The Kazhdan-Lusztig polynomial of a matroid

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Abstract. We associate to every matroid M a polynomial with integer coefficients, which we call the Kazhdan-Lusztig polynomial of M, in analogy with Kazhdan-Lusztig polynomials in representation theory. We conjecture that the coefficients are always non-negative, and we prove this conjecture for representable matroids by interpreting our polynomials as intersection cohomology Poincaré polynomials. We also introduce a q-deformation of the Möbius algebra of M, and use our polynomials to define a special basis for this deformation, analogous to the canonical basis of the Hecke algebra. We conjecture that the structure coefficients for multiplication in this special basis are non-negative, and we verify this conjecture in numerous examples.

1 Introduction

Our goal is to develop Kazhdan-Lusztig theory for matroids in analogy with the well-known theory for Coxeter groups. In order to make this analogy clear, we begin by summarizing the most relevant features of the usual theory.

Given a Coxeter group W along with a pair of elements $y, w \in W$, Kazhdan and Lusztig [KL79] associated a polynomial $P_{x,y}(t) \in \mathbb{Z}[t]$, which is non-zero if and only if $x \leq y$ in the Bruhat order. This polynomial has a number of different interpretations:

- **Combinatorics:** There is a purely combinatorial recursive definition of $P_{x,y}(t)$ in terms of more elementary polynomials, called R-polynomials. See [Lus83, Proposition 2], as well as [BB05, §5.5] for a more recent account.

- **Geometry:** If W is a finite Weyl group, then $P_{x,y}(t)$ may be interpreted as the Poincaré polynomial of a stalk of the intersection cohomology sheaf on a Schubert variety in the associated flag variety [KL80]. The Schubert variety is determined by y, and the point at

which one takes the stalk is determined by x. This proves that $P_{x,y}(t)$ has non-negative coefficients when W is a finite Weyl group. The non-negativity of the coefficients of $P_{x,y}(t)$ for arbitrary Coxeter groups was conjectured in [KL79], but was only recently proved by Williamson and the first author [EW14, 1.2(1)].

- **Algebra:** The polynomials $P_{x,y}(t)$ are the entries of the matrix relating the Kazhdan-Lusztig basis (or canonical basis) to the standard basis of the Hecke algebra of W, a q-deformation of the group algebra $\mathbb{C}[W]$. When W is a finite Weyl group, Kazhdan and Lusztig showed that the structure coefficients for multiplication in the Kazhdan-Lusztig basis are polynomials with non-negative coefficients. For general Coxeter groups, this is proved in [EW14, 1.2(2)].

In our analogy, the Coxeter group W is replaced by a matroid M, and the elements $x, y \in W$ are replaced by flats F and G of M. We only define a single polynomial $P_M(t)$ for each matroid, but one may associate to a pair $F \leq G$ the polynomial $P_{M_G^F}(t)$, where M_G^F is the matroid whose lattice of flats is isomorphic to the interval[3] $[F, G]$. The role of the R-polynomial is played by the characteristic polynomial of the matroid. The analogue of being a finite Weyl group is being a representable matroid; that is, the matroid $M_{\mathcal{A}}$ associated to a collection $\mathcal{A}$ of vectors in a vector space. The analogue of a Schubert variety is the reciprocal plane $X_{\mathcal{A}}$, also known as the spectrum of the Orlik-Terao algebra of $\mathcal{A}$. The analogue of the group algebra $\mathbb{C}[W]$ is the Möbius algebra $E(M)$; we introduce a q-defomation $E_q(M)$ of this algebra which plays the role of the Hecke algebra.

All of these analogies may be summarized as follows:

- **Combinatorics:** We give a recursive definition of the polynomial $P_M(t)$ in terms of the characteristic polynomial of a matroid (Theorem 2.2), and we conjecture that the coefficients are non-negative (Conjecture 2.3).

- **Geometry:** If M is representable over a finite field, we show that $P_M(t)$ is equal to the ℓ-adic étale intersection cohomology Poincaré polynomial of the reciprocal plane[4] (Theorem 3.10). Any matroid that is representable over some field is representable over a finite field, thus we obtain a proof of Conjecture 2.3 for all representable matroids (Corollary 3.11).

- **Algebra:** We use the polynomials $P_{M_G^F}(t)$ to define the Kazhdan-Lusztig basis of the q-deformed Möbius algebra $E_q(M)$. We conjecture that the structure constants for multiplication in this basis are polynomials in q with non-negative coefficients (Conjecture 4.2), and we verify this conjecture in a number of cases.

Remark 1.1. Despite these parallels, the behavior of the polynomials for matroids differs drastically from the behavior of ordinary Kazhdan-Lusztig polynomials for Coxeter groups. In particular,

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[3]We note that this shortcut has no analogue in ordinary Kazhdan-Lusztig theory, since the interval $[x, y]$ is not in general isomorphic to the Bruhat poset of some other Coxeter group. Furthermore, it is still an open question whether or not $P_{x,y}(t)$ is determined by the isomorphism type of the interval $[x, y]$.

[4]The reciprocal plane is a cone, so we could equivalently say that it is the Poincaré polynomial of the stalk of the intersection cohomology sheaf at the cone point.

one does not recover the classical Kazhdan-Lusztig polynomials for the Coxeter group S_n from the braid matroid. Polo [Pol99] has shown that any polynomial with non-negative coefficients and constant term 1 appears as a Kazhdan-Lusztig polynomial associated to some symmetric group, while Kazhdan-Lusztig polynomials of matroids are far more restrictive (see Proposition 2.14).

The original work of Kazhdan and Lusztig begins with an algebraic question (How can we find a basis for the Hecke algebra with certain nice properties?), which led them to both the combinatorics and the geometry. In our work, we began with a geometric question (What is the intersection cohomology of the reciprocal plane?), which led us naturally to the combinatorics. The algebraic facet of our work is somewhat more speculative and *ad hoc*, representing an attempt to trace backward the route of Kazhdan and Lusztig.

There is no known convolution product in the geometry of the reciprocal plane which would account for the q-deformed Möbius algebra $E_q(M)$, as the convolution product on flag varieties produces the Hecke algebra. Unlike in the Coxeter setting, the Kazhdan-Lusztig basis of $E_q(M)$ currently has no intrinsic definition, and the theory of this basis is far less satisfactory. For example, the basis is cellular, but in a trivial way: the cells are all one-dimensional. The identity is not an element of the basis.

Remark 1.2. When W is a finite Weyl group, yet another important interpretation of $P_{x,y}(t)$ is that it records the multiplicity space of a simple module in a Verma module in the graded lift of Berstein-Gelfand-Gelfand category $\mathcal{O}$ [BB81, BK81]. The analogous goal for matroids would be a categorification of the q-deformed Möbius algebra $E_q(M)$, or its regular representation. The Möbius algebra $E(M)$ is categorified by a monoidal category of "commuting" quiver representations [Bac79, Theorem 7], but we do not know how to modify this category to produce a categorification of $E_q(M)$.

Having made these caveats, the observed phenomenon of positivity indicates that our Kazhdan-Lusztig basis does hold interest. There are numerous other ways one could have used the Kazhdan-Lusztig polynomials of a matroid as a change of basis matrix, but the corresponding bases do not have positive structure coefficients. As seen in Remark 4.8, positivity is a subtle question, and would fail if all the Kazhdan-Lusztig polynomials were trivial.

We now give a more detailed summary of the contents of the paper. Section 2 (Combinatorics) is dedicated to the combinatorial definition of $P_M(t)$ along with basic properties and examples. In addition to our conjecture that the coefficients of $P_M(t)$ are non-negative (Conjecture 2.3), we also conjecture that they form a log concave sequence (Conjecture 2.5). We explicitly compute the coefficients of t and t^2 in terms of the Whitney numbers of the lattice of flats of M (Propositions 2.12 and 2.16). We prove non-negativity of the linear coefficient (Proposition 2.14), and we give formulas for the quadratic and cubic term (Propositions 2.16 and 2.18), though even in these cases we cannot prove non-negativity (Remark 2.17). We prove a product formula for direct sums (Proposition 2.7), which eliminates the possibility of "cheap" counterexamples to Conjecture 2.3 (Remark 2.8).

We also study in detail the cases of uniform matroids and braid matroids. For uniform matroids, we provide an even more explicit computation of the polynomial up to the cubic term (Corollary 2.20). For the braid matroid M_n corresponding to the complete graph on n vertices, we explain how to compute the coefficients of the Kazhdan-Lusztig polynomial using Stirling numbers. In an appendix, written jointly with Ben Young, we give tables of Kazhdan-Lusztig polynomials of uniform matroids and braid matroids of low rank. The polynomials that we see are unfamiliar; in particular, they do not appear to be related to any known matroid invariants. For both uniform matroids and braid matroids, we express the defining recursion in terms of a generating function identity (Propositions 2.21 and 2.27).

The purpose of Section 3 (Geometry) is to prove that, if $M_{\mathcal{A}}$ is the matroid associated to a vector arrangement $\mathcal{A}$ over a finite field, then the Kazhdan-Lusztig polynomial of $M_{\mathcal{A}}$ coincides with the ℓ-adic étale intersection cohomology Poincaré polynomial of the reciprocal plane $X_{\mathcal{A}}$ (Theorem 3.10). The key ingredient to our proof is Theorem 3.3, which says that, in an étale neighborhood of any point, $X_{\mathcal{A}}$ looks like the product of a vector space with a neighborhood of the cone point in the reciprocal plane of a certain smaller hyperplane arrangement. This improves upon a result of Sanyal, Sturmfels, and Vinzant [SSV13, Theorem 24], who prove the analogous statement on the level of tangent cones.

We conclude Section 3 with a digression in which we discuss a certain question of Li and Yong [LY11]. Given a point on a variety, they compare two polynomials: the local intersection cohomology Poincaré polynomial, and the numerator of the Hilbert series of the tangent cone. They are interested in the case of Schubert varieties, where the first polynomial is a Kazhdan-Lusztig polynomial. We consider the case of reciprocal planes, where the first polynomial is the Kazhdan-Lusztig polynomial of a matroid and the second polynomial is the h-polynomial of the broken circuit complex of the same matroid.

Section 4 (Algebra) deals with the Möbius algebra of a matroid, which has a $\mathbb{Z}$-basis given by flats with multiplication given by the join[5] operation: $\varepsilon_F \cdot \varepsilon_G = \varepsilon_{F \vee G}$. We introduce a q-deformation of this algebra; that is, a commutative, associative, unital $\mathbb{Z}[q, q^{-1}]$-algebra with basis given by flats, such that specializing q to 1 recovers the original Möbius algebra (Proposition 4.1). Using Kazhdan-Lusztig polynomials, we define a new basis whose relationship to the standard basis is analogous to the relationship between the canonical basis and standard basis for the Hecke algebra, and we conjecture that the structure coefficients for multiplication in the new basis lie in $\mathbb{N}[q]$ (Conjecture 4.2). We verify this conjecture for Boolean matroids (Proposition 4.5), for uniform matroids of rank at most 3 (Subsection 4.4), and for braid matroids of rank at most 3 (Subsection 4.5).

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[5]In the literature, one usually sees the multiplication given by meet rather than join. However, these two products are isomorphic; indeed, both are isomorphic to the coordinatewise product [Sol67]. The join product will be more natural for our purposes.

Addendum: After this paper was published, Ben Young discovered counterexamples to Conjecture 4.2. See Section 4.6.

2 Combinatorics

In this section we give a combinatorial definition of the Kazhdan-Lusztig polynomial of a matroid, we compute the first few coefficients, and we study the special cases of uniform matroids and braid matroids.

2.1 Definition

Let M be a matroid with no loops on a finite ground set $\mathcal{I}$. Let $L(M) \subset 2^{\mathcal{I}}$ denote the lattice of flats of M, ordered by inclusion, with minimum element $\emptyset$. Let μ be the Möbius function on $L(M)$, and let

$$\chi_M(t) = \sum_{F \in L(M)} \mu(\emptyset, F) \, t^{\mathrm{rk}\, M - \mathrm{rk}\, F}$$

be the **characteristic polynomial** of M. For any flat $F \in L(M)$, let $\mathcal{I}^F = \mathcal{I} \smallsetminus F$ and $\mathcal{I}_F = F$. Let M^F be the matroid on $\mathcal{I}^F$ consisting of subsets of $\mathcal{I}^F$ whose union with a basis for F are independent in M, and let M_F be the matroid on $\mathcal{I}_F$ consisting of subsets of $\mathcal{I}_F$ which are independent in M. We call the matroid M^F the **restriction** of M at F, and M_F the **localization** of M at F. (This terminology and notation comes from the corresponding constructions for arrangements; see Subsection 3.1.) We have $\mathrm{rk}\, M^F = \mathrm{rk}\, M - \mathrm{rk}\, F$ and $\mathrm{rk}\, M_F = \mathrm{rk}\, F$.

