A POSITIVE INTEGRAL PROPERTY ON THE GROUND STATE OF THE TWO-BOUNDARY TEMPERLEY–LIEB HAMILTONIAN

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ABSTRACT. We study the two-boundary Temperley–Lieb $O(n)$ loop model on Kazhdan–Lusztig bases of type A and B. We obtain explicit expressions of the ground state of the two-boundary Temperley–Lieb Hamiltonian by means of a coideal subalgebra of $U_q(\mathfrak{sl}_2)$. This ground state possesses a positive integral property. We conjecture that some components of the ground state are directly related to an enumeration of binary or permutation matrices.

1. INTRODUCTION

Razumov and Stroganov studied the ground state of the XXZ spin chain with the periodic boundary condition at the isotropic parameter $\Delta = -1/2$ [35, 37]. They observed that the largest component (under a certain normalization) is related to the total number of combinatorial objects called alternating sign matrices (see, e.g., [5] and references therein). They also obtained explicit expressions for some correlation functions. These striking observations open a new way to study quantum integral systems [1, 31, 36, 38, 45], quantum Knizhnik–Zamolodchikov equations [10, 14, 15, 16, 34] and combinatorics such as alternating sign matrices and plane partitions [2, 11, 15].

An alternative description of the Razumov–Stroganov correspondence (proven in [6]) is the one by the $O(n)$ loop model [1]. This model has the Temperley–Lieb Hamiltonian [29, 40, 46] which acts on the so-called link patterns [7, 12]. At $n = 1$, it was observed that the ground state on the link pattern bases counts the number of combinatorial objects such as alternating sign matrices and fully packed loop models [12]. There are several variants of the model by replacing the Temperley–Lieb algebra to the one- or two-boundary Temperley–Lieb algebras [8, 12, 30, 33]. These algebras act on link patterns with boundaries.

In the case of type A, the link pattern bases for the Temperley–Lieb algebra is equivalent to the parabolic Kazhdan–Lusztig bases of the Hecke algebra [13, 22] or to the (dual) canonical basis (of weight zero) [20, 21, 26, 27] in the tensor products of the fundamental representation of $U_q(\mathfrak{sl}_2)$ [17]. The coincidence of Kazhdan–Lusztig bases and canonical bases was shown in [18]. The Kazhdan–Lusztig bases for the Hermitian symmetric pair (B_N, A_{N-1}) are relevant to the representation of the one-boundary Temperley–Lieb algebra considered in this paper. The graphical description of these Kazhdan–Lusztig bases is revealed in [42]. A diagram for a Kazhdan–Lusztig basis has the flavour of a link pattern of type A and type B. The two-boundary Temperley–Lieb algebra can be regarded as a quotient algebra of the affine Hecke algebra of type C. Although the affine Hecke algebra is infinite dimensional, we impose a quotient relation (see Eqn.(10)) to obtain a finite dimensional representation of the two-boundary Temperley–Lieb algebra. Since the (one-boundary) Temperley–Lieb algebra is a subalgebras of the two-boundary Temperley–Lieb algebra, the representation theory of the former can be applicable to the latter.

In this paper, we investigate the two-boundary Temperley–Lieb $O(n)$ loop model on the Kazhdan–Lusztig bases of type A and B and on the standard bases. The Kazhdan–Lusztig bases of type B

can be regarded as a (dual) canonical basis of a coideal subalgebra of $U_q(\mathfrak{sl}_2)$ [41]. A key is the fact that the generator of the coideal subalgebra commutes with the one-boundary Temperley–Lieb Hamiltonian of the system. Thus an eigenfunction of the generator of the coideal subalgebra with the multiplicity one is also an eigenfunction of the one-boundary Temperley–Lieb Hamiltonian.

We have four types of Kazhdan–Lusztig bases which the two-boundary Temperley–Lieb algebra acts on. We call them type A, BI, BII and BIII respectively (see Section 3 for definitions). Type BII corresponds to a link pattern with a boundary with a suitable choice of the normalization (compare Type BII with ,e.g., [7]). We first consider the eigensystem of the generator X of a coideal subalgebra of $U_q(\mathfrak{sl}_2)$. From the explicit action of X on the Kazhdan–Lusztig bases, one can obtain all the eigenvalues of X and their multiplicities. For all types, there exists an eigenvector Ψ with the multiplicity one. One of the main results of this paper is explicit expressions of Ψ (see Definition 4.4, 4.11, 4.17 and 4.23). By construction, it is obvious that Ψ is a positive vector, *i.e.*, all the entries of Ψ are positive. We also show that this eigenfunction Ψ is an eigenvector of the generators of one-boundary Temperley–Lieb algebra with the eigenvalue zero. This implies that Ψ is the ground state of the one-boundary Temperley–Lieb Hamiltonian with the eigenvalue zero. Furthermore, this eigenfunction Ψ is the ground state of the two-boundary Temperley–Lieb Hamiltonian under an integrable condition $q^{N-1}QQ_0 = 1$ where N is the size of the system and q, Q and Q_0 are the Hecke parameters (see Section 2 for the definitions of parameters). This integrable condition can be viewed as a compatibility condition to embed the representation of the two-boundary Temperley–Lieb algebra (the affine Hecke algebra of type C) into the one of one-boundary Temperley–Lieb algebra (the Hecke algebra of type B).

From the explicit expressions of Ψ, we can compute correlation functions exactly. One can also show that Ψ is in $\mathbb{N}[q, q^{-1}, Q, Q^{-1}]$ for type A, BII and BIII and in $\mathbb{N}[q, q^{-1}]$ for type BI. This positive integral property appears not only on the Kazhdan–Lusztig bases but also on the standard bases (see Definition 4.27). The transition matrix from the standard bases to the Kazhdan–Lusztig bases is written in terms of the Kazhdan–Lusztig polynomials which also have another positive integral structure. Thus the origin of the positivity of Ψ may come from these two positivities. Since Ψ has a positive integral property, it is natural to ask whether the components of Ψ count combinatorial objects along the spirit of the Razumov–Stroganov correspondence. As a first step, we consider the sum of the components of Ψ. For type A, BI and BIII, the sums are conjectured to be the total number of symmetric binary/permutation matrices with appropriate conditions. In fact, some components are conjectured to be a (q, Q)-enumeration of symmetric binary matrices. We expect that these observations are a starting point of a connection of Kazhdan–Lusztig bases to enumerative combinatorics.

The paper is organized as follows. In Section 2, we briefly review the two-boundary Temperley–Lieb algebra and a coideal subalgebra of $U_q(\mathfrak{sl}_2)$. In Section 3, we introduce a diagrammatic presentation of the Kazhdan–Lusztig bases. We show the action of the two-boundary Temperley–Lieb algebra on the Kazhdan–Lusztig bases of type A and B. Section 4 is devoted to the analysis of the eigensystem of the generator X. We define an eigenfunction Ψ of X and show that this eigenfunction has the multiplicity one. Section 5 is devoted to the analysis of the action of the Hamiltonian on Ψ. We show that the generators of one-boundary Temperley–Lieb algebra acts zero on Ψ and that Ψ is the ground state of the two-boundary Temperley–Lieb Hamiltonian with the integrable condition. In Section 6, we compute correlation functions, show the positive integral property of Ψ and propose several conjectures on Ψ as a (q, Q)-enumeration of symmetric binary/permutation matrices. In Section A, we collect technical lemmas used in this paper.

2. Two-boundary Temperley–Lieb algebra

2.1. **Two-boundary Temperley–Lieb algebra.** The Temperley–Leib algebra [29, 40, 46] is an associative algebra over the ring $\mathbb{Z}[q, q^{-1}]$ and generated by $e_i, 1 \leq i \leq N - 1$ with the relations:

$$e_i^2 = -(q + q^{-1})e_i, \qquad 1 \leq i \leq N - 1, \tag{1}$$

$$e_i e_{i\pm 1} e_i = e_i, \tag{2}$$

$$e_i e_j = e_j e_i, \qquad |i - j| > 1. \tag{3}$$

The two-boundary Temperley–Lieb algebra [8, 9] is a generalization of the Temperley–Lieb algebra with extra generators e_n and e_0. The defining relations are relations (1)-(3) and

$$e_N^2 = -(Q + Q^{-1})e_N, \tag{4}$$

$$e_{N-1} e_N e_{N-1} = (qQ^{-1} + q^{-1}Q)e_{N-1}, \tag{5}$$

$$e_i e_N = e_N e_i, \qquad i \neq N - 1. \tag{6}$$

$$e_0^2 = -(Q_0 + Q_0^{-1})e_N, \tag{7}$$

$$e_1 e_0 e_1 = (qQ_0^{-1} + q^{-1}Q_0)e_1, \tag{8}$$

$$e_i e_0 = e_0 e_i, \qquad i \neq 1. \tag{9}$$

We call the subalgebra generated by $\{e_i : 1 \leq i \leq N\}$ the one-boundary Temperley–Lieb algebra. Note that the two-boundary Temperley–Lieb algebra is infinite dimensional. We impose the following two conditions to make the algebra finite dimensional:

$$I_N J_N I_N = \alpha I_N, \qquad J_N I_N J_N = \alpha J_N \tag{10}$$

where α is a parameter and

$$I_{2n} := \prod_{i=0}^{n-1} e_{2i+1}, \quad J_{2n} := e_0 \prod_{i=1}^{n-1} e_{2i} \cdot e_N,$$

$$I_{2n+1} := e_0 \prod_{i=1}^{n} e_{2i}, \quad J_{2n+1} := \prod_{i=0}^{n-1} e_{2i+1} \cdot e_N.$$

Let V_1 be a two-dimensional $\mathbb{C}$-vector space spanned by v_1 and v_{-1}. We have a representation of the two-boundary Temperley–Lieb algebra acting on $V_1^{\otimes N}$. The matrix representation of the generators are

$$e_i = \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{i-1} \otimes \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -q^{-1} & 1 & 0 \\ 0 & 1 & -q & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \otimes \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{N-i-1}, \qquad 1 \leq i \leq N - 1,$$

$$e_N = \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{N-1} \otimes \begin{pmatrix} -Q^{-1} & 1 \\ 1 & -Q \end{pmatrix},$$

$$e_0 = \begin{pmatrix} -Q_0 & 1 \\ 1 & -Q_0^{-1} \end{pmatrix} \otimes \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{N-1},$$

where the order of bases is (v_1, v_{-1}) for V_1 and $(v_1 \otimes v_1, v_1 \otimes v_{-1}, v_{-1} \otimes v_1, v_{-1} \otimes v_{-1})$ for $V_1 \otimes V_1$. A tensor product $v_{\epsilon_1} \otimes \cdots \otimes v_{\epsilon_N}$ with $\epsilon_i = 1$ or -1 for $1 \leq i \leq N$ is called a standard basis. In this

representation, one can show by a straightforward computation that

$$\alpha = \begin{cases} (Q^{-1} - q^{-1}Q_0)(Q - qQ_0^{-1}), & \text{for } N : \text{even,} \\ (1 + Q^{-1}Q_0)(1 + QQ_0^{-1}), & \text{for } N: \text{odd} \end{cases}$$

We consider the integrable Hamiltonian for the one- and two-boundary Temperley–Lieb algebras acting on $V_1^{\otimes N}$ defined by

$$H^{1B} = -\sum_{i=1}^{N-1} e_i - a_N e_N,$$
$$H^{2B} = H^{1B} - a_0 e_0,$$

where a_0 and a_N are parameters.

Remark 2.1. *The Temperley–Lieb Hamiltonian is rewritten as*

$$H^{2B} = -\frac{1}{2}\left(\sum_{i=1}^{N-1}\left(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y + \frac{q+q^{-1}}{2}\sigma_i^z\sigma_{i+1}^z \right) + \left(\frac{q-q^{-1}}{2} - a_0(Q_0 - Q_0^{-1}) \right)\sigma_1^z \right.$$

$$+ 2a_0(\sigma_1^+ + \sigma_1^-) + 2a_N(\sigma_N^+ + \sigma_N^-) - \left(\frac{q-q^{-1}}{2} - a_N(Q - Q^{-1}) \right)\sigma_N^z$$

$$\left. + \left(\frac{q+q^{-1}}{4}(N-1) + a_0\frac{Q_0 + Q_0^{-1}}{2} + a_N\frac{Q + Q^{-1}}{2} \right) \right).$$

where σ^x, σ^y and σ^z are the Pauli matrices and $\sigma^\pm = (\sigma^x \pm \sqrt{-1}\sigma^y)/2$. Thus the spectrum of H can be viewed as the one of the XXZ spin $1/2$ quantum chain with boundaries.

2.2. A coideal subalgebra of $U_q(\mathfrak{sl}_2)$.

The quantum group $\mathbf{U} := U_q(\mathfrak{sl}_2)$ is an associative algebra over $\mathbb{C}(q)$ with generators $E, F, K^{\pm 1}$ and relations

$$KK^{-1} = K^{-1}K = 1,$$
$$KEK^{-1} = q^2 E,$$
$$KFK^{-1} = q^{-2}F,$$
$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}}.$$

We introduce the quantum integer $[n] := \sum_{i=0}^{n-1} q^{n-1-2i}$, the quantum factorial $[n]! := \prod_{i=1}^{n}[i]$ and q-analogue of the binomial coefficient

$$\begin{bmatrix} n \\ m \end{bmatrix} := \frac{[n]!}{[n-m]![m]!}.$$

We define

$$[Q; n] := \frac{Qq^n - Q^{-1}q^{-n}}{q - q^{-1}}.$$

The comultiplication Δ is given by

$$\Delta(K^{\pm 1}) := K^{\pm 1} \otimes K^{\pm 1},$$
$$\Delta(E) = E \otimes K^{-1} + 1 \otimes E,$$
$$\Delta(F) = F \otimes 1 + K \otimes F.$$

We consider the two-dimensional representation in V_1. The action is given by

$$Ev_1 = 0, \qquad Ev_{-1} = v_1,$$
$$Fv_1 = v_{-1}, \qquad Fv_{-1} = 0,$$
$$Kv_{\pm 1} = q^{\pm 1}v_{\pm 1}.$$

We consider the Dynkin diagram of type A_1 and the identity involution. By a general theory of quantum symmetric space [23, 24, 25], one can obtain a coideal algebra of $\mathbf{U}$ associated with the involution. The coideal subalgebra $\mathbf{U}'$ of $U_q(\mathfrak{sl}_2)$ is a polynomial algebra in X, that is, $\mathbf{U}' := \mathbb{C}(q)[X]$. The injective $\mathbb{C}(q)$-algebra homomorphism $\iota : \mathbf{U}' \to \mathbf{U}$ is given by

$$X \mapsto F + cKE + sK$$

where c, s are indeterminates. The comultiplication Δ is given by

(11) $$\Delta(X) = K \otimes X + cKE \otimes 1 + F \otimes 1.$$

Note that $\mathbf{U}'$ is left coideal since $\Delta(X) \subset \mathbf{U} \times \mathbf{U}'$. In this paper, we consider

$$c = q^{-1}, \qquad s = \frac{Q - Q^{-1}}{q - q^{-1}}.$$

Theorem 2.2. $[H^{1B}, X] = 0$.

Proof. It is enough to show that $[e_i, X] = 0$ for all $1 \leq i \leq N$. When $N = 1$, we have $[e_1, X] = 0$ by a straightforward calculation. Since the comultiplication is given by Eqn.(11), we have $[e_N, X] = 0$ in general. Since the action of the Temperley–Lieb algebra commutes with the action of the quantum group $U_q(\mathfrak{sl}_2)$ in the tensor product of the fundamental representation, we have $[e_i, X] = 0$ for $1 \leq i \leq N - 1$. $\qquad\square$

3. Representations

The actions of the two-boundary Temperley–Lieb algebra on the standard bases are obvious through the matrix representation of the generators (See Section 2). In this section, we consider the action of the two-boundary Temperley–Lieb algebra on Kazhdan–Lusztig bases.

3.1. **Kazhdan–Lusztig bases.** The Hecke algebra of type A is a unital, associative algebra over $\mathbb{C}[q, q^{-1}]$ generated by the generators T_i, $1 \leq i \leq N-1$, satisfying the relations $(T_i - q^{-1})(T_i + q) = 0$, $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$ and $T_i T_j = T_j T_i$ for $|i - j| > 1$. The Temperley–Lieb algebra can be regarded as the Hecke algebra of type A with a quotient relation (2) through a relation $T_i = e_i + q^{-1}$. The representation of the Temperley–Lieb algebra in $V_1^{\otimes N}$ corresponds to the maximal parabolically induced representation of the Hecke algebra. The Hecke algebra of type B is generated by T_i, $1 \leq i \leq N$, with the relations of type A, $(T_N - Q^{-1})(T_N + Q) = 0$, $T_N T_{N-1} T_N T_{N-1} = T_{N-1} T_N T_{N-1} T_N$ and $T_i T_N = T_N T_i$ for $i \neq N - 1$. Similarly, the one-boundary Temperley–Lieb algebra can be regarded as the Hecke algebra of type B with a quotient relation (5). Therefore, one can apply the representation theory of the Hecke algebra of type B to the one-boundary Temperley–Lieb algebra. Since we consider the representation of the one-boundary Temperley–Lieb algebra in $V_1^{\otimes N}$ as in Section 2, the parabolic Kazhdan–Lusztig bases studied in [13] play a central role rather than original ones studied in [22]. More precisely, the Kazhan–Ludztig bases for the Hermitian symmetric pair (B_N, A_{N-1}) studied in [3, 4, 42] can be regarded as bases of the one-boundary Temperley–Lieb algebra in $V_1^{\otimes N}$. There are two types of parabolic Kazhdan–Lusztig bases according to the choice of a projection map (see, *e.g.*, Section 2.3 in [42]). In this paper, we will consider the parabolic

Kazhdan–Lusztig bases studied as C_x^- in [42]. Hereafter, a Kazhdan–Lusztig basis means this parabolic one.

