1}$ with $\kappa'_1 \prec_- \kappa_1$. Thus, if $\mathbf{k} \neq \mathbf{k}'$, we have $\mathbf{k}' \prec_- \mathbf{k}$. This completes the proof of the theorem. \square

4.4. Inversion formula. Let $\mathbf{k} = (k_1, \dots, k_n)$ and $\mathbf{l} = (l_1, \dots, l_n)$. We define

$$\begin{aligned} R_{\mathbf{k}, \mathbf{l}}(q^{-1}) &:= \langle v_{l_1} \heartsuit \dots \heartsuit v_{l_n}, v^{k_1} \otimes \dots \otimes v^{k_n} \rangle, & \text{for } \mathbf{k} \leq_+ \mathbf{l}, \\ R^{\mathbf{k}, \mathbf{l}}(q^{-1}) &:= \langle v_{k_1} \otimes \dots \otimes v_{k_n}, v^{l_1} \heartsuit \dots \heartsuit v^{l_n} \rangle, & \text{for } \mathbf{k} \leq_- \mathbf{l}. \end{aligned}$$

Then, the expansion of $v_{l_1} \heartsuit \dots \heartsuit v_{l_n}$ in terms of $v_{k_1} \otimes \dots \otimes v_{k_n}$ and its inverse is explicitly given by

$$\begin{aligned} v_{l_1} \heartsuit \dots \heartsuit v_{l_n} &= \sum_{\mathbf{k} \leq_+ \mathbf{l}} R_{\mathbf{k}, \mathbf{l}}(q^{-1}) v_{k_1} \otimes \dots \otimes v_{k_n}, \\ v_{l_1} \otimes \dots \otimes v_{l_n} &= \sum_{\mathbf{k} \leq_+ \mathbf{l}} R^{\mathbf{k}, \mathbf{l}}(q^{-1}) v_{k_1} \heartsuit \dots \heartsuit v_{k_n}. \end{aligned}$$

We have similar formulas for $v^{k_1} \heartsuit \dots \heartsuit v^{k_n}$. Thus, the knowledge of $R_{\mathbf{k}, \mathbf{l}}(q^{-1})$ and $R^{\mathbf{k}, \mathbf{l}}(q^{-1})$ determines the expansion of dual canonical basis in terms of standard basis and vice versa. We will consider $R_{\mathbf{k}, \mathbf{l}}(q^{-1})$ and $R^{\mathbf{k}, \mathbf{l}}(q^{-1})$ in the next section.

From Eqn.(5), we have an inversion formula:

$$(10) \quad \sum_{\mathbf{k}} R_{\mathbf{k}, \mathbf{l}}(q^{-1}) R^{\mathbf{k}, \mathbf{l}}(q^{-1}) = \delta_{l_1}^{l'_1} \dots \delta_{l_n}^{l'_n}.$$

5. PATH REPRESENTATION

5.1. Maps ζ and η . Let l be an integer satisfying $0 \leq l \leq m$. We define a map ζ' from an integer $(m - 2l)$ to a path of length m by

$$\zeta'(m - 2l) = \underbrace{+ \dots +}_l \underbrace{- \dots -}_{m-l}.$$

A map ζ from a sequence of integers $\mathbf{k} = (k_1, \dots, k_n) \in I_{\mathbf{m}}$ to a path of length $\sum_{i=1}^n m_i$ is given by

$$(11) \quad \zeta(\mathbf{k}) = \zeta'(k_1) \circ \dots \circ \zeta'(k_n).$$

Here, $\zeta'(k) \circ \zeta'(k')$ is concatenation of two paths $\zeta'(k)$ and $\zeta'(k')$.

Similarly, a map η' from an integer $(m - 2k)$ to a path of length m is defined by

$$\eta(m - 2k) = \underbrace{- \dots -}_{m-k} \underbrace{+ \dots +}_k.$$

We define a map η from a sequence of integers $\mathbf{k} = (k_1, \dots, k_n) \in I_{\mathbf{m}}$ to a path of length $\sum_{i=1}^n m_i$ by

$$\eta(\mathbf{k}) = \eta(k_1) \circ \dots \circ \eta(k_n).$$

Given $\mathbf{k} = (k_1, \dots, k_n)$ and $\mathbf{l} = (l_1, \dots, l_n)$, the condition $\mathbf{k} \leq_- \mathbf{l}$ is compatible with $\eta(\mathbf{k}) \leq \eta(\mathbf{l})$ at $\epsilon = -$, i.e., $\eta(\mathbf{k})$ is below $\eta(\mathbf{l})$. Similarly, $\mathbf{k} \leq_+ \mathbf{l}$ if and only if $\eta(\mathbf{k}) \leq \eta(\mathbf{l})$ at $\epsilon = +$, i.e., $\eta(\mathbf{k})$ is above $\eta(\mathbf{l})$.

5.2. Ballot strips. We make use of notations and terminologies for ballot strips used in [26, Section 3.1]. We recollect necessary definitions since the rules of stacking ballot strips are slightly different from [26].

Let $\alpha \in \mathcal{P}_N$. We denote by $S(\alpha)$ the integral points

$$S(\alpha) := \{(i, j) : (i, j) \text{ is above the path } \alpha, 0 < i \leq N, |j| < i, i + j - 1 \in 2\mathbb{Z}\}.$$

We put (45 degree rotated) squares of length $\sqrt{2}$ whose centers are all points in $S(\alpha)$. We regard this set of squares as a (45 degree rotated) shifted Ferrers diagram and denote it by $\lambda(\alpha)$. We denote by $|\alpha|$ the number of squares in the shifted Ferrers diagram $\lambda(\alpha)$. Given two paths $\alpha, \beta \in \mathcal{P}_N$ such that α is below β , we regard $\lambda(\alpha)/\lambda(\beta)$ as a shifted skew Ferrers diagram. The number of squares in the skew diagram is denoted by $|\alpha| - |\beta|$.

Let $\mathbf{k} = (k_1, \dots, k_n) \in I_{\mathbf{m}}$. Let $\alpha = \eta(\mathbf{k})$ and $\alpha' = \zeta(\mathbf{k})$ be two paths for \mathbf{k} . Let γ be a path inbetween α and α' . We call the shifted skew Ferrers diagram $\lambda(\alpha)/\lambda(\gamma)$ a *projection domain* (p -domain in short) for α . When we choose $\gamma = \alpha$ or all $m_i = 1$, there is no p -domain.

Two paths $\alpha = \eta(\mathbf{k})$ and $\beta = \eta(\mathbf{k}')$ (α is below β) characterize the domains, the shifted skew Ferrers diagram $\mu := \lambda(\alpha)/\lambda(\beta)$. Let P be a p -domain for α . We call the domain $P \cap \mu$ a p -domain for (α, β) .

Definition 5.1. We call a square whose centre is (N, j) with $|j| < N$ and $N - j - 1 \in 2\mathbb{Z}$ an anchor square.

We put ballot strips in a shifted skew Ferrers diagram. We have one constraint for a ballot strip: the rightmost square of a ballot strip of length (l, l') with $l' \geq 1$ is on an anchor square in the shifted skew Ferrers diagram.

Let $\mathcal{D}, \mathcal{D}'$ be two ballot strips. We pile \mathcal{D}' on top of \mathcal{D} by the following two rules in a shifted skew Ferrers diagram $\lambda(\alpha)/\lambda(\beta)$:

- Rule I (a) If there exists a square of \mathcal{D} just below a square of \mathcal{D}' , then all squares just below a square of \mathcal{D}' belong to \mathcal{D} .
 (b) Suppose $l' \geq 1$. The number of ballot strips of length (l, l') is even for $l' - 1 \in 2\mathbb{Z}$, and zero for otherwise.
 (c) There is no p -domain for (α, β) in the shifted skew Ferrers diagram.

- Rule II (a) If there exists a square of \mathcal{D}' just above, NW, or NE of a square of \mathcal{D} , then all squares just above, NW, and NE of a square of \mathcal{D} belong to \mathcal{D} or \mathcal{D}' except in a p -domain.
 (b) Suppose $l' \geq 1$. If there exists a ballot strip \mathcal{D} of length (l, l') with $l' - 1 \in 2\mathbb{Z}$ (resp. $l' \in 2\mathbb{Z}$), then there is a strip of length $(l'', l' + 1)$ with $l'' \geq l$ (resp. $(l'', l' - 1)$ with $l'' \leq l$) just above (resp. just below) \mathcal{D} .
 (c) There is no squares of a strip \mathcal{D} of length (l, l') with $l + l' \geq 1$ in a p -domain.
 (d) If a p -domain for (α, β) exists, fill it with unit squares.

Let $\alpha < \beta$ be two paths. We denote by $\text{Conf}(\alpha, \beta)$ the set of all possible configurations of ballot strips, and by $\text{Conf}^{I/II}(\alpha, \beta)$ the subset of configurations satisfying Rule I/II. In the case of Rule II, there may be several choices of a p -domain for (α, β) . However, given a p -domain for (α, β) , there exists at most one configuration satisfying Rule II.

The weight of a ballot strip is defined by

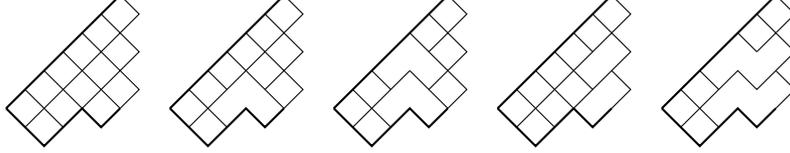
$$\begin{aligned} \text{wt}^I(\mathcal{D}) &:= \begin{cases} t^{-1}, & l' \text{ is even,} \\ 1, & l' \text{ is odd,} \end{cases} \\ \text{wt}^{II}(\mathcal{D}) &:= \begin{cases} -t^{-1}, & l' \text{ is even,} \\ 1, & l' \text{ is odd,} \\ t^{-1} & \text{a unit square in a } p\text{-domain.} \end{cases} \end{aligned}$$

Definition 5.2. The generating function of ballot strips for the paths $\alpha < \beta$ is defined by

$$Q_{\alpha,\beta}^{X,\epsilon} = \sum_{C \in \text{Conf}^X(\alpha,\beta)} \prod_{\mathcal{D} \in C} \text{wt}^X(\mathcal{D}),$$

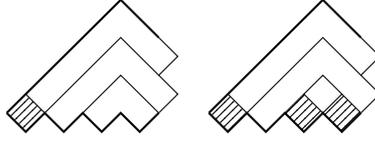
where $X = I, II$ and $\epsilon = \pm$. The order $\alpha < \beta$ is associated with the sign ϵ . We define $Q_{\alpha,\alpha}^{X,\epsilon} = 1$.

Example 5.3. Let $\alpha = - - + + - +$ and $\beta = + + + + + +$. The possible configurations satisfying Rule I are



The generating function is $Q_{\alpha,\beta}^{I,-} = t^{-13} + t^{-11} + 2t^{-9} + t^{-5}$.

Example 5.4. Let $\mathbf{m} = (2, 2, 2, 2)$, $\mathbf{k} = (0, 0, 0, 0)$ and $\mathbf{k}' = (-2, -2, -2, 2)$. Then, $\alpha := \eta(\mathbf{k}) = - - - - + + - +$ and $\beta := \eta(\mathbf{k}') = + + + + + + - -$. The possible configurations satisfying Rule II are



where a square \square is in a p -domain. The generating function is $Q_{\alpha,\beta}^{II,-} = t^{-3} + t^{-5}$.

From Lemma 4 in [27], we have

Lemma 5.5.

$$(12) \quad Q_{\alpha,\beta}^{X,+}(t^{-1}) = Q_{\beta,\alpha}^{X,-}(t^{-1}).$$

From Theorem 21 in [26], we have the following two lemmas:

Lemma 5.6.

$$(13) \quad Q_{\alpha,\beta}^{I,+}(t^{-1}) = P_{\alpha,\beta}^+(t^{-1}).$$

For given two paths α and β , we define a path γ by the following two conditions: 1) γ is below both α and β , and 2) there is no path above γ satisfying 1). We denote the path γ by $\min(\alpha, \beta)$.

For example, $\min(+ - - - +, - + + + -) = - - - - +$.

Lemma 5.7. Let $\alpha = \eta(\mathbf{k}), \beta = \eta(\mathbf{k}')$ and $\alpha' = \min(\zeta(\mathbf{k}), \beta)$.

$$(14) \quad Q_{\alpha,\beta}^{II,-}(t^{-1}) = \sum_{\alpha \leq \gamma \leq \alpha'} t^{-|\alpha|+|\gamma|} P_{\gamma,\beta}^-(t^{-1}).$$

Proof. Given γ ($\alpha \leq \gamma \leq \alpha'$), the domain inbetween α and γ is a p -domain for (α, β) . The number of squares in a p -domain is $|\alpha| - |\gamma|$. This gives the factor $t^{-|\alpha|+|\gamma|}$.