Lemma 2.1. *For any matroid M of positive rank,* $\displaystyle\sum_{F \in L(M)} t^{\mathrm{rk}\, F} \chi_{M_F}(t^{-1}) \chi_{M^F}(t) = 0.$

Proof. We have

$$
\begin{aligned}
\sum_F t^{\mathrm{rk}\, F} \chi_{M_F}(t^{-1}) \chi_{M^F}(t) &= \sum_F t^{\mathrm{rk}\, F} \sum_{E \leq F} \mu(\emptyset, E)\, t^{\mathrm{rk}\, E - \mathrm{rk}\, F} \sum_{G \geq F} \mu(F, G)\, t^{\mathrm{rk}\, M - \mathrm{rk}\, G} \\
&= \sum_{E \leq F \leq G} \mu(\emptyset, E) \mu(F, G)\, t^{\mathrm{rk}\, M + \mathrm{rk}\, E - \mathrm{rk}\, G} \\
&= t^{\mathrm{rk}\, M} \sum_{E \leq G} \mu(\emptyset, E)\, t^{\mathrm{rk}\, E - \mathrm{rk}\, G} \sum_{F \in [E, G]} \mu(F, G).
\end{aligned}
$$

The internal sum is equal to $\delta(E, G)$ [OT92, 2.38], thus our equation simplifies to

$$\sum_F t^{\text{rk } F} \chi_{M_F}(t^{-1}) \chi_{M^F}(t) = t^{\text{rk } M} \sum_F \mu(\emptyset, F).$$

This is 0 unless $\text{rk } M = 0$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

The following is our first main result.

Theorem 2.2. *There is a unique way to assign to each matroid M a polynomial $P_M(t) \in \mathbb{Z}[t]$ such that the following conditions are satisfied:*

1. *If $\text{rk } M = 0$, then $P_M(t) = 1$.*

2. *If $\text{rk } M > 0$, then $\deg P_M(t) < \frac{1}{2} \text{rk } M$.*

3. *For every M, $t^{\text{rk } M} P_M(t^{-1}) = \sum_F \chi_{M_F}(t) P_{M^F}(t)$.*

The polynomial $P_M(t)$ will be called the **Kazhdan-Lusztig polynomial** of M. Our proof of Theorem 2.2 closely follows Lusztig's combinatorial proof of the existence of the usual Kazhdan-Lusztig polynomials [Lus83, Proposition 2], which he attributes to Gabber.

Proof. Let M be a matroid of positive rank. We may assume inductively that $P_{M'}(t)$ has been defined for every matroid M' of rank strictly smaller than $\text{rk } M$; in particular, $P_{M^F}(t)$ has been defined for all $\emptyset \neq F \in L(M)$. Let

$$R_M(t) := \sum_{\emptyset \neq F} \chi_{M_F}(t) P_{M^F}(t);$$

then item 3 says exactly that

$$t^{\text{rk } M} P_M(t^{-1}) - P_M(t) = R_M(t).$$

It is clear that there can be at most one polynomial $P_M(t)$ of degree strictly less than $\frac{1}{2} \text{rk } M$ satisfying this condition. The existence of such a polynomial is equivalent to the statement $t^{\text{rk } M} R_M(t^{-1}) = -R_M(t)$.

We have

$$
\begin{aligned}
t^{\mathrm{rk}\,M} R_M(t^{-1}) \;&=\; t^{\mathrm{rk}\,M} \sum_{\emptyset \neq F} \chi_{M_F}(t^{-1}) P_{M^F}(t^{-1}) \\[2mm]
&=\; \sum_{\emptyset \neq F} t^{\mathrm{rk}\,F} \chi_{M_F}(t^{-1}) t^{\mathrm{rk}\,M^F} P_{M^F}(t^{-1}) \\[2mm]
&=\; \sum_{\emptyset \neq F \leq G} t^{\mathrm{rk}\,F} \chi_{M_F}(t^{-1}) \chi_{M_G^F}(t) P_{M^G}(t) \\[2mm]
&=\; \sum_{\emptyset \neq G} \left(-\chi_{M_G}(t) P_{M^G}(t) + P_{M^G}(t) \sum_{F \leq G} t^{\mathrm{rk}\,F} \chi_{M_F}(t^{-1}) \chi_{M_G^F}(t) \right).
\end{aligned}
$$

Since $\mathrm{rk}\,M_G = \mathrm{rk}\,G \neq 0$, Lemma 2.1 says that the internal sum is zero for all $G \neq \emptyset$, so our equation simplifies to $t^{\mathrm{rk}\,M} R_M(t^{-1}) = -\sum_{\emptyset \neq G} \chi_{M_G}(t) P_{M^G}(t) = -R_M(t)$. $\qquad\qquad\square$

Conjecture 2.3. *For any matroid M, the coefficients of the Kazhdan-Lusztig polynomial $P_M(t)$ are non-negative.*

Remark 2.4. In Section 3, we will prove Conjecture 2.3 for representable matroids by providing a cohomological interpretation of the polynomial $P_M(t)$ (Theorem 3.10).

Based on our computer computations for uniform matroids and braid matroids (see appendix), along with Proposition 2.14 and Remark 2.15, we make the following additional conjecture. A sequence $e_0, \ldots, e_r$ is called **log concave** if, for all $1 < i < r$, $e_{i-1} e_{i+1} \leq e_i^2$. It is said to have **no internal zeros** if the set $\{i \mid e_i \neq 0\}$ is an interval. Note that a log concave sequence of non-negative integers with no internal zeroes is always unimodal.

Conjecture 2.5. *For any matroid M, the coefficients of $P_M(t)$ form a log concave sequence with no internal zeroes.*

Remark 2.6. If M is representable, then Huh and Katz proved that the absolute values of the coefficients of $\chi_M(q)$ form a log concave sequence with no internal zeroes [HK12, 6.2], solving a conjure of Read for graphical matroids and the representable case of a conjecture of Rota-Heron-Walsh for arbitrary matroids.

2.2 Direct sums

The following proposition says that the Kazhdan-Lusztig polynomial is multiplicative on direct sums.

Proposition 2.7. *For any matroids M_1 and M_2, $P_{M_1 \oplus M_2}(t) = P_{M_1}(t) P_{M_2}(t)$.*

Proof. We proceed by induction. The statement is clear when $\operatorname{rk} M_1 = 0$ or $\operatorname{rk} M_2 = 0$. Now assume that the statement holds for M_1' and M_2' whenever $\operatorname{rk} M_1' \leq \operatorname{rk} M_1$ and $\operatorname{rk} M_2' \leq M_2$ with at least one of the two inequalities being strict.

We have $L(M_1 \oplus M_2) = L(M_1) \times L(M_2)$. The localization of $M_1 \oplus M_2$ at (F_1, F_2) is isomorphic to $(M_1)_{F_1} \oplus (M_2)_{F_2}$, and the restriction at (F_1, F_2) is isomorphic to $(M_1)^{F_1} \oplus (M_2)^{F_2}$. The characteristic polynomial of $(M_1)_{F_1} \oplus (M_2)_{F_2}$ is the product of the two characteristic polynomials, and our inductive hypothesis tells us that the Kazhdan-Lusztig polynomial of $(M_1)^{F_1} \oplus (M_2)^{F_2}$ is the product of the two Kazhdan-Lusztig polynomials, provided that $F_1 \neq \emptyset$ or $F_2 \neq \emptyset$. These two observations, along with the recursive definition of the Kazhdan-Lusztig polynomial, combine to tell us that

$$t^{\operatorname{rk} M_1 + \operatorname{rk} M_2} P_{M_1 \oplus M_2}(t^{-1}) - t^{\operatorname{rk} M_1} P_{M_1}(t^{-1}) \cdot t^{\operatorname{rk} M_2} P_{M_2}(t^{-1}) = -P_{M_1}(t) P_{M_2}(t) + P_{M_1 \oplus M_2}(t).$$

The left-hand side is concentrated in degree strictly greater than $\frac{1}{2} \operatorname{rk} M_1 + \frac{1}{2} \operatorname{rk} M_2$, while the right-hand side is concentrated in degree strictly less than $\frac{1}{2} \operatorname{rk} M_1 + \frac{1}{2} \operatorname{rk} M_2$. This tells us that both sides must vanish, and the proposition is proved. □

Remark 2.8. Proposition 2.7 rules out many potential counterexamples to Conjecture 2.3. That is, one cheap way to construct a non-representable matroid is to fix a prime p and let $M = M_1 \oplus M_2$, where M_1 is representable only in characteristic p and M_2 is representable only in characteristic $\neq p$. Proposition 2.7 will tells that the Kazhdan-Lusztig polynomial of $M_1 \oplus M_2$ is equal to the product of the Kazhdan-Lusztig polynomials of M_1 and M_2, each of which has non-negative coefficients because M_1 and M_2 are both representable.

Remark 2.9. Proposition 2.7 is also consistent with Conjecture 2.5, since the convolution of two non-negative log concave sequences with no internal zeroes is again log concave with no internal zeroes [Koo06, Theorem 1]. Note that the corresponding statement would be false without the no internal zeroes hypothesis.[6]

Corollary 2.10. *If M is the Boolean matroid on any finite set, then $P_M(t) = 1$.*

Proof. The Boolean matroid on a set of cardinality n is isomorphic to the direct sum of n copies of the unique rank 1 matroid on a set of cardinality 1. □

2.3 The first few coefficients

In this subsection we interpret the first few coefficients of $P_M(t)$ in terms of the doubly indexed Whitney numbers of M, introduced by Green and Zaslavsky [GZ83].

Proposition 2.11. *The constant term of $P_M(t)$ is equal to 1.*

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[6]We thank June Huh for pointing out this fact.

Proof. We proceed by induction on the rank of M. If $\operatorname{rk} M = 0$, then $P_M(t) = 1$ by definition. If $\operatorname{rk} M > 0$, we consider the recursion

$$t^{\operatorname{rk} M} P_M(t^{-1}) = \sum_F \chi_{M_F}(t) P_{M^F}(t).$$

Since $\deg P_M(t) < \operatorname{rk} M$, the left-hand side has no constant term, therefore we have

$$0 = \sum_F \chi_{M_F}(0) P_{M^F}(0).$$

By our inductive hypothesis, we may assume that $P_{M^F}(0) = 1$ for all nonempty flats F, and we therefore need to show that

$$0 = \sum_F \chi_{M_F}(0).$$

This follows from the fact that $\chi_{M_F}(0) = \mu(\emptyset, F)$ and $\operatorname{rk} M > 0$. $\square$

For all natural numbers i and j, let

$$w_{i,j} := \sum_{\operatorname{rk} E = i, \operatorname{rk} F = j} \mu(E, F) \qquad \text{and} \qquad W_{i,j} := \sum_{\operatorname{rk} E = i, \operatorname{rk} F = j} \zeta(E, F),$$

where $\zeta(E, F) = 1$ if $E \leq F$ and 0 otherwise. These are called **doubly indexed Whitney numbers** of the first and second kind, respectively. In the various propositions that follow, we let $d = \operatorname{rk} M$.

Proposition 2.12. *The coefficient of t in $P_M(t)$ is equal to $W_{0,d-1} - W_{0,1}$.*

Proof. We consider the defining recursion

$$t^{\operatorname{rk} M} P_M(t^{-1}) = \sum_F \chi_{M_F}(t) P_{M^F}(t)$$

and compute the coefficient of $t^{\operatorname{rk} M - 1}$ on the right-hand side. The flat $F = \mathcal{I}$ contributes $-W_{0,1}$, and each of the $W_{0,d-1}$ flats of rank $d - 1$ contributes 1. $\square$

Remark 2.13. If M is the matroid associated to a hyperplane arrangement, Proposition 2.12 says that the coefficient of t in $P_M(t)$ is equal to the number of lines in the lattice of flats minus the number of hyperplanes.

Proposition 2.14. *The coefficient of t in $P_M(t)$ is always non-negative, and the following are equivalent:*

(i) $P_M(t) = 1$

(ii) the coefficient of t is zero

(iii) the lattice $L(M)$ is modular.

Proof. Non-negativity of the linear term follows from Proposition 2.12 along with the hyperplane theorem [Aig87, 8.5.1 & §8.5]. The hyperplane theorem also states that the linear term is zero if and only if $L(M)$ is modular. The first item obviously implies the second, so it remains only to show that $P_M(t) = 1$ whenever $L(M)$ is modular.