We have four types of Kazhdan–Lusztig bases associated with the Temperley–Lieb algebra and the one-boundary Temperley–Lieb algebras. The first one is the Kazhdan–Lusztig basis of Temperley–Lieb algebra of type A, the second is the Kazhdan–Lusztig basis of the one-boundary Temperley–Lieb algebra for $Q = q^M$ with $M \in \mathbb{N}_+$ and the third and the fourth are the Kazhdan–Lusztig bases of the one-boundary Temperley–Lieb algebra for $q \neq Q$ where q and Q are algebraically independent. Note that we will consider the representation of the two-boundary Temperley–Lieb algebra on Kazhdan–Lusztig bases of Type A in the first case and of type B in the second, third and fourth cases. The difference between the third and the fourth bases is the total order with respect to q and Q. We call these bases type A, BI, BII and BIII respectively. We index a Kazhdan–Lusztig basis by a binary string $\{+, -\}^N$. Given two binary strings $\epsilon := \epsilon_1 \ldots \epsilon_N$ and $\epsilon' := \epsilon'_1 \ldots \epsilon'_N$, we denote $\epsilon < \epsilon'$ if $\epsilon_j = \epsilon'_j$ for $1 \leq j \leq i-1$ and $\epsilon_i < \epsilon'_i$. All types of Kazhdan–Lusztig bases are characterized by the following two conditions [13, 22, 28]: (1) a Kazhdan–Lusztig basis is invariant under the involutive ring automorphism known as "bar involution" where $T_i \to T_i^{-1}$, $q \to q^{-1}$ and $Q \to Q^{-1}$. On the module $V^{\otimes N}$, we define $\overline{v_\epsilon} = v_\epsilon$ where $\epsilon_i = 1$ for $1 \leq i \leq N$. (2) The expansion of a Kazhdan–Lusztig basis w indexed by a binary string $\epsilon = \epsilon_1 \ldots \epsilon_N$ in terms of standard basis has the leading term $v_{\kappa_1} \otimes \cdots \otimes v_{\kappa_N}$ where $\kappa_i = 1$ if $\epsilon_i = +$ and $\kappa_i = -1$ if $\epsilon_i = -$. The vector $w - v_{\kappa_1} \otimes \cdots \otimes v_{\kappa_N}$ is a linear combination of $v_{\kappa'_1} \otimes \cdots \otimes v_{\kappa'_N}$, $\kappa < \kappa'$, with a coefficient in $\mathbb{Z}(\Gamma_-^X)$ for Type X. Here $\Gamma_-^A = \Gamma_-^{BI} = \{q^{-i} | i \in \mathbb{N}_+\}$, $\Gamma_-^{BII} = \{q^{-i}Q^j | i \in \mathbb{N}_+, j \in \mathbb{Z}\} \cup \{Q^{-i} | i \in \mathbb{N}_+\}$ and $\Gamma_-^{BIII} = \{q^i Q^{-j} | i \in \mathbb{Z}, j \in \mathbb{N}_+\} \cup \{q^{-i} | i \in \mathbb{N}_+\}$.

Since $V_1^{\otimes N}$ can be viewed as the tensor products of fundamental representation of $U_q(\mathfrak{sl}_2)$, a Kazhdan–Lusztig basis of type A is nothing but the dual canonical basis of $U_q(\mathfrak{sl}_2)$ considered in [17] (see also [43]). A Kazhdan–Lusztig basis of type BI is considered in [42] and can be viewed as the dual canonical basis of a coideal subalgebra of $U_q(\mathfrak{sl}_2)$ [41]. A Kazhdan–Lusztig basis of type BII is studied in [42]. One can easily show that a basis of type BIII satisfies the criteria for a Kazhdan–Lusztig basis.

We briefly review the graphical presentation of a Kazhdan–Lusztig basis following [17, 42, 43]. Let $b = b_1 \cdots b_N \in \{\pm\}^N$ be a binary string. We place an up arrow (resp. a down arrow) from left to right according to $b_i = +$ (resp. $b_i = -$). We have the following two rules.

(A) We make a pair between adjacent down arrow and up arrow in this order. Then connect this pair into a simple arc.
(B) Repeat the procedure (A) until all the up arrows are to the left of all down arrows.

Type A. The Kazhdan–Lusztig basis of Type A follows rules (A) and (B).

Type BI. In addition to rules (A) and (B), we have three more rules:

(C) Put an integer $p, 2 \leq p \leq M$, on the $(M + 1 - p)$-th down arrow from right.
(D) Put a star ($\bigstar$) on the M-th down arrow from right if it exists.
(E) For remaining down arrows, we make a pair of adjacent down arrows from right to left. Then connect this pair into a simple dashed arc.

After applying rules (A)-(E), we may have an unpaired down arrow which does not form a dashed arc.

Type BII. After applying rules (A) and (B) to a diagram, we have unpaired up arrows and unpaired down arrows. We call the $(2i - 1)$-th (resp. $2i$-th) unpaired down arrow from right an o-unpaired (resp. e-unpaired) down arrow. We have an additional rule:

(F) We put a vertical line with a mark e (resp. o) on an e-unpaired (resp. o-unpaired) down arrow.

Type BIII. We apply rules (A) and (B) to a diagram. We have unpaired up arrows and unpaired down arrows. We enumerate unpaired down arrows from right to left by $1, 2, \ldots$. Then, we have an additional rule:

(G) We put a vertical line with a circled integer i on the i-th unpaired down arrow.

A diagram corresponds to a vector in $V_1 \otimes \cdots \otimes V_1$ as follows. An unpaired up arrow (resp. down arrow) in a diagram is v_1 (resp. v_{-1}) in a tensor product. Each building block (a simple arc, a dashed arc, a down arrow with a star, mark e or mark o and a down arrow with an integer or with a circled integer) is a vector in V_1 or $V_1 \otimes V_1$:

$$\bigcup \quad = \quad v_{-1} \otimes v_1 - q^{-1} v_1 \otimes v_{-1},$$

$$\overset{\cdot\;\cdot}{\underset{\,}{\bigcup}} \quad = \quad v_{-1} \otimes v_{-1} - q^{-1} v_1 \otimes v_1,$$

$$\underset{\bigstar}{\big\downarrow} \quad = \quad v_{-1} - q^{-1} v_1,$$

$$\underset{p}{\big|} \quad = \quad v_{-1} - q^{-p} v_1, \quad \text{for } 2 \le p \le M,$$

$$\underset{o}{\big|} \quad = \quad v_{-1} - Q^{-1} v_1,$$

$$\underset{e}{\big|} \quad = \quad v_{-1} + q^{-1} Q v_1.$$

$$\underset{\textcircled{p}}{\big|} \quad = \quad v_{-1} - q^{p-1} Q^{-1} v_1.$$

An unpaired up (resp. down) arrow corresponds to v_1 (resp. v_{-1}). A vector in $V^{\otimes N}$ corresponding to a diagram is given by a tensor product of a vector corresponding to a building block.

Example 3.1. *Let $b = + - - + - -$. The Kazhdan–Lusztig basis indexed by b are*

for Type A, BI ($M = 1$), BII and BIII respectively. The diagram of type A corresponds to a vector

$$v_1 \otimes v_{-1} \otimes v_{-1} \otimes v_1 \otimes v_{-1} \otimes v_{-1} - q^{-1} v_1 \otimes v_{-1} \otimes v_1 \otimes v_{-1} \otimes v_{-1} \otimes v_{-1}$$

in $V_1^{\otimes 6}$.

3.2. **Action of the Temperley–Lieb algebra on Kazhdan–Lusztig bases.** Recall that a Kazhdan–Lusztig basis is expressed as a tensor product of building blocks. Thus it is enough to consider the action of Temperley–Lieb algebra on partial diagrams. We list up all the partial

diagrams for the action of e_i, $1 \leq i \leq N-1$.

$$e_i(\downarrow\downarrow) = 0, \qquad e_i\left(\begin{smallmatrix} i & i+1 \\ & \smallsmile \end{smallmatrix} \right) = 0, \qquad e_i\left(\begin{smallmatrix} i & i+1 \\ \downarrow & \smallsmile \end{smallmatrix} \right) = \smile \uparrow,$$

$$e_i\left(\begin{smallmatrix} i & i+1 \\ \downarrow & \smile \end{smallmatrix} \right) = \smile \uparrow, \qquad e_i\left(\begin{smallmatrix} i & i+1 \\ \downarrow & | \\ & \bigstar \end{smallmatrix} \right) = \smile ,$$

$$e_i\left(\begin{smallmatrix} i\ i+1 \\ \smile\ \smile \end{smallmatrix} \right) = \smile\!\!\bigcup\!\!\smile, \qquad e_i\left(\begin{smallmatrix} i\ i+1 \\ \smile\ \cup \end{smallmatrix} \right) = \smile\!\!\bigcup\!\!\smile,$$

$$e_i\left(\begin{smallmatrix} i\ i+1 \\ \bigcup \end{smallmatrix} \right) = \smile\ \smile, \qquad e_i\left(\begin{smallmatrix} i\ i+1 \\ \cup\ | \\ \bigstar \end{smallmatrix} \right) = \bigstar\ \smile ,$$

$$e_i\left(\begin{smallmatrix} | & | \\ p & p+1 \end{smallmatrix} \right) = 0, \quad e_i\left(\begin{smallmatrix} i\ i+1 \\ |\ \cup \\ p \end{smallmatrix} \right) = \smile\ \underset{p}{|} ,$$

$$e_i\left(\begin{smallmatrix} i\ i+1 \\ |\ \cup \\ \circled{p} \end{smallmatrix} \right) = \smile\ \underset{\circled{p}}{|} , \quad e_i\left(\begin{smallmatrix} | & | \\ \circled{p+1} & \circled{p} \end{smallmatrix} \right) = 0, \quad e_i\left(\begin{smallmatrix} i\ i+1 \\ |\ \cup \\ e \end{smallmatrix} \right) = \smile\ \underset{e}{|} ,$$

$$e_i\left(\begin{smallmatrix} i\ i+1 \\ |\ \cup \\ o \end{smallmatrix} \right) = \smile\ \underset{o}{|} , \quad e_i\left(\begin{smallmatrix} | & | \\ e & o \end{smallmatrix} \right) = (q^{-1}Q + qQ^{-1})\ \smile ,$$

$$e_i\left(\begin{smallmatrix} | & | \\ o & e \end{smallmatrix} \right) = -(Q + Q^{-1})\ \smile , \quad e_i\left(\smile \right) = -[2]\ \smile , \quad e_i\left(\begin{smallmatrix} i & i+1 \\ \uparrow & \smile \end{smallmatrix} \right) = \smile \downarrow,$$

$$e_i\left(\begin{smallmatrix} i\,i+1 \\ \smile\ \smile \end{smallmatrix} \right) = \smile\!\!\bigcup\!\!\smile, \quad e_i\left(\begin{smallmatrix} i\ i+1 \\ \uparrow\ \cup \end{smallmatrix} \right) = \smile \uparrow,$$

$$e_i\left(\begin{smallmatrix} & i\ i+1 \\ \smile & \bigcup \end{smallmatrix} \right) = \smile\ \smile, \quad e_i\left(\begin{smallmatrix} i\,i+1 \\ \bigcup\ | \\ p \end{smallmatrix} \right) = \underset{p}{|}\ \smile ,$$

$$e_i\left(\begin{smallmatrix} i\,i+1 \\ \bigcup\ | \\ \circled{p} \end{smallmatrix} \right) = \underset{\circled{p}}{|}\ \smile , \quad e_i\left(\uparrow\ \underset{p}{|} \right) = \smile , \quad e_i\left(\uparrow\ \underset{\circled{p}}{|} \right) = \smile ,$$

$$e_i\left(\uparrow\ \underset{\bigstar}{|} \right) = \smile , \quad e_i\left(\uparrow\ \underset{e}{|} \right) = \smile , \quad e_i\left(\uparrow\ \underset{o}{|} \right) = \smile ,$$

$$e_i\left(\bigcup \overset{i\,i+1}{\underset{e}{|}}\right) = \underset{e}{|}\ \bigcup\,, \quad e_i\left(\bigcup \overset{i\,i+1}{\underset{o}{|}}\right) = \underset{o}{|}\ \bigcup\,,$$

$$e_i(\uparrow\downarrow) = \bigcup\,, \quad e_i\left(\bigcup\overset{i\ i+1}{\underset{\downarrow}{}}\right) = \downarrow\ \bigcup\,, \quad e_i\left(\bigcup\overset{i\ i+1}{\underset{\uparrow}{}}\right) = \uparrow\ \bigcup\,,$$

$$e_i\left(\overset{i\ i+1}{\bigcup}\right) = \bigcup\ \bigcup\,, \qquad e_i\left(\overset{i\ i+1}{\bigcup}\right) = \bigcup\ \bigcup\,, \qquad e_i(\uparrow\uparrow) = 0.$$

Example 3.2. *Let D be a diagram of type BI ($M = 2$) depicted as*

$$D = \uparrow \overset{}{\underset{\bigstar}{\bigcup}} \,|\, \bigcup \,\underset{2}{|}\ .$$

Then, we have

$$e_1(D) = \bigcup\ \bigcup\ \downarrow\,|\,\underset{\bigstar}{}\bigcup\,\underset{2}{|}\,, \qquad e_2(D) = \uparrow\ \bigcup\ \underset{\bigstar}{\,\dashv\,}\,|\,\bigcup\,\underset{2}{|}\,,$$

$$e_5(D) = \uparrow\,|\ \underset{\bigstar}{\bigcup}\ \bigcup\ \bigcup\,\underset{2}{|}\,, \qquad e_6(D) = \uparrow\,\underset{}{\bigcup}\ \bigcup\,\underset{\bigstar\,2}{|\,|}$$

and $e_3(D) = e_7(D) = -[2]D$.

3.3. Action of e_N on Kazhdan–Lusztig bases.

The action of e_N on a Kazhdan–Lusztig basis is given as follows.

Type A. Let D be a diagram of type A and $N_\uparrow$ (resp. $N_\downarrow$) be the number of up (resp. down) arrows. We have three cases for D: 1) the rightmost arrow of D is an up arrow, *i.e.*, $N_\downarrow = 0$, 2) the rightmost arrow is a down arrow and 3) the rightmost arrow is an up arrow forming an arc.

Case 1. Let D' be a diagram obtained from D by changing the rightmost up arrow to a down arrow. Then, the action of e_N on D is given by

$$e_N(D) = D' - Q^{-1}D.$$

Case 2. We enumerate down arrows in D from right to left by $1, 2, \ldots, N_\downarrow$. For each i, $1 \le i \le N_\downarrow - 1$, we denote by $A_{(i)}(D)$ a diagram obtained from D by connecting the i-th and the $(i+1)$-th down arrows via an arc. We denote by $A_{(N_\downarrow)}(D)$ a diagram obtained from D by changing the $N_\downarrow$-th arrow to an up arrow. The action of e_N on D is given by

$$e_N(D) = \sum_{1 \le i \le N_\downarrow} q^{-(i-1)} A_{(i)}(D) - QD.$$

Case 3. Let $\widetilde{D}$ be a diagram obtained from D by changing the rightmost arc to two down arrows. We enumerate unpaired down arrows of $\widetilde{D}$ from right to left by $1, 2, \ldots N_\downarrow + 2$. For each i, $2 \le i \le N_\downarrow + 1$, we denote by $A_{(i)}(\widetilde{D})$ obtained from $\widetilde{D}$ by connecting the i-th and the $(i+1)$-th down arrows via an arc. We denote by $A_{(N_\uparrow + 2)}(\widetilde{D})$ a diagram obtained from $\widetilde{D}$ by changing the $(N_\uparrow + 2)$-th down arrow to an up arrow. For each $1 \le i < j \le N_\downarrow + 1$ with $j - i \ge 2$, we denote by $B_{(j,i)}(\widetilde{D})$ a diagram obtained from $\widetilde{D}$ by connecting the i-th and the $(i+1)$-th down arrows via an arc and the j-th and

the $(j+1)$-th arrows via an arc. For each i, $1 \leq i \leq N_\downarrow - 1$, we denote by $B_{(i+1,i)}(\widetilde{D})$ a diagram obtained from $\widetilde{D}$ by connecting the i-th and the $(i+3)$-th down arrows via an arc and the $(i+1)$-th and the $(i+2)$-th down arrows via an arc. We denote by $B_{(N_\downarrow+1,N_\downarrow)}(\widetilde{D})$ a diagram obtained from $\widetilde{D}$ by connecting the $(N_\downarrow + 2)$-th and the $(N_\downarrow + 1)$-th arrows via an arc and putting an up arrow at the $N_\downarrow$-th site. For each i, $1 \leq i \leq N_\downarrow$, we denote by $B_{(N_\downarrow+2,i)}$ the diagram obtained from $\widetilde{D}$ by changing the $(N_\downarrow + 2)$-th down arrow to an up arrow and connecting the i-th and the $i+1$-th down arrows via an arc. We denote by $B_{(N_\downarrow+2,N_\downarrow+1)}(\widetilde{D})$ a diagram obtained from $\widetilde{D}$ by changing the $(N_\downarrow + 2)$-th and $(N_\downarrow + 1)$-th down arrows to two up arrows. The action of e_N on an arc is

$$e_N(\,\smile\,) = \downarrow\downarrow \; -Q^{-1} \smile + q^{-1}(Q - Q^{-1}) \uparrow\downarrow \; -q^{-1} \uparrow\uparrow \,.$$