The domain inbetween γ and β is filled with ballot strips satisfying Rule II-(a) and II-(b). Since these two rules are the same as the ones for Kazhdan–Lusztig polynomials (Section 3.1 in [26]), the contribution is nothing but the Kazhdan–Lusztig polynomial $P_{\gamma,\beta}^-(-t^{-1})$ (Corollary 20 in [26]). The sum over all possible γ gives the desired equation. \square

5.3. Path representation of R .

Theorem 5.8. *Let $\mathbf{k}, \mathbf{l} \in I_{\mathbf{m}}$, $\alpha = \eta(\mathbf{k})$ and $\beta = \eta(\mathbf{l})$.*

$$(15) \quad R^{\mathbf{k}, \mathbf{l}}(q^{-1}) = Q_{\alpha, \beta}^{II, -}(q^{-1}).$$

Proof. From Lemma 4.6, the dual canonical basis in $V_1^{\otimes N}$ is expanded as

$$(16) \quad v^{\kappa_1} \dot{\smile} \dots \dot{\smile} v^{\kappa_N} = \sum_{\gamma_1 \leq \gamma_2} P_{\gamma_1, \gamma_2}^-(-q^{-1}) v^{\kappa'_1} \otimes \dots \otimes v^{\kappa'_N}$$

where $\gamma_1 = \eta(\kappa')$ and $\gamma_2 = \eta(\kappa)$. Given κ' , we define $k_i, 1 \leq i \leq n$, as $k_i = \kappa'_{d_{i-1}+1} + \dots + \kappa'_{d_i}$, where $d_i := \sum_{l=1}^i m_l$ and $d_0 = 0$. The action of $\pi = \pi_1 \otimes \dots \otimes \pi_n$ on $v^{\kappa'_1} \otimes \dots \otimes v^{\kappa'_N}$ is

$$\pi(v^{\kappa'_1} \otimes \dots \otimes v^{\kappa'_N}) = q^{-D} v^{k_1} \otimes \dots \otimes v^{k_n}$$

where $D = |\eta(\mathbf{k})| - |\eta(\kappa')|$.

From Theorem 4.7, the dual canonical basis is obtained by multiplying π on $v^{\kappa_1} \dot{\smile} \dots \dot{\smile} v^{\kappa_N}$ for an appropriate choice of κ . For such κ , the multiplication of π on Eqn.(16) yields

$$v^{\mathbf{l}_1} \dot{\smile} \dots \dot{\smile} v^{\mathbf{l}_N} = \sum_{\alpha \leq \beta} \sum_{\alpha \leq \gamma \leq \alpha'} q^{-|\alpha|+|\gamma|} P_{\gamma, \beta}^-(-q^{-1}) v^{k_1} \otimes \dots \otimes v^{k_n}$$

where $\alpha = \eta(\mathbf{k}), \beta = \eta(\mathbf{l})$ and $\alpha' = \min(\alpha, \beta)$. Together with Lemma 5.7, we complete the proof. \square

5.4. Inversion formula. Fix \mathbf{m} . Suppose that a path α is below a path β and written as $\alpha = \eta(\mathbf{k})$ and $\beta = \eta(\mathbf{k}')$ with $\mathbf{k}, \mathbf{k}' \in I_{\mathbf{m}}$. A path γ is said to be *admissible* if and only if there exists $\mathbf{k}'' \in I_{\mathbf{m}}$ satisfying $\gamma = \eta(\mathbf{k}'')$ and γ is above α and below β . We denote by $\text{Adm}(\alpha, \beta)$ the set of all admissible paths inbetween two paths α and β .

Theorem 5.9.

$$(17) \quad \sum_{\beta \in \text{Adm}(\alpha, \gamma)} Q_{\alpha, \beta}^{I, -}(t^{-1}) Q_{\beta, \gamma}^{II, -}(t^{-1}) = \delta_{\alpha, \gamma}.$$

Proof. We prove the theorem along [27, Theorem 6] and [26, Theorem 10]. The main differences are the existence of p -domains and admissibility of a path.

When $\alpha = \gamma$, Theorem holds true. Hereafter, we assume $\alpha \neq \gamma$. Fix two paths α, γ and a configuration $C \in \text{Conf}(\alpha, \gamma)$ such that there exists a path β satisfying $C = C_I(\beta) \sqcup C_{II}(\beta)$ where $C_I(\beta) \in \text{Conf}^I(\alpha, \beta)$ and $C_{II}(\beta) \in \text{Conf}^{II}(\beta, \gamma)$. We denote by $P(C)$ the set of such paths β 's. If there exist p -domains in $C_{II}(\beta)$, we define a path $\bar{\beta}$ such that the shifted skew Ferrers diagram $\lambda(\beta)/\lambda(\bar{\beta})$ is the p -domains for (β, γ) . If there is no p -domain, we define $\bar{\beta} = \beta$. We denote by $\bar{P}(C)$ the set of such $\bar{\beta}$. Notice that $\bar{\beta}$ may not be admissible. Reversely, one can obtain an admissible path β from $\bar{\beta}$ as follows. If $\bar{\beta}$ is admissible, then $\beta = \bar{\beta}$. If a path $\bar{\beta}$ is not admissible, one can obtain an admissible path β below $\bar{\beta}$ such that the number of squares filling the region $\lambda(\beta)/\lambda(\bar{\beta})$ is minimum. Therefore, there is a one-to-one correspondence between an admissible path in $P(C)$ and a path in $\bar{P}(C)$. Since a unit square can be piled on top of any other ballot strip by Rule I, the region inbetween α and $\bar{\beta}$ satisfies Rule I. The configuration C is written as $C = C_I(\bar{\beta}) \sqcup C_{II}(\bar{\beta})$. We define the set of ballot strips $\mathcal{I}(C)$ as

$$\mathcal{I}(C) := \left(\bigcup_{\bar{\beta} \in \bar{P}(C)} C_I(\bar{\beta}) \right) \cap \left(\bigcup_{\bar{\beta} \in \bar{P}(C)} C_{II}(\bar{\beta}) \right).$$

An element in $\mathcal{I}(C)$ is a single ballot strip or ballot strips which are on top of each other and glued together. The element of $\mathcal{I}(C)$ is on the border of $C_I(\bar{\beta})$ and $C_{II}(\bar{\beta})$. By a similar argument to Proposition 4 in [27], one can show $\mathcal{I}(C) \neq \emptyset$. Further, Lemma 5 in [27] holds true as well. From the definition of $\mathcal{I}(C)$, a path

$\bar{\beta} \in \bar{P}(C)$ is characterized by being above or below an element in $\mathcal{I}(C)$. Thus, the cardinality of $\bar{P}(C)$ is $2^{|\mathcal{I}(C)|}$.

We denote by $\text{wt}(C)$ the weight of a configuration C . The weight of a unit square in a p -domain for Rule II is t^{-1} , which is the same value as the weight of a unit square for Rule I. We have

$$\begin{aligned} \sum_{\beta \in \text{Adm}(\alpha, \gamma)} Q_{\alpha, \beta}^{I, -} Q_{\beta, \gamma}^{II, -} &= \sum_C \sum_{\beta \in P(C)} \text{wt}(C) \\ &= \sum_C |\text{wt}(C)| \sum_{\bar{\beta} \in \bar{P}(C)} \text{sign}(C). \end{aligned}$$

To prove the theorem, it is enough to show $\sum_{\bar{\beta} \in \bar{P}(C)} \text{sign}(C) = 0$. We first show this equality in the case of two paths. Let $\beta_1, \beta_2 \in \bar{P}(C)$ be two paths such that there exists an element \mathcal{D} belonging to $C_I(\beta_1)$ and $C_{II}(\beta_2)$. In other words, this means β_1 is above β_2 . We have two cases for \mathcal{D} :

Case 1. \mathcal{D} is of length $(l, 0)$. The sign of \mathcal{D} for Rule I is plus, whereas minus for Rule II. Thus the contributions cancel each other.

Case 2. \mathcal{D} is glued ballot strips. For Rule I, \mathcal{D} consists of three elements: two ballot strips of same length (l, l') with $l' - 1 \in 2\mathbb{Z}$ and a ballot strip of length $(l'', 0)$ for some l'' . On the other hand, for Rule II, \mathcal{D} consists of the ballot strip of length (l, l') and a ballot strip of length $(l + l'', l' + 1)$. The sign is plus for Rule I and minus for Rule II. Thus the contributions cancel each other.

Finally, $\sum_{\bar{\beta} \in \bar{P}(C)} \text{sign}(C)$ is reduced to the sum of contributions by local sum, which is zero. This completes the proof. \square

Theorem 5.10. *We have*

$$(18) \quad R_{\mathbf{k}, l}(q^{-1}) = Q_{\eta(\mathbf{k}), \eta(l)}^{I, +}(q^{-1})$$

Proof. From inversion relations (10) and Theorem 5.9 together with Lemma 5.5, we have $R_{\mathbf{k}, l} = Q_{\eta(l), \eta(\mathbf{k})}^{I, -} = Q_{\eta(\mathbf{k}), \eta(l)}^{I, +}$. \square

From Lemma 5.6, it is obvious that

Corollary 5.11.

$$(19) \quad R_{\mathbf{k}, l}(q^{-1}) = P_{\eta(\mathbf{k}), \eta(l)}^+(q^{-1}).$$

5.5. Canonical bases of U. We first prove two lemmas used later.

Lemma 5.12. *Let $l \in I_{(m)}$ and $\epsilon = +$. We have*

$$(20) \quad \sum_{\alpha' \leq \gamma \leq \alpha} q^{2|\gamma| - |\alpha| - |\alpha'|} = \begin{bmatrix} m \\ (m-l)/2 \end{bmatrix}$$

where $\alpha = \eta(l)$ and $\alpha' = \zeta(l)$.

Proof. We prove the lemma by induction. When $m = 1$ and $m = 2$, the lemma is obviously true. We assume that the lemma holds true for all $m' < m$ and all $l \in I_{(m')}$. A path γ starts with $+$ or $-$. Let P_+ (resp. P_-) be the set of γ , $\alpha' \leq \gamma \leq \alpha$, starting $+$ (resp. $-$). We define two paths: $\alpha_1 := + \underbrace{- \cdots -}_{(m+l)/2} \underbrace{+ \cdots +}_{(m-l)/2-1}$ and

$\alpha'_1 := -\underbrace{+\cdots+}_{(m-l)/2} -\underbrace{\cdots-}_{(m+l)/2-1}$. Then we have

$$\begin{aligned} \sum_{\gamma \in P_+} q^{2|\gamma| - |\alpha| - |\alpha'|} &= q^{-(m+l)/2} \sum_{\gamma \in P_+} q^{2|\gamma| - |\alpha_1| - |\alpha'|} \\ &= q^{-(m+l)/2} \begin{bmatrix} m-1 \\ (m-l)/2 - 1 \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} \sum_{\gamma \in P_-} q^{2|\gamma| - |\alpha| - |\alpha'|} &= q^{(m-l)/2} \sum_{\gamma \in P_-} q^{2|\gamma| - |\alpha| - |\alpha'_1|} \\ &= q^{(m-l)/2} \begin{bmatrix} m-1 \\ (m-l)/2 \end{bmatrix} \end{aligned}$$

where we have used the induction assumption. The sum of two contributions is the right hand side of Eqn.(20). \square

Lemma 5.13. *Let $\alpha = \eta(\mathbf{l})$, $\alpha' = \zeta(\mathbf{l})$ and $\beta = \eta(\mathbf{k})$ with $\mathbf{k}, \mathbf{l} \in I_{\mathbf{m}}$. We have*

$$(21) \quad P_{\alpha, \beta}^+(q^{-1}) = \sum_{\alpha' \leq \gamma \leq \alpha} q^{|\gamma| - |\alpha'|} \prod_{i=1}^n \begin{bmatrix} m_i \\ (m_i - l_i)/2 \end{bmatrix}^{-1} P_{\gamma, \beta}^+(q^{-1}).$$

Proof. We first show that if $\alpha' \leq \gamma \leq \alpha$ ($\epsilon = +$), $P_{\gamma, \beta}^+ = q^{-|\alpha| + |\gamma|} P_{\alpha, \beta}^+$. Recall that $P_{\alpha, \beta}^+ = Q_{\alpha, \beta}^{l, +}$ (Lemma 5.6). The region between α and β is filled with ballot strips by Rule I. One can pile a ballot strip on top of a ballot strip of length (l, l') with $l + l' \geq 1$ if and only if there is a convex path $+ -$ below the strip. We consider the region between α' and β . From the definition of η , there is no convex path below the region $\lambda(\alpha)/\lambda(\gamma)$. Thus there is no ballot strips of length (l, l') with $l + l' \geq 1$ in this region. Since the weight of a unit square is t^{-1} , we have $P_{\gamma, \beta}^+ = q^{-|\alpha| + |\gamma|} P_{\alpha, \beta}^+$.

The right hand side of Eqn.(21) is

$$\begin{aligned} \sum_{\alpha' \leq \gamma \leq \alpha} q^{|\gamma| - |\alpha'|} \prod_{i=1}^n \begin{bmatrix} m_i \\ (m_i - l_i)/2 \end{bmatrix}^{-1} P_{\gamma, \beta}^+(q^{-1}) &= \prod_{i=1}^n \begin{bmatrix} m_i \\ (m_i - l_i)/2 \end{bmatrix}^{-1} P_{\alpha, \beta}^+(q^{-1}) \sum_{\alpha' \leq \gamma \leq \alpha} q^{2|\gamma| - |\alpha| - |\alpha'|} \\ &= P_{\alpha, \beta}^+ \end{aligned}$$

where we have used Lemma 5.12. This completes the proof. \square

The following theorem is the dual statement of Theorem 4.7. Given $\mathbf{k} \in I_{\mathbf{m}}$, let $(\kappa_1, \dots, \kappa_N)$ be the sequence defined just above Theorem 4.7.