We proceed by induction on $d = \operatorname{rk} M$. The base case is trivial. Assume the statement holds for all matroids of rank smaller than d, and that $L(M)$ is modular. In particular, for any flat F, $L(M^F)$ is also modular, so we may assume that $P_{M^F}(t) = 1$ for all $F \neq \emptyset$. Thus the defining recursion says that

$$t^d P_M(t^{-1}) - P_M(t) = \sum_{F \neq \emptyset} \chi_{M_F}(t),$$

and we need only show that the right-hand side is equal to $t^d - 1$. Equivalently, we need to show that $\sum_F \chi_{M_F}(t)$ is equal to t^d.

Since $L(M)$ is modular, there exists another matroid M' such that $L(M')$ is dual to $L(M)$; that is, there exists an order-reversing and rank-reversing bijection between $L(M)$ and $L(M')$. (This is simply the statement that the dual of $L(M)$ is again a geometric lattice, which follows from modularity.) This implies that

$$\sum_F \chi_{M_F}(t) = \sum_{F,G} \mu_M(F, G) t^{d - \operatorname{rk}_M G} = \sum_{F,G} \mu_{M'}(G, F) t^{\operatorname{rk}_{M'} G}.$$

By Möbius inversion, this sum vanishes in all degrees less than d, and the coefficient of t^d is equal to $\mu_{M'}(\mathcal{I}, \mathcal{I}) = 1$. $\square$

Remark 2.15. Note that the implication of (i) by (ii) in Proposition 2.14 provides evidence for the lack of internal zeroes in the sequence of coefficients of $P_M(t)$ (Conjecture 2.5).

Proposition 2.16. *The coefficient of t^2 in $P_M(t)$ is equal to*

$$w_{0,2} - W_{1,d-1} + W_{0,d-2} - W_{d-3,d-2} + W_{d-3,d-1}.$$

Proof. We again consider the defining recursion, and this time we compute the coefficient of $t^{\operatorname{rk} M - 2}$ on the right-hand side. The flat $F = \mathcal{I}$ contributes $w_{0,2}$, each flat F of rank $d - 1$ contributes $-W_{0,1}(M_F)$, and $-\sum_{\operatorname{rk} F = d-1} W_{0,1}(M_F) = -W_{1,d-1}$. Each of the $W_{0,d-2}$ flats of rank $d - 2$ contributes 1. Each flat F of rank $d - 3$ contributes the linear term of $P_{M^F}(t)$, which is equal to $W_{0,2}(M^F) - W_{0,1}(M^F)$ by Proposition 2.12. Summing over all such flats, we obtain the final two terms $W_{d-3,d-1} - W_{d-3,d-2}$. $\square$

Remark 2.17. We have $w_{0,2} = W_{1,2} - W_{0,2}$,

$$W_{1,2} - W_{1,d-1} = \sum_{\operatorname{rk} F = 1} \left(W_{0,1}(M^F) - W_{0,d-2}(M^F) \right),$$

and

$$W_{d-3,d-1} - W_{d-3,d-2} = \sum_{\mathrm{rk}\,G=d-3} \left(W_{0,2}(M^G) - W_{0,1}(M^G) \right),$$

thus the coefficient of t^2 is equal to

$$\sum_{\mathrm{rk}\,F=1} \left(W_{0,1}(M^F) - W_{0,d-2}(M^F) \right) \;+\; \sum_{\mathrm{rk}\,G=d-3} \left(W_{0,2}(M^G) - W_{0,1}(M^G) \right) \;+\; \left(W_{0,d-2} - W_{0,2} \right).$$

The hyperplane theorem says that each of the summands in the first sum is non-positive and each of the summands in the second sum is non-negative. The statement that $W_{0,d-2} - W_{0,2}$ is non-negative as long as $d \geq 4$ is a long standing conjecture in matroid theory, called the "top-heavy conjecture" [DW75], [Kun86, 2.5.2]. (Note that if $d \leq 4$, then the coefficient of t^2 in $P_M(t)$ is automatically zero.) Thus a comparison (in either direction) between the absolute values of the two sums would yield a logical implication (in the corresponding direction) between Conjecture 2.3 (for quadratic terms) and the top-heavy conjecture.

The next proposition, whose proof we omit, indicates the difficulty with finding a closed formula for these coefficients. On the other hand, [Wak, 5.5] presents a formula for all coefficients, albeit recursively defined, in the same vein as Proposition 2.16.

Proposition 2.18. *The coefficient of t^3 in $P_M(t)$ is equal to*

$$w_{0,3} - W_{d-4,d-3} + W_{d-4,d-1} - W_{1,d-2} + W_{0,d-3}$$

$$+ \quad \sum_{\mathrm{rk}\,F=d-1} w_{0,2}(M_F) - \sum_{\mathrm{rk}\,F=d-3} W_{0,1}(M_F)\Big[W_{0,2}(M^F) - W_{0,1}(M^F) \Big]$$

$$+ \quad \sum_{\mathrm{rk}\,F=d-5} \Big[w_{0,2}(M^F) - W_{1,4}(M^F) + W_{0,3}(M^F) + W_{2,4}(M^F) - W_{2,3}(M^F) \Big].$$

2.4   Uniform matroids

Given non-negative integers d and m, let $M_{m,d}$ be the uniform matroid of rank d on a set of cardinality $m + d$, and write

$$P_{M_{m,d}}(t) = P_{m,d}(t) = \sum_i c^i_{m,d}\, t^i.$$

The values of $P_{m,d}(t)$ for small m and d appear in the appendix.

For any flat F of rank strictly less than d, the localization $(M_{m,d})_F$ is a Boolean matroid, and the restriction $M^F_{m,d}$ is isomorphic to $M_{m,d-\mathrm{rk}\,F}$, thus our recursive definition will give us a recursive relation among the coefficients $c^i_{m,d}$ for a single fixed m. Specifically, we have the following result (the factor before $c^j_{m,k}$ is a trinomial coefficient).

Proposition 2.19. *For any m, d, and i, we have*

$$c_{m,d}^i = (-1)^i \binom{m+d}{i} + \sum_{j=0}^{i-1} \sum_{k=2j+1}^{i+j} (-1)^{i+j+k} \binom{m+d}{m+k, i+j-k, d-i-j} c_{m,k}^j.$$

We can use Proposition 2.19 obtain explicit formulas for the first few coefficients. In general, the formula for $c_{m,d}^i$ will be a signed sum of $(i+1)$-nomial coefficients, each with $m+d$ on top. We omit the proof of Corollary 2.20 because it is a straightforward application of the proposition.

Corollary 2.20. *We have*

$$c_{m,d}^0 \;=\; 1$$

$$c_{m,d}^1 \;=\; \binom{m+d}{m+1} - \binom{m+d}{1}$$

$$c_{m,d}^2 \;=\; \binom{m+d}{m+1,d-3,2} - \binom{m+d}{m+1,d-2,1} + \binom{m+d}{m+2,d-2,0} - \binom{m+d}{m+2,d-3,1} + \binom{m+d}{2}$$

$$
\begin{aligned}
c_{m,d}^3 \;=\;\; & \binom{m+d}{m+1,d-3,2,0} - \binom{m+d}{m+1,d-4,2,1} + \binom{m+d}{m+1,d-4,3,0} - \binom{m+d}{m+1,d-5,3,1} + \binom{m+d}{m+1,d-5,2,2} \\
& - \binom{m+d}{m+2,d-3,1,0} + \binom{m+d}{m+2,d-4,1,1} - \binom{m+d}{m+2,d-5,2,1} + \binom{m+d}{m+2,d-5,3,0} + \binom{m+d}{m+3,d-3,0,0} \\
& - \binom{m+d}{m+3,d-4,1,0} + \binom{m+d}{m+3,d-5,2,0} - \binom{m+d}{3}.
\end{aligned}
$$

We can also express our recursion in terms of a generating function identity. Let

$$\Phi_m(t, u) = \sum_{d=1}^{\infty} P_{m,d}(t) u^d.$$

Proposition 2.21. *We have*

$$\Phi_m(t^{-1}, tu) = \frac{tu - u}{(1 - tu + u)(1 + u)^m} + \frac{1}{(1 - tu + u)^{m+1}} \Phi_m\left(t, \frac{u}{1 - tu + u}\right).$$

Proof. Our defining recursion tells us that

$$
\begin{aligned}
\Phi_m(t^{-1}, tu) \;=\;\; & \sum_{d=1}^{\infty} P_{m,d}(t^{-1}) t^d u^d \\
=\;\; & \sum_{d=1}^{\infty} \left[\sum_{i=0}^{d} (-1)^i \binom{m+d}{i} (t^{d-i} - 1) + \sum_{k=1}^{d} \binom{m+d}{d-k} (t-1)^{d-k} P_{m,k}(t) \right] u^d.
\end{aligned}
$$

If we introduce new dummy indices $e = d - i$ and $f = d - k$, we may rewrite this equation as

$$\Phi_m(t^{-1}, tu) = \sum_{e=0}^{\infty} (t^e - 1) u^e \sum_{i=0}^{\infty} \binom{m+e+i}{i} (-u)^i + \sum_{k=1}^{\infty} P_{m,k}(t) u^k \sum_{f=0}^{\infty} \binom{m+k+f}{f} (ut - u)^f.$$

Next, we recall that
$$\sum_{\ell=0}^{\infty} \binom{r+\ell}{\ell} x^\ell = \frac{1}{(1-x)^{r+1}}.$$

We will use this formula with $r = m+e$ and $x = -u$, and then again with $r = m+k$ and $x = tu-u$, to get

$$
\begin{aligned}
\Phi_m(t^{-1}, tu) &= \sum_{e=0}^{\infty} \frac{(t^e-1)u^e}{(1+u)^{m+e+1}} + \sum_{k=1}^{\infty} \frac{P_{m,k}(t)u^k}{(1-tu+u)^{m+k+1}} \\
&= \frac{1}{(1+u)^{m+1}} \sum_{e=0}^{\infty} \frac{(t^e-1)u^e}{(1+u)^e} + \frac{1}{(1-tu+u)^{m+1}} \sum_{k=1}^{\infty} P_{m,k}(t) \left(\frac{u}{1-tu+u} \right)^k \\
&= \frac{tu-u}{(1-tu+u)(1+u)^m} + \frac{1}{(1-tu+u)^{m+1}} \Phi_m \left(t, \frac{u}{1-tu+u} \right).
\end{aligned}
$$

This completes the proof. □

Remark 2.22. A general formula for $c^i_{m,d}$ can be obtained from [GPY, 3.1] and the ensuing remarks.

2.5 Braid matroids

Let M_n be the braid matroid of rank $n-1$; this is the matroid associated with the complete graph on n vertices, or with the braid arrangement (Example 3.2). The lattice $L(M_n)$ is isomorphic to the lattice of set-theoretic partitions of the set $[n]$. Let $P_n(t) = P_{M_n}(t)$. Values of $P_n(t)$ for $n \le 20$ appear in the appendix.

For any partition λ of the number n, let

$$m(\lambda) := \frac{n!}{\prod_{i=1}^{\ell(\lambda)} \lambda_i! \cdot \prod_{j=1}^{\lambda_1} (\lambda_j^t - \lambda_{j+1}^t)!}$$

be the number of flats of type λ, where λ^t denotes the transpose partition and $\ell(\lambda)$ is the number of parts of λ. For such a flat F, the localization $(M_n)_F$ is isomorphic to $M_{\lambda_1} \oplus \cdots \oplus M_{\lambda_{\ell(\lambda)}}$, and has characteristic polynomial

$$\chi(t) = \prod_{i=1}^{\ell(\lambda)} (t-1) \cdots (t-\lambda_i+1) = \prod_{j=1}^{\lambda_1-1} (t-j)^{\lambda_{j+1}^t}.$$

The restriction M_n^F is isomorphic (after simplification) to $M_{\ell(\lambda)}$.