Then, the action of e_N on D is given by

$$e_N(D) = \widetilde{D} - Q^{-1}D + q^{-1}(Q - Q^{-1}) \sum_{2 \leq i \leq N_\downarrow+2} q^{-(i-2)} A_{(i)}(\widetilde{D}) - q^{-1} \sum_{1 \leq i < j \leq N_\downarrow+2} c_{(j,i)} B_{(j,i)}(\widetilde{D}).$$

where

(12)
$$c_{(j,i)} := \begin{cases} q^{-j+2}, & \text{for } i = 1, \\ q^{-2i+2}, & \text{for } j = i+1, \\ q^{-i-j+3}(1+q^2), & \text{otherwise.} \end{cases}$$

Example 3.3. *Let D be a diagram depicted as*

$$D = \uparrow \; \underbrace{\smile\smile} \; \uparrow\downarrow\downarrow \; \smile \,.$$

Then, we have

$$A_{(2)}(\widetilde{D}) = \uparrow \; \underbrace{\smile\smile} \; \uparrow\downarrow \; \smile \; \downarrow, \qquad A_{(3)}(\widetilde{D}) = \uparrow \; \underbrace{\smile\smile} \; \uparrow \; \smile\downarrow\downarrow,$$

$$A_{(4)}(\widetilde{D}) = \uparrow \; \underbrace{\smile\smile} \; \uparrow\uparrow \; \downarrow\downarrow\downarrow, \qquad B_{(3,1)}(\widetilde{D}) = \uparrow \; \underbrace{\smile\smile} \; \uparrow \; \smile \; \smile,$$

$$B_{(2,1)}(\widetilde{D}) = \uparrow \; \underbrace{\smile\smile} \; \uparrow \; \underbrace{\smile\smile}, \qquad B_{(3,2)}(\widetilde{D}) = \uparrow \; \underbrace{\smile\smile} \; \uparrow \; \smile \; \uparrow\downarrow,$$

$$B_{(4,1)}(\widetilde{D}) = \uparrow \; \underbrace{\smile\smile} \; \uparrow\uparrow\downarrow \; \smile, \qquad B_{(4,2)}(\widetilde{D}) = \uparrow \; \underbrace{\smile\smile} \; \uparrow\uparrow \; \smile \; \downarrow,$$

$$B_{(4,3)}(\widetilde{D}) = \uparrow \; \underbrace{\smile\smile} \; \uparrow\uparrow\uparrow\downarrow\downarrow \,.$$

Type BI. When the rightmost arrow in D is an up arrow or a down arrow with the integer M, the action of e_N on partial diagrams is given by

$$e_N(\uparrow) \;\; = \;\; \underset{\text{M}}{\big|} \,,$$

$$e_N \left(\underset{\text{M}}{\big|} \right) \;\; = \;\; -(q^M + q^{-M}) \underset{\text{M}}{\big|} \,.$$

Below, we consider the case where the rightmost arrow of D is an up arrow forming an arc. The other arcs and up arrows are irrelevant for the action of e_N since a Kazhdan–Lusztig basis is tensor

products of building blocks. The action of e_N on an arc is given by

$$e_N\left(\smile\right) \;=\; \downarrow\,\big|\; +\; \uparrow\,\big| \;,\quad \text{for } M=1,$$

$$e_N\left(\smile\right) \;=\; \underset{\text{M-1 \ M}}{\big|\;\big|} \;+\; \langle M-1\rangle\underset{\text{M}}{\downarrow\,\big|} \;,\quad \text{for } M\geq 2,$$

where $\langle k\rangle := q^k + q^{-k}$ for $k\in\mathbb{N}_+$. Then, the action of e_N on partial diagrams is given by (see [42])

$$e_N\left(\underset{\text{r}}{\big|}\cdots\underset{\text{M}}{\big|}\smile\right) \;=\; \underset{\text{r-2}}{\big|}\cdots\underset{\text{M}}{\big|} \;+\; \langle r-2\rangle\underset{\text{r-1}}{\overset{\uparrow}{\big|}}\cdots\underset{\text{M}}{\big|}$$

$$+\sum_{r\leq k\leq M}\langle k-1\rangle\;\underset{\text{r}}{\big|}\cdots\underset{\text{k-1}}{\big|}\underset{\text{k}}{\smile}\cdots\underset{\text{M}}{\big|}\;,\quad \text{for } r\geq 3,$$

$$e_N\left(\underset{2}{\big|}\cdots\underset{\text{M}}{\big|}\smile\right) \;=\; \overset{\uparrow}{\underset{\bigstar}{\big|}}\cdots\underset{\text{M}}{\big|} \;+\; \overset{\downarrow}{\underset{\bigstar}{\big|}}\cdots\underset{\text{M}}{\big|}$$

$$+\sum_{2\leq k\leq M}\langle k-1\rangle\;\underset{2}{\big|}\cdots\underset{\text{k-1}}{\big|}\underset{\text{k}}{\smile}\cdots\underset{\text{M}}{\big|}\;,$$

$$e_N\left(\underset{\bigstar}{\big|}\cdots\underset{\text{M}}{\big|}\smile\right) \;=\; \underset{\bigstar}{\overset{\frown}{}}\,\big|\cdots\underset{\text{M}}{\big|} \;+\; \underset{\bigstar}{\smile}\,\big|\cdots\underset{\text{M}}{\big|}$$

$$+\sum_{2\leq k\leq M}\langle k-1\rangle\;\underset{\bigstar}{\big|}\cdots\underset{\text{k-1}}{\big|}\underset{\text{k}}{\smile}\cdots\underset{\text{M}}{\big|}\;.$$

Example 3.4. *Let D be a diagram depicted as*

$$D = \uparrow\,\overset{\frown}{\underset{\bigstar}{\smile}}\,\big|\underset{2}{\smile}\,\big|\smile\,,$$

where $M=2$. Then, the action of e_{13} on D is

$$e_{13}(D) \;=\; \uparrow\,\overset{\frown}{\smile}\,\overset{\frown}{\smile}\,\big|\underset{2}{\underset{\bigstar}{\smile}}\,\big| \;+\; \uparrow\,\overset{\frown}{\smile}\,\smile\,\big|\underset{2}{\underset{\bigstar}{\smile}}\,\big|$$

$$+\langle 1\rangle\,\uparrow\,\overset{\frown}{\underset{\bigstar}{\smile}}\,\big|\smile\,\smile\,\smile\,\big|_{2}$$

Type BII. The actions of e_N on partial diagrams are given by

$$e_N(\uparrow) \;=\; \underset{\text{o}}{\big|}\;,$$

$$e_N\left(\underset{\text{o}}{\big|}\right) \;=\; -(Q+Q^{-1})\underset{\text{o}}{\big|}\;,$$

$$e_N\left(\smile\right) \;=\; \underset{\text{e \ o}}{\big|\;\big|}\;.$$

Type BIII. We have three cases for the rightmost arrow a of D: 1) an arrow a is an up arrow, 2) a is a down arrow with the circled integer one, and 3) a is an up arrow forming an arc.

In the case 1 and 2, we have

$$e_N(\uparrow) \quad = \quad \underset{\textcircled{1}}{\big|} \, ,$$

$$e_N\left(\underset{\textcircled{1}}{\big|}\right) \quad = \quad -(Q + Q^{-1})\underset{\textcircled{1}}{\big|} \, .$$

In the case 3, the action of e_N on an arc is given by

$$e_N\left(\,\cup\,\right) = \underset{\textcircled{2}}{\big|}\,\underset{\textcircled{1}}{\big|} + \langle\langle 1 \rangle\rangle \uparrow \underset{\textcircled{1}}{\big|} \, ,$$

where $\langle\langle k \rangle\rangle := Qq^{-k} + Q^{-1}q^{k}$. The action of e_N on a partial diagram is given by

$$e_N\left(\underset{\textcircled{r}}{\big|}\,\cdots\,\underset{\textcircled{1}}{\big|}\,\cup\right) \quad = \quad \underset{\textcircled{r+2}}{\big|}\,\cdots\,\underset{\textcircled{1}}{\big|} + \langle\langle r+1 \rangle\rangle \uparrow \underset{\textcircled{r+1}}{\big|}\,\cdots\,\underset{\textcircled{1}}{\big|}$$

$$+ \sum_{1 \le k \le r} \langle\langle k \rangle\rangle \underset{\textcircled{r}}{\big|}\,\cdots\,\underset{\textcircled{k+1}}{\big|}\underset{\textcircled{k}}{\cup}\,\cdots\,\underset{\textcircled{1}}{\big|} \, .$$

Example 3.5. *Let D be a diagram of type BIII depicted as*

$$D = \cup\,\uparrow\big|\underset{\textcircled{3}}{}\cup\underset{\textcircled{2}}{\big|}\cup\underset{\textcircled{1}}{\big|}\left(\cup\right).$$

Then we define

$$D_1 := \cup\,\uparrow\big|\underset{\textcircled{5}}{}\cup\underset{\textcircled{4}}{\big|}\cup\underset{\textcircled{3}\textcircled{2}}{\big|\big|}\cup\underset{\textcircled{1}}{\big|}\,, \quad D_2 := \cup\,\uparrow\big|\underset{\textcircled{3}}{}\cup\underset{\textcircled{2}}{\big|}\cup\,\cup\,\cup\underset{\textcircled{1}}{\big|}\,,$$

$$D_3 := \cup\,\uparrow\big|\underset{\textcircled{3}}{}\cup\left(\cup\right)\underset{\textcircled{2}}{\big|}\cup\underset{\textcircled{1}}{\big|}\,, \quad D_4 := \cup\,\uparrow\left(\cup\right)\cup\underset{\textcircled{3}\textcircled{2}}{\big|\big|}\cup\underset{\textcircled{1}}{\big|}\,.$$

$$D_5 := \cup\,\uparrow\uparrow\cup\underset{\textcircled{4}}{\big|}\cup\underset{\textcircled{3}\textcircled{2}}{\big|\big|}\cup\underset{\textcircled{1}}{\big|}\,.$$

The action of e_{14} on D is

$$e_{14}(D) = D_1 + \langle\langle 1 \rangle\rangle D_2 + \langle\langle 2 \rangle\rangle D_3 + \langle\langle 3 \rangle\rangle D_4 + \langle\langle 4 \rangle\rangle D_5.$$

3.4. Action of e_0 on Kazhdan–Lusztig bases.

Type A. We denote by u the bijection of sets $u : D \to D$ defined by reflecting a diagram about a vertical axis and reversing orientations of all arrows. The action of e_0 on a diagram D is given by

$$(13) \qquad\qquad\qquad e_0(D) = u(e_N(u(D))),$$

with a change of the parameter $Q \to Q_0$.

For example, the action of e_0 on an arc is

$$e_0(\,\cup\,) = \uparrow\uparrow - Q_0^{-1}\,\cup\, + q^{-1}(Q_0 - Q_0^{-1})\uparrow\downarrow - q^{-1}\downarrow\downarrow\,.$$

Type BI. Let D be a diagram of type BI and $N_\uparrow$ be the number of (unpaired) up arrows. Let r be a smallest integer attached to down arrows with an integer p, $1 \le p \le M$. If D has a down arrow with a star, then we define $r = 1$. If there is no down arrow with an integer in D, we define $r = M + 1$. We enumerate (unpaired) up arrows from left to right by $1, 2, \ldots, N_\uparrow$. We denote by $A_{(i)}(D)$, $1 \le i \le N_\uparrow - 1$, a diagram obtained from D by connecting the i-th and $(i+1)$-th up arrows via an arc. We have three cases for D: 1) the leftmost arrow is an up arrow, 2) the leftmost arrow is an unpaired down arrow, 3) the leftmost arrow is a down arrow forming an arc.

Case 1. We have two cases for D: a) D does not have an unpaired down arrow, and b) D has an unpaired down arrow.

Case 1-a. We denote by $A_{(N_\uparrow)}$ a diagram obtained from D by changing the $N_\uparrow$-th up arrow to a down arrow with the integer $r - 1$ for $r \ge 2$. If $r = 1$, denote by $A_{(N_\uparrow)}$ a diagram obtained from D by changing the $N_\uparrow$-th up arrow to an unpaired down arrow. The action of e_0 on D is given by

$$e_0(D) = \sum_{1 \le i \le N_\uparrow} q^{-(i-1)} A_{(i)}(D) + (q^{-(r+N_\uparrow - 2)}(1 - \delta_{1,r}) - Q_0)D,$$

where $\delta_{i,j}$ is the delta function, that is, $\delta_{i,j}$ is one if $i = j$ and zero otherwise.

Case 1-b. We denote by $A_{(N_\uparrow)}(D)$ a diagram obtained from D by connecting the $N_\uparrow$-th up arrow and the unpaired down arrow via a dashed arc, and by $A_{(N_\uparrow+1)}(D)$ a diagram obtained from D by changing the unpaired down arrow to an up arrow. The action of e_0 on D is given by

$$e_0(D) = \sum_{1 \le i \le N_\uparrow+1} q^{-(i-1)} A_{(i)} - Q_0 D.$$

Case 2. Let $\tilde{D}$ be a diagram obtained from D by changing the unpaired down arrow to an up arrow. The action of e_0 on D is given by

$$e_0(D) = \tilde{D} - Q_0^{-1} D.$$

Case 3. Let $\tilde{D}$ be a diagram obtained from D by changing the leftmost arc to two up arrows. We enumerate (unpaired) up arrows of $\tilde{D}$ from left to right by $1, 2 \ldots, N_\uparrow + 2$. We denote by $A_{(N_\uparrow+2)}(\tilde{D})$ a diagram obtained from $\tilde{D}$ by changing the $(N_\uparrow + 2)$-th up arrow to a down arrow. For each $1 \le i < j \le N_\uparrow + 1$ with $j - i \ge 2$, we denote by $B_{(i,j)}(\tilde{D})$ a diagram obtained from $\tilde{D}$ by connecting the i-th and the $(i + 1)$-th up arrows via an arc and the j-th and the $(j + 1)$-th up arrows via an arc. For each $1 \le i \le N_\uparrow + 1$, we denote by $B_{(i,i+1)}$ a diagram obtained from $\tilde{D}$ by connecting the i-th and the $(i + 3)$-th up arrows via an arc and the $(i + 1)$-th and the $(i + 2)$-th up arrows via an arc. We denote by $B_{(N_\uparrow,N_\uparrow+1)}(\tilde{D})$ a diagram obtained from $\tilde{D}$ by connecting the $(N_\uparrow + 1)$-th and the $(N_\uparrow + 2)$-th up arrows via an arc and putting a down arrow at the $(N_\uparrow + 2)$-th site. For each $1 \le i \le N_\uparrow$, we denote by $B_{(i,N_\uparrow+2)}(\tilde{D})$ a diagram obtained from $\tilde{D}$ by connecting the i-th and the $(i + 1)$-th up arrows via an arc and putting a down arrow at $(N_\uparrow + 2)$-th site. Finally, we denote by $B_{(N_\uparrow+1,N_\uparrow+2)}(\tilde{D})$ a diagram obtained from $\tilde{D}$ by putting two down arrows at the $(N_\uparrow + 1)$-th and the $(N_\uparrow + 2)$-th sites.

We have three cases for D: a) $r = 1$ and D does not have an unpaired down arrow, b) $r \ge 2$, and c) D has an unpaired down arrow.

Case 3-a. The action of e_0 on D is given by

$$e_0(D) = (1 - q^{-2N_\uparrow - 2})\tilde{D} - Q_0^{-1}D + q^{-1}(Q_0 - Q_0^{-1}) \sum_{2 \leq i \leq N_\uparrow + 2} q^{-(i-2)}A_{(i)}(\tilde{D})$$
$$-q^{-1} \sum_{1 \leq i < j \leq N_\uparrow + 2} c_{(j,i)}B_{(i,j)}(\tilde{D}),$$

where $c_{(j,i)}$ is defined in Eqn. (12).

Case 3-b. The action of e_0 on D is given by

$$e_0(D) = (1 + q^{-N_\uparrow - r}(Q_0 - Q_0^{-1}) - q^{-2N_\uparrow - 2r})\tilde{D} - (Q_0^{-1} + q^{-N_\uparrow - r})D$$
$$+ \sum_{2 \leq i \leq N_\uparrow + 2} \tilde{c}_{(i)}A_{(i)}(\tilde{D}) - q^{-1} \sum_{1 \leq i < j \leq N_\uparrow + 2} c_{(j,i)}B_{(i,j)}(\tilde{D})$$

where

$$\tilde{c}_{(i)} := q^{-(i-1)}(Q_0 - Q_0^{-1}) - q^{-N_\uparrow - r - i + 1}(1 + (1 - \delta_{2,r}\delta_{i,N_\uparrow + 2})q^2).$$

Case 3-c. For each $1 \leq i \leq N_\uparrow + 1$, we denote by $B_{(i,N_\uparrow + 3)}(\tilde{D})$ a diagram obtained from $\tilde{D}$ by connecting the i-th and $(i+1)$-th up arrows via an arc and changing the unpaired down arrow to an up arrow. We also denote by $A'_{(N_\uparrow + 2)}(\tilde{D})$ a diagram obtained from $\tilde{D}$ by connecting the $(N_\uparrow + 2)$-th up arrow and the unpaired down arrow via an arc. Then, the action of e_0 on D is given by

$$e_0(D) = (1 - q^{-2N_\uparrow - 4})\tilde{D} - Q_0^{-1}D + (Q_0 - Q_0^{-1}) \sum_{2 \leq i \leq N_\uparrow + 3} q^{-(i-1)}A_{(i)}(\tilde{D})$$
$$-q^{-2N_\uparrow - 3}A'_{(N_\uparrow + 2)} - q^{-1} \sum_{1 \leq i < j \leq N_\uparrow + 2} c_{(j,i)}B_{(i,j)}(\tilde{D})$$
$$-q^{-1} \sum_{1 \leq i \leq N_\uparrow + 1} q^{-N_\uparrow - i}(1 + (1 - \delta_{1,i})q^2)B_{(i,N_\uparrow + 3)}(\tilde{D}).$$

Type BII. We have three cases for the leftmost arrow a of D: 1) a is an up arrow, 2) a is an e- or o-unpaired down arrow, and 3) a is a down arrow forming an arc.