Theorem 5.14. *The canonical basis $v_{m_1-2k_1} \uplus \cdots \uplus v_{m_n-2k_n}$ is given by*

$$(22) \quad v_{m_1-2k_1} \uplus \cdots \uplus v_{m_n-2k_n} = (\pi_1 \otimes \cdots \otimes \pi_n) v_{\kappa_1} \uplus \cdots \uplus v_{\kappa_N}.$$

Proof. We compute the right hand side of Eqn.(22) briefly. We expand $v_{\kappa_1} \uplus \cdots \uplus v_{\kappa_N}$ in terms of the standard basis $v_{\kappa'_1} \otimes \cdots \otimes v_{\kappa'_N}$. We also expand $v_{m_1-2k_1} \uplus \cdots \uplus v_{m_n-2k_n}$ in terms of the standard basis $v_{m_1-2l_1} \otimes \cdots \otimes v_{m_n-2l_n}$. Let $\alpha = \eta(m_1 - l_1, \dots, m_n - l_n)$, $\alpha' = \zeta(m_1 - l_1, \dots, m_n - l_n)$, $\beta = \eta(\kappa)$ and $\gamma = \eta(\kappa')$. The coefficient is nothing but the Kazhdan–Lusztig polynomials $P_{\gamma, \beta}^+$. The projection (1) gives the factor $q^{|\gamma| - |\alpha'|} \prod_{i=1}^n \begin{bmatrix} m_i \\ (m_i - l_i)/2 \end{bmatrix}^{-1}$. Thus the sum of all the contributions is reduced to the right hand side of Eqn.(21). From Lemma 5.13 and Corollary 5.11, we obtain Eqn.(22). \square

6. INTEGRAL STRUCTURE

6.1. Integral structure of tensor products. From the definitions of an arc and a dashed arc, we have the following lemma.

Lemma 6.1. *We have two identities:*

$$(23) \quad \downarrow \uparrow = \cup + q^{-1} \uparrow \downarrow$$

$$(24) \quad \downarrow \downarrow = \text{dashed arc} + q^{-1} \uparrow \uparrow$$

Theorem 6.2. *The coefficients of the decomposition*

$$v^{l_1} \heartsuit \cdots \heartsuit v^{l_N} = \sum_{\mathbf{k}} c_{\mathbf{k}}^1 v^{k_1} \spadesuit \cdots \spadesuit v^{k_N}$$

belong to $q^{-1}\mathbb{N}[q^{-1}]$ except $c_{\mathbf{1}}^1 = 1$.

Proof. The diagram D of $v^{l_1} \heartsuit \cdots \heartsuit v^{l_N}$ consists of arcs, up arrows and down arrows. The down arrows are right to the up arrows in D . To obtain the expansion of D in terms of $D' := v^{k_1} \spadesuit \cdots \spadesuit v^{k_N}$, we successively use Lemma 6.1. We change two adjacent down arrows into a dashed arc, and move up arrows to the left of down arrows. The coefficients in the right hand sides of Eqns.(23) and (24) are one for an arc and a dashed arc, and are in $q^{-1}\mathbb{N}[q^{-1}]$ for otherwise. Thus, the coefficient $c_{\mathbf{1}}^1 = 1$ and other coefficients are in $q^{-1}\mathbb{N}[q^{-1}]$. This completes the proof. \square

Remark 6.3. *In the proof of Theorem 6.2, the numbers of arcs and up arrows in D' may be more than those in D . However, if D has an arc (resp. an up arrow), then D' has also an arc (resp. an up arrow) in the same position. The coefficients of decomposition of $v^{l_1} \heartsuit \cdots \heartsuit v^{l_N}$ is equivalent to the decomposition of $v^{-1} \otimes \cdots \otimes v^{-1}$ where the number of v^{-1} is the same as the number of down arrows in D . This implies that $c_{\mathbf{1}}^{\mathbf{k}} = R_{\mathbf{k}_0, \mathbf{l}_0}$ for some \mathbf{k}_0 and \mathbf{l}_0 .*

Theorem 6.4. *The coefficients of the decomposition*

$$v^{l_1} \spadesuit \cdots \spadesuit v^{l_N} = \sum_{\mathbf{k}} c_{\mathbf{k}}^1 v^{k_1} \heartsuit \cdots \heartsuit v^{k_N},$$

belong to $q^{-1}\mathbb{Z}[q^{-1}]$ except $c_{\mathbf{1}}^1 = 1$.

Proof. We expand a dashed arc and a down arrow with a star in terms of up and down arrows where the coefficients of up arrows are in $q^{-1}\mathbb{Z}[q^{-1}]$. We move up arrows to the left by using Eqn.(23) in Lemma 6.1. We have $c_{\mathbf{1}}^1 = 1$ and $c_{\mathbf{k}}^{\mathbf{l}} \in q^{-1}\mathbb{Z}[q^{-1}]$. \square

Remark 6.5. *By a similar argument to Remark 6.3 and Theorem 6.4, the decomposition of $v^{l_1} \spadesuit \cdots \spadesuit v^{l_N}$ is divided into two steps: (1) decompose $v^{-1} \spadesuit \cdots \spadesuit v^{-1}$ into $v^{k_1} \otimes \cdots \otimes v^{k_N}$. (2) Decomposition of $v^{k_1} \otimes \cdots \otimes v^{k_N}$ into $v^{\sigma_1} \heartsuit \cdots \heartsuit v^{\sigma_N}$. The contribution of step (1) in $c_{\mathbf{k}}^{\mathbf{l}}$ is $R_{\mathbf{k}_0, \mathbf{l}_0}$ for some \mathbf{k}_0 and \mathbf{l}_0 . Similarly, the contribution of step (2) is $P_{\mathbf{k}_0, \mathbf{l}_1}^{A,+}$ where $P_{\mathbf{k}_0, \mathbf{l}_1}^{A,+}$ is the Kazhdan–Lusztig polynomial of type A (see, e.g., [27]). Thus, $c_{\mathbf{k}}^{\mathbf{l}} = R_{\mathbf{k}_0, \mathbf{l}_0} P_{\mathbf{k}_0, \mathbf{l}_1}^{A,+}$ for some $\mathbf{k}_0, \mathbf{l}_0$ and \mathbf{l}_1 .*

Let M (resp. N) be a product of finitely many irreducible representations of $U_q(\mathfrak{sl}_2)$ (resp. U). Let $\{b_{\mathbf{k}} | \mathbf{k} \in I_m\}$ be dual canonical bases in M and $\{b'_{\mathbf{k}'} | \mathbf{k}' \in I_{m'}\}$ be dual canonical bases in N .

Theorem 6.6. *Consider the decomposition*

$$b_{\mathbf{k}} \otimes b'_{\mathbf{k}'} = \sum_{\mathbf{l} \in I_m, \mathbf{l}' \in I_{m'}} c_{\mathbf{k}, \mathbf{k}'}^{\mathbf{l}, \mathbf{l}'} b_{\mathbf{l}} \spadesuit b_{\mathbf{l}'}$$

where $b_1 \diamond b_V$ is a dual canonical base of U in $M \otimes N$. The coefficients $c_{\mathbf{k}, \mathbf{k}'}^{\mathbf{l}, \mathbf{l}'}$ belong to $q^{-1}\mathbb{N}[q^{-1}]$ unless $c_{\mathbf{k}, \mathbf{k}'}^{\mathbf{k}, \mathbf{k}'} = 1$.

Proof. The diagram $b \otimes b'$ is obtained by placing the diagrams b and b' in parallel. Note that the down arrows in b is left to the up arrows in b' . To obtain a diagram for a dual canonical basis of U , we use Lemma 6.1 successively. We can move an up arrow in b' to the right of a down arrow in b . The coefficients in the right hand sides of Eqns.(23) and (24) are in $\mathbb{N}[q^{-1}]$. The coefficient of the arc and the dashed arc is one, that is, not in $q^{-1}\mathbb{N}[q^{-1}]$. Thus, we have $c_{\mathbf{k}, \mathbf{k}'}^{\mathbf{k}, \mathbf{k}'} = 1$ and $c_{\mathbf{k}, \mathbf{k}'}^{\mathbf{l}, \mathbf{l}'} \in q^{-1}\mathbb{N}[q^{-1}]$. \square

6.2. Action of Y on standard bases and dual canonical bases. We consider the action of Y on a standard basis $v^{\mathbf{k}} := v^{\kappa_1} \otimes \cdots \otimes v^{\kappa_N}$ where $\kappa_i = \pm 1$. For each $1 \leq i \leq N$, we define $Y_{(i)}(v^{\mathbf{k}}) := v^{\kappa_1} \otimes \cdots \otimes v^{\kappa_{i-1}} \otimes v^{-\kappa_i} \otimes v^{\kappa_{i+1}} \otimes \cdots \otimes v^{\kappa_N}$. Set $d_i := \sum_{j=1}^i \kappa_j$. The action of Y is defined by

$$(25) \quad Y(v^{\mathbf{k}}) := \sum_{i=1}^N q^{d_{i-1}} Y_{(i)}(v^{\mathbf{k}}) + q^{d_N} v^{\mathbf{k}}.$$

Proposition 6.7. *The definition (25) provides the action of Y on a standard basis.*

Proof. We prove Proposition by induction on N . When $N = 1$, Proposition holds true by a straightforward calculation. We assume that Proposition is true up to some $N \geq 1$. A standard basis $v^{\mathbf{k}}$ is written as $v^{\mathbf{k}} = v^{\kappa_1} \otimes v'$ where v' is a standard basis of length $N - 1$. Let $d'_i := \sum_{j=2}^i \kappa_j$. From the induction assumption, we have

$$Y(v') = \sum_{i=2}^N q^{d'_{i-1}} Y_{(i-1)}(v') + q^{d'_N} v'.$$

From Eqns.(3) and (25), we have

$$\begin{aligned} Y(v^{\mathbf{k}}) &= q^{\kappa_1} v^{\kappa_1} \otimes Y(v') + v^{-\kappa_1} \otimes v' \\ &= \sum_{i=2}^N q^{\kappa_1 + d'_{i-1}} v^{\kappa_1} \otimes Y_{(i-1)}(v') + q^{\kappa_1 + d'_N} v' \\ &= \sum_{i=1}^N q^{d_{i-1}} Y_{(i)}(v) + q^{d_N} v, \end{aligned}$$

where we have used $v^{\kappa_1} \otimes Y_{(i-1)}(v') = Y_{(i)}(v)$. \square

Recall that the diagram D for a dual canonical basis of U consists of arcs, dashed arcs, up arrows, down arrow with a star and at most one unpaired down arrow. We enumerate all the up arrows from left to right by $1, 2, \dots, n_{\uparrow}$ where n_{\uparrow} is the number of up arrows of D .

If $n_{\uparrow} = 0$, we have three cases for a diagram D :

- (1) D has no down arrow with a star. Define the action of Y by

$$Y(D) := D.$$

- (2) D has a down arrow with a star and an unpaired down arrow. Let D' be a diagram obtained from D by changing the unpaired down arrow of D to an up arrow. Define the action of Y by

$$Y(D) := D'.$$

- (3) D has a down arrow with a star but no unpaired down arrow. Define the action of Y by

$$Y(D) := 0.$$

Suppose $n_{\uparrow} > 0$. For each i , $1 \leq i < n_{\uparrow}$, we denote by $Y_{(i)}(D)$ a diagram obtained from D by connecting the i -th and $(i + 1)$ -th up arrows of D via an unoriented arc.

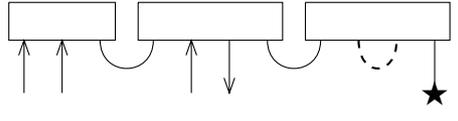
We denote by D' a diagram obtained from D by changing the n_{\uparrow} -th up arrow to a down arrow. We construct $Y_{(n_{\uparrow})}(D)$ from D' as follows. If D has an unpaired down arrow d , $Y_{(n_{\uparrow})}(D)$ is obtained from D' by connecting the reversed down arrow of D' and d via a dashed arc. If D has no unpaired down arrow, then $Y_{(n_{\uparrow})}(D) = D'$.

If there is no down arrow with a star in D , we define $Y_{(n_{\uparrow}+1)}(D) = D$. If there is an unpaired down arrow in D , we denote by $Y_{(n_{\uparrow}+1)}(D)$ a diagram obtained from D by changing the unpaired down arrow to an up arrow. Define $Y_{(n_{\uparrow}+1)}(D) = 0$ for otherwise.

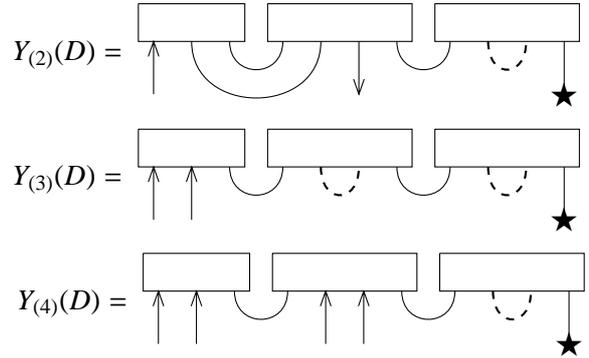
Define the action of Y by

$$(26) \quad Y(D) := \sum_{1 \leq i \leq n_{\uparrow}+1} [i]Y_{(i)}(D).$$

Example 6.8. Let $\mathbf{k} = (1, 0, -2) \in I_{(3,4,4)}$. The diagram D is



Then



We have $Y_{(1)}(D) = 0$. Therefore,

$$Y(D) = [2]Y_{(2)}(D) + [3]Y_{(3)}(D) + [4]Y_{(4)}(D).$$

Theorem 6.9. The definition (26) provides the action of Y on a canonical basis.