The Whitney numbers of the M_n can be interpreted in terms of Stirling numbers of the first and second kind, respectively. By definition,

$$s(n,k) := w_{0,n-k} \qquad \text{and} \qquad S(n,k) := W_{0,n-k}.$$

Lemma 2.23. *For all $i \leq j$, $W_{i,j} = S(n, n-i)S(n-i, n-j)$.*

Proof. A flat of rank i corresponds to a partition of $[n]$ into $n-i$ blocks, and there are $W_{0,i} = S(n, n-i)$ such flats. For each such flat, a flat of rank j lying above it corresponds to a partition of the set of blocks into $n-j$ blocks, and there are $S(n-i, n-j)$ such flats. □

Corollary 2.24. *The coefficient of t in $P_n(t)$ is equal to $S(n, 2) - S(n, n-1)$, and the coefficient of t^2 is equal to $s(n, n-2) - S(n, n-1)S(n-1, 2) + S(n, 3) + S(n, 4)$.*

Proof. This follows from Proposition 2.12, Proposition 2.16, and Lemma 2.23, along with the observation that $w_{0,2} = s(n, n-2)$. □

Lemma 2.25. *For any matroids M and M',*

$$W_{i,j}(M \oplus M') = \sum_{k,\ell} W_{k,\ell}(M) W_{i-k, j-\ell}(M').$$

Proof. This follows from the fact that $L(M \oplus M') = L(M) \times L(M')$ as ranked posets. □

The following proposition, which may be derived from Proposition 2.18, Lemma 2.23, and Lemma 2.25, expresses the cubic term of $P_n(t)$ in terms of Stirling numbers and binomial coefficients. More generally, since any restriction of a braid matroid is another braid matroid and any localization of a braid matroid is a direct sum of braid matroids, it would be possible to express every coefficient of $P_n(t)$ in terms of Stirling numbers and binomial coefficients.

Proposition 2.26. *The coefficient of t^3 in $P_n(t)$ is equal to*

$$s(n, n-3) \quad + \sum_{\lambda \vdash n, \ell(\lambda)=4} m(\lambda)\Big[S(\lambda_1, \lambda_1 - 1)S(\lambda_1 - 1, \lambda_1 - 2) + S(\lambda_2, \lambda_2 - 1)S(\lambda_1, \lambda_1 - 1)$$

$$+ S(\lambda_2, \lambda_2 - 1)S(\lambda_2 - 1, \lambda_2 - 2) - S(\lambda_1, \lambda_1 - 2) - S(\lambda_2, \lambda_2 - 2)\Big]$$

$$- \quad S(n, n-1)S(n-1, 3) + S(n, 4)$$

$$+ \quad \sum_{\lambda \vdash n, \ell(\lambda)=4} m(\lambda)\left(\binom{\lambda_1}{2} + \binom{\lambda_2}{2} + \binom{\lambda_3}{2} + \binom{\lambda_4}{2}\right)$$

$$+ \quad 5S(n, 5) + 15S(n, 6).$$

Finally, we express the recursion for the polynomials $P_n(t)$ as a generating function identity, just as we did for uniform matroids. Let

$$\Psi(t, u) = \sum_{n=1}^{\infty} P_n(t) u^{n-1}.$$

For any partition ν (of any number), let $\tilde{\nu}$ be the partition of $|\nu| + \ell(\nu)$ obtained by adding 1 to each of the parts of ν.

Proposition 2.27. *We have*

$$\Psi(t^{-1}, tu) = \sum_{\nu} m(\tilde{\nu}) u^{|\tilde{\nu}|-1} \prod_{j=1}^{\nu_1} (t-j)^{\nu_j^t} \cdot \frac{\partial_u^{|\tilde{\nu}|}}{|\tilde{\nu}|!} \left(u^{|\nu|+1} \Psi(t, u) \right),$$

where the sum is over all partitions ν of any size.

Proof. Our defining recursion tells us that

$$\Psi(t^{-1}, tu) = \sum_{n=1}^{\infty} P_n(t^{-1}) t^{n-1} u^{n-1} = \sum_{n=1}^{\infty} \left[\sum_{\lambda \vdash n} m(\lambda) P_{\ell(\lambda)}(t) \prod_{j=1}^{\lambda_1 - 1} (t-j)^{\lambda_{j+1}^t} \right] u^{n-1}$$

$$= \sum_{\lambda} m(\lambda) P_{\ell(\lambda)}(t) u^{|\lambda|-1} \prod_{j=1}^{\lambda_1 - 1} (t-j)^{\lambda_{j+1}^t}.$$

(We adopt the convention that $P_0(t) = 0$ so that the empty partition contributes nothing to the sum.) For any partition ν, let $\tilde{\nu}_k$ be the partition obtained by adding k new parts of size 1 to $\tilde{\nu}$. We will replace the sum over λ with a sum over ν and k, with $\lambda = \tilde{\nu}_k$. Note that we have

$$m(\lambda) = \binom{|\tilde{\nu}_k|}{|\tilde{\nu}|} m(\tilde{\nu}), \qquad \ell(\lambda) = \ell(\nu) + k, \qquad \lambda_1 - 1 = \nu_1, \qquad \text{and} \qquad \lambda_{j+1}^t = \nu_j^t,$$

thus we can rewrite our equation as

$$\Psi(t^{-1}, tu) = \sum_{\nu} \sum_{k=0}^{\infty} \binom{|\tilde{\nu}_k|}{|\tilde{\nu}|} m(\tilde{\nu}) P_{\ell(\nu)+k}(t) u^{|\tilde{\nu}_k|-1} \prod_{j=1}^{\nu_1} (t-j)^{\nu_j^t}$$

$$= \sum_{\nu} m(\tilde{\nu}) u^{|\tilde{\nu}|} \prod_{j=1}^{\nu_1} (t-j)^{\nu_j^t} \cdot \sum_{k=0}^{\infty} \binom{|\tilde{\nu}_k|}{|\tilde{\nu}|} P_{\ell(\nu)+k}(t) u^{\ell(\nu)+k-1}.$$

Next, we observe that

$$\binom{|\tilde{\nu}_k|}{|\tilde{\nu}|} u^{\ell(\nu)+k-1} = u^{\ell(\nu)-1} \frac{\partial_u^{|\tilde{\nu}|}}{|\tilde{\nu}|!} u^{|\tilde{\nu}_k|},$$

so

$$\Psi(t^{-1}, tu) = \sum_{\nu} m(\tilde{\nu}) u^{|\tilde{\nu}|} \prod_{j=1}^{\nu_1} (t-j)^{\nu_j^t} \cdot \sum_{k=0}^{\infty} u^{\ell(\nu)-1} \frac{\partial_u^{|\tilde{\nu}|}}{|\tilde{\nu}|!} u^{|\tilde{\nu}_k|} P_{\ell(\nu)+k}(t)$$

$$= \sum_{\nu} m(\tilde{\nu}) u^{|\tilde{\nu}|-1} \prod_{j=1}^{\nu_1} (t-j)^{\nu_j^t} \cdot \frac{\partial_u^{|\tilde{\nu}|}}{|\tilde{\nu}|!} \left(u^{|\nu|+1} \sum_{k=0}^{\infty} u^{\ell(\nu)+k-1} P_{\ell(\nu)+k}(t) \right)$$

$$= \sum_{\nu} m(\tilde{\nu}) u^{|\tilde{\nu}|-1} \prod_{j=1}^{\nu_1} (t-j)^{\nu_j^t} \cdot \frac{\partial_u^{|\tilde{\nu}|}}{|\tilde{\nu}|!} \left(u^{|\nu|+1} \Psi(t, u) \right).$$

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This completes the proof. □

3   Geometry

In this section we give a cohomological interpretation of the polynomial $P_M(t)$ whenever the matroid M is representable; this interpretation is analogous to the interpretation of Kazhdan-Lusztig polynomials associated to Weyl groups as local intersection cohomology groups of Schubert varieties [KL80]. In particular, we prove Conjecture 2.3 for representable matroids.

3.1   The reciprocal plane

Let k be a field. An **arrangement** $\mathcal{A}$ over k is a triple $(\mathcal{I}, V, a)$, where $\mathcal{I}$ is a finite set, V is a finite dimensional vector space over k, and a is a map from $\mathcal{I}$ to $V^* \smallsetminus \{0\}$ such that the image of a spans V^*. Let

$$U_\mathcal{A} := \{v \in V \mid \langle a(i), v \rangle \neq 0 \text{ for all } i \in \mathcal{I}\};$$

this variety is called the **complement** of $\mathcal{A}$. We have a natural inclusion of $U_\mathcal{A}$ into $(k^\times)^\mathcal{I}$ whose i^{th} coordinate is given by $a(i)$. Consider the involution of $(k^\times)^\mathcal{I}$ obtained by inverting every coordinate, and let $U_\mathcal{A}^{-1}$ be the image of $U_\mathcal{A}$ under this involution. The **reciprocal plane** $X_\mathcal{A}$ is defined to be the closure of $U_\mathcal{A}^{-1}$ inside of $k^\mathcal{I}$. Its coordinate ring $k[X_\mathcal{A}]$ is isomorphic to the subalgebra of $k(V)$ generated by $\{a(i)^{-1} \mid i \in \mathcal{I}\}$; this ring is called the **Orlik-Terao algebra**.

Consider the polynomial ring $k[u]_\mathcal{I}$ with generators $\{u_i \mid i \in \mathcal{I}\}$. For all $S \subset \mathcal{I}$, let

$$u_S := \prod_{i \in S} u_i.$$

Consider the surjective map $\rho : k[u]_\mathcal{I} \to k[X_\mathcal{A}]$ taking u_i to $a(i)^{-1}$. Suppose that $c \in k^\mathcal{I}$ has the property that $\sum c_i a(i) = 0$; we call such a vector a **dependency** for $\mathcal{A}$. Let $S_c := \{i \in \mathcal{I} \mid c_i \neq 0\}$ be the support of c, and for all $i \in S_c$, let $S_c^i = S_c \smallsetminus \{i\}$. Then we obtain an element

$$f_c(u) := \sum_{i \in S_c} c_i u_{S_c^i} \in \ker(\rho).$$

Indeed, if we take the polynomials f_c associated to vectors c of minimal support, we obtain a universal Gröbner basis for the kernel of ρ [PS06, Theorem 4]. Note that the kernel of ρ is a homogeneous ideal, thus inducing a grading on $k[X_\mathcal{A}]$.

Let $M_\mathcal{A}$ be the matroid with ground set $\mathcal{I}$ consisting of subsets of $\mathcal{I}$ on which a is injective with linearly independent image. We say that $\mathcal{A}$ **represents** $M_\mathcal{A}$ over k. Given a flat F, let $\mathcal{I}^F = \mathcal{I} \smallsetminus F$ and $\mathcal{I}_F = F$. Let

$$V^F := \mathrm{Span}\{a(i) \mid i \in F\}^\perp \subset V \qquad \text{and} \qquad V_F := V/V^F,$$

and consider the natural maps

$$a^F : \mathcal{I}^F \to (V^F)^* \qquad \text{and} \qquad a_F : \mathcal{I}_F \to V_F^*.$$

We define the **restriction** $\mathcal{A}^F := (\mathcal{I}^F, V^F, a^F)$ and the **localization** $\mathcal{A}_F := (\mathcal{I}_F, V_F, a_F)$. Then we have

$$M_{\mathcal{A}^F} = M^F \qquad \text{and} \qquad M_{\mathcal{A}_F} = M_F.$$

For any subset $F \subset \mathcal{A}$, let $X_{\mathcal{A},F}$ be the subvariety of $X_{\mathcal{A}} \subset k^{\mathcal{I}}$ consisting of points whose i^{th} coordinate vanishes if and only if $i \notin F$. The following result is proved in [PS06, Proposition 5].

Proposition 3.1. *The subvariety $X_{\mathcal{A},F} \subset X_{\mathcal{A}}$ is nonempty if and only if F is a flat, in which case it is isomorphic to $U_{\mathcal{A}_F}$, and its closure is isomorphic to $X_{\mathcal{A}_F}$.*

Example 3.2. Let $V = k^n/k_{\Delta}$, and let $\mathcal{A}$ be the **braid arrangement** consisting of all linear functionals of the form $x_i - x_j$, where $i < j$. Flats of $\mathcal{A}$ correspond to set-theoretic partitions of $[n]$; the restrictions $\mathcal{A}^F$ are smaller braid arrangements (with multiplicities), while the localizations $\mathcal{A}_F$ are products of smaller braid arrangements.

The complement $U_{\mathcal{A}}$ is the set of distinct ordered n-tuples of points in k up to simultaneous translation. In the closure of $U_{\mathcal{A}}$, distances between points may go to zero (that is, the points are allowed to collide). When they do, you see the complement of a restriction of $\mathcal{A}$. In the closure of $U_{\mathcal{A}}^{-1}$, distances between points may go to infinity, which means that our set of n points may split into a disjoint union of smaller sets, each of which lives in a "far away" copy of k. When they do, you see the complement of a localization of $\mathcal{A}$.

3.2 Local geometry of the reciprocal plane

For any flat F of $\mathcal{A}$, let $W_{\mathcal{A},F} \subset X_{\mathcal{A}}$ be the open subvariety defined by the nonvanishing of u_i for all $i \in F$. Equivalently, $W_{\mathcal{A},F}$ is the preimage of $X_{\mathcal{A},F}$ along the canonical projection $\pi : X_{\mathcal{A}} \to X_{\mathcal{A}_F}$ given by setting the coordinates in $\mathcal{I} \setminus F$ to zero. The following theorem will be the main ingredient in our proof of Theorem 3.10, which gives a cohomological interpretation of the Kazhdan-Lusztig polynomial of a representable matroid. It says roughly that the reciprocal plane $X_{\mathcal{A}^F}$ associated to the restriction $\mathcal{A}^F$ is an "étale slice" to the stratum $X_{\mathcal{A},F} \subset X_{\mathcal{A}}$.