Case 1. Let $N_\uparrow$ be the number of up arrows of a diagram D. We enumerate up arrows from left to right by $1, 2, \ldots, N_\uparrow$. For each $1 \leq i \leq N_\uparrow - 1$, we denote by $A_{(i)}(D)$ a diagram obtained from D by connecting the i-th and $(i+1)$-th up arrows via an arc. We denote by $A_{(N_\uparrow)}(D)$ a diagram obtained from D by changing the $N_\uparrow$-th up arrow to an e- or o-unparied down arrow.

Suppose that the leftmost down arrow of D is an o-unpaired down arrow. Then, the action of e_0 on D is given by

$$e_0(D) = -(Q_0 + q^{-N_\uparrow})D + \sum_{1 \leq i \leq N_\uparrow} q^{-(i-1)}A_{(i)}.$$

Suppose that the leftmost down arrow of D is an e-unpaired down arrow or D does not have a down arrow. The action of e_0 on D is given by

$$e_0(D) = (q^{-N_\uparrow + 1}Q - Q_0)D + \sum_{1 \leq i \leq N_\uparrow} q^{-(i-1)}A_{(i)}.$$

Case 2. The action of e_0 on a partial diagram of D is given by

$$e_0\left(\begin{array}{c}|\\ \mathrm{e}\end{array}\right) = (q^{-1}Q - Q_0^{-1}) \begin{array}{c}|\\ \mathrm{e}\end{array} - (q^{-2}Q^2 - q^{-1}QQ_0^{-1} + q^{-1}QQ_0 - 1) \uparrow,$$

$$e_0\left(\begin{array}{c}|\\ \mathrm{o}\end{array}\right) = -(Q^{-1} + Q_0^{-1}) \begin{array}{c}|\\ \mathrm{o}\end{array} + (1 + Q^{-1}Q_0 - Q^{-1}Q_0^{-1} - Q^{-2}) \uparrow,$$

Case 3. We have two cases for D:

(a) The leftmost down arrow is an o-unpaired down arrow. The action of e_0 on D is given by

$$e_0\left(\smile\right) = -q^{-1} \begin{array}{cc}|&|\\ \mathrm{o}&\mathrm{e}\end{array} - (Q_0^{-1} - q^{-2}Q) \smile - (q^{-1}(Q_0^{-1} + Q^{-1}) - q^{-3}Q - q^{-1}Q_0) \uparrow \begin{array}{c}|\\ \mathrm{e}\end{array}$$
$$+ (1 - q^{-4}Q^2 + q^{-2}Q_0^{-1}Q - q^{-2}QQ_0) \uparrow\uparrow.$$

(b) The leftmost down arrow is an e-unpaired down arrow or D does not have a down arrow. The action of e_0 on D is given by

$$e_0\left(\smile\right) = -q^{-1} \begin{array}{cc}|&|\\ \mathrm{e}&\mathrm{o}\end{array} - (Q_0^{-1} + q^{-1}Q^{-1}) \smile - (q^{-1}Q_0^{-1} + q^{-2}(Q^{-1} - Q) - q^{-1}Q_0) \uparrow \begin{array}{c}|\\ \mathrm{o}\end{array}$$
$$+ (1 - q^{-2}Q^{-2} - q^{-1}Q^{-1}Q_0^{-1} + q^{-1}Q^{-1}Q_0) \uparrow\uparrow.$$

Type BIII. Let $N_\uparrow$ be the number of up arrows of a diagram D and r be the largest integer attached to down arrows with a circled integer. If there is no down arrow in D, we define $r = 0$. We enumerate up arrow from left to right. For each $1 \le i \le N_\uparrow - 1$, we denote by $A_{(i)}(D)$ a diagram obtained from D by connecting the i-th and $(i+1)$-th arrows via an arc. We denote by $A_{(N_\uparrow)}(D)$ a diagram obtained from D by changing the $N_\uparrow$-th up arrow to a down arrow with a circled integer $r + 1$. We have three cases for D: 1) the leftmost arrow is an up arrow, 2) the leftmost arrow is a down arrow with a circled integer r, and 3) the leftmost arrow is a down arrow forming an arc.

Case 1. The action of e_0 on D is given by

$$e_0(D) = (q^{-N_\uparrow + r + 1}Q^{-1} - Q_0)D + \sum_{1 \le i \le N_\uparrow} q^{-(i-1)} A_{(i)}(D).$$

Case 2. The action of e_0 on D is given by

$$e_0\left(\begin{array}{c}|\\ \textcircled{r}\end{array}\right) = (-Q_0^{-1} - q^{r-1}Q^{-1}) \begin{array}{c}|\\ \textcircled{r}\end{array} + (1 + q^{r-1}Q^{-1}(Q_0 - Q_0^{-1}) - q^{2r-2}Q^{-2}) \uparrow.$$

Case 3. Let $\tilde{D}$ be a diagram obtained from D by changing the leftmost arc to two up arrows. For each $1 \le i < j \le N_\uparrow + 1$ with $j - i \ge 2$, we denote by $B_{(i,j)}(\tilde{D})$ a diagram obtained from $\tilde{D}$ by connecting the i-th and the $(i+1)$-th up arrows via an arc and the j-th and $(j+1)$-th up arrows via an arc. For each $1 \le i \le N_\uparrow - 1$, we denote by $B_{(i,i+1)}(\tilde{D})$ a diagram obtained from $\tilde{D}$ by connecting the i-th and the $(i+3)$-th up arrows via an arc and the $(i+1)$-th and $(i+2)$-th up arrows via an arc. We denote by $B_{(N_\uparrow, N_\uparrow+1)}(\tilde{D})$ a diagram obtained from $\tilde{D}$ by connecting the $(N_\uparrow + 1)$-th and the $(N_\uparrow + 2)$-th up arrows via an arc and putting a down arrow with the circled integer $r + 1$ at the $N_\uparrow$-th site. For each $1 \le i \le N_\uparrow$, we denote by $B_{(i, N_\uparrow+2)}(\tilde{D})$ a diagram obtained from $\tilde{D}$ by connecting the i-th and the $(i + 1)$-th up arrows via an arc and putting a down arrow with the circled integer $r + 1$ at the $N_\uparrow + 2$-th site. We denote by $B_{(N_\uparrow+1, N_\uparrow+2)}(\tilde{D})$ a diagram obtained from

$\tilde{D}$ by putting two down arrows with circled integers $r+1$ and $r+2$ at the $(N_\uparrow + 1)$-th and the $(N_\uparrow + 2)$-th sites. The action of e_0 on D is given by

$$
\begin{aligned}
e_0(D) \;=\; & (1 + q^{-N_\uparrow + r - 1} Q^{-1}(Q_0 - Q_0^{-1}) - q^{-2N_\uparrow + 2r - 2} Q^{-2})\tilde{D} + (-Q_0^{-1} - q^{-N_\uparrow + r - 1} Q^{-1})D \\
& + \sum_{2 \le i \le N_\uparrow + 2} \tilde{c}_{(i)} A_{(i)}(\tilde{D}) - q^{-1} \sum_{1 \le i < j \le N_\uparrow + 2} c_{(j,i)} B_{(i,j)}(\tilde{D})
\end{aligned}
$$

where $c_{(j,i)}$ is defined in Eqn.(12) and

$$
\tilde{c}_{(i)} := q^{-(i-1)}(Q_0 - Q_0^{-1}) - q^{-N_\uparrow + r - i}(1 + q^2)Q^{-1}.
$$

4. Eigensystem of X

Since X commutes with the Hamiltonian H^{1B} (Theorem 2.2), an eigenvector of X with the multiplicity one is also an eigenvector of H^{1B}. We will first find an eigenvector of X with the multiplicity one.

4.1. Type A.

We consider the action of X on the Kazhdan–Lusztig basis of type A. Let D be a diagram of type A, $n_\uparrow$ be the number of (unpaired) up arrows and $n_\downarrow$ be the number of (unpaired) down arrows. We define the weight of D by $\mathrm{wt}(D) = n_\uparrow - n_\downarrow$. We enumerate the (unpaired) up arrows from left to right by $1, 2, \ldots, n_\uparrow$. For each i, $1 \le i < n_\uparrow$, we denote by $E_{(i)}(D)$ a diagram obtained from D by connecting the i-th and $(i+1)$-th up arrows via an arc. We denote by $E_{(n_\uparrow)}$ a diagram obtained from D by changing the $n_\uparrow$-th up arrow to a down arrow. Similarly, we enumerate (unpaired) down arrows from right to left by $1, 2, \ldots, n_\downarrow$. For each i, $1 \le i < n_\downarrow$, we denote by $F_{(i)}(D)$ a diagram obtained from D by connecting the i-th and $(i+1)$-th down arrows via an arc. We denote by $F_{(n_\downarrow)}$ a diagram obtained from D by changing the $n_\downarrow$-th down arrow to an up arrow.

We define the action of X by

$$
(14) \qquad X(D) := \sum_{1 \le i \le n_\uparrow} [i] E_{(i)}(D) + \sum_{1 \le i \le n_\downarrow} q^{\mathrm{wt}(D)+1} [i] F_{(i)}(D) + q^{\mathrm{wt}(D)} \frac{Q - Q^{-1}}{q - q^{-1}} D.
$$

Example 4.1. *Let D be a diagram depicted as*

$$
D = \; \uparrow \uparrow \; \underset{\smile}{\overset{\frown}{\cup}} \; \uparrow \downarrow \downarrow \; \cup \; \downarrow
$$

Then, we have $\mathrm{wt}(D) = 0$ *and*

$$
E_{(1)} = \; \cup \; \underset{\smile}{\overset{\frown}{\cup}} \; \uparrow \downarrow \downarrow \; \cup \; \downarrow \, , \qquad E_{(2)} = \; \uparrow \; \underset{\smile}{\overset{\frown}{\cup}} \; \downarrow \downarrow \; \cup \; \downarrow \, ,
$$

$$
E_{(3)} = \; \uparrow \; \uparrow \; \underset{\smile}{\overset{\frown}{\cup}} \; \downarrow \downarrow \downarrow \; \cup \; \downarrow \, , \qquad F_{(1)} = \; \uparrow \; \uparrow \; \underset{\smile}{\overset{\frown}{\cup}} \; \uparrow \downarrow \; \underset{\smile}{\overset{\frown}{\cup}} \, ,
$$

$$
F_{(2)} = \; \uparrow \; \uparrow \; \underset{\smile}{\overset{\frown}{\cup}} \; \uparrow \; \cup \; \cup \; \downarrow \, , \qquad F_{(3)} = \; \uparrow \; \uparrow \; \underset{\smile}{\overset{\frown}{\cup}} \; \uparrow \; \uparrow \downarrow \; \cup \; \downarrow \, .
$$

Therefore, the action of X on D is given by

$$
X(D) = E_{(1)} + q F_{(1)} + [2](E_{(2)} + q F_{(2)}) + [3](E_{(3)} + q F_{(3)}) + [Q; 0]D.
$$

Theorem 4.2. *The above definition provides the action of X on the Kazhdan–Lusztig basis of type A.*

Proof. We prove Theorem by induction. When $N = 1$ or 2, Theorem holds true by a direct computation. Suppose that Theorem is true up to some $N \geq 2$. We have two cases for the leftmost arrow: 1) an up arrow and 2) a down arrow.

Case 1. Let D be a diagram $\uparrow D'$ where D' is a diagram of length $N - 1$. By using the comultiplication, we have

$$
\begin{aligned}
X(D) &= (K \otimes X + q^{-1}KE \otimes 1 + F \otimes 1)(\uparrow D') \\
&= q \uparrow X(D') + \downarrow D'.
\end{aligned}
$$

From the assumption, we have

$$
\begin{aligned}
\uparrow X(D') &= \sum_{1 \leq i \leq n'_\uparrow} [i] \uparrow E_{(i)}(D') + \sum_{1 \leq i \leq n'_\downarrow} q^{\mathrm{wt}(D')+1}[i] \uparrow F_{(i)}(D') + q^{\mathrm{wt}(D')}\frac{Q - Q^{-1}}{q - q^{-1}} \uparrow D' \\
&= \sum_{2 \leq i \leq n_\uparrow} [i - 1]E_{(i)}(D) + \sum_{1 \leq i \leq n_\downarrow} q^{\mathrm{wt}(D)}[i]F_{(i)}(D) + q^{\mathrm{wt}(D)-1}\frac{Q - Q^{-1}}{q - q^{-1}}D
\end{aligned}
$$

where $n'_\uparrow$ (resp. $n'_\downarrow$) is the number of up (resp. down) arrows in D' and we have used $\mathrm{wt}(D) = \mathrm{wt}(D') + 1$, $n_\uparrow = n'_\uparrow + 1$ and $n_\downarrow = n'_\downarrow$. We also have

$$
\downarrow D' = \sum_{1 \leq i \leq n_\uparrow} q^{-(i-1)} E_{(i)}(D)
$$

By $q[i - 1] + q^{-(i-1)} = [i]$, the sum $q \uparrow X(D') + \downarrow D'$ gives a desired expression.

Case 2. We have two cases for D: a) D has no up arrows and b) the leftmost arrow forms an arc.

Case 2-a. The diagram D is written as $\downarrow D'$. We want to compute $X(D) = q^{-1} \downarrow X(D') + \uparrow D'$. We have

$$
\text{(15)} \qquad \downarrow X(D') = \sum_{1 \leq i \leq n'_\downarrow} q^{\mathrm{wt}(D')+1}[i] \downarrow F_{(i)}(D') + q^{\mathrm{wt}(D')}\frac{Q - Q^{-1}}{q - q^{-1}} \downarrow D'.
$$

Note that $\uparrow D' = F_{(n_\downarrow)}(D)$, $\downarrow F_{(i)}(D') = F_{(i)}(D)$ for $1 \leq i \leq n_\downarrow - 1$ and $\downarrow F_{(n'_\downarrow)}(D') = F_{(n_\downarrow-1)}(D) + q^{-1}F_{(n_\downarrow)}(D)$. We also have $\mathrm{wt}(D') = \mathrm{wt}(D) + 1$ and $[i]q^{-i-1} + 1 = q^{-i}[i + 1]$. Inserting these into Eqn.(15), we obtain Eqn.(14).

Case 2-b. Let D' be a diagram obtained from D by removing arcs and $D'' = X(D')$. From the definition of the action of X, the action of X on D is obtained by inserting the removed arcs of D into diagrams D'' at the same position as D. Thus, without loss of generality, we assume

$$
D = \bigcup \underbrace{\uparrow \ldots \uparrow}_{x_1}\underbrace{\downarrow \ldots \downarrow}_{x_2}.
$$

The action of X on D is given by

$$
\begin{aligned}
X(D) &= X(\downarrow \underbrace{\uparrow \ldots \uparrow}_{x_1+1}\underbrace{\downarrow \ldots \downarrow}_{x_2}) - q^{-1}X(\uparrow\downarrow \underbrace{\uparrow \ldots \uparrow}_{x_1}\underbrace{\downarrow \ldots \downarrow}_{x_2}) \\
&= q^{-1} \downarrow X(\underbrace{\uparrow \ldots \uparrow}_{x_1+1}\underbrace{\downarrow \ldots \downarrow}_{x_2}) + \underbrace{\uparrow \ldots \uparrow}_{x_1+2}\underbrace{\downarrow \ldots \downarrow}_{x_2} - \uparrow X(\downarrow \underbrace{\uparrow \ldots \uparrow}_{x_1}\underbrace{\downarrow \ldots \downarrow}_{x_2}) - q^{-1} \downarrow\downarrow \underbrace{\uparrow \ldots \uparrow}_{x_1}\underbrace{\downarrow \ldots \downarrow}_{x_2} \\
&= \bigcup X(\underbrace{\uparrow \ldots \uparrow}_{x_1}\underbrace{\downarrow \ldots \downarrow}_{x_2}).
\end{aligned}
$$

Thus we have a desired expression (14). $\qquad \square$

Theorem 4.3. *X has the eigenvalue $[Q; N - 2i]$, $0 \leq i \leq N$, of multiplicity $\binom{N}{i}$.*

Proof. We consider the matrix representation of $X = (X_{D,D'})$ on the Kazhdan–Lusztig bases. We will construct eigenvectors of X.