Proof. It is enough to show the case of $\mathbf{m} = (1, \dots, 1)$ since the actions of generators of $U_q(\mathfrak{sl}_2)$ and that of projector commute with each other. Hereafter, we set $\mathbf{m} = (1, \dots, 1)$. We prove Theorem by induction. In the case of $L = 1, \dots, 4$, we can verify Theorem by a direct computation. Suppose that Theorem is true up to some $L \geq 4$.

Suppose the leftmost arrow is an up arrow. Let D be a diagram $\uparrow \pi_0$ where π_0 is a diagram of length $L - 1$.

$$(27) \quad \begin{aligned} Y(D) &= (K \otimes t + q^{-1}KE \otimes 1 + F \otimes 1) \uparrow \pi_0 \\ &= q \uparrow Y(\pi_0) + \downarrow \pi_0 \end{aligned}$$

We have the following six cases:

(A) π_0 has no up arrows.

(A1) π_0 has no down arrow with a star. The diagram π_0 consists of arcs.

- (A2) π_0 has a down arrow with a star and an unpaired arrow. The diagram π_0 is written as $\pi_0 = \pi_1 \downarrow \pi_2$ where π_1 consists of arcs and π_2 consists of arcs, dashed arcs and down arrow with a star.
- (A3) π_0 has a down arrow with a star but no unpaired arrow. The diagram π_0 consists of arcs, dashed arcs and a down arrow with a star.
- (B) π_0 has up arrows.
- (B1) π_0 has no down arrow with a star. The diagram π_0 is written as $\pi_0 = \pi_1 \uparrow \pi_2 \uparrow \dots \uparrow \pi_l$ where $\pi_i, 1 \leq i \leq l$, consists of arcs.
- (B2) π_0 has a down arrow with a star and an unpaired arrow. The diagram π_0 is written as $\pi_0 = \pi_1 \uparrow \pi_2 \uparrow \dots \uparrow \pi_l \downarrow \pi_{l+1}$ where $\pi_i, 1 \leq i \leq l$, consists of arcs and π_{l+1} consists of arcs, dashed arcs and a down arrow with a star.
- (B3) π_0 has a down arrow with a star but no unpaired arrow. The diagram π_0 is written as $\pi_0 = \pi_1 \uparrow \pi_2 \uparrow \dots \uparrow \pi_l$ where $\pi_i, 1 \leq i \leq l-1$, consists of arcs and π_l consists of arcs, dashed arcs and a down arrow with a star.

We consider (B2) case since all other cases can be similarly proven. From the assumption, we have

$$(28) \quad Y(\pi_0) = \sum_{2 \leq i \leq l-1} [i-1] \cdot \pi_1 \uparrow \dots \uparrow \pi_{i-1} \frown \pi_i \pi_{i+1} \uparrow \dots \uparrow \pi_l \downarrow \pi_{l+1} \\ + [l-1] \cdot \pi_1 \uparrow \dots \uparrow \pi_{l-1} \frown \pi_l \pi_{l+1} + [l] \cdot \pi_1 \uparrow \dots \uparrow \pi_{l+1}.$$

We also have

$$(29) \quad \downarrow \pi_1 \uparrow \dots \uparrow \pi_l \downarrow \pi_{l+1} = \sum_{1 \leq i \leq l-1} q^{-(i-1)} \cdot \uparrow \pi_1 \dots \frown \pi_i \dots \pi_l \downarrow \pi_{l+1} \\ + q^{-(l-1)} \cdot \uparrow \pi_1 \dots \pi_{l-1} \frown \pi_l \pi_{l+1} + q^{-l} \cdot \uparrow \pi_1 \uparrow \dots \uparrow \pi_{l+1}.$$

Substituting Eqns.(28) and (29) into Eqn.(27), we obtain

$$Y(D) = \sum_{1 \leq i \leq l-1} [i] \cdot \uparrow \pi_1 \uparrow \dots \frown \pi_i \dots \pi_l \downarrow \pi_{l+1} \\ + [l] \cdot \uparrow \pi_1 \dots \pi_{l-1} \frown \pi_l \pi_{l+1} + [l+1] \cdot \uparrow \pi_1 \uparrow \dots \uparrow \pi_{l+1},$$

where we have used $q[i] + q^{-i} = [i+1]$. This is the desired expression for $Y(D)$.

Suppose the leftmost arrow a is a down arrow. Then D has the following four possibilities:

- (A) a is connected with an up arrow by an arc. The diagram D is written as $\frown \pi_0 \pi_1$ where π_0 consists of arcs and π_1 consists of arcs, dashed arcs, a down arrow with a star and an unpaired down arrow.
- (B) a is a down arrow with a star. D is written as $\star \downarrow \pi_0$ where π_0 consists of arcs.
- (C) a is connected with an down arrow by a dashed arc. The diagram D is written as $\frown \pi_0 \pi_1$ where π_0 consists of arcs and π_1 consists of arcs, dashed arcs and a down arrow with a star.
- (D) a is an unpaired down arrow. The diagram D is depicted as $\downarrow \pi_0$ where π_0 consists of arcs, dashed arc and a down arrow with a star.

We consider the case (C) since other cases are similarly proven. We have

$$Y(D) = Y(\downarrow \pi_0 \downarrow \pi_1 - q^{-1} \uparrow \pi_0 \uparrow \pi_1) \\ = q^{-1} \downarrow Y(\pi_0 \downarrow \pi_1) + \uparrow \pi_0 \downarrow \pi_1 - \uparrow Y(\pi_0 \downarrow \pi_1) - q^{-1} \downarrow \pi_0 \downarrow \pi_1 \\ = 0.$$

Here we have used $Y(\pi_0 \downarrow \pi_1) = \pi_0 \uparrow \pi_1$ since the down arrow in $\pi_0 \downarrow \pi_1$ is an unpaired down arrow in the diagram $\pi_0 \downarrow \pi_1$. This is the desired expression for $Y(D)$. \square

Corollary 6.10. *All coefficients of the action of Y on the dual canonical basis in a tensor product $V_{m_1} \otimes \dots \otimes V_{m_n}$ belong to $\mathbb{N}[q, q^{-1}]$. Especially, they are quantum integers.*

6.3. Eigensystem of Y . Let $\mathbf{k} \in I_{\mathbf{m}}$ and D be a diagram for a dual canonical base $v^{k_1} \diamond \dots \diamond v^{k_n}$. The integer n_{\uparrow} is the number of up arrows in D . We define an integer $N_{\mathbf{k}}$ as follows:

- (1) If D does not have a down arrow with a star, $N_{\mathbf{k}} = n_{\uparrow} + 1$.
- (2) If D has a down arrow with a star but no unpaired down arrow, $N_{\mathbf{k}} = n_{\uparrow}$.
- (3) If D has a down arrow with a star and an unpaired down arrow, $N_{\mathbf{k}} = -(n_{\uparrow} + 1)$.

Note that $|N_{\mathbf{k}}|$ is the maximum integer which appears in the expansion of $Y(D)$. For an integer $j \in \mathbb{Z}$, we define

$$M_j := \#\{\mathbf{k} \mid N_{\mathbf{k}} = j \text{ and } \mathbf{k} \in I_{\mathbf{m}}\}.$$

For example, consider $I_{(2,2)}$. Then, $N_{(2,-2)} = -3$ and $N_{(0,-2)} = 1$. We have $M_5 = 1, M_3 = 2, M_1 = 3, M_{-1} = 2$ and $M_{-3} = 1$.

We consider the matrix representation of Y in $V_{m_1} \otimes \dots \otimes V_{m_n}$ with respect to dual canonical bases.

Theorem 6.11. *The generator Y has an eigenvalue $[j]$, $j \in \mathbb{Z}$, with the multiplicity M_j .*

Before proceeding to the proof of Theorem 6.11, we introduce notations and lemmas.

We define the lexicographic order for $\mathbf{k} \in I_{\mathbf{m}}$: $\mathbf{k}' <_{\text{lex}} \mathbf{k}$ if and only if there exists i such that $k'_j = k_j$ for $1 \leq j \leq i$ and $k'_{i+1} < k_{i+1}$.

Let Γ be a graph whose vertices are indexed by $\mathbf{k} \in I_{\mathbf{m}}$. We connect vertices \mathbf{k} and \mathbf{k}' by an arrow from \mathbf{k} to \mathbf{k}' if $v^{k'_1} \diamond \dots \diamond v^{k'_n}$ appears in the expansion of $Y(v^{k_1} \diamond \dots \diamond v^{k_n})$. If there exists an arrow from \mathbf{k} to \mathbf{k}' , then we denote it by $\mathbf{k} \rightarrow \mathbf{k}'$. Suppose D and D' are the diagrams associated with \mathbf{k} and \mathbf{k}' . We write $D \rightarrow D'$ if $\mathbf{k} \rightarrow \mathbf{k}'$.

It is obvious that

Lemma 6.12. *The vertex $(m_1, \dots, m_n) \in I_{\mathbf{m}}$ has a unique incoming arrow from itself. A vertex \mathbf{k} with $\mathbf{k} \neq (m_1, \dots, m_n)$ has at least one incoming arrows.*

Lemma 6.13. *In Γ , suppose $\mathbf{k}' \rightarrow \mathbf{k}$ with $\mathbf{k}' <_{\text{lex}} \mathbf{k}$ and $\mathbf{k}, \mathbf{k}' \neq (m_1, \dots, m_n)$. Then we have*

- (1) *The number of such \mathbf{k}' is at most one for each vertex \mathbf{k} .*
- (2) *The vertex \mathbf{k}' has a incoming arrow from the vertex \mathbf{k} , that is, $\mathbf{k} \rightarrow \mathbf{k}'$.*
- (3) *The vertices \mathbf{k}, \mathbf{k}' do not have an arrow from itself.*
- (4) *There is no \mathbf{k}'' such that $\mathbf{k}'' \rightarrow \mathbf{k}'$ and $\mathbf{k}'' <_{\text{lex}} \mathbf{k}'$*

Proof. Recall the explicit action of Y in Theorem 6.9. The relation $\mathbf{k}' \rightarrow \mathbf{k}$ with $\mathbf{k}' <_{\text{lex}} \mathbf{k}$ means that the diagram D of $v^{k_1} \diamond \dots \diamond v^{k_n}$ has a down arrow with a star but not an unpaired down arrow. Consider the diagram D' obtained from D by changing the rightmost up arrow to an unpaired down arrow. We have $\mathbf{k}' <_{\text{lex}} \mathbf{k}$ and D appears in the expansion of $Y(D')$. Similarly, let D'' be a diagram obtained from D by changing the leftmost down arrow (which forms a dashed arc or a down arrow with a star) to an up arrow, or by changing an outer arc to two up arrows. In these two cases, we have $\mathbf{k} <_{\text{lex}} \mathbf{k}''$. Thus (1) follows. The statements (2) is obvious since D' appears in the expansion $Y(D)$. The statement (3) follows from the fact that D (resp. D') does not appear in the expansion of $Y(D)$ (resp. $Y(D')$). We can show the statement (4) by a similar argument on D' . \square

Proof of Theorem 6.11. We will construct an eigenvector characterized by \mathbf{k} with the eigenvalue $N_{\mathbf{k}}$.

Let $a_{\mathbf{k}}$ be an indeterminate and take a linear combination of the dual canonical basis,

$$\chi := \sum_{\mathbf{k}} a_{\mathbf{k}} v^{k_1} \diamond \cdots \diamond v^{k_n}.$$

From Corollary 6.10, the action of Y on a dual canonical basis is written as

$$Y(v^{k_1} \diamond \cdots \diamond v^{k_n}) = \sum_{\mathbf{k}'} [n_{\mathbf{k}', \mathbf{k}}] v^{k'_1} \diamond \cdots \diamond v^{k'_n}$$

with a non-negative integer $n_{\mathbf{k}', \mathbf{k}}$. We want to solve the eigenvalue problem $Y\chi = y\chi$ (with an eigenvalue y). The eigenvalue problem is equivalent to

$$(30) \quad \sum_{\mathbf{k}': \mathbf{k}' \rightarrow \mathbf{k}} [n_{\mathbf{k}, \mathbf{k}'}] a_{\mathbf{k}'} = ya_{\mathbf{k}}.$$

Let D be the diagram for $v^{k_1} \diamond \cdots \diamond v^{k_n}$ and Γ' be a graph obtained from Γ by deleting an outgoing arrow from a vertex \mathbf{k}' if $a_{\mathbf{k}'} = 0$. Suppose $a_{\mathbf{k}} \neq 0$. We have three cases for D : (A) D has no down arrow with a star, (B) D has a down arrow with a star but not an unpaired down arrow, and (C) D has a down arrow with a star and an unpaired down arrow.

Case A. We set $a_{\mathbf{k}'} = 0$ if $|N_{\mathbf{k}'}| \geq N_{\mathbf{k}}$. In Γ' , there exists no incoming arrow on a vertex \mathbf{k} except from itself. The graph Γ' does not have an arrow from \mathbf{k}' with $\mathbf{k} <_{\text{lex}} \mathbf{k}'$. We solve the eigenvalue problem by induction in the lexicographic order.

The \mathbf{k} -component of Eqn.(30) is equal to

$$[N_{\mathbf{k}}]a_{\mathbf{k}} = ya_{\mathbf{k}}.$$

The solution is $y = [N_{\mathbf{k}}]$ since $a_{\mathbf{k}} \neq 0$.