Theorem 3.3. *Let F be a flat of $\mathcal{A}$ and let $x \in X_{\mathcal{A},F} \subset X_{\mathcal{A}}$. Then there exists an open subscheme $\mathring{W}_{\mathcal{A},F} \subset W_{\mathcal{A},F}$ containing x and a map $\Phi : \mathring{W}_{\mathcal{A},F} \to X_{\mathcal{A}^F} \times X_{\mathcal{A},F}$ such that $\Phi(x) = (0,x)$ and Φ is étale at x.*

Proof. Consider the natural projection from V to V_F, and choose a splitting $\sigma : V_F \to V$ of this projection. Let $\iota : X_{\mathcal{A},F} \to U_{\mathcal{A}_F}$ be the isomorphism mentioned in Proposition 3.1. Concretely, $X_{\mathcal{A},F}$ and $U_{\mathcal{A}_F}$ are both subschemes of $(k^{\times})^F$, and ι is given by inverting all of the coordinates. For all $j \in \mathcal{I} \setminus F$, let

$$b_j := \pi^* \iota^* \sigma^* u_j \in k[W_{\mathcal{A},F}].$$

Here we regard $u_j \in k[u]_{\mathcal{I}}$ as a function on $V \subset k^{\mathcal{I}}$, so that $\sigma^* u_j$ is a function on V_F, and therefore on $U_{\mathcal{A}_F} \subset V_F$. Then $\iota^* \sigma^* u_j$ is a function on $X_{\mathcal{A},F}$, and b_j is its pullback to $W_{\mathcal{A},F}$. By construction of b_j, we have

$$\sum_{i \in F} c_i u_i^{-1} + \sum_{j \in \mathcal{I} \smallsetminus F} c_j b_j = 0 \in k[W_{\mathcal{A},F}] \tag{1}$$

for any dependency c of $\mathcal{A}$.

Let $\mathring{W}_{\mathcal{A},F}$ be the open subscheme of $W_{\mathcal{A},F}$ defined by the nonvanishing of $1 - b_j u_j$ for all $j \in \mathcal{I} \smallsetminus F$. Since u_j vanishes at x for all $j \in \mathcal{I} \smallsetminus F$, we have $x \in \mathring{W}_{\mathcal{A},F}$. Recall that

$$k[X_{\mathcal{A}}] \cong k[u]_{\mathcal{I}} \Big/ \left\langle f_c(u) \mid c \in k^{\mathcal{I}} \text{ a dependency} \right\rangle.$$

For any dependency c, let $\bar{c}$ be the projection of c onto $k^{\mathcal{I}} \smallsetminus F$. Then $\bar{c}$ is a dependency for $\mathcal{A}^F$, and all dependencies for $\mathcal{A}^F$ arise in this way, thus

$$k[X_{\mathcal{A}^F}] \cong k[u]_{\mathcal{I} \smallsetminus F} \Big/ \left\langle f_{\bar{c}}(u) \mid c \in k^{\mathcal{I}} \text{ a dependency} \right\rangle.$$

We define the map

$$\varphi : k[X_{\mathcal{A}^F}] \to k[\mathring{W}_{\mathcal{A},F}]$$

by putting

$$\varphi(u_j) = \frac{u_j}{1 - b_j u_j}$$

for all $j \in \mathcal{I} \smallsetminus F$. To show that this is well-defined, we must show that $f_{\bar{c}}(u)$ maps to zero. Indeed, we have

$$
\begin{aligned}
\varphi(f_{\bar{c}}(u)) &= \sum_{j \in S_{\bar{c}}} c_j \prod_{k \in S_{\bar{c}}^j} \frac{u_k}{1 - b_k u_k} \\
&= \frac{\sum_{j \in S_{\bar{c}}} c_j u_{S_{\bar{c}}^j}(1 - b_j u_j)}{\prod_{k \in S_{\bar{c}}}(1 - b_k u_k)} \\
&= \frac{f_{\bar{c}}(u) - \sum_{j \in \mathcal{I} \smallsetminus F} c_j b_j \, u_{S_{\bar{c}}}}{\prod_{k \in S_{\bar{c}}}(1 - b_k u_k)} \\
&= \frac{f_{\bar{c}}(u) + \sum_{i \in F} c_i u_i^{-1} \, u_{S_{\bar{c}}}}{\prod_{k \in S_{\bar{c}}}(1 - b_k u_k)} \\
&= \frac{f_c(u)}{u_{S_c \cap F} \prod_{k \in S_{\bar{c}}}(1 - b_k u_k)}.
\end{aligned}
$$

Since $f_c(u)$ vanishes on $X_{\mathcal{A}}$, it vanishes on $\mathring{W}_{\mathcal{A},F} \subset X_{\mathcal{A}}$, as well.

Now consider the map $\Phi : \mathring{W}_{\mathcal{A},F} \to X_{\mathcal{A}^F} \times X_{\mathcal{A},F}$ induced by φ on the first factor and given by π on the second factor. Since $\pi(x) = x$ and u_j vanishes on x for all $j \in \mathcal{I} \smallsetminus F$, we have $\Phi(x) = (0, x)$. The statement that Φ is étale at x is equivalent to the statement that Φ induces an isomorphism

on tangent cones. Indeed, the tangent cone of $X_{\mathcal{A}F} \times X_{\mathcal{A},F}$ at $(0,x)$ is isomorphic to $X_{\mathcal{A}F} \times V_F$, and the same is true of the tangent cone of $X_{\mathcal{A}}$ at x [SSV13, Theorem 24]. The fact that Φ induces an isomorphism follows from the fact that, for all $i \in F$, $\pi^*(u_i) = u_i$, and for all $j \in \mathcal{I} \smallsetminus F$, $\varphi(u_j) = u_j + O(u_j^2)$. □

3.3 Intersection cohomology

The purpose of this subsection is to introduce and prove Theorem 3.7. This is a slight reformulation of [PW07, 4.1], which was in turn based on the work in [KL80, §4]. See also [Let13, 3.3.3] for a similar result, formulated in Hodge theoretic terms, with a slightly different set of hypotheses.

Let X be a variety over a finite field $\mathbb{F}_q$. Fix a prime number ℓ not dividing q, and consider the ℓ-adic étale intersection cohomology group $\mathrm{IH}^*(X; \overline{\mathbb{Q}}_\ell) := \mathrm{H}^{*-\dim X}(X; \mathrm{IC}_X)$. Let Fr be the Frobenius automorphism of X, and let Fr^i be the induced automorphism of $\mathrm{IH}^i(X; \overline{\mathbb{Q}}_\ell)$. We say that X is **pure** if the eigenvalues of Fr^i all have absolute value equal to $q^{i/2}$. We say that X is **chaste** if $\mathrm{IH}^i(X; \overline{\mathbb{Q}}_\ell) = 0$ for all odd i and Fr^{2i} acts by multiplication by $q^i \in \mathbb{Z} \subset \overline{\mathbb{Q}}_\ell$ on $\mathrm{IH}^{2i}(X; \overline{\mathbb{Q}}_\ell)$. If X is chaste, then we define

$$P_X(t) := \sum_{i \geq 0} \dim \mathrm{IH}^{2i}(X; \overline{\mathbb{Q}}_\ell)\, t^i,$$

so that $P_X(q^s) = \mathrm{tr}\big((\mathrm{Fr}^*)^s\big)$.

Given a point $x \in X$, we will also be interested in the local intersection cohomology groups $\mathrm{IH}^*_x(X; \overline{\mathbb{Q}}_\ell) := \mathrm{H}^{*-\dim X}(\mathrm{IC}_{X,x})$. We say that X is **pointwise pure** or **pointwise chaste** at x if the analogous properties hold for the local intersection cohomology groups at x. If X is pointwise chaste at x, we define

$$P_{X,x}(t) := \sum_{i \geq 0} \dim \mathrm{IH}^{2i}_x(X; \overline{\mathbb{Q}}_\ell)\, t^i.$$

We say that X is an **affine cone** if it is affine and its coordinate ring $\mathbb{F}_q[X]$ admits a non-negative grading with only scalars in degree zero. The **cone point** of X is the closed point defined by the vanishing of all functions of positive degree. If X is an affine cone with cone point x, then $\mathrm{IH}^*(X; \overline{\mathbb{Q}}_\ell)$ is canonically isomorphic to $\mathrm{IH}^*_x(X; \overline{\mathbb{Q}}_\ell)$ [Spr84, Corollary 1].

Proposition 3.4. *If X is an affine cone of positive dimension, then X is pure and $\mathrm{IH}^i(X; \overline{\mathbb{Q}}_\ell) = 0$ for all $i \geq \dim X$.*

Proof. Let $U \subset X$ be the complement of the cone point, and let $Z = U/\mathbb{G}_m = \mathrm{Proj}\,\mathbb{F}_q[X]$. Let $j : U \to X$ be the inclusion; then $\mathrm{IC}_X = j_{!*}\,\mathrm{IC}_U = \tau^{<0}Rj_*\,\mathrm{IC}_U$, so

$$\mathrm{IH}^i(X; \overline{\mathbb{Q}}_\ell) = \mathrm{H}^{i-\dim X}(X; \mathrm{IC}_X) = \mathrm{H}^{i-\dim X}(X; \tau^{<0}Rj_*\,\mathrm{IC}_U)$$

vanishes when $i \geq \dim X$, and it is equal to $\mathrm{IH}^i(U; \overline{\mathbb{Q}}_\ell)$ when $i < \dim X$.

By the Leray-Serre spectral sequence applied to the $\mathbb{G}_m$-bundle $U \to Z$, combined with the hard Lefschetz theorem for $\mathrm{IH}^*(Z; \overline{\mathbb{Q}}_\ell)$, $\mathrm{IH}^i(U; \overline{\mathbb{Q}}_\ell)$ is isomorphic to the space of primitive vectors in

$\text{IH}^*(Z; \overline{\mathbb{Q}}_\ell)$ for all $i < \dim X$. Thus purity of X follows from purity of the projective variety Z. $\square$

Remark 3.5. Proposition 3.4 is well-known to experts; in particular, a version of the argument above can also be found in [BJ04, 4.2] and [dCM09, 3.1].

The following combinatorial lemma will be needed in the proof of Theorem 3.7; the statement and proof of this lemma were communicated to us by Ben Webster. Let k be a field of characteristic zero. For all positive integers m, n, s, consider the super power sum polynomial

$$p_{m,n,s}(x, y) := x_1^s + \cdots + x_m^s - y_1^s - \cdots - y_n^s,$$

where $x = (x_1, \ldots, x_m) \in k^m$ and $y = (y_1, \ldots, y_n) \in k^n$.

Lemma 3.6. *Suppose that* $p_{m,n,s}(x, y) = p_{m',n',s}(x', y')$ *for all* $s \geq 0$, *that* $x_i \neq 0 \neq y_i$ *for all* i, *and that* $x_i \neq y_j$ *and* $x_i' \neq y_i'$ *for all* i, j. *Then* $m = m'$, $n = n'$, *and* (x, y) *may be taken to* (x', y') *by an element of* $S_m \times S_n$.

Proof. Consider the rational function

$$f(z) := \frac{(1 - x_1 z) \cdots (1 - x_m z)}{(1 - y_1 z) \cdots (1 - y_n z)} \in k(z).$$

We have

$$
\begin{aligned}
f(z) &= \exp \log f(z) \\[2ex]
&= \exp \Big(\log(1 - x_1 z) + \cdots + \log(1 - x_m z) - \log(1 - y_1 z) - \cdots - \log(1 - y_n z) \Big) \\[2ex]
&= \exp \sum_{s=1}^{\infty} \left(\frac{(x_1 z)^s}{s} + \cdots + \frac{(x_m z)^s}{s} - \frac{(y_1 z)^s}{s} - \cdots - \frac{(y_n z)^s}{s} \right) \\[2ex]
&= \exp \sum_{s=1}^{\infty} \frac{p_{m,n,s}(x, y)}{s} z^s.
\end{aligned}
$$

A rational function over a field of characteristic zero is determined by its Taylor expansion at zero, thus the values of the super power sums determine the rational function $f(z)$. By looking at zeros and poles of $f(z)$ with multiplicity, they determine m, n, x (up to permutation), and y (up to permutation). $\square$

We say that a variety Y over $\mathbb{F}_q$ has **polynomial count** if there exists a polynomial $\nu_Y(t) \in \mathbb{Z}[t]$ such that, for all $s \geq 1$, $|Y(\mathbb{F}_{q^s})| = \nu(q^s)$. Let X be an affine cone, and let $X = \bigsqcup X_\beta$ be a stratification such that X_0 is the only zero-dimensional stratum, consisting only of the cone point.