Let $\mathcal{D}_n$, $0 \leq n \leq \lfloor N/2 \rfloor$, be the set of diagrams with n arcs, $\mathcal{D}_{\overline{n}}^{\leq} := \bigcup_{0 \leq i \leq n} \mathcal{D}_i$ and $\mathcal{D}_{\overline{n}}^{\geq} := \bigcup_{n \leq i \leq \lfloor N/2 \rfloor} \mathcal{D}_i$. The cardinality of $\mathcal{D}_n$, $|\mathcal{D}_n|$, is given by

$$|\mathcal{D}_n| = (N - 2n + 1)\left(\binom{N}{n} - \binom{N}{n-1}\right).$$

Let I be a set of the positions of arcs from left in a diagram $D \in \mathcal{D}_n$ and denote by $\mathcal{D}_n^I$ the set of diagrams with arcs located as I. Then, the set $\mathcal{D}_n$ is a direct sum of $\mathcal{D}_n^I$, that is, $\mathcal{D}_n = \bigsqcup_I \mathcal{D}_n^I$. The cardinality of $\mathcal{D}_n^I$ is given by $|\mathcal{D}_n^I| = N - 2n + 1$.

We define a vector $\psi := \sum_D \psi_D D$ with the following property. We set $\psi_D = 0$ for all $D \in \mathcal{D}_{\overline{n}}^{\leq}$ except some $D \in \mathcal{D}_n^I$. Let $A = (X_{D,D'})_{D,D' \in \mathcal{D}_n^I}$ be a submatrix of X. If there exists an eigenvector ψ of X with the above property, the eigenvalues of A coincides with the ones of X. This is because an element of $\mathcal{D}_{\overline{n-1}}^{\leq}$ cannot be appeared in the expansion of $X(D)$ for $D \in \mathcal{D}_{\overline{n}}^{\geq}$ (see Eqn.(14)). The submatrix A is of size $N - 2n + 1$ and tridiagonal whose entries are

$$A_{i,i} = q^{N-2n+2-2i}\frac{Q - Q^{-1}}{q - q^{-1}}, \qquad A_{i,i-1} = [N - 2n + 1 - i], \qquad A_{i,i+1} = q^{N-2n-2i}[i].$$

From Lemma A.1, the eigenvalues are $[Q; N - 2n - 2\lambda]$, $\lambda = 0, 1, \ldots, N - 2n$ and the multiplicities are one. For each eigenvalue of A, there exists a unique eigenvector and we set ψ_D, $D \in \mathcal{D}^I$, as this eigenvector. Given an eigenvalue a of A and ψ_D, $D \in \mathcal{D}_{\overline{n}}^{\leq}$, other components ψ_D, $D \in \mathcal{D}_{\overline{n+1}}^{\geq}$ are determined by solving the eigenvalue problem. If the multiplicity of a (as the eigenvalue of X) is not one, then ψ may not be determined uniquely. However, we have at least one eigenvector of X and this eigenvector is characterized by n, a and I. Since the eigenvalues are of the form $[Q; N - 2j]$, $0 \leq j \leq N$, the multiplicity is given by

$$\sum_{i=0}^{\min(j,N-j)} |\mathcal{D}_n|/|\mathcal{D}_n^I| = \sum_{i=0}^{\min(j,N-j)} \binom{N}{i} - \binom{N}{i-1}$$
$$= \binom{N}{j}.$$

This completes the proof. $\square$

Let D be a diagram of Type A. We define S as the set of arcs, $S_\uparrow$ as the set of unpaired up arrows and $S_\downarrow$ as the set of unpaired down arrows. We define

$$N_1 = q^{d(d-1)/2}Q^d$$

where $d = |S_\uparrow| + |S|$.

We enumerate up arrows, down arrows and arcs from left. If there are arcs inside of an arc, we increase an integer one by one from outside to inside. Let N_A be an integer assigned to $A \in S \cup S_\uparrow \cup S_\downarrow$. We define

$$N_2 := \prod_{A \in S \cup S_\downarrow} [N_A].$$

If the i-th down arrow and the j-th $(j > i)$ up arrow form an arc, we define the size of arc as $(j - i + 1)/2$. Let B be an arc and m_B be its size. We define

$$N_3 := \prod_{B \in S} [m_B]^{-1}.$$

Similarly, we enumerate down arrows and arcs from right. Let N_C be an integer assigned to $C \in S \cup S_\downarrow$. We define

$$N_4 := \prod_{C \in S_\downarrow} [N_C]^{-1}.$$

In the above notation, we define the vector $\Psi := \sum_D \Psi_D |D\rangle$:

Definition 4.4. $\Psi_D = N_1 \cdot N_2 \cdot N_3 \cdot N_4$.

Example 4.5. *Let D be a diagram depicted as*

$$\uparrow \uparrow \; \cup \!\! \cup \; \uparrow \downarrow \; \cup \downarrow .$$

We have

$$N_1 = q^{15}Q^6, \qquad N_2 = \frac{[8]!}{[2][5]}, \qquad N_3 = [2]^{-1}, \qquad N_4 = [3]^{-1}.$$

Theorem 4.6. Ψ *is the eigenvector of X with the eigenvalue $[Q; N]$.*

Proof. Let D be a diagram starting with n_1 up arrows, followed by an outer arc of size m_1, followed by n_2 up arrows, followed by an outer arc of size m_2, $\cdots$, followed by n_{I+1} up arrows, followed by n'_{J+1} down arrows, followed by an outer arc of size m'_J, followed by n'_J down arrows, $\cdots$, and ending with n'_1 down arrows. As a diagram, D is

(16) $$\underbrace{\uparrow \ldots \uparrow}_{n_1} \underbrace{\cup}_{\text{size } m_1} \uparrow \ldots \uparrow \underbrace{\cup}_{\text{size } m_I} \underbrace{\uparrow \ldots \uparrow}_{n_{I+1}} \underbrace{\downarrow \ldots \downarrow}_{n'_{J+1}} \underbrace{\cup}_{\text{size } m'_J} \downarrow \ldots \downarrow \underbrace{\cup}_{\text{size } m'_1} \underbrace{\downarrow \ldots \downarrow}_{n'_1}$$

where the inside of an outer arc is filled with arcs.

Set $N_\uparrow = \sum_{i=1}^{I+1} n_i$, $N_\downarrow = \sum_{i=1}^{J+1} n'_i$, $M = \sum_{i=1}^{I} m_i$ and $M' = \sum_{i=1}^{J} m'_i$. The component Ψ_D is explicitly given by

(17) $$\Psi_D = q^{d(d-1)/2} Q^d \prod_{i=1}^{I} \frac{[\sum_{j=1}^{i}(n_j + m_j)]!}{[n_i + \sum_{j=1}^{i-1}(n_j + m_j)]!} \cdot \frac{[N_\uparrow + M + N_\downarrow + M']!}{[N_\uparrow + M]!} \cdot \prod_{A \in S} [m_A]^{-1}$$

$$\times \prod_{i=1}^{J+1} \frac{[\sum_{j=1}^{i-1}(n'_j + m'_j)]!}{[n'_i + \sum_{j=1}^{i-1}(n'_j + m'_j)]!}$$

where $d = N_\uparrow + M + M'$.

Let $X_{D,D'}$ be the matrix representation of the action of X on the Kazhdan–Lusztig bases, that is, $X(D) = \sum_{D'} X_{D',D} D'$. Note that the explicit formulae for $X_{D',D}$ is given by Eqn.(14). We want to show that

(18) $$\sum_{D'} X_{D,D'} \Psi_{D'} = [Q; N] \Psi_D.$$

We have five cases for $X_{D,D'} \neq 0$: 1) D' does not have an outer arc of size m_i, 2) D' does not have an outer arc of size m'_i, 3) D' has $n_{I+1} + 1$ up arrows and $n'_{J+1} - 1$ down arrows instead of n_{I+1}

up arrows and n_{J+1} down arrows, 4) D' has $n_{I+1} - 1$ up arrows and $n'_{J+1} + 1$ down arrows instead of n_{I+1} up arrows and n_{J+1} down arrows, and 5) $D' = D$.

In the first case, we have $X_{D,D'} = [1 + \sum_{j=1}^{i} n_j]$. The contribution to the left hand side of Eqn.(18) is

$$q^d Q \Psi_D \sum_{i=1}^{I} \left[1 + \sum_{j=1}^{i} n_j \right] [m_i] \frac{[1 + N_\uparrow + N_\downarrow + M + M']}{[1 + N_\uparrow + M]} \frac{\prod_{j \geq i+1}^{I}[1 + \sum_{k=1}^{j}(n_k + m_k)]}{\prod_{j \geq i}^{I}[1 + n_j + \sum_{k=1}^{j-1}(n_k + m_k)]}.$$

Inserting Lemma A.2 into the above expression, we obtain

(19) $$q^d Q \Psi_D \frac{[1 + N_\uparrow + N_\downarrow + M + M'][M]}{[1 + M + N_\uparrow]}.$$

In the second case, we have $X_{D,D'} = q^{N_\uparrow - N_\downarrow - 1}[1 + \sum_{j=1}^{i} n'_j]$. The contribution to the left hand side of Eqn.(18) is

$$q^{-d'} Q^{-1} \Psi_D \sum_{i=1}^{J} \left[1 + \sum_{j=1}^{i} n'_j \right] [m'_i][1 + N_\uparrow + d'] \frac{\prod_{j \geq i+2}[1 + \sum_{k=1}^{j-1}(n'_{k-1} + m'_{k-1})]}{\prod_{j \geq i}[1 + n'_j + \sum_{k=1}^{j-1}(n'_{k-1} + m'_{k-1})]}$$

where $d' = M + M' + N_\downarrow$. Inserting Lemma A.3 into the above expression, we obtain

(20) $$q^{-d'} Q^{-1} \Psi_D \frac{[1 + N_\uparrow + M + N_\downarrow + M'][M']}{[1 + N_\downarrow + M']}.$$

In the third case, we have $X_{D,D'} = [N_\uparrow + 1]$. The contribution is

(21) $$q^d Q \Psi_D \frac{[N_\downarrow + M'][N_\uparrow + 1]}{[N_\uparrow + M + 1]}.$$

In the fourth case, we have $X_{D,D'} = [N_\downarrow + 1]$. The contribution is

(22) $$q^{-d'} Q^{-1} \Psi_D \frac{[N_\uparrow + M][N_\downarrow + 1]}{[N_\downarrow + M' + 1]}.$$

In the fifth case, we have $X_{D,D} = q^{N_\uparrow - N_\downarrow}[Q; 0]$. The contribution is

(23) $$q^{N_\uparrow - N_\downarrow}[Q; 0]\Psi_D.$$

Note that $N = N_\uparrow + N_\downarrow + 2M + 2M'$. We obtain the right hand side of Eqn.(18) as the sum of Eqns.(19) to (23). This completes the proof.   □

4.2. Type BI. We consider the action of X on the Kazhdan–Lusztig basis of type BI.

Let D be a diagram of type BI and $N_\uparrow$ be the number of unpaired up arrows. Recall that D consists of up arrows, arcs, at most one unpaired down arrow, dashed arcs and down arrows with an integer p, $1 \leq p \leq M$. We enumerate the (unpaired) up arrows from left to right by $1, 2 \ldots, N_\uparrow$. For each i, $1 \leq i < N_\uparrow$, we denote by $X_{(i)}(D)$ a diagram obtained from D by connecting the i-th up arrow and $(i + 1)$-th up arrow via an arc.

Suppose D has an unpaired down arrow. We denote by $X_{(N_\uparrow)}(D)$ a diagram obtained from D by connecting $N_\uparrow$-th up arrow and the unpaired down arrow via a dashed arc. We denote by $X_{(N_\uparrow+1)}(D)$ a diagram obtained from D by changing the unpaired down arrow to an up arrow. We define the action of X on D by

(24) $$X(D) := \sum_{i=1}^{N_\uparrow + 1} [i] X_{(i)}(D).$$

Suppose D does not have an unpaired down arrow. We regard the down arrow with a star as the down arrow with the integer one. Let r be the smallest integer attached to down arrows with an integer $1 \leq p \leq M$. If there is no down arrow with an integer, we define $r = M + 1$. We denote by $X_{(N_\uparrow)}(D)$ a diagram obtained from D by changing the $N_\uparrow$-th up arrow to a down arrow. The action of X on D is defined by

$$(25) \qquad X(D) := \sum_{i=1}^{N_\uparrow} [i] X_{(i)}(D) + (1 - \delta_{1,r})[N_\uparrow + r - 1]D.$$

where $\delta_{i,j}$ is the Delta function satisfying $\delta_{i,i} = 1$ and $\delta_{i,j} = 0$ for $i \neq j$.

Example 4.7. *Let D be a diagram depicted as*

$$D = \; \uparrow \; \downarrow \; \overset{\smile}{\underset{\bigstar}{\cup}} \; | \; \underset{2}{\cup} \; | \; \cup \; ,$$

where $M = 2$. We have

$$X_{(1)} = \; \smile \; \overset{\smile}{\underset{\bigstar}{\cup}} \; | \; \underset{2}{\cup} \; | \; \cup \; , \qquad X_{(2)} = \; \uparrow \; \uparrow \; \overset{\smile}{\underset{\bigstar}{\cup}} \; | \; \underset{2}{\cup} \; | \; \cup \; .$$

The action of X on D is

$$X(D) = X_{(1)}(D) + [2]X_{(2)}(D).$$

Example 4.8. *Let D be a diagram depicted as*

$$D = \; \uparrow \; \cup \; \uparrow \; \underset{2}{|} \; \underset{3}{\cup\cup} \; | $$

where $M = 3$ and $r = 2$. We have

$$X_{(1)} = \; \cup\cup \; \underset{2}{|} \; \underset{3}{\cup\cup} \; | \; , \qquad X_{(2)} = \; \uparrow \; \cup\cup \; \underset{\bigstar 2}{|} | \; \underset{3}{\cup\cup} \; | \; .$$

The action of X on D is

$$X(D) = X_{(1)}(D) + [2]X_{(2)}(D) + [3]D.$$

Theorem 4.9. *The above definitions (24) and (25) provides the action of X on the Kazhdan–Lusztig basis of type BI.*

Proof. We prove Theorem by induction. When $N = 1$, Theorem is true by a straightforward calculation. We assume that Theorem holds true up to $N - 1 \geq 1$. Let D be a diagram of length N. We have two cases for the leftmost arrow of D: 1) an up arrow and 2) a down arrow.

Case 1. In this case, a diagram D is written as $D = \uparrow D'$. By using the comultiplication, we have

$$(26) \qquad X(\uparrow D') = q \uparrow X(D') + \downarrow D'.$$

We have two cases for D': a) D' has an unpaired down arrow and b) D' does not have an unpaired down arrow.

Case 1-a. Inserting Eqn.(24) and $\downarrow D' = \sum_{i=1}^{N_\uparrow+1} q^{-(i-1)} X_{(i)}(D)$ into Eqn.(26), we obtain

$$
\begin{aligned}
X(\uparrow D') &= q \sum_{i=2}^{N_\uparrow+1} [i-1] X_{(i)}(D) + \sum_{i=1}^{N_\uparrow+1} q^{-(i-1)} X_{(i)}(D) \\
&= \sum_{i=1}^{N_\uparrow+1} [i] X_{(i)}(D),
\end{aligned}
$$

where we have used $q[i-1] + q^{-(i-1)} = [i]$.

Case 1-b. We have

(27)
$$
\downarrow D' = \sum_{i=1}^{N_\uparrow} q^{-(i-1)} X_{(i)}(D) + (1-\delta_{1,r}) q^{N_\uparrow+r-2} D.
$$

Inserting Eqns.(25) and (27) into Eqn.(26), we obtain

$$
\begin{aligned}
X(\uparrow D') &= q \sum_{i=2}^{N_\uparrow} [i-1] X_{(i)}(D) + q(1-\delta_{1,r})[N_\uparrow+r-2]D + \sum_{i=1}^{N_\uparrow} q^{-(i-1)} X_{(i)}(D) \\
&\qquad + (1-\delta_{1,r}) q^{N_\uparrow+r-2} D \\
&= \sum_{i=1}^{N_\uparrow} [i] X_{(i)}(D) + (1-\delta_{1,r})[N_\uparrow+r-1]D.
\end{aligned}
$$

Case 2. We have four cases for the leftmost down arrow a of D: a) the arrow a is an unpaired arrow, b) the arrow a forms a dashed arc, c) the arrow a is a down arrow with the integer r, $1 \leq r \leq M$, and d) the arrow a forms an arc.

Case 2-a. The diagram D is written as $D = \downarrow D'$. We have $X(\downarrow D') = q^{-1} \downarrow X(D') + \uparrow D'$. Since the diagram D' has no unpaired up arrows and $r = 1$, we obtain $X(D') = 0$ by using Eqn.(25). Therefore, we have $X(\downarrow D') = \uparrow D'$.

Case 2-b. Let E be a diagram obtained from D by removing arcs and $E' = X(E)$. From the definition of the action of X, the action of X on D is obtained by inserting the removed arcs of D into diagrams of E' at the same position as D. Thus, without loss of generality, the diagram D is written as $D = \overset{\frown}{} D'$ where D' is a diagram of length $N-2$. We have

$$
\begin{aligned}
X(D) &= X(\downarrow\downarrow D') - q^{-1} X(\uparrow\uparrow D') \\
&= q^{-1} \downarrow X(\downarrow D') + \uparrow\downarrow D' - \uparrow X(\uparrow D') - q^{-1} \downarrow\uparrow D' \\
&= 0,
\end{aligned}
$$

where we have used $X(\uparrow D') = \downarrow D'$ and $X(\downarrow D') = \uparrow D'$.

Case 2-c. The diagram D is graphically written as $D = \underset{r}{\big|} D'$ where D' is a diagram of length $N-1$.

We have

$$
\begin{aligned}
X(D) &= X(\downarrow D') - q^{-r} X(\uparrow D') \\
&= q^{-1} \downarrow X(D') + \uparrow D' - q^{-r+1} \uparrow X(D') - q^{-r} \downarrow D' \\
&= [r-1] \downarrow D' - q^{-r} [r-1] \uparrow D' \\
&= [r-1] D,
\end{aligned}
$$

where we have used $X(D') = [r]D'$.