Suppose that we have a partial solution $\{a_{\mathbf{l}} \mid \mathbf{l} <_{\text{lex}} \mathbf{l}' \leq_{\text{lex}} \mathbf{k}\}$ for some \mathbf{l} . We will show that $a_{\mathbf{l}}$ is uniquely obtained from the partial solution. We have two cases from Lemma 6.13.

- (1) The vertex \mathbf{l} has no incoming arrow from \mathbf{l}' with $\mathbf{l}' <_{\text{lex}} \mathbf{l}$. Equation (30) is equal to

$$\sum_{\substack{\mathbf{l}' \neq \mathbf{l} \\ \mathbf{l}' \rightarrow \mathbf{l}}} [n_{\mathbf{l}, \mathbf{l}'}] a_{\mathbf{l}'} = (y - [n_{\mathbf{l}, \mathbf{l}}]) a_{\mathbf{l}}.$$

Since the integer $n_{\mathbf{l}, \mathbf{l}}$ is zero or a positive integer less than $N_{\mathbf{k}}$, $a_{\mathbf{l}}$ is uniquely written in terms of the partial solution $\{a_{\mathbf{l}'} \mid \mathbf{l}' <_{\text{lex}} \mathbf{l}\}$.

- (2) The vertex \mathbf{l} has a incoming arrow from \mathbf{l}' with $\mathbf{l}' <_{\text{lex}} \mathbf{l}$. Equation (30) is equal to

$$\begin{aligned} \sum_{\substack{\mathbf{l}' \neq \mathbf{l} \\ \mathbf{l}' \rightarrow \mathbf{l}}} [n_{\mathbf{l}, \mathbf{l}'}] a_{\mathbf{l}'} &= ya_{\mathbf{l}} - [n_{\mathbf{l}, \mathbf{l}}] a_{\mathbf{l}}, \\ \sum_{\substack{\mathbf{l}' \neq \mathbf{l} \\ \mathbf{l}' \rightarrow \mathbf{l}'}} [n_{\mathbf{l}, \mathbf{l}'}] a_{\mathbf{l}'} &= ya_{\mathbf{l}'} - [n_{\mathbf{l}', \mathbf{l}}] a_{\mathbf{l}}. \end{aligned}$$

where $\mathbf{l} <_{\text{lex}} \mathbf{l}'$. Since $n_{\mathbf{k}, \mathbf{k}'}, n_{\mathbf{k}', \mathbf{k}}$ are less than $N_{\mathbf{k}}$, the rank of the equations is two. We can solve these two equations simultaneously with respect to $a_{\mathbf{l}}$ and $a_{\mathbf{l}'}$.

In both cases, $a_{\mathbf{l}}$ is expressed in terms of the partial solution. An eigenvector with an eigenvalue $N_{\mathbf{k}}$ is constructed.

Case B. Let D' be the diagram for $v^{k'_1} \diamond \cdots \diamond v^{k'_n}$ obtained from D by changing the rightmost up arrow to an unpaired down arrow. We set $a_{\mathbf{k}'} \neq 0$ and $a_{\mathbf{k}''} = 0$ if $|N_{\mathbf{k}''}| \geq N_{\mathbf{k}}$. In Γ' , the vertices \mathbf{k}, \mathbf{k}' have only one incoming arrow, namely, $\mathbf{k} \leftrightarrow \mathbf{k}'$. The \mathbf{k} - and \mathbf{k}' -components in Eqn.(30) are equal to

$$\begin{aligned} [N_{\mathbf{k}'}]a_{\mathbf{k}'} &= ya_{\mathbf{k}}, \\ [N_{\mathbf{k}}]a_{\mathbf{k}} &= ya_{\mathbf{k}'}. \end{aligned}$$

Note $N_{\mathbf{k}} > 0$. Since $N_{\mathbf{k}'} = -N_{\mathbf{k}}$, we have the solution $y = \pm[N_{\mathbf{k}}]$. We take $y = [N_{\mathbf{k}}]$ since $y = -[N_{\mathbf{k}}] = [N_{\mathbf{k}'}]$ will be considered in Case C. By a similar argument to Case A, we can construct an eigenvector with an eigenvalue $[N_{\mathbf{k}}]$.

Case C. Let D' be the diagram for $v^{k'_1} \diamond \cdots \diamond v^{k'_n}$ such that D' is obtained from D by changing the unpaired down arrow to an up arrow. We set $a_{\mathbf{k}'} \neq 0$ and $a_{\mathbf{k}''} = 0$ if $|N_{\mathbf{k}''}| \geq N_{\mathbf{k}}$. Note that $N_{\mathbf{k}} < 0$. By a similar argument to Case (B), we can construct an eigenvector with an eigenvalue $N_{\mathbf{k}}$.

In the cases B and C, an eigenvector is characterized by \mathbf{k} and \mathbf{k}' . However, they are distinguished by the sign of an eigenvalue. Therefore, in all cases, an eigenvector is characterized by \mathbf{k} by construction. This completes the proof of Theorem. \square

Set $L = \sum_{i=1}^n m_i$ for $\mathbf{m} = (m_1, \dots, m_n)$.

Corollary 6.14. *The generator Y has the eigenvector with the eigenvalue $[L + 1]$ and the multiplicity is one.*

Proof. Choose $\mathbf{k} = (m_1, \dots, m_n)$. We have $N_{\mathbf{k}} = L + 1$. The number of up arrows in the diagram for $\mathbf{k}' \neq \mathbf{k}$ is less than L . Thus, we have $M_{L+1} = 1$. \square

Corollary 6.15. *The eigenvalue $[L + 1]$ is the largest at $q = 1$.*

Let $\Psi := \sum_{\mathbf{k} \in I_{\mathbf{m}}} \Psi_{\mathbf{k}} v^{k_1} \diamond \cdots \diamond v^{k_n}$ be the eigenvector of Y with the eigenvalue $[L + 1]$.

Lemma 6.16. *The $(-m_1, \dots, -m_n)$ -component of Ψ is non-zero.*

Proof. We make use of the notation in the proof of Theorem 6.11. There exists a sequence of \mathbf{k}_i such that

$$\mathbf{k}_0 = (m_1, \dots, m_n) \rightarrow \mathbf{k}_1 \rightarrow \mathbf{k}_2 \rightarrow \cdots \rightarrow \mathbf{k}_L = (-m_1, \dots, -m_n).$$

Note that this sequence is unique. For some \mathbf{k}_i , we have $\mathbf{k}_i \leftarrow \mathbf{k}_{i+1}$. For each \mathbf{k}_i , there is no \mathbf{k}' such that $\mathbf{k}' \neq \mathbf{k}_{i-1}, \mathbf{k}_{i+1}$ and $\mathbf{k}' \rightarrow \mathbf{k}_i$. From Lemma 6.13, there exists no triplet $(i, i + 1, i + 2)$ such that $\mathbf{k}_i \leftarrow \mathbf{k}_{i+1} \leftarrow \mathbf{k}_{i+2}$. There is no incoming arrow from itself except \mathbf{k}_0 .

From Eqn.(30), when $\mathbf{k}_{i-1} \rightarrow \mathbf{k}_i \rightarrow \mathbf{k}_{i+1}$, we have

$$[n_{\mathbf{k}_{i-1}, \mathbf{k}_i}]a_{\mathbf{k}_{i-1}} = [L + 1]a_{\mathbf{k}_i}.$$

When $\mathbf{k}_{i-1} \rightarrow \mathbf{k}_i \leftrightarrow \mathbf{k}_{i+1} \rightarrow \mathbf{k}_{i+2}$, we have

$$\begin{aligned} [n_{\mathbf{k}_{i-1}, \mathbf{k}_i}]a_{\mathbf{k}_{i-1}} + [n_{\mathbf{k}_{i+1}, \mathbf{k}_i}]a_{\mathbf{k}_{i+1}} &= [L + 1]a_{\mathbf{k}_i}, \\ [n_{\mathbf{k}_i, \mathbf{k}_{i+1}}]a_{\mathbf{k}_i} &= [L + 1]a_{\mathbf{k}_{i+1}}. \end{aligned}$$

In both cases, $a_{\mathbf{k}_i}$ is proportional to $a_{\mathbf{k}_{i-1}}$. Therefore, $a_{(-m_1, \dots, -m_n)} \neq 0$ if $a_{(m_1, \dots, m_n)} \neq 0$. The unique eigenvector Ψ has non-zero $a_{(m_1, \dots, m_n)}$. This completes the proof. \square

Definition 6.17. *We normalize Ψ such that $\Psi_{(-m_1, \dots, -m_n)} = 1$.*

Let D be a diagram and E be a diagram obtained from D by removing the projectors. The graph Γ for D is obtained from the graph Γ' for E by deleting some vertices. The eigenvalue problem for Ψ_D is the same as the one for Ψ_E . Thus this implies

Lemma 6.18. *We have $\Psi_D = \Psi_E$.*

6.4. Eigenvector of Y . We first consider the eigenfunction of Y with the eigenvalue $[N + 1]$ on standard bases. Then, we will show an explicit expression of $\Psi_{\mathbf{k}}$ on the Kazhdan–Lusztig bases.

Let $v^\kappa := v^{\kappa_1} \otimes \cdots \otimes v^{\kappa_N}$ with $\kappa_i = \pm 1$, and $J := \{i | \kappa_i = 1\}$. We define a vector indexed by a binary string κ by

$$(31) \quad \Psi_\kappa^0 := \prod_{i \in J} q^{N+1-i},$$

and $\Psi^0 := \sum_\kappa \Psi_\kappa^0 v^\kappa$.

Proposition 6.19. *We have $Y\Psi^0 = [N + 1]\Psi^0$.*

Proof. In general, a binary string κ is written as

$$\underbrace{+\dots+}_{n_1} \underbrace{-\dots-}_{n'_1} \underbrace{+\dots+}_{n_2} \cdots \underbrace{+\dots+}_{n_I} \underbrace{-\dots-}_{n'_I}.$$

We set $N_\uparrow := \sum_{i=1}^I n_i$, $N_\downarrow := \sum_{i=1}^I n'_i$ and $J_i := \{j \in \mathbb{N} | 1 + \sum_{k=1}^{i-1} (n_k + n'_k) \leq j \leq n_i + \sum_{k=1}^{i-1} (n_k + n'_k)\}$. We will show $\sum_{\kappa'} Y_{\kappa, \kappa'} \Psi_{\kappa'}^0 = [N + 1]\Psi_\kappa^0$ where $Y_{\kappa, \kappa'}$ is a matrix representation of Y . From the action of Y on v^κ , v^κ appears in the expansion of $Y(v^{\kappa'})$ if and only if there exists i such that $\kappa'_i = -\kappa_i$ and $\kappa_j = \kappa'_j$ for $j \neq i$, or $\kappa' = \kappa$.

The binary string κ' with $\kappa \neq \kappa'$ and $Y_{\kappa, \kappa'} \neq 0$ can be obtained from κ by changing κ_i to $-\kappa_i$ for $1 \leq i \leq N$. We have two cases for κ_i : 1) $\kappa_i = +$ and 2) $\kappa_i = -$.

Case 1. When $j \in J_i$, $1 \leq i \leq I$, we have $Y_{\kappa, \kappa'} = q^{j-1-2\sum_{k=1}^{i-1} n'_k}$ and $\Psi_{\kappa'} = q^{-(N+1-j)}\Psi_\kappa$. The sum of contributions of these κ' 's is given by

$$\begin{aligned} \Psi_\kappa^0 \sum_{i=1}^I \sum_{j \in J_i} q^{-N-2-2j-2\sum_{k=1}^{i-1} n'_k} &= \sum_{i=1}^{N_\uparrow} q^{-N-2+2j} \Psi_\kappa^0 \\ &= q^{-N+N_\uparrow-1} [N_\uparrow] \Psi_\kappa^0 \end{aligned}$$

Case 2. Let $J'_i := \{j \in \mathbb{N} | 1 + n_i + \sum_{k=1}^{i-1} (n_k + n'_k) \leq j \leq \sum_{k=1}^i (n_k + n'_k)\}$. By a similar argument to Case 1, the sum of contributions is given by

$$\Psi_\kappa^0 \sum_{i=1}^I \sum_{j \in J'_i} q^{N+2-2j+2\sum_{k=1}^i n_k} = q^{N-N_\downarrow+1} [N_\downarrow] \Psi_\kappa^0.$$

Since $Y_{\kappa, \kappa} = q^{N_\uparrow - N_\downarrow}$, we have a contribution from κ itself, which is $q^{N_\uparrow - N_\downarrow} \Psi_\kappa^0$. The sum of these three contributions gives $[N + 1]\Psi_\kappa^0$. This completes the proof. \square

Below, we will consider the eigenvector Ψ on the Kazhdan–Lusztig bases. From Lemma 6.18, it is enough to consider $\Psi_{\mathbf{k}}$ with $k_i \in \{1, -1\}$ for all $1 \leq i \leq n$ instead of $\Psi_{\mathbf{k}'}$ with $\mathbf{k}' \in I_{\mathbf{m}}$.