Theorem 3.7. *Suppose that* X_β *has polynomial count for all* β, *and that* $X \smallsetminus X_0$ *is everywhere pointwise chaste with local intersection cohomology Poincaré polynomial* $P_{X,x}(t) = P_\beta(t)$ *for all*

$x \in X_\beta$. *Then X is chaste (and therefore also pointwise chaste at the cone point), and*

$$t^{\dim X} P_X(t^{-1}) = \sum_\beta \nu_{X_\beta}(t) P_\beta(t).$$

Proof. Consider the Frobenius automorphism Fr_c^* of $\mathrm{IH}_c^*(X; \overline{\mathbb{Q}}_\ell)$, the compactly supported intersection cohomology group. By Poincaré duality [KW01, II.7.3], we have

$$q^{s \dim X} \mathrm{tr}\Big((\mathrm{Fr}^*)^{-s} \curvearrowright \mathrm{IH}^{2 \dim X - i}(X; \overline{\mathbb{Q}}_\ell)\Big) = \mathrm{tr}\Big((\mathrm{Fr}_c^*)^s \curvearrowright \mathrm{IH}_c^i(X; \overline{\mathbb{Q}}_\ell)\Big).$$

By the Lefschetz formula [KW01, III.12.1(4)], we have

$$\sum_{i \geq 0} (-1)^i \mathrm{tr}\Big((\mathrm{Fr}_c^*)^s \curvearrowright \mathrm{IH}_c^i(X; \overline{\mathbb{Q}}_\ell)\Big) = \sum_{x \in X(\mathbb{F}_{q^s})} \sum_{i \geq 0} (-1)^i \mathrm{tr}\Big((\mathrm{Fr}_x^*)^s \curvearrowright \mathrm{IH}_x^*(X; \overline{\mathbb{Q}}_\ell)\Big).$$

If $x \in X_\beta(\mathbb{F}_{q^s})$ for some $\beta \neq 0$, then x contributes $P_\beta(q^s)$ to this sum. If x is the cone point, then $\mathrm{IH}_x^*(X; \overline{\mathbb{Q}}_\ell) \cong \mathrm{IH}^*(X; \overline{\mathbb{Q}}_\ell)$, so x contributes $\sum (-1)^i \mathrm{tr}((\mathrm{Fr}^*)^s \curvearrowright \mathrm{IH}^i(X; \overline{\mathbb{Q}}_\ell))$. Thus we have

$$\sum_{i \geq 0} (-1)^i \left(q^{s \dim X} \mathrm{tr}\Big((\mathrm{Fr}^*)^{-s} \curvearrowright \mathrm{IH}^{2 \dim X - i}(X; \overline{\mathbb{Q}}_\ell)\Big) - \mathrm{tr}\Big((\mathrm{Fr}^*)^s \curvearrowright \mathrm{IH}^i(X; \overline{\mathbb{Q}}_\ell)\Big) \right) = \sum_{\beta \neq 0} \nu_{X_\beta}(q^s) P_\beta(q^s).$$

$$(2)$$

Let $r_i = \dim \mathrm{IH}^i(X; \overline{\mathbb{Q}}_\ell)$, and let $\alpha_{1,i}, \cdots, \alpha_{r_i,i}$ be the eigenvalues of $\mathrm{Fr}^* \curvearrowright \mathrm{IH}^i(X; \overline{\mathbb{Q}}_\ell)$, counted with multiplicity. Write

$$\alpha_i := (\alpha_{i,1}, \cdots, \alpha_{i,r_i}) \qquad \text{and} \qquad q^{\dim X}/\alpha_i := (q^{\dim X}/\alpha_{i,1}, \cdots, q^{\dim X}/\alpha_{i,r_i}),$$

so that the left-hand side of Equation (2) is equal to

$$\sum_{i \geq 0} (-1)^i p_{r_i, r_i, s}(q^{\dim X}/\alpha, \alpha).$$

The right-hand side is a polynomial in q^s with integer coefficients, and therefore can be written in the form $p_{m,n,s}(x,y)$, where the entries of x and y are all non-negative powers of q and the powers that appear in x are distinct from the powers that appear in y. Assuming that $\dim X > 0$, Proposition 3.4 tells us that the entries of α_i are disjoint from the entries of $q^{\dim X}/\alpha_i$, thus the hypotheses of Lemma 3.6 are satisfied and each $\alpha_{i,j}$ is equal to a power of q. Since we already know that X is pure, this implies that X is chaste, and Equation (2) becomes

$$q^{s \dim X} P_X(q^{-s}) - P_X(q^s) = \sum_{\beta \neq 0} \nu_{X_\beta}(q^s) P_\beta(q^s).$$

Since this holds for all s, we may replace q^s with the variable t. Moving $P_X(t) = P_0(t)$ to the right-hand side, and noting that $\nu_{X_0}(t) = 1$, we obtain the desired equality. $\square$

3.4 Cohomological interpretation of Kazhdan-Lusztig polynomials

We now combine the results of Subsections 3.1, 3.2, and 3.3 to give a cohomological interpretation of the polynomial $P_{M_{\mathcal{A}}}(t)$. Let $\mathcal{A}$ be an arrangement over a finite field $\mathbb{F}_q$, and let ℓ be a prime that does not divide q.

Lemma 3.8. *For every flat F and every element $x \in X_{\mathcal{A},F}$, $\mathrm{IH}_x^*(X_{\mathcal{A}};\overline{\mathbb{Q}}_\ell) \cong \mathrm{IH}^*(X_{\mathcal{A}^F};\overline{\mathbb{Q}}_\ell)$.*

Proof. By Theorem 3.3, we have an étale map from a neighborhood of $x \in X_{\mathcal{A}}$ to a neighborhood of $(0,x) \in X_{\mathcal{A}^F} \times X_{\mathcal{A},F}$. It follows that the local intersection cohomology of $X_{\mathcal{A}}$ at x is isomorphic to the local intersection cohomology of $X_{\mathcal{A}^F}$ at the cone point times the local intersection cohomology of $X_{\mathcal{A},F}$ at x. By the contraction lemma [Spr84, Corollary 1], the local intersection cohomology of $X_{\mathcal{A}^F}$ at the cone point is isomorphic to the the global intersection cohomology of $X_{\mathcal{A}^F}$. Since $X_{\mathcal{A},F}$ is smooth, the local intersection cohomology of $X_{\mathcal{A},F}$ is trivial. $\square$

Let $\chi_{\mathcal{A}}(t) = \chi_{M_{\mathcal{A}}}(t)$ be the characteristic polynomial of $\mathcal{A}$. The variety $U_{\mathcal{A}}$ is polynomial count with $\nu_{U_{\mathcal{A}}}(t) = \chi_{\mathcal{A}}(t)$ [OT92, 2.69]. For any arrangement $\mathcal{A}$ in V, let

$$\mathrm{rk}\,\mathcal{A} := \mathrm{rk}\,M_{\mathcal{A}} = \dim V = \dim X_{\mathcal{A}}.$$

Proposition 3.9. *The reciprocal plane $X_{\mathcal{A}}$ is chaste, and*

$$t^{\mathrm{rk}\,\mathcal{A}}P_{X_{\mathcal{A}}}(t^{-1}) = \sum_F \chi_{\mathcal{A}_F}(t)P_{X_{\mathcal{A}^F}}(t).$$

Proof. We proceed by induction on the rank of $\mathcal{A}$. If $\mathrm{rk}\,\mathcal{A} = 0$, the statement is trivial. Now assume that the proposition holds for all arrangements of smaller rank. In particular, this means that $X_{\mathcal{A}^F}$ is chaste for all nonempty flats F. By Lemma 3.8, this implies that $X_{\mathcal{A}}$ is pointwise chaste away from the cone point, with $P_{X_{\mathcal{A}},x}(t) = P_{X_{\mathcal{A}^F}}(t)$ for all F nonempty and $x \in X_{\mathcal{A},F}$. The statement then follows from Theorem 3.7. $\square$

As a consequence, we find that the intersection cohomology Poincaré polynomial of a reciprocal plane over a finite field coincides with the Kazhdan-Lusztig polynomial of the corresponding matroid.

Theorem 3.10. *If $\mathcal{A}$ is an arrangement over a finite field, then $P_{X_{\mathcal{A}}}(t) = P_{M_{\mathcal{A}}}(t)$.*

Proof. This follows from Proposition 3.4, Theorem 3.9, and the uniqueness of Theorem 2.2. $\square$

Corollary 3.11. *If a matroid M is representable, then $P_M(t)$ has non-negative coefficients.*

Proof. If M is representable over some field, then it is representable over a finite field [Rad57, Theorems 4 & 6], and the corollary follows from Theorem 3.10. $\square$

Let $\mathcal{A}$ be an arrangement over $\mathbb{C}$. Theorem 3.10 says that we may interpret $P_{M_{\mathcal{A}}}(t)$ geometrically by choosing a representation of $M_{\mathcal{A}}$ over a finite field and considering the ℓ-adic étale intersection

cohomology of the resulting reciprocal plane. However, one might prefer to think about the topological intersection cohomology groups of $X_\mathcal{A}(\mathbb{C})$. Let $P_{X_\mathcal{A}}(t) = \sum_{i\geq0} \dim \mathrm{IH}^{2i}(X_\mathcal{A}(\mathbb{C}); \mathbb{C})\, t^i$.

Proposition 3.12. *If $\mathcal{A}$ is an arrangement over $\mathbb{C}$, then the topological intersection cohomology of $X_\mathcal{A}(\mathbb{C})$ vanishes in odd degree, and $P_{X_\mathcal{A}}(t) = P_{M_\mathcal{A}}(t)$. Furthermore, the topological analogue of Lemma 3.8 holds.*

Proof. Choose a spreading out of $X_\mathcal{A}$ and then base change to a finite field $\mathbb{F}_q$ of sufficiently large characteristic. The fact that the topological intersection cohomology of $X_\mathcal{A}(\mathbb{C})$ coincides with the graded dimension of the ℓ-adic étale intersection cohomology of $X_\mathcal{A}(\overline{\mathbb{F}}_q)$ after tensoring with $\mathbb{C}$ follows from [BBD82, 6.1.9] (see also [Con, 1.4.8.1]). The same goes for local intersection cohomology groups. □

Remark 3.13. For $\mathcal{A}$ an arrangement over a finite field or $\mathbb{C}$, the isomorphism class of the variety $X_\mathcal{A}$ is not determined by the matroid $M_\mathcal{A}$. However, Theorem 3.10 and Proposition 3.12 imply that the intersection cohomology Poincaré polynomial $P_{X_\mathcal{A}}(t)$ is determined by $M_\mathcal{A}$.

3.5 Relation to the work of Li and Yong

Li and Yong [LY11] associate to any variety Y over a field k and any closed point $p \in Y$ two polynomials:

$$P_{p,Y}(t) := \sum_{i\geq0} \dim \mathrm{H}^{2i-\dim Y}(Y; \mathrm{IC}_{Y,p})\, t^i \qquad \text{and} \qquad H_{p,Y}(t) := (1-t)^{\dim Y}\mathrm{Hilb}(k[TC_pY]; t).$$

If Y is a Schubert variety, then $P_{p,Y}(t)$ is an ordinary Kazhdan-Lusztig polynomial. If the Schubert variety is covexillary, they prove that $\deg P_{p,Y}(t) = \deg H_{p,Y}(t)$, and that the coefficients of $H_{p,Y}(t)$ are greater than or equal to the corresponding coefficients of $P_{p,Y}(t)$ [LY11, 1.2]. They conclude by asking for what pairs (p, Y) this same statement holds [LY11, 7.1].

If $\mathcal{A}$ is an arrangement over $k = \mathbb{F}_q$ or $\mathbb{C}$, $Y = X_\mathcal{A}$, and $p \in X_{\mathcal{A},F}$, then Lemma 3.8 and Theorem 3.10 (if $k = \mathbb{F}_q$) or Proposition 3.12 (if $k = \mathbb{C}$) tell us that

$$P_{p,Y}(t) = P_{M_{\mathcal{A}^F}}(t).$$

Furthermore, Lemma 3.8 and [Ber10, 4.3][7] tell us that

$$H_{p,Y}(t) = h^{\mathrm{bc}}_{M_{\mathcal{A}^F}}(t),$$

the h-polynomial of the broken circuit complex of $M_{\mathcal{A}^F}$.