Case 2-d. By a similar argument to Case 2-b, without loss of generality, we assume that the diagram D is written as $D = \bigcup D'$ where D' is a diagram of length $N - 2$. We have

$$
\begin{aligned}
X(D) &= X(\downarrow\uparrow D') - q^{-1}X(\uparrow\downarrow D') \\
&= q^{-1}\downarrow X(\uparrow D') + \uparrow\uparrow D' - \uparrow X(\downarrow D') - q^{-1}\downarrow\downarrow D' \\
&= \downarrow\uparrow X(D') + q^{-1}\downarrow\downarrow D' + \uparrow\uparrow D' - q^{-1}\uparrow\downarrow X(D') - \uparrow\uparrow D' - q^{-1}\downarrow\downarrow D' \\
&= \bigcup X(D').
\end{aligned}
$$

In both Case 1 and 2, $X(D)$ coincides with the definitions (24) and (25). This completes the proof. $\square$

Let $N_\uparrow$ be the number of up arrows in D. Let r be the smallest integer attached to down arrows with an integer $1 \le p \le M$. If there is no down arrow with an integer, we define $r = M + 1$. We define an integer E_D as follows:

(1) If D has an unpaired down arrow, $E_D = -(N_\uparrow + 1)$.
(2) If D does not have an unpaired down arrow, $E_D = N_\uparrow + r - 1$.

Note that $|E_D|$ is the maximum integer which appears in the expansion of $X(D)$. We denote by $\mathcal{D}$ the set of diagrams of length N. For an integer $i \in \mathbb{Z}$, we define

$$Z_i := \#\{E_D | E_D = i \text{ and } D \in \mathcal{D}\}.$$

Theorem 4.10. *X has an eigenvalue $[N + M - 2i]$, $0 \le i \le N$, of multiplicity $\binom{N}{i}$.*

We omit the proof since one can apply the same method as [41, Theorem 6.11]. As a corollary, we have $Z_{N+M-2i} = \binom{N}{i}$. For each eigenvalue $[N + M - 2i]$, an eigenvector is characterized by a diagram D with $E_D = N + M - 2i$. See [41] for $M = 1$ case.

Let D be a diagram of Type BI, $N_\uparrow$ be the number of up arrows (excluding up arrows forming arcs), N_1 be the number of the unpaired down arrow (N_1 is either 0 or 1). Let S be the set of all arcs of D. If D has the down arrow with the integer M, S_R is defined as the set of arcs right to the down arrow with the integer M. Otherwise, S_R is the empty set. If D has the down arrow with a star, $S_{W'}$ is defined as the set of arcs which are left to the down arrow with the integer M and right to the down arrow with a star. Otherwise, $S_{W'}$ is the empty set. If D has the down arrow with a star, S_W is defined as the set of arcs which are left to the down arrow with a star and either right to the unpaired down arrow for $N_1 = 1$ or right to the leftmost down arrow forming a dashed arc for $N_1 = 0$. Otherwise, S_W is the empty set. If D has the down arrow with a star and up arrows, S_L is defined as the set of arcs which are left to the leftmost down arrow (which is an unpaired down arrow for $N_1 = 1$, the leftmost down arrow forming a dashed arc for $N_1 = 0$, or the down arrow with a star for $N_1 = 0$ and D without dashed arcs) and right to the rightmost up arrow. If D has the down arrow with a star but no up arrows, S_L is defined as the set of arcs which are left to the leftmost down arrow. Otherwise, S_L is the empty set. An arc A is called an *outer arc* if there are no arcs and no dashed arc outside of A. We denote the set of outer arc by S^+. We define $S_R^+ := S^+ \cap S_R$, $S_W^+ := S^+ \cap S_W$ and $S_L^+ := S^+ \cap S_L$.

Let T be the set of dashed arcs, U be the set of down arrows with integers $p, 2 \le p \le M$. We define T' as the set of dashed arcs except the leftmost one and U' as the set of down arrows with integers $p, 2 \le p \le M - 1$. Then, V (resp. V') is given by the union of U (resp. U') and the down arrow with a star if it exists.

We define the following values:

$$
\begin{aligned}
N_2 &:= N_\uparrow + N_1 + |S| + |T|, \\
N_3 &:= |S_W| + |T|, \\
N_4 &:= |S_{W'}| + |S_W| + |T| + M \\
N_5 &:= N - |S| + |S_L| + |S_W| + |S_{W'}| + |S_R| + M \\
N_6 &:= \begin{cases} [N_5]/[N_4], & \text{for } |V| = M \text{ and } N_1 = 0, \\ 1, & \text{otherwise.} \end{cases}
\end{aligned}
$$

We enumerate all arrows from left to right. Let s_1 be the integer assigned to the down arrow with a star and s_p be the integer assigned to the down arrow with the integer p, $2 \le p \le M$. If i-th down arrow and j-th $(i < j)$ up arrow forms an arc A, then the size of A is $(j - i + 1)/2$ and denoted by m_A. Similarly, if k-th down arrow and l-th $(k < l)$ down arrow forms a dashed arc B, then the size of B is $(l - k + 1)/2$. Let C be the down arrow with the integer p or the down arrow with a star. Let E be a dashed arc. We define

$$
\begin{aligned}
d_{1,A,C} &:= (i - s_p + M - p + 1)/2, \qquad \text{for } 2 \le p \le M, \\
d_{2,A} &:= (i - s_1 + M)/2 \\
d_{3,A} &:= N - j, \\
N_7 &:= \prod_{C \in U} \prod_{A \in S_R^+} \frac{[d_{1,A,C}]}{[d_{1,A,C} + m_A]}, \\
N_8 &:= \begin{cases} \displaystyle\prod_{A \in S_R^+} \prod_{i=0}^{N_3} \frac{[d_{2,A} + i]}{[d_{2,A} + m_A + i]} \prod_{A' \in S_W} \frac{[d_{2,A} + m_A + h_{A'}]}{[d_{2,A} + h_{A'}]}, & \text{for } |V| = M, \\ 1 & \text{otherwise,} \end{cases} \\
N_9 &:= \begin{cases} \displaystyle\prod_{A \in S_W^+} \frac{[d_{3,A} + m_A + M]}{[d_{3,A} + 2m_A + M]} & \text{for } N_1 = 1, \\ \displaystyle\prod_{A \in S_W^+ \cup S_L^+} \frac{[d_{3,A} + m_A + M]}{[d_{3,A} + 2m_A + M]} & \text{for } N_1 = 0, \end{cases}
\end{aligned}
$$

where $h_{A'}$, $A' \in S_W$ is the sum of the number of arcs in S_W right to A' or outside of A' (including A'), and the number of dashed arcs right to A'. We also define

$$
\begin{aligned}
d_{4,C} &:= (s_M - s_p + M - p)/2 + 1, \\
d_{5,E} &:= (s_M - k + M + 1)/2, \\
N_{10} &:= \begin{cases} \displaystyle\prod_{C \in U'} [d_{4,C}]^{-1}, & \text{for } N_1 = 0 \text{ and } T = \emptyset \\ \displaystyle\prod_{C \in V'} [d_{4,C}]^{-1} \prod_{E \in T'} [d_{5,E}]^{-1}, & \text{for } N_1 = 0 \text{ and } T \ne \emptyset, \\ \displaystyle\prod_{C \in V'} [d_{4,C}]^{-1} \prod_{E \in T} [d_{5,E}]^{-1} & \text{for } N_1 = 1, \\ 1 & \text{otherwise,} \end{cases}
\end{aligned}
$$

We enumerate up arrows, an unpaired down arrow (if it exists), arcs, dashed arcs and down arrows with the integer p, $2 \le p \le M$ from left to right. If there exist arcs inside of an arc or a dashed arc, we increase the integer one by one from outside to inside. Let N_A be the integer

assigned to an arc, a dashed arc or a down arrow with the integer p, and $N_\downarrow$ be the integer assigned to the unpaired down arrow. We define

$$N_{11} := \begin{cases} \prod\limits_{A \in S \cup T \cup U} [N_A], & N_1 = 0, \\ [N_\downarrow] \cdot \prod\limits_{A \in S \cup T \cup U} [N_A], & N_1 = 1. \end{cases}$$

We enumerate arcs, dashed arcs, down arrows with the integer $p, 2 \le p \le M$, and the down with a star from right to left. If there exists arcs inside of an arc or a dashed arc, we increase the integer one by one from inside to outside. Let N_B be the integer assigned to a dashed arc or a down arrow with a star. Then, we define

$$N_{12} := \frac{\prod\limits_{i=1}^{N_2} (q^{i+M-1} + q^{-(i+M-1)})}{\prod\limits_{B \in T \cup (V \setminus U)} (q^{N_B} + q^{-N_B})}.$$

Definition 4.11.

$$\Psi_D := \prod_{A \in S} [m_A]^{-1} \cdot N_6 \cdot N_7 \cdot N_8 \cdot N_9 \cdot N_{10} \cdot N_{11} \cdot N_{12}.$$

Example 4.12. *Let D be a diagram depicted as*
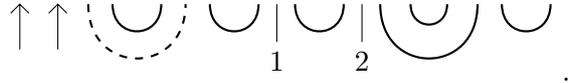
We have

$$N_1 = 0, \quad N_6 = [20]/[6], \quad N_7 = 1/[4],$$
$$N_8 = [3]/[9], \quad N_9 = [13]/[14], \quad N_{10} = 1/[3],$$
$$N_{11} = [10]!/[2], \quad N_{12} = \prod_{i \in I}(q^{i+1} + q^{-i-1}),$$
$$\prod_{A \in S}[m_A]^{-1} = [2]^{-1},$$

where $I = \{1,2,3,4,6,7,9\}$.

Theorem 4.13. Ψ *is the eigenvector of X with the eigenvalue $[N+M]$. The multiplicity is one.*

Proof. From Theorem 4.10, the multiplicity of the eigenvalue $[N+M]$ is obviously one. Let $(X_{D,D'})$ be the matrix representation of X. We will show that

$$(28) \qquad\qquad \sum_{D'} X_{D,D'} \Psi_{D'} = [N + M]\Psi_D.$$

We have four cases for D: 1) D does not have down arrows, 2) $2 \le r \le M$, 3) $N_1 = 1$ and 4) $r = 1$ and $N_1 = 0$.

Case 1. Let D be a diagram starting with n_1 up arrows, followed by an outer arc of size m_1, followed by n_2 up arrows, followed by an outer arc of size m_2, $\cdots$ and ending with n_{I+1} up arrows. Inside of an outer arc of size m_i, $1 \le i \le I$, is filled with arcs. Set $M' = \sum_{i=1}^{I} m_i$ and $N_\uparrow = \sum_{i=1}^{I+1} n_i$. Let D' be a diagram obtained from D by changing the outer arc of size m_i to two up arrows. We have $X_{D,D'} = [1 + \sum_{j=1}^{i} n_j]$. The contribution of these D''s to the left hand side of Eqn.(28) is

$$\sum_{i=1}^{I} \left[1 + \sum_{j=1}^{i} n_j \right] [m_i] \frac{\prod_{j \ge i+1}[1 + \sum_{k=1}^{j}(n_k + m_k)]}{\prod_{j \ge i}[1 + n_j + \sum_{k=1}^{j-1}(n_k + m_k)]} (q^{N_\uparrow + M' + M} + q^{-(N_\uparrow + M' + M)}) \Psi_D.$$

Inserting Lemma A.2, we obtain $[M'](q^{N_\uparrow + M' + M} + q^{-(N_\uparrow + M' + M)})\Psi_D$.

From Eqn.(25), the contribution of the diagonal term is $[N_\uparrow + M]\Psi_D$. Note $N = N_\uparrow + 2M'$. The sum of these two contributions is the right hand side of Eqn.(28).

Case 2. In this case, we have $S_R = S_W = S_L = \emptyset$. Let D be a diagram starting with n_1 up arrows, followed by an outer arc of size m_1, followed by n_2 up arrows, followed by an outer arc of size m_2, $\cdots$, followed by n_{I+1} up arrows, followed by a down arrow with the integer r and ending with G arcs, $|T|$ dashed arcs and down arrows with the integer p, $r + 1 \le p \le M$. Set $M' = \sum_{i=1}^{I} m_i$ and $N_\uparrow = \sum_{i=1}^{I+1} n_i$. Let D' be a diagram obtained from D by changing the outer arc of size m_i to two up arrows. By a similar argument to Case 1, the contribution of these D''s to the left hand side of Eqn.(28) is

$$\frac{[M'][1 + N_\uparrow + M' + G + M - r + 1]}{[1 + M' + N_\uparrow]} (q^{N_\uparrow + M' + G + M} + q^{-(N_\uparrow + M' + G + M)}) \Psi_D.$$

Let D' be a diagram obtained form D by changing the down arrow with the integer r to an up arrow. We have $X_{D,D'} = [N_\uparrow + 1]$. The contribution of this diagram is

$$[N_\uparrow + 1] \frac{[G + M - r + 1]}{[1 + N_\uparrow + M']} (q^{N_\uparrow + M' + G + M} + q^{-(N_\uparrow + M' + G + M)}) \Psi_D.$$

The contribution of the diagonal term is $[N_\uparrow + r - 1]\Psi_D$. Note that $N = N_\uparrow + 2M' + 2G + M - r + 1$. The sum of these three contributions is the right hand side of Eqn.(28).

Case 3. Let D be diagram starting with n_1 up arrows, followed by an outer arc of size m_1, followed by n_2 up arrows, followed by an outer arc of size m_2, $\cdots$, followed by n_{I+1} up arrows, followed by an unpaired down arrow and ending with $|S_W| + |S_{W'}| + |S_R|$ arcs, $|T|$ dashed arcs, a down arrow with a star and down arrows with an integer p, $2 \le p \le M$. Set $M' = \sum_{i=1}^{I} m_i$ and $N_\uparrow = \sum_{i=1}^{I+1} n_i$. Let D' be a diagram obtained from D by changing the outer arc of size m_i to two up arrows. By a similar argument to Case 1, the contribution of these D''s to the left hand side of (28) is

$$[M'] \frac{[1 + d]}{[1 + M' + N_\uparrow]} (q^{d+1} + q^{-(d+1)}),$$

where $d = N_\uparrow + |S| + |T| + M$. Let D' be a diagram obtained from D by changing the unpaired down arrow to an up arrow. We have $X_{D,D'} = [N_\uparrow + 1]$. The contribution is

$$[N_\uparrow + 1] \frac{[1 + N_\uparrow + 2|S| - M' + 2|T| + 2M]}{[1 + N_\uparrow + M']}.$$

Note that $N = N_\uparrow + 2|S| + 2|T| + M + 1$. The sum of two contributions gives the right hand side of Eqn.(28).

Case 4. Let D be diagram starting with n_1 up arrows, followed by an outer arc of size m_1, followed by n_2 up arrows, followed by an outer arc of size m_2, $\cdots$, followed by n_{I+1} up arrows and ending with $|S_W| + |S_{W'}| + |S_R|$ arcs, $|T|$ dashed arcs, a down arrow with a star and down arrows with an integer p, $2 \leq p \leq M$. We have two cases for D: a) $n_{I+1} \neq 0$, that is, $|S_L| = \emptyset$ and b) for a given H satisfying $1 \leq H \leq I$, $n_{H+1} \neq 0$ and $n_i = 0$ for $H + 2 \leq i \leq I + 1$. Set $N_\uparrow = \sum_{i=1}^{I+1} n_i$, $M' = \sum_{i=1}^{I} m_i$ and $d = N_\uparrow + |S| + |T| + M$.

Case 4-a. Set $M' = \sum_{i=1}^{I} m_i$ and $L_1 := N_\uparrow + 2|S| - M' + 2|T| + 2M$. Let D' be a diagram obtained form D by changing the outer arc of size m_i, $1 \leq i \leq I$ to two up arrows. We have $X_{D,D'} = [1 + \sum_{j=1}^{i} n_j]$. By a similar argument to Case 1, the contribution to the left hand side of (28) is

$$(29) \qquad [M'] \frac{[d][L_1 + 1]}{[N_\uparrow + M' + 1][L_1]} (q^d + q^{-d}) \Psi_D.$$

Let D' be a diagram obtained from D by changing the leftmost dashed arc to an up arrow and an unpaired down arrow. We have $X_{D,D'} = [N_\uparrow + 1]$. The contribution is

$$(30) \qquad [N_\uparrow + 1] \frac{[|S_{W'}| + |S_W| + |T| + |S_R| + M][d]}{[L_1][1 + N_\uparrow + M']} (q^d + q^{-d})(q^{d'} + q^{-d'}) \Psi_D,$$

where $d' = d - N_\uparrow - M'$. Let D' be a diagram obtained from D by changing the rightmost up arrow to an unpaired down arrow. We have $X_{D,D'} = [N_\uparrow]$. The contribution is

$$(31) \qquad [N_\uparrow] \frac{[N_\uparrow + M']}{[L_1]} \Psi_D.$$

Note $N = N_\uparrow + 2|S| + 2|T| + M$. The sum of three contributions (29), (30) and (31) gives the right hand side of Eqn.(28).