Let D be a diagram for a dual canonical basis, N be the total number of all arrows and N_\downarrow is the number of an unpaired down arrow (N_\downarrow is either 0 or 1). Let S be the set of all arcs of D . We define the sets of arcs

S_R, S_M and S_L as follows. If D has a down arrow with a star, we define

$$\begin{aligned} S_R &:= \{A \in S \mid A \text{ is right to the down arrow with a star}\}, \\ S_M &:= \begin{cases} \{A \in S \setminus S_R \mid A \text{ is right to the unpaired down arrow}\}, & \text{for } N_1 = 1, \\ \{A \in S \setminus S_R \mid A \text{ is right to the leftmost down arrow forming a dashed arc}\} & \text{for } N_1 = 0, \end{cases} \\ S_L &:= \begin{cases} \{A \in S \setminus (S_R \cup S_M) \mid A \text{ is right to the rightmost up arrow}\}, & \text{if } D \text{ has an up arrow,} \\ \{A \in S \setminus (S_R \cup S_M)\}, & \text{if } D \text{ has no up arrows} \end{cases} \end{aligned}$$

and otherwise S_R, S_M and S_L are defined as the empty set. An arc A is said to be an *outer arc* if there exists neither an arc nor a dashed arc outside of A . We denote by S^+ the set of outer arcs and define $S_R^+ := S_R \cap S^+, S_M^+ := S_M \cap S^+$ and $S_L^+ := S_L \cap S^+$. We define T as the set of all dashed arcs of D .

We define

$$\begin{aligned} N_2 &:= \begin{cases} |S|, & D \text{ does not have a down arrow with a star,} \\ |S| + |T| + 1 & D \text{ has a down arrow with a star,} \end{cases} \\ N_3 &:= |S_M| + |T|, \\ N_4 &:= N - |S| + |S_R| + |S_M| + |S_L| + 1, \\ N_5 &:= \begin{cases} [N_4]/[N_3] & D \text{ has a down arrow with a star and } N_1 = 0, \\ 1 & \text{otherwise.} \end{cases} \end{aligned}$$

We enumerate all arrows (including arrows forming an arc, a dashed arc and a down arrow with a star) from left to right. Let s be the integer assigned to the down arrow with a star. If the i -th down arrow and the j -th ($i < j$) up arrow form an arc A , then A is said to be size $(j - i + 1)/2$. We denote by m_A the size of an arc A . For an arc A , we define the following values:

$$\begin{aligned} d_{1,A} &:= N - j, & d_{2,A} &:= \frac{i - s + 1}{2}. \\ N_6 &:= \begin{cases} \prod_{A \in S_L^+ \cup S_M^+} \frac{[1 + m_A + d_{1,A}]}{[1 + 2m_A + d_{1,A}]}, & N_1 = 0, \\ \prod_{A \in S_M^+} \frac{[1 + m_A + d_{1,A}]}{[1 + 2m_A + d_{1,A}]}, & N_1 = 1, \end{cases} \\ N_7 &:= \prod_{A \in S_R^+} \prod_{i=0}^{N_3} \frac{[d_{2,A} + i]}{[d_{2,A} + m_A + i]} \prod_{B \in S_M} \frac{[d_{2,A} + m_A + c_B]}{[d_{2,A} + c_B]} \end{aligned}$$

where $c_B, B \in S_M$, is the sum of the number of arcs in S_M right to B or outside of B (including B), and the number of dashed arcs right to B . Suppose that k -th down arrow and the l -th ($k < l$) down arrow form a dashed arc A . Then A is said to be size $(l - k + 1)/2$. Let T' be the set of dashed arcs except the leftmost one. We define

$$N_8 := \begin{cases} \prod_{A \in T} [d_{3,A} + m_A]^{-1}, & \text{for } N_1 = 1, \\ \prod_{A \in T'} [d_{3,A} + m_A]^{-1}, & \text{for } N_1 = 0, \end{cases}$$

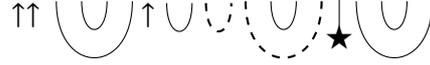
where $d_{3,A} := (s - l + 1)/2$. Let U be the union of T and a down arrow with a star. We define

$$\begin{aligned} d_{4,A} &:= \begin{cases} 1 + (N - s)/2, & \text{for a down arrow with a star,} \\ 1 + (N - k)/2, & \text{for a dashed arc,} \end{cases} \\ N_9 &:= \frac{\prod_{i=1}^{N-N_2} (q^i + q^{-i})}{\prod_{A \in U} (q^{d_{4,A}} + q^{-d_{4,A}})} \end{aligned}$$

We enumerate up arrows, arcs and dashed arcs and an unpaired down arrow from left. If there are arcs inside an arc or a dashed arc, we increase the integer one by one. Let N_A be the integer assigned to an arc or a dashed arc A , and N_\downarrow be the integer assigned to an unpaired down arrow. We define

$$N_{10} := \begin{cases} \prod_{A \in S \cup T} [N_A], & \text{for } N_1 = 0, \\ [N_\downarrow] \prod_{A \in S \cup T} [N_A] & \text{for } N_1 = 1. \end{cases}$$

Example 6.20. Let D be a diagram depicted as



Then we have $N_2 = 9$, $N_5 = [19]/[4]$, $N_6 = [13]/[14]$, $N_7 = [4][5]^{-1}[6]^{-1}$, $N_8 = [3]^{-1}$, $N_{10} = [3] \cdot [4] \cdot [6] \cdot [7] \cdot [8] \cdot [9] \cdot [10] \cdot [11]$ and

$$N_9 = (q + q^{-1})(q^2 + q^{-2})(q^4 + q^{-4}) \prod_{i=7}^{11} (q^i + q^{-i}).$$

Theorem 6.21. In the above notation, we have

$$(32) \quad \Psi_D = \prod_{A \in S} [m_A]^{-1} \cdot N_5 \cdot N_6 \cdot N_7 \cdot N_8 \cdot N_9 \cdot N_{10}.$$

Before proceeding to the proof of Theorem 6.21, we prove five lemmas used later.

Let $\mathbf{k}_i = (\underbrace{1, \dots, 1}_{N-i+1}, \underbrace{-1, \dots, -1}_{i-1})$ for $1 \leq i \leq N+1$. Then, $\{\mathbf{k}_i | 1 \leq i \leq n\}$ satisfies a unique sequence

$$\mathbf{k}_1 = (1, \dots, 1) \rightarrow \mathbf{k}_2 \leftrightarrow \mathbf{k}_3 \rightarrow \dots \rightarrow \mathbf{k}_{N+1} = (-1, \dots, -1).$$

Lemma 6.22. The components $\Psi_i := \Psi_{\mathbf{k}_i}$, $1 \leq i \leq N+1$, are

$$(33) \quad \Psi_{2m+1} = \frac{\prod_{i=1}^{N-m} (q^i + q^{-i}) [N-m]}{\prod_{i=1}^m (q^i + q^{-i})} \begin{bmatrix} N-m \\ m \end{bmatrix},$$

$$(34) \quad \Psi_{2m} = \frac{\prod_{i=1}^{N-m} (q^i + q^{-i}) [N+1]}{\prod_{i=1}^m (q^i + q^{-i})} \frac{[N-m]}{[m]} \begin{bmatrix} N-m \\ m-1 \end{bmatrix}.$$

Proof. We use the notation in the proof of Theorem 6.11. The sequence is locally equivalent to

$$\mathbf{k}_{2m-1} \rightarrow \mathbf{k}_{2m} \leftrightarrow \mathbf{k}_{2m+1}.$$

Since Ψ is the eigenvector with the eigenvalue $[N+1]$, the components $\Psi_i := \Psi_{\mathbf{k}_i}$, $1 \leq i \leq N+1$, satisfy the following eigenvalue problem:

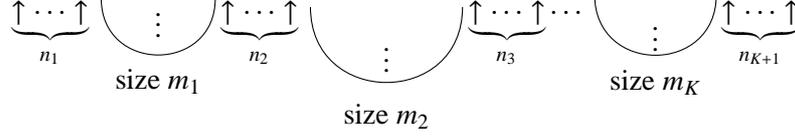
$$\begin{aligned} [N-2m+2]\Psi_{2m-1} + [N-2m+1]\Psi_{2m+1} &= [N+1]\Psi_{2m}, \\ [N-2m+1]\Psi_{2m} &= [N+1]\Psi_{2m+1}. \end{aligned}$$

The solution is

$$\begin{aligned} \Psi_{2m} &= \frac{[N+1][N-2m+2]}{[2m][2N-2m+2]} \Psi_{2m-1}, \\ \Psi_{2m+1} &= \frac{[N-2m+1][N-2m+2]}{[2m][2N-2m+2]} \Psi_{2m-1}. \end{aligned}$$

Suppose $\Psi_1 = \prod_{i=1}^N (q^i + q^{-i})$. The coefficients Ψ_k satisfy Eqns.(33) and (34). We have $\Psi_{N+1} = 1$. This normalization is compatible with Definition 6.17. This completes the proof. \square

Let D be a diagram



where the region inside an arc of size m_i , $1 \leq i \leq K$, is filled with arcs. Let S_i be the set of all arcs inside the arc of size m_i .

Lemma 6.23. *Let D be a diagram depicted above. The component Ψ_D is*

$$\Psi_D = \prod_{i=1}^d (q^i + q^{-i}) \prod_{j=1}^K \frac{[\sum_{p=1}^j (n_p + m_p)]!}{[n_j + \sum_{p=1}^{j-1} (n_p + m_p)]!} \prod_{A \in S_j} [m_A]^{-1}$$

where $d = \sum_{i=1}^{K+1} (m_i + n_i)$ with $m_{K+1} = 0$.

Proof. We prove Theorem by induction. When D has no arc, that is, $n_1 \neq 0$, $n_i = 0$, $2 \leq i \leq K+1$, and $m_j = 0$, $1 \leq j \leq K$, we have $\Psi_D = \prod_{i=1}^{n_1} (q^i + q^{-i})$. Theorem holds true since this is compatible with Definition 6.17 and Lemma 6.22.

We assume that Theorem is true for D' where D' has one less arcs than D . From Eqn.(30), the eigenvalue problem is equal to

$$(35) \quad \sum_{i=1}^K \prod_{j=1}^{d+1} (q^j + q^{-j}) \cdot \left[1 + \sum_{j=1}^i n_j \right] \left(\prod_{j=1}^{i-1} L_j \right) L'_i \left(\prod_{j=i+1}^K L''_j \right) + \left[1 + \sum_{p=1}^{K+1} n_p \right] \Psi_D = \left[1 + \sum_{p=1}^{K+1} (n_p + 2m_p) \right] \Psi_D$$

where $m_{K+1} = 0$ and

$$\begin{aligned} L_j &= \frac{[\sum_{p=1}^j (n_p + m_p)]!}{[n_j + \sum_{p=1}^{j-1} (n_p + m_p)]!} \prod_{A \in S_j} [m_A]^{-1}, \\ L'_j &= \frac{[\sum_{p=1}^j (n_p + m_p)]! \cdot [m_j]}{[1 + n_j + \sum_{p=1}^{j-1} (n_p + m_p)]!} \prod_{A \in S_j} [m_A]^{-1}, \\ L''_j &= \frac{[1 + \sum_{p=1}^j (n_p + m_p)]!}{[1 + n_j + \sum_{p=1}^{j-1} (n_p + m_p)]!} \prod_{A \in S_j} [m_A]^{-1}. \end{aligned}$$

The first term of the left hand side of Eqn.(35) is rewritten as

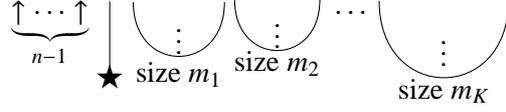
$$(36) \quad \prod_{j=1}^{d+1} (q^j + q^{-j}) \left(\prod_{j=1}^K L_j \right) \cdot \sum_{i=1}^K \left[1 + \sum_{j=1}^i n_j \right] [m_i] \frac{\prod_{j:j>i} [1 + \sum_{p=1}^j m_p + n_p]}{\prod_{j:j \geq i} [1 + n_j + \sum_{p=1}^{j-1} m_p + n_p]}.$$

Substituting Eqn.(36) and Lemma A.1 into Eqn.(35), we obtain

$$\Psi_D = \prod_{i=1}^d (q^i + q^{-i}) \cdot \prod_{i=1}^K L_i.$$

This completes the proof. \square

Let D be a diagram depicted as



and D' be the diagram obtained from D by changing the rightmost up arrow of D to a down arrow. The region inside an arc of size m_i , $1 \leq i \leq K$, is filled with arcs. Let S be the set of all arcs. We denote by m_A the size of an arc A .

Lemma 6.24. *In the above notation, we have*

$$\begin{aligned}\Psi_D &= \frac{\prod_{i=1}^{n-1+M} (q^i + q^{-i})}{q^{M+1} + q^{-(M+1)}} \frac{[n-1+M]!}{[n-1]! \cdot \prod_{A \in S} [m_A]} \frac{[1+n+2M]}{[1+M]}, \\ \Psi_{D'} &= \frac{[n-1]}{[1+n+2M]} \Psi_D,\end{aligned}$$

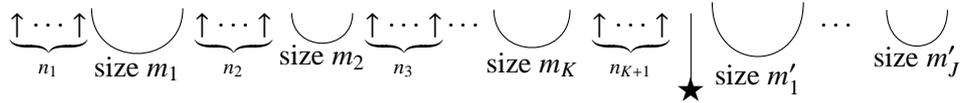
where $M = \sum_{i=1}^K m_i$.