Both properties studied by Li and Yong fail in general for $X_\mathcal{A}$; for example, if $M_{\mathcal{A}^F}$ is the

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[7]The Hilbert series of the Orlik-Terao algebra in characteristic zero was computed in [Ter02, 1.2] and independently in [PS06, Proposition 7]; Berget's proof is the first one that works in positive characteristic.

uniform matroid of rank d on a set of cardinality $d + 1$, we have

$$h^{\mathrm{bc}}_{M_{\mathcal{A}^F}}(t) = 1 + t + t^2 + \cdots + t^{d-1},$$

while $P_{M_{\mathcal{A}^F}}(t)$ is a polynomial of degree less than $\frac{d}{2}$ with linear coefficient equal to $\binom{d+1}{2} - (d+1)$ (Corollary 2.20). It would be interesting to determine whether there is a nice class of "covexillary matroids" for which $h^{\mathrm{bc}}_M(t)$ dominates $P_M(t)$.

4 Algebra

In this section we define a q-deformation of the Möbius algebra of a matroid, use Kazhdan-Lusztig polynomials to define a special basis for this algebra, and conjecture that the structure coefficients for this basis are non-negative. We then verify the conjecture for Boolean matroids, and for uniform matroids and braid matroids of rank at most 3.

4.1 The deformed Möbius algebra

Fix a matroid M. The **Möbius algebra** is defined to be the free abelian group

$$E(M) := \mathbb{Z}\{\varepsilon_F \mid F \in L(M)\}$$

equipped with the multiplication $\varepsilon_F \cdot \varepsilon_G := \varepsilon_{F \vee G}$. We define a deformation

$$E_q(M) := \mathbb{Z}[q, q^{-1}]\{\varepsilon_F \mid F \in L(M)\}$$

with multiplication

$$\varepsilon_F \cdot \varepsilon_G := \sum_{H \geq I \geq F \vee G} \mu(I, H)\, q^{\mathrm{crk}\, I}\, \varepsilon_H,$$

where $\mathrm{crk}\, I := \mathrm{rk}\, M - \mathrm{rk}\, I$ is the corank of I. The fact that we recover our original multiplication when $q = 1$ follows from the fact that $\sum_{H \geq I \geq F \vee G} \mu(I, H) = \delta(H, F \vee G)$.

Proposition 4.1. *The $\mathbb{Z}[q, q^{-1}]$-algebra $E_q(M)$ is commutative, associative, and unital, with unit equal to*

$$\sum_{F \leq G} \mu(F, G)\, q^{-\,\mathrm{crk}\, F}\, \varepsilon_G.$$

Proof. Commutativity is immediate from the definition. For associativity, we note that

$$\varepsilon_F \cdot \varepsilon_G = \sum_{H \geq I \geq F \vee G} \mu(I, H)\, q^{\mathrm{crk}\, I}\, \varepsilon_H$$

$$= \sum_{H, I} \zeta(F, I)\zeta(G, I)\mu(I, H)\, q^{\mathrm{crk}\, I}\, \varepsilon_H,$$

24

and therefore

$$
\begin{aligned}
(\varepsilon_F \cdot \varepsilon_G) \cdot \varepsilon_J &= \sum_{H,I,K,L} \zeta(F,I)\zeta(G,I)\zeta(H,L)\zeta(J,L)\mu(I,H)\mu(L,K)\, q^{\operatorname{crk} I + \operatorname{crk} L}\, \varepsilon_K \\[2mm]
&= \sum_{I,K,L} \zeta(F,I)\zeta(G,I)\zeta(J,L)\mu(L,K)\, q^{\operatorname{crk} I + \operatorname{crk} L} \sum_{H} \mu(I,H)\zeta(H,L)\, \varepsilon_K \\[2mm]
&= \sum_{I,K,L} \zeta(F,I)\zeta(G,I)\zeta(J,L)\mu(L,K)\, q^{\operatorname{crk} I + \operatorname{crk} L} \delta(I,L)\, \varepsilon_K \\[2mm]
&= \sum_{I,K} \zeta(F,I)\zeta(G,I)\zeta(J,I)\mu(I,K)\, q^{2\,\operatorname{crk} I}\, \varepsilon_K.
\end{aligned}
$$

This expression is clearly symmetric in F, G, and J, hence our product is associative.

For the statement about the unit, we observe that

$$
\begin{aligned}
\left(\sum_{F \leq G} \mu(F,G)\, q^{-\operatorname{crk} F}\, \varepsilon_G \right) \cdot \varepsilon_H &= \sum_{F \leq G} \mu(F,G)\, q^{-\operatorname{crk} F} \sum_{I \geq J \geq G \vee H} \mu(J,I)\, q^{\operatorname{crk} J}\, \varepsilon_I \\[2mm]
&= \sum_{F,G,I,J} \mu(F,G)\zeta(G,J)\zeta(H,J)\mu(J,I)\, q^{\operatorname{crk} J - \operatorname{crk} F}\, \varepsilon_I \\[2mm]
&= \sum_{F,I,J} \zeta(H,J)\mu(J,I)\, q^{\operatorname{crk} J - \operatorname{crk} F} \left(\sum_G \mu(F,G)\zeta(G,J) \right) \varepsilon_I \\[2mm]
&= \sum_{F,I,J} \zeta(H,J)\mu(J,I)\, q^{\operatorname{crk} J - \operatorname{crk} F} \delta(F,J)\, \varepsilon_I \\[2mm]
&= \sum_{I,J} \zeta(H,J)\mu(J,I)\, \varepsilon_I \\[2mm]
&= \sum_I \delta(H,I)\, \varepsilon_I \\[2mm]
&= \varepsilon_H.
\end{aligned}
$$

This completes the proof. $\qquad\square$

4.2   The Kazhdan-Lusztig basis

We now define a new basis for $E_q(M)$ in terms of the standard basis, using Kazhdan-Lusztig polynomials to define the matrix coefficients. The definition is analogous to that of the Kazhdan-Lusztig basis for the Hecke algebra, and we therefore call our new basis the **Kazhdan-Lusztig basis**.

For all $F \in L(M)$, let
$$x_F := \sum_{G \geq F} q^{\operatorname{rk} G - \operatorname{rk} F} P_{M_G^F}(q^{-2})\, \varepsilon_G.$$

It is clear that
$$x_F \in \varepsilon_F + q\mathbb{Z}[q]\{\varepsilon_G \mid G > F\},$$

and therefore that $\{x_F \mid F \in L(M)\}$ is a $\mathbb{Z}[q, q^{-1}]$-basis for $E_q(M)$. Even better, it is a $\mathbb{Z}[q]$-basis for the (non-unital) subring $\mathbb{Z}[q]\{\varepsilon_F \mid F \in L(M)\} \subset E_q(M)$.

Consider the structure constants for multiplication in this basis. That is, for all F, G, H, define $C_{FG}^H(q) \in \mathbb{Z}[q]$ by the equation
$$x_F \cdot x_G = \sum_H C_{FG}^H(q)\, x_H.$$

We conjecture that this polynomial has non-negative coefficients.

Conjecture 4.2. *For all $F, G, H \in L(M)$, $C_{FG}^H(q) \in \mathbb{N}[q]$.*

4.3 Boolean matroids

In this subsection we will prove Conjecture 4.2 for Boolean matroids by producing an explicit formula for multiplication in the Kazhdan-Lusztig basis. We first need the following two lemmas.

Lemma 4.3. *Fix subsets $F, G, L \subset [n]$ with $F \cup G \subset L$. Let $F \Delta G := F \cup G \smallsetminus F \cap G$ be the symmetric difference of F and G. Then*

$$\sum_{\substack{H \supset F \\ I \supset G \\ H \cup I = L}} q^{|H| + |I| - |F| - |G|} = (1 + q)^{|F \Delta G|} (2q + q^2)^{|L \smallsetminus F \cup G|}.$$

Proof. The trick is to write

$$H = F \sqcup H' \sqcup H'' \sqcup J \qquad \text{and} \qquad I = G \sqcup I' \sqcup I'' \sqcup J,$$

where

$H' = F^c \cap G \cap H$, $H'' = F^c \cap G^c \cap H \cap I^c$, $I' = F \cap G^c \cap I$, $I'' = F^c \cap G^c \cap H^c \cap I$, and $J = F^c \cap G^c \cap H \cap I$.

Then the left-hand side becomes

$$\sum_{H', H'', I', I'', J} q^{|H'| + |H''| + |I'| + |I''| + 2|J|},$$

where the sum is over $H' \subset F^c \cap G$, $I' \subset F \cap G^c$, and $H'', I'', J \subset F^c \cap G^c \cap L$ with

$$H'' \sqcup I'' \sqcup J = F^c \cap G^c \cap L.$$

We have

$$\sum_{H',I'} q^{|H'|+|I'|} = (1+q)^{|F\Delta G|},$$

and, for each fixed J,

$$\sum_{H'',I''} q^{|H''|+|I''|} = (2q)^{|F^c \cap G^c \cap L \cap J^c|}.$$

Thus the left-hand side is equal to

$$(1+q)^{|F\Delta G|} \sum_{J \subset F^c \cap G^c \cap L} q^{2|J|}(2q)^{|F^c \cap G^c \cap L \cap J^c|} = (1+q)^{|F\Delta G|}(2q+q^2)^{|F^c \cap G^c \cap L|},$$

where the last equality is an application of the binomial theorem. □

Lemma 4.4. *Fix subsets $F \subset G \subset [n]$. Then for any polynomials $f(q)$ and $g(q)$, we have*

$$\sum_{F \subset H \subset G} f(q)^{|G \smallsetminus H|} g(q)^{|H \smallsetminus F|} = \left(f(q) + g(q) \right)^{|G \smallsetminus F|}.$$

Proof. This is simply a reformulation of the binomial theorem. □

Proposition 4.5. *Let M be the Boolean matroid on the ground set $[n]$. Then for any subsets $F, G \subset [n]$, we have*

$$x_F \cdot x_G = \sum_{K \supset F \cup G} q^{n-|K|}(1+q)^{|K|-|F \cap G|} x_K.$$

Proof. For each $F \subset G$, M_G^F is again Boolean, so $P_{M_G^F}(t) = 1$ by Corollary 2.10. This means that

$$x_F = \sum_{G \supset F} q^{|G \smallsetminus F|} \varepsilon_G,$$

and, by Möbius inversion,

$$\varepsilon_F = \sum_{G \supset F} (-q)^{|G \smallsetminus F|} x_G.$$

We therefore have

$$
\begin{aligned}
x_F \cdot x_G &= \left(\sum_{H \supset F} q^{|H \smallsetminus F|} \varepsilon_H \right) \cdot \left(\sum_{I \supset G} q^{|I \smallsetminus G|} \varepsilon_I \right) \\[2mm]
&= \sum_{\substack{H \supset F \\ I \supset G}} q^{|H|+|I|-|F|-|G|} \varepsilon_H \cdot \varepsilon_I \\[2mm]
&= \sum_{\substack{H \supset F \\ I \supset G}} q^{|H|+|I|-|F|-|G|} \sum_{J \supset H \cup I} q^{n-|J|}(1-q)^{|J \smallsetminus H \cup I|} \varepsilon_J \\[2mm]
&= \sum_{\substack{H \supset F \\ I \supset G}} q^{|H|+|I|-|F|-|G|} \sum_{J \supset H \cup I} q^{n-|J|}(1-q)^{|J \smallsetminus H \cup I|} \sum_{K \supset J} (-q)^{|K \smallsetminus J|} x_K.
\end{aligned}
$$

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By Lemma 4.3, this equation becomes

$$x_F \cdot x_G = \sum_{K \supset J \supset L \supset F \cup G} (1+q)^{|F \Delta G|} (2q+q^2)^{|L \smallsetminus F \cup G|} q^{n-|J|} (1-q)^{|J \smallsetminus L|} (-q)^{|K \smallsetminus J|} x_K.$$

By writing $n - |J| = n - |K| + |K \smallsetminus J|$, we may rewrite our equation as

$$x_F \cdot x_G = (1+q)^{|F \Delta G|} \sum_{K \supset J \supset L \supset F \cup G} q^{n-|K|} (2q+q^2)^{|L \smallsetminus F \cup G|} (1-q)^{|J \smallsetminus L|} (-q^2)^{|K \smallsetminus J|} x_K.$$

Applying Lemma 4.4 first to the sum over J and then to the sum over L, this becomes

$$
\begin{aligned}
x_F \cdot x_G &= (1+q)^{|F \Delta G|} \sum_{K \supset L \supset F \cup G} q^{n-|K|} (2q+q^2)^{|L \smallsetminus F \cup G|} (1-q-q^2)^{|K \smallsetminus L|} x_K \\
&= (1+q)^{|F \Delta G|} \sum_{K \supset F \cup G} q^{n-|K|} (1+q)^{|K \smallsetminus F \cup G|} x_K \\
&= \sum_{K \supset F \cup G} q^{n-|K|} (1+q)^{|K|-|F \cap G|} x_K.
\end{aligned}
$$

This completes the proof. $\qquad\square$

4.4 Uniform matroids

In this subsection we give the multiplication table for $E_q(M)$ in terms of the Kazhdan-Lusztig basis when M is a uniform matroid of rank at most 3. The rank 1 case is covered by Proposition 4.5 with $n = 1$.