Case 4-b. We have $S_L \neq \emptyset$ and $|S_L| = \sum_{i=H+1}^{I} m_i$. Set $M' = \sum_{i=1}^{H} m_i$ and $L_1 := N_\uparrow + 2|S| - M' + 2|T| + 2M$. Let D' be a diagram obtained from D by changing the outer arc of size m_i, $1 \leq i \leq H$, to two up arrows. By a similar argument to Case 1, the contribution of these D''s to the left hand side of (28) is

$$(32) \qquad [M'] \frac{[L_1 + 1]}{[L_1]} \frac{[d]}{[1 + N_\uparrow + M']} (q^d + q^{-d}) \Psi_D.$$

Let D' be a diagram obtained from D by changing the outer arc of size m_i, $H + 1 \leq i \leq I$, to two up arrows. We have $X_{D,D'} = [N_\uparrow + 1]$. Thus, the contribution of these D''s is

$$(33) \quad \Psi_D \frac{[N_\uparrow + 1][d]}{[1 + N_\uparrow + M' + |S_L|][L_1]} (q^d + q^{-d}) \sum_{i=H+1}^{I} [m_i][l_i] \frac{\prod_{j \geq i+1}[w_j]}{\prod_{j \geq i}[w_{j-1}]} \prod_{j=H+1}^{i} \frac{[g_j]}{[g_j - m_j]}$$

where

$$l_i \quad := \quad 1 + N_\uparrow + 2|S| - M' + 2|T| + 2M - \sum_{j=H+1}^{i} m_j,$$

$$w_j \quad := \quad 1 + N_\uparrow + M' + \sum_{k=H+1}^{j} m_k,$$

$$g_i \quad := \quad 2|S_R| + 2|S_{W'}| + 2|S_W| + 2M + 2|T| + 2\sum_{j=i}^{I} m_j.$$

Let D' be a diagram obtained from D by changing the leftmost dashed arc to an up arrow and a down arrow. The contribution of D' is

$$(34) \qquad \Psi_D[N_\uparrow + 1]\frac{[2L_2][N_\uparrow + |S| + |T| + M]}{[L_1][1 + N_\uparrow + M' + |S_L|]}(q^d + q^{-d})\prod_{i=H+1}^{I}\frac{[g_i]}{[g_i - m_i]},$$

where $L_2 := |S_{W'}| + |S_W| + |T| + |S_R| + M$. By Lemma A.9 with $x = N_\uparrow + M'$ and $z = |S_{W'}| + |S_W| + |S_R| + |T| + M - 1$, the sum of contributions (33) and (34) is given by

$$(35) \qquad \frac{[N_\uparrow + 1][N_\uparrow + |S| + |T| + M][2L_2 + 2|S_L|]}{[L_1][1 + N_\uparrow + M']}(q^d + q^{-d})\Psi_D.$$

Finally, let D' be a diagram obtained from D by changing the rightmost up arrow to an unpaired down arrow. The contribution of this D' is

$$(36) \qquad \frac{[N_\uparrow][N_\uparrow + M']}{[L_1]}\Psi_D.$$

The sum of the contributions (32), (35) and (36) gives the right hand side of Eqn.(28). This completes the proof. □

4.3. Type BII.
We consider the action of X on the Kazhdan–Lusztig basis of type BII.

Let D be a diagram of type BII and $N_\uparrow$ be the number of (unpaired) up arrows. we enumerate the up arrows from left to right by $1, 2, \ldots, N_\uparrow$. For each i, $1 \le i < N_\uparrow$, we denote by $X_{(i)}(D)$ a diagram obtained from D by connecting the i-th up arrow and $(i+1)$-th up arrow via an arc. We denote by $X_{(N_\uparrow)}(D)$ a diagram obtained from D by changing the $N_\uparrow$-th up arrow to an e-unpaired or an o-unpaired down arrow. Suppose that D does not have down arrows or the leftmost down arrow of D is an e-unpaired down arrow. We define the action of X by

$$(37) \qquad X(D) := \sum_{i=1}^{N_\uparrow}[i]X_{(i)}(D) + [Q; N_\uparrow]D.$$

Suppose that the leftmost down arrow of D is an o-unpaired down arrow. Then, we define the action of X by

$$(38) \qquad X(D) := \sum_{i=1}^{N_\uparrow}[i]X_{(i)}(D) + [Q; -N_\uparrow - 1]D.$$

Example 4.14. *Let D be a diagram depicted as*

$$D = \uparrow \ \uparrow \ | \ \underset{e}{\bigcup \bigcup} \ | \ \underset{o}{\bigcup}.$$

We have

$$X_{(1)} = \underset{e}{\bigcup} \ | \ \underset{o}{\bigcup \bigcup} \ | \ \underset{o}{\bigcup}, \qquad X_{(2)} = \uparrow \ | \ \underset{o}{|} \ \underset{e}{|} \ \underset{o}{\bigcup \bigcup} \ | \ \underset{o}{\bigcup}.$$

The action of X on D is

$$X(D) = X_{(1)}(D) + [2]X_{(2)}(D) + [Q; 2]D.$$

Theorem 4.15. *The action of X defined in Eqns.(37) and (38) provides the action on the Kazhdan–Lusztig bases of type BII.*

Proof. We prove Theorem by induction. When $N = 1$, Theorem is true by a straightforward calculation. We assume that Theorem holds true for diagrams of length up to $N - 1$. Let D be a diagram of length N. We have two cases for the leftmost arrow a of D: 1) the arrow a is an up arrow and 2) the arrow a is a down arrow.

Case 1. A diagram D is written as $\uparrow D'$ where D' is a diagram of length $N - 1$. The action of X on D is

$$X(\uparrow D') = q \uparrow X(D') + \downarrow D'. \tag{39}$$

We have two cases for the leftmost down arrow a' of D': a) the arrow a' is an e-unpaired down arrow and b) the arrow a' is an o-unpaired down arrow.

Case 1-a. From Eqn.(37) for D', we have

$$q \uparrow X(D') \;=\; \sum_{i=2}^{N_\uparrow} q[i-1] X_{(i)}(D) + q[Q; N_\uparrow - 1]D, \tag{40}$$

$$\downarrow D' \;=\; \sum_{i=1}^{N_\uparrow} q^{-(i-1)} X_{(i)}(D) + q^{-(N_\uparrow - 1)} Q^{-1} D. \tag{41}$$

From $q[i-1] + q^{-(i-1)} = [i]$ and $q[Q; N_\uparrow - 1] + q^{-(N_\uparrow - 1)} Q^{-1} = [Q; N_\uparrow]$, the sum of Eqns.(40) and (41) is equal to the right hand side of Eqn.(37).

Case 1-b. Similarly, we have

$$q \uparrow X(D') \;=\; \sum_{i=2}^{N_\uparrow} q[i-1] X_{(i)}(D) + q[Q; -N_\uparrow]D,$$

$$\downarrow D' \;=\; \sum_{i=1}^{N_\uparrow} q^{-(i-1)} X_{(i)}(D) - q^{-N_\uparrow} Q D$$

From $q[Q; -N_\uparrow] - q^{-N_\uparrow} Q = [Q; -N_\uparrow - 1]$, the sum of $q \uparrow X(D')$ and $\downarrow D'$ is equal to the right hand side of Eqn.(38).

Case 2. We have three cases for the leftmost down arrow a of D: a) the arrow a is an e-unpaired down arrow, b) the arrow a is an o-unpaired down arrow and c) the arrow a forms an arc.

Case 2-a. The diagram D is written as $\underset{\text{e}}{\big|} D'$ where D' is a diagram of length $N - 1$. We have

$$
\begin{aligned}
X(D) &= X(\downarrow D') + q^{-1} Q X(\uparrow D') \\
&= q^{-1} \downarrow X(D') + \uparrow D' + Q \uparrow X(D') + q^{-1} Q \downarrow D' \\
&= [Q; 0]D
\end{aligned}
$$

where we have used $X(D') = [Q; -1]D'$.

Case 2-b. The diagram D is written as $\underset{\text{o}}{\big|} D'$ where D' is a diagram of length $N - 1$. We have

$$
\begin{aligned}
X(D) &= X(\downarrow D') - Q^{-1} X(\uparrow D') \\
&= q^{-1} \downarrow X(D') + \uparrow D' - q Q^{-1} \uparrow X(D') - Q^{-1} \downarrow D' \\
&= [Q; -1]D
\end{aligned}
$$

where we have used $X(D') = [Q; 0]D'$.

Case 2-c. By a similar argument to Case 2-b in the proof of Theorem 4.9, we can assume D is written as $D = \bigcup D'$ where D' is a diagram of length $N - 2$. We have

$$
\begin{aligned}
X(D) &= X(\downarrow\uparrow D') - q^{-1}X(\uparrow\downarrow D') \\
&= q^{-1}\downarrow X(\uparrow D') + \uparrow\uparrow D' - \uparrow X(\downarrow D') - q^{-1}\downarrow\downarrow D' \\
&= \downarrow\uparrow X(D') + q^{-1}\downarrow\downarrow + \uparrow\uparrow D' - q^{-1}\uparrow\downarrow X(D') - \uparrow\uparrow D' - q^{-1}\downarrow\downarrow D' \\
&= \bigcup X(D').
\end{aligned}
$$

In both Case 1 and 2, $X(D)$ coincides with the definitions (37) and (38). This completes the proof. $\square$

Theorem 4.16. X has the eigenvalue $[Q; N - 2i]$, $0 \leq i \leq N$, of multiplicity $\binom{N}{i}$.

Proof. Recall that a diagram D is constructed from a binary string b. The lexicographic order of binary strings induces a natural lexicographic order of diagrams. We consider the matrix representation of X on the Kazhdan–Lusztig bases. In the lexicographic order of bases, X is a lower triangular matrix. From Eqns.(37) and (38), the diagonal entries of X are $[Q; N_\uparrow]$ or $[Q; -N_\uparrow - 1]$. Thus an eigenvalue of X is of the form $[Q; n]$ with some n. The multiplicity of an eigenvalue $[Q; n]$, $n \in \mathbb{N}$, is the number of diagrams which has n up arrows and the leftmost down arrow is e-unpaired down arrow. Similarly, the multiplicity of an eigenvalue $[Q; -n - 1]$, $n \in \mathbb{N}$, is the number of diagrams which has n up arrows and the leftmost down arrow is o-unpaired down arrow. Set $\lambda_D = n$ (resp. $\lambda_D = -n - 1$) if the leftmost down arrow is e-unpaired (resp. o-unpaired) down arrow. If D does not have down arrows, we set $\lambda_D = n$.

Suppose that a diagram D has m arcs. We denote by D' a diagram obtained from D by removing all the arcs. The diagram D' is of length $N - 2m$ and without arcs. We denote by $\mathcal{D}'$ the set of such D''s. The cardinality of $\mathcal{D}'$ is $N - 2m + 1$. Reversely, if we have a diagram $D' \in \mathcal{D}'$, one can construct a diagram D by inserting arcs into D'. Given D', the number of possible D's is given by $\binom{N}{m} - \binom{N}{m-1}$. Obviously, we have $\lambda_{D'} = \lambda_D$. Given a diagram $D' \in \mathcal{D}'$, we have $\lambda_{D'} = N - 2m - 2j$ with some j, $0 \leq j \leq N - 2m$. Note that there is a one-to-one correspondence between j and D'. Thus the number of diagrams for $\lambda_D = N - 2i$, $0 \leq i \leq N$, is

$$
\begin{aligned}
\sum_{k=0}^{\min(i, N-i)} \binom{N}{k} - \binom{N}{k-1} &= \binom{N}{\min(i, N-i)} \\
&= \binom{N}{i}
\end{aligned}
$$

This completes the proof. $\square$

Let D be a diagram of Type BII. We denote by S the set of arcs, by $S_\uparrow$ the set of up arrows, by S_{e} the set of e-unpaired down arrows and by S_{o} the set of o-unpaired down arrows. We enumerate arcs, up arrows and e-unpaired down arrows from left. Let N_A be the integer assigned to $A \in S \cup S_\uparrow \cup S_{\mathrm{e}}$. We define

$$
\begin{aligned}
N_1 &:= \prod_{A \in S \cup S_{\mathrm{e}}} [N_A], \\
N_2 &:= \prod_{B \in S} [m_B]^{-1}.
\end{aligned}
$$

where m_B is the size of $B \in S$.

We enumerate arcs and e-unpaired from right to left by $1, 2, \ldots$. If there exist arcs inside of an arc or a dashed arc, we increase the integer one by one from inside to outside. Let N_C be the integer assigned to $C \in S \cup S_e$. We define

$$N_3 := \prod_{C \in S_e} [N_C]^{-1}.$$

We enumerate arcs, up arrows and o-unpaired down arrows from right. Let N_E be the integer assigned to $E \in S \cup S_\uparrow \cup S_o$. We define

$$N_4 := \prod_{E \in S \cup S_\uparrow} (q^{N_E - 1} Q + q^{-(N_E - 1)} Q^{-1}).$$

In the above notation, we define

Definition 4.17. $\Psi_D = N_1 \cdot N_2 \cdot N_3 \cdot N_4.$

Example 4.18. *Let D be a diagram depicted as*

$$\uparrow \uparrow \smile | \underbrace{\smile} | \atop{e} \quad {o}$$

We have

$$N_1 = \frac{[6]!}{[2]}, \qquad N_2 = [2]^{-1}, \qquad N_3 = [3]^{-1}, \qquad N_4 = \prod_{i=2}^{6} (q^{i-1} Q + q^{-(i-1)} Q^{-1}).$$

Theorem 4.19. *Ψ is the eigenvector of X with the eigenvalue $[Q; N]$. The multiplicity is one.*

Proof. Let $X = (X_{D,D'})$ be the matrix representation of X. We show that

$$\sum_{D'} X_{D,D'} \Psi_{D'} = [Q; N] \Psi_D. \tag{42}$$

We have three cases for D: 1) D does not have down arrows, 2) the leftmost down arrow of D is an e-unpaired down arrow and 3) the leftmost down arrow of D is an o-unpaired down arrow.

Case 1. Let D be a diagram starting with n_1 up arrows, followed by an outer arc of size m_1, followed by n_2 up arrows, followed by an outer arc of size m_2, $\cdots$ and ending with n_{I+1} up arrows. Inside of an outer arc of size m_i, $1 \leq i \leq I$, is filled with arcs. As a diagram, D is

$$\underbrace{\uparrow \ldots \uparrow}_{n_1} \underbrace{\smile}_{\text{size } m_1} \uparrow \ldots \uparrow \underbrace{\smile}_{\text{size } m_I} \underbrace{\uparrow \ldots \uparrow}_{n_{I+1}}. \tag{43}$$

Set $N_\uparrow = \sum_{i=1}^{I+1} n_i$ and $M = \sum_{i=1}^{I} m_i$. The component Ψ_D is explicitly written as

$$\Psi_D = \prod_{i=1}^{I} \frac{[\sum_{j=1}^{i}(n_j + m_j)]}{[n_i + \sum_{j=1}^{i-1}(n_j + m_j)]} \cdot \prod_{B \in S} [m_B]^{-1} \cdot \prod_{i=1}^{N_\uparrow + M} (Q q^{i-1} + Q^{-1} q^{-(i-1)}).$$

Let D' be a diagram obtained from D by changing the arc of size m_i to two up arrows. Then, we have $X_{D,D'} = [1 + \sum_{j=1}^{i} n_j]$. The contribution to the left hand side of Eqn.(42) is

$$\Psi_D \sum_{i=1}^{I} \left[1 + \sum_{j=1}^{i} n_j \right] [m_i] \frac{\prod_{j \geq i+1}^{I}[1 + \sum_{k=1}^{j}(n_k + m_k)]}{\prod_{j \geq i}^{I}[1 + n_j + \sum_{k=1}^{j-1}(n_k + m_k)]} (q^{N_\uparrow + M} Q + q^{-(N_\uparrow + M)} Q^{-1}).$$

Inserting Lemma A.2, the above expression is reduced to

$$[M](q^{N_\uparrow+M}Q + q^{-(N_\uparrow+M)}Q^{-1})\Psi_D.$$

The contribution of the diagonal term is $[Q; N_\uparrow]\Psi_D$. Therefore, the left hand side of Eqn.(42) is

$$
\begin{aligned}
[M](q^{N_\uparrow+M}Q + q^{-(N_\uparrow+M)}Q^{-1})\Psi_D + [Q; N_\uparrow]\Psi_D &= [Q; N_\uparrow + 2M]\Psi_D \\
&= [Q; N]\Psi_D.
\end{aligned}
$$

Case 2. Let D be a diagram starting with n_1 up arrows, followed by an outer arc of size m_1, followed by n_2 up arrows, followed by an outer arc of size m_2, $\cdots$, followed by n_{I+1} up arrows, followed by e-unpaired down arrow and ending with $G - 1$ e-unpaired down arrows, M' arcs and G o-unpaired down arrows. Set $N_\uparrow := \sum_{i=1}^{I+1} n_i$ and $M = \sum_{i=1}^{I} m_i$. Let D' be a diagram obtained from D by changing the outer arc of size m_i to two up arrows. We have $X_{D,D'} = [1 + \sum_{j=1}^{i} n_j]$. By a similar calculation to Case 1, the contribution to the left hand side of Eqn.(42) is

$$\frac{[M][d+1]}{[1 + M + N_\uparrow]}(Qq^d + Q^{-1}q^{-d})\Psi_D$$

where $d = G + M + M' + N$.

Let D' be a diagram obtained from D by changing the leftmost down arrow to an up arrow. We have $X_{D,D'} = [N_\uparrow + 1]$. The contribution to the left hand side of Eqn.(42) is

$$(44) \qquad \frac{[N_\uparrow + 1][G + M']}{[1 + M + N_\uparrow]}(Qq^d + Q^{-1}q^{-d})\Psi_D.$$

The contribution of the diagonal term is $[Q; N]\Psi_D$. Note that $N = N_\uparrow + 2M + 2M' + 2G$. The sum of three contributions gives the right hand side of (42).