Proof. There exists only one diagram D'' , $D <_{\text{lex}} D''$, such that the relation $D'' \rightarrow D$ holds. The diagram D'' is obtained from D by changing the down arrow with a star to an up arrow. We have $\Psi_{D''}$ from Lemma 6.23. The eigenvalue problem (30) is equivalent to

$$\begin{aligned}\prod_{i=1}^{n+M} (q^i + q^{-i}) \cdot [n] \frac{[n+M]!}{[n]! \prod_{A \in S} [m_A]} + [n-1] \Psi_{D'} &= [n+1+2M] \Psi_D, \\ [n-1] \Psi_D &= [n+1+2M] \Psi_{D'}.\end{aligned}$$

Solving the above equations, we have a desired expression. \square

Let D be a diagram depicted as



and D' be the diagram obtained from D by changing the rightmost up arrow of D to a down arrow. Let S be the set of all arcs inside of an arc of size m_i , $1 \leq i \leq K$ or m'_j , $1 \leq j \leq J$. We consider two cases: (1) $n_{K+1} \neq 0$ (Lemma 6.25) and (2) $n_{I+1} \neq 0$ and $n_i = 0$, $I+2 \leq i \leq K+1$ (Lemma 6.26).

Lemma 6.25. *Let $n_{K+1} \neq 0$. Set $N_i = \sum_{j=1}^i n_j$, $M_i = \sum_{j=1}^i m_j$, $N = N_{K+1}$, $M = M_K$ and $M' = \sum_{i=1}^J m'_i$. We have*

$$(37) \quad \Psi_D = \frac{\prod_{i=1}^{N+M+M'} (q^i + q^{-i})}{(q^{M'+1} + q^{-(M'+1)})} \frac{[2+N+M+2M']}{\prod_{A \in S} [m_A] \cdot [M'+1]} \prod_{i=1}^K \frac{[N_i + M_i]!}{[N_i + M_{i-1}]!} \frac{[N+M+M']!}{[N+M]!},$$

$$(38) \quad \Psi_{D'} = \frac{[N+M]}{[2+N+M+2M']} \Psi_D.$$

Proof. We prove Lemma by induction. The case where $m_i = 0$ for $1 \leq i \leq K$ is reduced to Lemma 6.24. We assume that Lemma holds true for a diagram which has one less arcs than D . Let D_1 be a diagram obtained from D by changing the outer arc of size m_i to two up arrows. Then, we have $D_1 \rightarrow D$. If D_2 is obtained from D by changing the down arrow with a star to an up arrow, we have $D_2 \rightarrow D$. Let D' be a diagram obtained from D by changing the rightmost up arrow of D to an unpaired down arrow. We have $D \leftrightarrow D'$. Let D_3 be a diagram obtained from D' by changing the outer arc of size m_i to two up arrows. We have $D_3 \rightarrow D'$.

Let A be the right hand side of Eqn.(37). We have an explicit expression for Ψ_{D_1} from the induction assumption, and Ψ_{D_2} from Lemma 6.24. Therefore, the eigenvalue problem (30) for the D - and D' -components is reduced to

$$\begin{aligned} & \sum_{i=1}^K A(q^d + q^{-d})[1 + N_i][m_i] \frac{\prod_{j \geq i+1} [1 + N_j + M_j] [N + M + M' + 1][N + M + 2M' + 3]}{\prod_{j \geq i} [1 + N_j + M_{j-1}] [N + M + 1][N + M + 2M' + 2]} \\ & \quad + A(q^d + q^{-d})(q^{M'+1} + q^{-(M'+1)}) \frac{[N + M + M' + 1][N + 1][M' + 1]}{[N + M + 1][N + M + 2M' + 2]} + [N]\Psi_{D'} \\ & \hspace{15em} = [N + 2M + 2M' + 2]\Psi_D, \end{aligned}$$

$$\begin{aligned} & \sum_{i=1}^K A(q^d + q^{-d})[1 + N_i][m_i] \frac{\prod_{j \geq i+1} [1 + N_j + M_j] [N + M + M' + 1]}{\prod_{j \geq i} [1 + N_j + M_{j-1}] [2 + N + M + 2M']} \\ & \hspace{15em} + [N]\Psi_D = [N + 2M + 2M' + 2]\Psi_{D'}. \end{aligned}$$

where $d = N + M + M' + 1$. Substituting Lemma A.1 into the above equations, we obtain

$$\Psi_D = A, \quad \Psi_{D'} = \frac{[N + M]}{[2 + N + M + 2M']}A.$$

This completes the proof. \square

Lemma 6.26. *Let $n_{I+1} \neq 0$ and $n_i = 0$ for $I+2 \leq i \leq K+1$. Set $N = \sum_{i=1}^{I+1} n_i$, $M_1 = \sum_{i=1}^I m_i$, $M_2 = \sum_{i=I+1}^K m_i$ and $M' = \sum_{i=1}^J m'_i$. We have*

$$(39) \quad \Psi_D = \frac{\prod_{i=1}^d (q^i + q^{-i})}{q^{M'+1} + q^{-(M'+1)}} \prod_{A \in \mathcal{S}} [m_A]^{-1} \prod_{i=1}^I \frac{[\sum_{j=1}^i n_j + m_j]!}{[\sum_{j=1}^i n_j + \sum_{j=1}^{i-1} m_j]!} \frac{[d]!}{[M_1 + N]!} \frac{[L_1]}{[M' + 1]} \prod_{i=I+1}^K \frac{[g_i - m_i]}{[g_i]},$$

$$(40) \quad \Psi_{D'} = \frac{[N + M_1]}{[L_1]} \Psi_D,$$

where $d = N + M_1 + M_2 + M'$, $L_1 = N + M_1 + 2M_2 + 2M' + 2$ and $g_i = 2 + 2M' + 2 \sum_{j=i}^K m_j$.

Proof. We prove Lemma by induction. We consider the case where $m_i = 0$ for $1 \leq i \leq K-1$ and $m_K \neq 0$. If D_1 has one less arcs than D (D_1 does not have the outer arc of size m_K), we have $D_1 \rightarrow D$. If D_2 does not have a down arrow with a star but all arcs are in the same position as D , then $D_2 \rightarrow D$. We have a formula for Ψ_{D_1} and Ψ_{D_2} from Lemma 6.23 and Lemma 6.25. On the other hand, a diagram D_3 such that $D_3 \rightarrow D'$ is only $D_3 = D$. Let A be the right hand side of Eqn.(39). The eigenvalue problem is reduced to

$$\begin{aligned} & A(q^{d+1} + q^{-(d+1)})[m_K] \frac{[N + m_K + M' + 1] [2 + 2m_K + 2M'] [N + m_K + 2M' + 3]}{[N + m_K + 1] [2 + m_K + 2M'] [N + 2m_K + 2M' + 2]} \\ & \quad + A(q^{d+1} + q^{-(d+1)})(q^{M'+1} + q^{-(M'+1)}) \frac{[N + 1][N + m_K + M' + 1][M' + 1][2 + 2m_K + 2M']}{[N + m_K + 1][N + 2m_K + 2M' + 2][2 + m_K + 2M']} \\ & \hspace{15em} + [N]\Psi_{D'} = [N + 2 + 2m_K + 2M']\Psi_D, \\ & \hspace{15em} [N]\Psi_D = [N + 2 + 2m_K + 2M']\Psi_{D'} \end{aligned}$$

The solution is

$$\Psi_D = A, \quad \Psi_{D'} = \frac{[N]}{[N + 2 + 2m_K + 2M']}A$$

which is a desired expression.

We assume that Lemma holds true for a diagram which has one less arcs than D . If D_1 has one less arcs than D , $D_1 \rightarrow D$. If D_2 does not have a down arrow with a star and all arcs are in the same position as D ,

$D_2 \rightarrow D$. We also have $D' \rightarrow D$. From the assumption, Lemma 6.23 and Lemma 6.25, we have a formula for the components Ψ_{D_1} and Ψ_{D_2} . Similarly, if D_3 has one less arcs than D' , $D_3 \rightarrow D'$. We have $D \rightarrow D'$. Let A be the right hand side of Eqn.(39). The eigenvalue problem (30) is reduced to

$$(41) \quad \sum_{i=1}^I A(q^{d+1} + q^{-(d+1)}) \left[1 + \sum_{j=1}^i n_j \right] \frac{\prod_{j \geq i+1} [1 + \sum_{k=1}^j n_k + m_k]}{\prod_{j \geq i} [1 + \sum_{k=1}^j n_k + \sum_{k=1}^{j-1} m_k]} \frac{[m_i][d+1]}{[N+M_1+1]} \frac{[L_1+1]}{[L_1]} \\ + \sum_{i=I+1}^K (q^{d+1} + q^{-(d+1)}) A \frac{\prod_{j \geq i+1} [1 + N + M_1 + \sum_{k=I+1}^j m_k]}{\prod_{j \geq i} [1 + N + M_1 + \sum_{k=I+1}^{j-1} m_k]} \frac{[m_i][N+1][d+1]}{[1+N+M_1+M_2]} \frac{[h_i]}{[L_1]} \prod_{j=I+1}^i \frac{[g_i]}{[g_i - m_i]} \\ + A(q^{d+1} + q^{-(d+1)}) (q^{M'+1} + q^{-(M'+1)}) [N+1] \frac{[d+1]}{[N+M_1+M_2+1]} \frac{[M'+1]}{[L_1]} \prod_{j=I+1}^K \frac{[g_j]}{[g_j - m_j]} \\ + [N]\Psi_{D'} = [L_1 + M_1]\Psi_D$$

where $h_i = 3 + N + \sum_{j=I+1}^i m_j + 2 \sum_{j=i+1}^K m_j + 2M'$ and

$$(42) \quad \sum_{i=1}^I A(q^{d+1} + q^{-(d+1)}) \left[1 + \sum_{j=1}^{i-1} n_j \right] \frac{\prod_{j=i+1}^I [1 + \sum_{k=1}^j n_k + m_k]}{\prod_{j=i}^I [1 + \sum_{k=1}^j n_k + \sum_{k=1}^{j-1} m_k]} \frac{[d+1]}{[L_1]} + [N]\Psi_D = [L_1 + M_1]\Psi_{D'}.$$

We apply Lemma A.1 to the first term of Eqn.(41), Lemma A.2 to the second and third terms with $x = N + M_1$ and $z = M'$, and Lemma A.1 to Eqn.(42). We obtain

$$(q^{d+1} + q^{-(d+1)}) \frac{[d+1][L_1 - N]}{[L_1]} A + [N]\Psi_{D'} = [L_1 + M_1]\Psi_D, \\ (q^{d+1} + q^{-(d+1)}) A [M_1] \frac{[d+1]}{[L_1]} + [N]\Psi_D = [L_1 + M_1]\Psi_{D'}.$$

The solution is

$$\Psi_D = A, \quad \Psi_{D'} = \frac{[N + M_1]}{[L_1]} A.$$

This completes the proof. \square

Proof of Theorem 6.21. We prove Theorem by induction on the numbers of arcs and dashed arcs. From Lemma 6.22, Lemma 6.25 and Lemma 6.26, Theorem holds true when the number of dashed arcs or arcs is zero.

Let D_0 be a diagram which has a down arrow with a star and $N_1 = 0$. The diagram D_0 starts with n_1 up arrows, followed by an outer arc of size m_1 , followed by n_2 up arrows, followed by an outer arc of size m_2, \dots , followed by n_{I+1} up arrows, followed by a dashed arc, \dots , followed by a down arrow with a star, followed by an arc of size m'_1, \dots and ends with an arc of size m'_j . Let D'_0 be a diagram obtained from D_0 by changing the rightmost up arrow of D_0 to an unpaired down arrow. We assume that Eqn.(32) holds true for a diagram which has one less dashed arcs or one less arcs than D_0 . Let $N' = \sum_{i=1}^{I+1} n_i$, $M = \sum_{i=1}^I m_i$, $L_1 = N' + M + 2|S_M| + 2|T| + 2|S_R| + 2$, $L_2 = N' + M + |S_M| + |T| + |S_R| + 1$ and A be the right hand side of Eqn.(32) for D_0

We consider the two cases: 1) $n_{I+1} \neq 0$ and 2) $n_{H+1} \neq 0, n_i = 0, H + 2 \leq i \leq I + 1$.

Case 1. If D_1 has one less arcs than D_0 , then $D_1 \rightarrow D_0$. If D_2 has one less dashed arcs than D_0 , then $D_2 \rightarrow D_0$. We have $D'_0 \rightarrow D_0$. Similarly, if D_3 has one less arcs than D'_0 , then $D_3 \rightarrow D'_0$. We have $D_0 \rightarrow D'_0$. The eigenvalue problem is equivalent to

$$(43) \quad (q^{L_2} + q^{-L_2})A \sum_{i=1}^I \left[1 + \sum_{j=1}^i n_j \right] [m_i] \frac{\prod_{j=i+1}^I [1 + \sum_{k=1}^j (n_k + m_k)]}{\prod_{j=i}^I [1 + \sum_{k=1}^j n_j + \sum_{k=1}^{j-1} m_k]} \frac{[L_2][1 + L_1]}{[1 + N' + M][L_1]} \\ + (q^{L_2} + q^{-L_2})(q^d + q^{-d})[N' + 1]A \frac{[L_2][1 + |S_R| + |S_M| + |T|]}{[L_1][N' + M + 1]} + [N']\Psi_{D'_0} = [L_1 + M]\Psi_{D_0},$$

where $d = |S_R| + |S_M| + |T| + 1$ and

$$(q^{L_2} + q^{-L_2})A[M] \frac{[L_2]}{[L_1]} + [N']\Psi_{D_0} = [L_1 + M]\Psi_{D'_0},$$

where we have used the fact that the weight of the leftmost dashed arc in D'_0 is $(1 + |S_R| + |T|)^{-1}$. Applying Lemma A.1 to Eqn.(43), we obtain

$$\Psi_{D_0} = A, \quad \Psi_{D'_0} = \frac{[N' + M]}{[L_1]}A.$$

This completes the proof of Case 1.