Example 4.6. Let M be the uniform matroid of rank 2 on the ground set $[n] = \{1, \ldots, n\}$. In this case, $P_{M_G^F}(t) = 1$ for all $F \le G$ (since $\operatorname{rk} M = 2$), and we have the following multiplication table:

$$
\begin{aligned}
x_{[n]}^2 &= x_{[n]} \\
x_{[n]} \cdot x_{\{i\}} &= (1+q) x_{[n]} \\
x_{[n]} \cdot x_\emptyset &= (1+nq+q^2) x_{[n]} \\
x_{\{i\}}^2 &= q x_{\{i\}} + (1+q) x_{[n]} \\
x_{\{i\}} \cdot x_{\{j\}} &= (1+q)^2 x_{[n]} \qquad (i \ne j) \\
x_{\{i\}} \cdot x_\emptyset &= q(1+q) x_{\{1\}} + \left(1 + nq + (n-1)q^2\right) x_{[n]} \\
x_\emptyset^2 &= q^2 x_\emptyset + q(1+q) \sum_i x_{\{i\}} + \left(1 + nq + (n-1)^2 q^2\right) x_{[n]}.
\end{aligned}
$$

Example 4.7. Let M be the uniform matroid of rank 3 on the ground set $[n]$. In this case, $P_{M_G^F}(t) = 1$ for all $F \le G$ unless $F = \emptyset$ and $G = [n]$, in which case Corollary 2.20 tells us that

$$P_{M_G^F}(t) = P_M(t) = 1 + \left(\tbinom{n}{2} - n\right) t.$$

We have the following multiplication table:

$$
\begin{aligned}
x_{[n]}^2 &= x_{[n]} \\
x_{[n]} \cdot x_{\{i,j\}} &= (1+q)x_{[n]} \\
x_{[n]} \cdot x_i &= \left(1 + (n-1)q + q^2\right)x_{[n]} \\
x_{[n]} \cdot x_\emptyset &= \left(1 + \tbinom{n}{2}q + \tbinom{n}{2}q^2 + q^3\right)x_{[n]} \\
x_{\{i,j\}} \cdot x_{\{i,j\}} &= qx_{\{i,j\}} + (1+q)x_{[n]} \\
x_{\{i,j\}} \cdot x_{\{i,k\}} &= (1+q)^2 x_{[n]} \qquad (j \neq k) \\
x_{\{i,j\}} \cdot x_{\{k,\ell\}} &= (1+q)^2 x_{[n]} \qquad (\{i,j\} \cap \{k,\ell\} = \emptyset) \\
x_{\{i,j\}} \cdot x_{\{i\}} &= q(1+q)x_{\{i,j\}} + \left(1 + (n-1)q + (n-2)q^2\right)x_{[n]} \\
x_{\{i,j\}} \cdot x_k &= (1 + nq + nq^2 + q^3)x_{[n]} \qquad (k \notin \{i,j\}) \\
x_{\{i,j\}} \cdot x_\emptyset &= q(1+q)^2 x_{\{i,j\}} + \left(1 + \tbinom{n}{2}q + (n^2 - n - 3)q^2 + \left(\tbinom{n}{2} - 2\right)q^3\right)x_{[n]} \\
x_{\{i\}}^2 &= q^2 x_{\{i\}} + q(1+q)\sum_{j\neq i} x_{\{i,j\}} + \left(1 + (n-1)q + (n-2)^2 q^2\right)x_{[n]} \\
x_{\{i\}}x_{\{j\}} &= q(1+q)^2 x_{\{i,j\}} + \left(1 + (2n-3)q + n(n-2)q^2 + (2n-5)q^3\right)x_{[n]} \qquad (i \neq j) \\
x_{\{i\}} \cdot x_\emptyset &= q^2(1+q)x_i + q(1+q)^2 \sum_{j\neq i} x_{\{i,j\}} \\
&\quad + \left(1 + \tbinom{n}{2}q + \tfrac{1}{2}(n-1)(n^2-6)q^2 + \tfrac{1}{2}(n-1)(n^2-8)q^3 + \tfrac{1}{2}n(n-3)q^4\right)x_{[n]} \\
x_\emptyset^2 &= q^3 x_\emptyset + q^2(1+q)\sum_i x_{\{i\}} + q(1+q)^2 \sum_{i<j} x_{\{i,j\}} \\
&\quad + \left(1 + \tbinom{n}{2}q + \tfrac{1}{4}n(n^3 - 2n^2 - n - 2)q^2 + \tfrac{1}{2}(n-1)(n^3 - n^2 - 5n - 2)q^3\right. \\
&\quad \left. + \tfrac{1}{4}n^2(n+1)(n-3)q^4 + \tfrac{1}{2}n(n-3)q^5\right)x_{[n]}.
\end{aligned}
$$

Note that each coefficient is non-negative for all $n \geq 3$, and when $n = 3$, this multiplication table agrees with Proposition 4.5.

Remark 4.8. It is reasonable to ask if Conjecture 4.2 would still hold if we were to redefine the Kazhdan-Lusztig basis by putting $x_F := \sum_{G \geq F} q^{\mathrm{rk}\,G - \mathrm{rk}\,F}\, \varepsilon_G$; that is, if we were to pretend that all Kazhdan-Lusztig polynomials were equal to 1. If we did this, the linear term of $C_{\emptyset\emptyset}^{[n]}(q)$ in Example 4.7 would be equal to $2n - \binom{n}{2}$, which is negative when $n > 5$. Thus the Kazhdan-Lusztig polynomials truly play a necessary role in Conjecture 4.2.

4.5 Braid matroids

Finally, we consider braid matroids of small rank. The braid matroid M_2 is isomorphic to the Boolean matroid of rank 1. The braid matroid M_3 is isomorphic to the uniform matroid of rank 2 on the ground set [3].

Example 4.9. Let M_4 be the braid arrangement of rank 3. The ground set $\mathcal{I}$ has cardinality $\binom{4}{2} = 6$, and $P_{M_4}(t) = 1 + t$ (see appendix). Flats correspond to set theoretic partitions of [4].

Flats of rank 1 all have cardinality 1, corresponding to partitions of [4] into one set of cardinality 2 and two singletons. Flats of rank 2 come in two types: those of cardinality 2 (partitions of [4] into two subsets of cardinality 2), and those of cardinality 3 (partitions of [4] into one subset of cardinality 3 and one singleton). We omit the full multiplication table, but give the single most interesting product:

$$x_\emptyset^2 = q^3 x_\emptyset + q^2(1+q) \sum_{|F|=1} x_F + q(1+q)^2 \sum_{|F|=2} x_F + q(1+3q+4q^2) \sum_{|F|=3} x_F$$

$$+ \left(q^5 + 13q^4 + 38q^3 + 32q^2 + 7q + 1 \right) x_\mathcal{I}.$$

4.6 Update

After this paper was published, Ben Young wrote software to compute the polynomials $C_{FG}^H(q)$ in SAGE. Using this software, we discovered that that Conjecture 4.2 is false for a number of examples in rank 4 and higher, including the uniform matroid of rank 4 on 6 elements, the uniform matroid of rank 6 on 7 elements, and the supersolvable matroid represented by all vectors in $\mathbb{F}_2^4$ with at least two coordinates equal to 1. We were not able to find any braid matroids for which the conjecture fails (we computed all of the structure coefficients for M_5, M_6, and M_7, and some of the structure coefficients for M_8).

In addition, we used this software to identify and correct minor errors in Examples 4.7 and 4.9. In Example 4.7, we corrected the coefficient of q^4 in $C_{\emptyset\emptyset}^{[n]}(q)$. In Example 4.9, we corrected the coeffiecients of q, q^2, and q^3 in $C_{\emptyset\emptyset}^{\mathcal{I}}(q)$.

A   Appendix (with Ben Young)

We include here computer generated computations of Kazhdan-Lusztig polynomials of uniform matroids and braid matroids of small rank. Individual Kazhdan-Lusztig polynomials are to be read vertically; for example, Table A.1 tells us that the Kazhdan-Lusztig polynomial of $M_{1,8}$ is equal to $1 + 27t + 120t^2 + 84t^3$.

We see some interesting patterns in the tables. First, we find experimental evidence for Conjecture 2.5. Also, with the help of the On-Line Encyclopedia of Integer Sequences [Slo14], we can find formulas for specific coefficients. For example, we observe that the leading coefficient of the Kazhdan-Lusztig polynomial of the uniform matroid $M_{1,2k-1}$ is equal to the Catalan number $C_k = \frac{1}{k+1}\binom{2k}{k}$, and the leading coefficient of the Kazhdan-Lusztig polynomial of the braid matroid M_{2k} is equal to $(2k-3)!! \, (2k-1)^{(k-2)}$. The former statement, along with a combinatorial description of all coefficients of Kazhdan-Lusztig polynomials of uniform matroids, is proved in [GPY].

The sage code which was used to compute these tables is available at `https://github.com/benyoung/kl-matro`

A.1 Uniform matroids

Table 1: Kazhdan-Lusztig polynomials for the uniform matroid $M_{1,d}$

$d =$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
t			2	5	9	14	20	27	35	44	54	65	77	90	104
t^2					5	21	56	120	225	385	616	936	1365	1925	2640
t^3							14	84	300	825	1925	4004	7644	13650	23100
t^4									42	330	1485	5005	14014	34398	76440
t^5											132	1287	7007	28028	91728
t^6													429	5005	32032
t^7															1430

Table 2: Kazhdan-Lusztig polynomials for the uniform matroid $M_{2,d}$

$d =$	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	1	1	1	1	1	1	1	1	1	1	1	1
t	5	14	28	48	75	110	154	208	273	350	440	544	663
t^2			21	98	288	675	1375	2541	4368	7098	11025	16500	23936
t^3					84	552	2145	6380	16016	35672	72618	137760	246840
t^4							330	2805	13585	49049	146510	382200	899640
t^5									1287	13442	78078	331968	1150968
t^6											5005	62062	420784
t^7													19448

Table 3: Kazhdan-Lusztig polynomials for the uniform matroid $M_{3,d}$

$d =$	3	4	5	6	7	8	9	10	11	12	13	14
1	1	1	1	1	1	1	1	1	1	1	1	1
t	9	28	62	117	200	319	483	702	987	1350	1804	2363
t^2			56	288	927	2365	5214	10374	19110	33138	54720	86768
t^3					300	2145	9020	28886	77714	184730	399840	803760
t^4							1485	13585	70499	271635	862680	2384760
t^5									7007	78078	482118	2171988
t^6											32032	420784

A.2 Braid matroids

Table 4: Kazhdan-Lusztig polynomials for the braid matroid M_n

$n =$	1	2	3	4	5	6	7	8	9	10	11	12	13
1	1	1	1	1	1	1	1	1	1	1	1	1	1
t				1	5	16	42	99	219	466	968	1981	4017
t^2						15	175	1225	6769	32830	147466	632434	2637206
t^3								735	16065	204400	2001230	16813720	128172330
t^4										76545	2747745	56143395	864418555
t^5												13835745	746080335

$n =$	14	15	16	17
1	1	1	1	1
t	8100	16278	32647	65399
t^2	10811801	43876001	176981207	711347303
t^3	915590676	6252966720	41362602281	267347356003
t^4	11200444255	129344350135	1377269949055	13819966094935
t^5	22495833360	502627875750	9305666915545	151395489770525
t^6	3859590735	293349030975	12290930276625	376566883537845
t^7			1539272109375	157277996100225

$n =$	18	19	20
1	1	1	1
t	130918	261972	524097
t^2	2853229952	11430715476	45762931992
t^3	1698735206324	10656703437054	66208557177786
t^4	132618161185510	1229703907984734	11100857399288280
t^5	2242336712846230	30941776173508200	404180066561961690
t^6	9443716601138820	205809448675350520	4042252614171772000
t^7	8758018896026400	352844128436870070	11522756204094885750
t^8	831766748637825	110176255068905025	7879824460254822075
t^9			585243816844111425

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