Case 2. Set $M_1 = \sum_{i=1}^H m_i$. By a similar argument to Case 1, the eigenvalue problem for Ψ_D is equal to

$$(44) \quad (q^{L_2} + q^{-L_2})A \sum_{i=1}^H \left[1 + \sum_{j=1}^i n_j \right] [m_i] \frac{\prod_{j \geq i+1} [1 + \sum_{k=1}^j n_k + m_k]}{\prod_{j \geq i} [1 + \sum_{k=1}^j n_k + \sum_{k=1}^{j-1} m_k]} \frac{[L_2][1 + L_1 + M - M_1]}{[1 + N' + M_1][L_1 - M_1 + M]} \\ + (q^{L_2} + q^{-L_2})A \sum_{i=H+1}^K \frac{\prod_{j \geq i+1} [1 + M_1 + N' + \sum_{k=H+1}^j m_k]}{\prod_{j \geq i} [1 + M_1 + N' + \sum_{k=H+1}^{j-1} m_k]} \frac{[N' + 1][L_2][L_{3,i}]}{[1 + M + N'][L_1 - M_1 + M]} \prod_{j=H+1}^i \frac{[g_j]}{[g_j - m_j]} \\ + (q^{L_2} + q^{-L_2})(q^d + q^{-d})A[N' + 1] \frac{[L_2][1 + |S_R| + |S_M| + |T|]}{[N' + M + 1][L_1 - M_1 + M]} \prod_{i=H+1}^K \frac{[g_i]}{[g_i - m_i]} \\ + [N']\Psi_{D'} = [L_1 + M]\Psi_D,$$

where $L_{3,i} = 1 + L_1 + \sum_{j=i+1}^K m_j$, $g_i = 2 + 2 \sum_{j=i}^K m_j + 2|S_R| + 2|S_M| + 2|T|$ and $d = |S_R| + |S_M| + |T| + 1$. Applying Lemma A.1 to the first term and Lemma A.2 to the second and third terms of Eqn.(44), we obtain

$$(45) \quad \frac{A[2L_2][2d + 2M - M_1]}{[L_1 + M - M_1]} + [N']\Psi_{D'} = [L_1 + M]\Psi_D.$$

The eigenvalue problem for $\Psi_{D'}$ is

$$A \sum_{i=1}^H [m_i] \frac{\prod_{j \geq i+1} [1 + \sum_{k=1}^j n_k + m_k]}{\prod_{j \geq i} [1 + \sum_{k=1}^j n_k + \sum_{k=1}^{j-1} m_k]} \frac{[2L_2]}{[L_1 + M - M_1]} + [N']\Psi_{D'} = [L_1 + M]\Psi_{D'}.$$

Inserting Lemma A.1, we obtain

$$(46) \quad A \frac{[M_1][2L_2]}{[L_1 + M - M_1]} + [N']\Psi_{D'} = [L_1 + M]\Psi_{D'}.$$

The solution of Eqns.(45) and (46) is

$$\Psi_D = A, \quad \Psi_{D'} = \frac{[N' + M_1]}{[L_1 + M - M_1]}A.$$

This completes the proof. \square

We have two propositions regarding the eigenfunction Ψ .

Proposition 6.27. *All components of Ψ belong to $\mathbb{N}[q, q^{-1}]$ and invariant under $q \rightarrow q^{-1}$.*

Proof. Let $\mathbf{k} \in I_{\mathbf{m}}$. From Theorem 6.21, the eigenvector $\Psi_{\mathbf{k}}$ contains only quantum numbers and $(q^i + q^{-i})$ for some i . Therefore, Ψ_D is obviously invariant under $q \rightarrow q^{-1}$. We abbreviate $v^{k_1} \otimes \dots \otimes v^{k_n}$ as $v^{\mathbf{k}}$. Recall that Ψ_{κ}^0 , $\kappa = \{\pm 1\}^N$, is the eigenvector of Y with the eigenvalue $[N + 1]$. Since the action of Y commutes with the one of the projection $\pi := \pi_{m_1} \otimes \dots \otimes \pi_{m_n}$, the eigenvector $\Psi_{\mathbf{k}}^0$ on the standard basis is written as $\Psi_{\mathbf{k}}^0 v^{\mathbf{k}} = \sum_{\kappa} \pi(\Psi_{\kappa}^0 v^{\kappa})$ where the sum is all over κ 's satisfying $v^{\mathbf{k}} \propto \pi(v^{\kappa})$. The transition matrix from the standard basis to the Kazhdan–Lusztig basis is nothing but $R := R_{\mathbf{k}, \mathbf{l}}$. The two vectors Ψ and Ψ^0 are a unique eigenvector of Y with the multiplicity one. Therefore, we have $\Psi = R\Psi^0$. From Corollary 5.11, we have $R_{\mathbf{k}, \mathbf{l}} \in q^{-1}\mathbb{N}[q^{-1}]$ for $\mathbf{k} \neq \mathbf{l}$ and $R_{\mathbf{k}, \mathbf{k}} = 1$. Since the vector Ψ^0 is in $\mathbb{N}[q]$, we have $\Psi_{\mathbf{k}} \in \mathbb{N}[q, q^{-1}]$. \square

Remark 6.28. *In the proof of Proposition 6.27, we show that $\Psi = R\Psi^0$. This gives highly non-trivial relations among the Kazhdan–Lusztig polynomials.*

Let $\mathbf{k} \in I_{\mathbf{m}}$. Recall the diagram for a standard base $v^{k_1} \otimes \dots \otimes v^{k_n}$. Let $J_{\mathbf{k}}$ be the set of positions (from the right end) of up arrows of the diagram. We define

$$d_{\mathbf{k}} := \sum_{i \in J_{\mathbf{k}}} i.$$

For example, when $\mathbf{k} = (0, 2)$ with $\mathbf{m} = (2, 2)$, $J_{\mathbf{k}} = \{1, 2, 4\}$ and $d_{\mathbf{k}} = 1 + 2 + 4 = 7$.

Proposition 6.29. *The component $\Psi_{\mathbf{k}}$ has the leading term $q^{d_{\mathbf{k}}}$ with the leading coefficient one.*

Proof. Notice that if $\mathbf{k} <_{\text{lex}} \mathbf{k}'$, we have $d_{\mathbf{k}'} < d_{\mathbf{k}}$. The vector $\Psi_{\mathbf{k}}^0$ has a leading term $q^{d_{\mathbf{k}}}$ with the coefficient one. The matrix representation of R is an upper triangular matrix whose diagonal entries are one and non-zero entries are in $q^{-1}\mathbb{N}[q^{-1}]$. In the proof of Proposition 6.27, we show that $\Psi = R\Psi^0$. Therefore, the leading term of $\Psi_{\mathbf{k}}$ is $q^{d_{\mathbf{k}}}$ and the leading coefficient is one. \square

6.5. Sum rule. Below, we set $m_i = m$, $1 \leq i \leq n$. We denote by $s_{m,L} := \sum_{\mathbf{k}} \Psi_{\mathbf{k}}$ the sum of components of Ψ . We are interested in the case of $q = 1$ since we expect that the sum of the eigenvector Ψ is related to the total number of some combinatorial objects. In Table 1, we list up first few values of $s_{m,L}$ at $q = 1$.

The Pell numbers P_n are defined by the recurrence relation

$$P_n = 2P_{n-1} + P_{n-2},$$

with $P_0 = 1$ and $P_1 = 1$. Let c_n be the sequence of integers A094723 in [28]. The sequence c_n is given by

$$c_n = P_{n+2} - 2^n.$$

Theorem 6.30 (Sum Rule I). *The sum $s_{m,1}$ at $q = 1$ satisfy*

$$s_{m,1} = c_m.$$

Proof. Since we have $L = 1$, diagrams for the dual canonical bases have no arcs. The number of diagrams is $m + 1$. The graph Γ is equal to

$$\begin{array}{c} \circlearrowleft \\ m \rightarrow m - 1 \leftrightarrow m - 2 \rightarrow \dots \rightarrow -m. \end{array}$$

The graph Γ is isomorphic to the sequence considered in Lemma 6.22. Therefore, the component $\Psi_k, k \in I_m$, is the same as in Lemma 6.22. The sum $s_{m,1}$ is written as

$$\begin{aligned} s_{m,1} &= \sum_{n=0}^{\lfloor m/2 \rfloor} 2^{m-2n} \binom{m-n}{n} + \sum_{n=1}^{\lfloor (m+1)/2 \rfloor} 2^{m-2n} \frac{m+1}{n} \binom{m-n}{n-1} \\ &= -2^m + \sum_{n=0}^{\infty} 2^{m-2n+1} \binom{m-n+1}{n}. \end{aligned}$$

Let

$$A_m = \sum_{n=0}^{\infty} 2^{m-2n+1} \binom{m-n+1}{n}.$$

Then A_m satisfies the recurrence relation $A_{m+1} = 2A_m + 2A_{m-1}$ with $A_0 = 2$ and $A_1 = 5$. This implies $A_m = P_{m+2}$. Therefore, $s_{m,1} = P_{m+2} - 2^m$. This completes the proof. \square

TABLE 1. First few values of the sum $s_{m,L}$ at $q = 1$

	Length L					
	1	2	3	4	5	6
1	3	10	38	156	692	3256
2	8	92	1408	26576	594432	
3	21	832	52736	4700592	549144752	
4	54	7276	1924040	817051024	505001670752	...
m	5	137	62756	69395300	141326485016	4184410893902752
	6	344	534416	2479324096	24339640457600	430183061610221568
	7	857	4514352	88070572208	4184410893902752	398477790183643039008
	8	2122		⋮		

Let c_n be the sequence of integers A00902 in [28] (see also [25]). The sequence c_n satisfies the following recurrence relation:

$$c_n = 2c_{n-1} + (2n-2)c_{n-2},$$

with the initial condition $c_1 = 1$ and $c_2 = 3$.

Conjecture 6.31 (Sum Rule II). *The sum $s_{1,L}$ at $q = 1$ satisfy*

$$s_{1,L} = c_{L+1}$$

for all $L \geq 1$.

We have checked the conjecture up to $L = 24$. The sum is conjectured to be equal to the total number of arrangements of bishops in $2n \times 2n$ with a certain symmetry. It would be interesting if we can find a combinatorial meaning of each component $\Psi_{\mathbf{k}}$.

APPENDIX A.

Lemma A.1.

$$(47) \quad \sum_{i=1}^K \left[1 + \sum_{j=1}^i n_j \right] [m_i] \frac{\prod_{j \geq i+1} [1 + \sum_{k=1}^j (n_k + m_k)]}{\prod_{j \geq i} [1 + \sum_{k=1}^j n_k + \sum_{k=1}^{j-1} m_k]} = \left[\sum_{i=1}^K m_i \right]$$

Proof. We prove Lemma by induction on K . Let f^K be the left hand side of Eqn.(47). It is obvious that $f^1 = [m_1]$. We assume Eqn.(47) is true for $K - 1$. Then, we have

$$\begin{aligned} f^K &= f^{K-1} \frac{[1 + \sum_{j=1}^K (n_j + m_j)]}{[1 + \sum_{j=1}^K n_j + \sum_{j=1}^{K-1} m_j]} + \frac{[m_K][1 + \sum_{j=1}^K n_j]}{[1 + \sum_{j=1}^K n_j + \sum_{j=1}^{K-1} m_j]} \\ &= \left[\sum_{i=1}^K m_i \right]. \end{aligned}$$

This completes the proof. \square

Lemma A.2.

$$(48) \quad \sum_{i=1}^K I_i \cdot J_i + \frac{[2z + 2]}{[x + 1 + \sum_{i=1}^K m_i]} I_K = [1 + x]^{-1} \left[2 + 2z + 2 \sum_{i=1}^K m_i \right],$$

where

$$\begin{aligned} I_i &:= \prod_{j=1}^i \frac{[2 + 2m_j + 2z + 2 \sum_{k=j+1}^K m_k]}{[2 + m_j + 2z + 2 \sum_{k=j+1}^K m_k]}, \\ J_i &:= \frac{[m_i][x + 3 + \sum_{j=1}^i m_j + 2 \sum_{j=i+1}^K m_j + 2z]}{[1 + x + \sum_{j=1}^{i-1} m_j][1 + x + \sum_{j=1}^i m_j]}. \end{aligned}$$

Proof. We prove Lemma by induction on K . Let $f^K(z)$ be the left hand side of Eqn.(48). We have $f^1(z) = [1 + x]^{-1}[2 + 2z + 2m_1]$. We assume that Lemma holds true for $f^{K-1}(z)$.

$$\begin{aligned} f^K(z) &= f^{K-1}(z + 2m_K) - \frac{[2z + 2m_K + 2]}{[x + 1 + \sum_{i=1}^{K-1} m_i]} I_{K-1} + J_K I_K + \frac{[2z + 2]}{[x + 1 + \sum_{i=1}^K m_i]} I_K \\ &= [1 + x]^{-1} \left[2 + 2z + 2 \sum_{i=1}^K m_i \right]. \end{aligned}$$

This completes the proof. \square

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