Case 3. Let D be a diagram starting with n_1 up arrows, followed by an outer arc of size m_1, followed by n_2 up arrows, followed by an outer arc of size m_2, $\cdots$, followed by n_{I+1} up arrows, followed by o-unpaired down arrow and ending with $G - 1$ e-unpaired down arrows, M' arcs and $G - 1$ o-unpaired down arrows. We set $N_\uparrow := \sum_{i=1}^{I+1} n_i$ and $M := \sum_{i=1}^{I} m_i$. Let D' be a diagram obtained from D by changing the arc of size m_i to two up arrows. By a similar calculation to Case 2, the contribution to the left hand side of (42) is

$$\frac{[M][d]}{[1 + M + N_\uparrow]}(Qq^d + Q^{-1}q^{-d})\Psi_D.$$

where $d = M + M' + N_\uparrow + G$. Let D' be a diagram obtained from D by changing the leftmost down arrow to an up arrow. The contribution of this diagram is

$$\frac{[N_\uparrow + 1][d]}{[1 + M + N_\uparrow]}(Qq^{M'+G-1} + Q^{-1}q^{-(M'+G-1)})\Psi_D.$$

The contribution of the diagonal term is $[Q; -N_\uparrow - 1]\Psi_D$. Note that $N = N_\uparrow + 2M + 2M' + 2G - 1$. The sum of three contributions gives the right hand side of Eqn.(42). This completes the proof. $\qquad \square$

4.4. **Type BIII.** We consider the action of X on Kazhdan–Lusztig bases of type BIII.

Let D be a diagram of type BIII, $N_\uparrow$ be the number of up arrows and $N_\downarrow$ be the number of down arrows with a circled integer. We define the weight of D by $\mathrm{wt}(D) := N_\uparrow - N_\downarrow$. We enumerate the up arrows from left to right by $1, 2, \ldots, N_\uparrow$. For each $1 \le i < N_\uparrow$, we denote by $X_{(i)}(D)$ a diagram obtained from D by connecting the i-th up arrow and the $(i+1)$-th up arrow via an arc. We denote

by $X_{(N_\uparrow)}(D)$ a diagram obtained from D by changing the $N_\uparrow$-th up arrow to the down arrow with a circled integer $N_\downarrow + 1$. We define the action of X by

(45)
$$X(D) := \sum_{i=1}^{N_\uparrow}[i]X_{(i)}(D) + [Q; \mathrm{wt}(D)]D.$$

Example 4.20. *Let D be a diagram depicted as*

$$D = \uparrow \smile \uparrow \underset{\textcircled{2}}{\mid} \overbrace{\smile} \underset{\textcircled{1}}{\mid} \smile.$$

We have

$$X_{(1)}(D) = \overbrace{\smile} \underset{\textcircled{2}}{\mid} \overbrace{\smile} \underset{\textcircled{1}}{\mid} \smile, \qquad X_{(2)}(D) = \uparrow \smile \underset{\textcircled{3}\textcircled{2}}{\mid \mid} \overbrace{\smile} \underset{\textcircled{1}}{\mid} \smile.$$

Then, the action of X on D is

$$X(D) = X_{(1)}(D) + [2]X_{(2)}(D) + [Q; 0]D.$$

Theorem 4.21. *The action of X defined in Eqn.(45) provides the action of X on the Kazhdan–Lusztig basis of Type BIII.*

Proof. We prove Theorem by induction. When $N = 1$, Theorem is true by a straightforward calculation. We assume that Theorem holds true for diagrams of length up to $N - 1$. Let D be a diagram of length N. We have two cases for the leftmost arrow a of D: 1) the arrow a is an up arrow and 2) the arrow a is a down arrow.

Case 1. A diagram D is written as $\uparrow D'$ where D' is a diagram of length $N - 1$. The action of X on D is

(46)
$$X(\uparrow D') = q \uparrow X(D') + \downarrow D.$$

From the assumption, we have

$$\uparrow X(D') = \sum_{i=2}^{N_\uparrow}[i-1]t_{(i)}(D) + [Q; \mathrm{wt}(D) - 1],$$

$$\downarrow D' = \sum_{i=1}^{N_\uparrow}q^{-(i-1)}Q^{-1}X_{(i)}(D) + q^{-N_\uparrow + N_\downarrow + 1}.$$

Note that $q[Q; d] + q^{-d}Q^{-1} = [Q; d + 1]$. Inserting these two expressions into Eqn.(46), we obtain Eqn.(45).

Case 2. We have two cases for the leftmost arrow a: i) the arrow a is a down arrow with a circled integer r and ii) the arrow a forms an arc.

Case 2-i. A diagram D is written as $\underset{\textcircled{r}}{\mid} D'$ where D' is a diagram of length $N - 1$. The weight of D is $-r$. We have

$$X(\underset{\textcircled{r}}{\mid} D') = q^{-1} \downarrow X(D') + \uparrow D' - q^r Q^{-1} \uparrow X(D') - q^{r-1}Q^{-1} \downarrow D',$$

$$= [Q; -r]D$$

where we have used $X(D') = [Q; -(r - 1)]D'$, $q^{-1}[Q; -(r - 1)] - q^{r-1}Q^{-1} = [Q; -r]$ and $1 - q^r Q^{-1}[Q; -(r - 1)] = -q^{r-1}Q^{-1}[Q; -r]$.

Case 2-ii. By a similar argument to Case 2-b in the proof of Theorem 4.9, we can assume that a diagram D is written as $\bigcup D'$ where D' is a diagram of length $N - 2$. We have

$$
\begin{aligned}
X(\bigcup D') &= X(\downarrow\uparrow D' - q^{-1}\uparrow\downarrow D') \\
&= \bigcup X(D').
\end{aligned}
$$

In both Case 1 and 2, we have Eqn.(45) for a diagram D. This completes the proof. □

Theorem 4.22. *X has the eigenvalue $[Q; N - 2i]$, $1 \le i \le N$, of multiplicity $\binom{N}{i}$.*

Proof. Let $X_{D,D'}$ be a matrix representation of X with respect to diagrams. Recall that there is a one-to-one correspondence between a binary string and a diagram. Thus we have a natural order for diagrams induced from the lexicographic order for binary strings. From Eqn.(45), the matrix $X_{D,D'}$ is a lower triangular matrix. Since the diagonal terms are $[Q; \mathrm{wt}(D)]$, the eigenvalues are of the form $[Q; N - 2i]$. The multiplicities are equal to the number of D such that $\mathrm{wt}(D) = N - 2j$. By a similar argument to the second paragraph in the proof of Theorem 4.16, the cardinality of D satisfying $\mathrm{wt}(D) = N - 2j$ is $\binom{N}{j}$.

□

Let D be a diagram of type BIII, S be the set of arcs, $S_\uparrow$ be the set of (unpaired) up arrows and $S_\downarrow$ be the set of down arrows with a circled integer. We enumerate up arrows, arcs and down arrows with a circled integer from left to right by $1, 2, \ldots$. Let N_A be the integer assigned to $A \in S \cup S_\uparrow \cup S_\downarrow$. We define

$$
N_1 := \prod_{A \in S \cup S_\downarrow} [N_A].
$$

Let B be an arc of size m_B. We define

$$
N_2 := \prod_{B \in S} [m_B]^{-1}.
$$

Similarly, we enumerate arcs and down arrows with a circled integer from right to left by $1, 2, \ldots$. Let N_C be the integer assigned to $C \in S \cup S_\downarrow$. We define

$$
\begin{aligned}
N_3 &:= \prod_{S \in S_\downarrow} [N_C]^{-1}, \\
N_4 &:= \prod_{i=1}^{d} (Qq^{i-1} + Q^{-1}q^{-(i-1)}),
\end{aligned}
$$

where $d = |S_\uparrow| + |S|$.

Definition 4.23. $\Psi_D := N_1 \cdot N_2 \cdot N_3 \cdot N_4$.

Example 4.24. *Let D be a diagram depicted as*

$$
\uparrow \ \uparrow \ \underset{\text{③}}{\downarrow} \ \bigcup\!\!\bigcup \ \underset{\text{②}}{\downarrow} \ \bigcup \ \underset{\text{①}}{\downarrow} \ .
$$

We have

$$
N_1 = \frac{[8]!}{[2]}, \qquad N_2 = [2]^{-1}, \qquad N_3 = [3]^{-1}[6]^{-1}, \qquad N_4 = \prod_{i=1}^{5}(Qq^{i-1} + Q^{-1}q^{-(i-1)}).
$$

Theorem 4.25. *The vector Ψ is the eigenvector of X with the eigenvalue $[Q;N]$.*

Proof. Let $X = (X_{D,D'})$ be a matrix representation of X. We will show $\sum_{D'} X_{D,D'}\Psi_{D'} = [Q;N]\Psi_D$ by computing the left hand side. We have two cases for a diagram D: 1) D has no down arrows and 2) D has down arrows with a circled integer.

Case 1. A diagram D is depicted as in Eqn.(43) where m_i, $1 \le i \le I$, is the size of an outer arc. Set $M' = \sum_{i=1}^{I} m_i$ and $N' = \sum_{i=1}^{I} n_i$. Since $X_{D,D} = [Q;N']$, we have a contribution from D itself. That is $[Q;N']\Psi_D$. Let D' be a diagram obtained from D by changing an outer arc of size m_i to two up arrows. Since $X_{D,D'} = [1 + \sum_{j=1}^{i} n_j]$, the contribution of such D''s is

$$\Psi_D \sum_{i=1}^{I} (Qq^d + Q^{-1}q^{-d})[m_i] \left[1 + \sum_{j=1}^{i} n_j \right] \frac{\prod_{j=i+1}^{I}[1 + \sum_{k=1}^{j}(n_k + m_k)]}{\prod_{j=i}^{I}[1 + n_j + \sum_{k=1}^{j-1}(n_k + m_k)]}.$$

where $d = N' + M'$. From Lemma A.2, the above expression is reduced to $[M'](Qq^d + Q^{-1}q^{-d})\Psi_D$. Therefore, the sum of contributions is $[Q;N' + 2M']$.

Case 2. A diagram D is locally depicted as
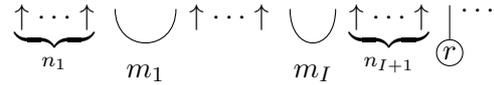
where the region inside of an outer arc of size m_i is filled with smaller arcs. Set $M' = \sum_{i=1}^{I} m_i$ and $N' = \sum_{i=1}^{I+1} n_i$. Let S_R be the set of arcs which are right to the down arrow with a circled integer r. We have three types of contributions: a) the diagonal term, *i.e.*, $D' = D$, b) D' is obtained from D by changing an outer arc of size m_i to two up arrows and c) D' is obtained from D by changing the down arrow with a circled integer r to an up arrow.

The contribution of case a is $[Q;N' - r]\Psi_D$. By a similar argument to Case 1, the contribution of case b is

$$[M'](Qq^d + Q^{-1}q^{-d})\frac{[M' + N' + |S_R| + r + 1]}{[M' + N' + 1]}\Psi_D.$$

where $d = N' + M' + |S_R|$. For case c, the contribution is

$$(47) \qquad \frac{[N' + 1][|S_R| + r]}{[N' + M' + 1]}(Qq^d + Q^{-1}q^{-d})\Psi_D.$$

The sum of three contributions becomes $[Q;N' + 2M' + 2|S_R| + r]\Psi_D = [Q;N]\Psi_D$. This completes the proof. $\square$

4.5. **Standard bases.** We consider the action of X on the standard basis $v := v_{\epsilon_1} \otimes \ldots \otimes v_{\epsilon_N}$ where $\epsilon_i = \pm 1$. Let $d_i := \sum_{j=1}^{i} \epsilon_i$. For each i, $1 \le i \le N$, we define

$$X_{(i)}(v) := v_{\epsilon_1} \otimes \ldots \otimes v_{\epsilon_{i-1}} \otimes v_{-\epsilon_i} \otimes v_{\epsilon_{i+1}} \otimes \ldots \otimes v_{\epsilon_N}.$$

The action of X is defined by

$$(48) \qquad X(v) := \sum_{i=1}^{N} q^{d_{i-1}} X_{(i)}(v) + q^{d_N}[Q;0]v.$$

Proposition 4.26. *The definition (48) provides the action of X on the standard basis.*

Proof. We prove Proposition by induction on N. When $N = 1$, Proposition holds true by a straightforward calculation. We assume that Proposition is true up to some $N \geq 2$. A standard basis $v := v_{\epsilon_1} \otimes \ldots \otimes v_{\epsilon_N}$ is written as $v = v_{\epsilon_1} \otimes v'$ where v' is a standard basis of length $N - 1$. Let $d'_i := \sum_{j=2}^{i} \epsilon_j$. From the induction assumption, we have

$$(49) \qquad X(v') = \sum_{i=2}^{N} q^{d'_{i-1}} X_{(i-1)}(v') + q^{d'_N}[Q; 0]v'$$

From Eqns.(11) and (49), we have

$$
\begin{aligned}
X(v) &= q^{\epsilon_1} v_{\epsilon_1} \otimes X(v') + v_{-\epsilon_1} \otimes v' \\
&= \sum_{i=2}^{N} q^{\epsilon_1 + d'_{i-1}} v_{\epsilon_1} \otimes X_{(i-1)}(v') + v_{-\epsilon_1} \otimes v' + q^{\epsilon_1 + d'_N}[Q; 0]v \\
&= \sum_{i=1}^{N} q^{d_{i-1}} X_{(i)}(v) + q^{d_N}[Q; 0]v,
\end{aligned}
$$

where we have used $v_{\epsilon_1} \otimes X_{(i-1)}(v') = X_{(i)}(v)$.   □

For a binary string $\epsilon \in \{\pm\}^N$, let I_ϵ be the set of positions of pluses from right. We define

$$
\begin{aligned}
N_1 &:= q^{d_\epsilon}, \\
N_2 &:= Q^{d'}
\end{aligned}
$$

where $d_\epsilon := \sum_{i \in I_\epsilon}(i - 1)$ and d' is the number of pluses in the binary string ϵ. We define a vector $\Psi^0 := \sum_\epsilon \Psi^0_\epsilon |\epsilon\rangle$ as follows.

Definition 4.27. $\Psi^0_\epsilon := N_1 N_2$.

Example 4.28. *Let $\epsilon_1 := + + -+$, $\epsilon_2 := - - ++$ and $\epsilon_3 := - - +-$. We have*

$$\Psi_{\epsilon_1} = q^5 Q^3, \quad \Psi_{\epsilon_2} = qQ^2, \quad \Psi_{\epsilon_3} = qQ.$$

Proposition 4.29. *The vector Ψ is the eigenvector of X with the eigenvalue $[Q; N]$.*

Proof. Let $X = (X_{\epsilon, \epsilon'})$ be a matrix representation of X. We will show $\sum_{\epsilon'} X_{\epsilon, \epsilon'} \Psi_{\epsilon'} = [Q; N]\Psi_\epsilon$ by computing the left hand side. We make use of a diagram D of type A associated with ϵ. The diagram D is depicted as in Eqn.(16). Set $N_\uparrow = \sum_{i=1}^{I+1} n_i$, $N_\downarrow = \sum_{i=1}^{J+1} n'_i$, $M = \sum_{i=1}^{I} m_i$ and $M' = \sum_{i=1}^{I} m'_i$.

We enumerate all arrows from left to right by $1, 2, \ldots, N$. From Eqn.(48), a diagram D' satisfying $X_{D,D'} \neq 0$ can be obtained from D by reversing an up (resp. down) arrow to a down (resp. up) arrow. The number of reversed arrows in D' is at most one. Let $N_\uparrow$ (resp. $N_\downarrow$) be the number of unpaired up (down) arrows in D. Since we have $X_{D,D} = q^{N_\uparrow - N_\downarrow}[Q; 0]$, the contribution from the diagonal term is $q^{N_\uparrow - N_\downarrow}[Q; 0]\Psi_D$.

Firstly, we reverse an up arrow a of D to obtain D'. We have two cases: 1) a is an unpaired up arrow and 2) a is an up arrow forming an arc.

Case 1. Let j be the position of the arrow a. Then, j satisfies $\sum_{k=1}^{i-1} n_k + 2\sum_{k=1}^{i-1} m_k + 1 \leq j \leq \sum_{k=1}^{i} n_k + 2\sum_{k=1}^{i-1} m_k$ for some $1 \leq i \leq I + 1$. We have $X_{D,D'} = q^{j - 2\sum_{k=1}^{i-1} m_k - 1}$ and $\Psi_{D'} = q^{-(N-j)} Q^{-1} \Psi_D$. Thus the contribution is given by

$$(50) \qquad q^{-N + 2j - 2\sum_{k=1}^{i-1} m_k - 1} Q^{-1} \Psi_D.$$

Case 2. We have two cases for the arc b containing the arrow a: a) b is in the region inside of an outer arc of size m_i and b) b is in the region inside of an outer arc of size m_i'.

Case 2-a. Suppose that there are l_1 arcs outside of a (including a itself), $n_\uparrow$ up arrows and l_2 arcs left to the arrow a. Then, a is the $(n_\uparrow + l_1 + 2l_2 + 1)$-th arrow from left. We have $X_{D,D'} = q^{n_\uparrow - l_1}$ and $\Psi_{D'} = q^{-(N - n_\uparrow - l_1 - 2l_2 - 1)}Q^{-1}\Psi_D$. Thus the contribution is given by

$$(51) \qquad q^{-N + 2n_\uparrow + 2l_2 + 1}Q^{-1}\Psi_D.$$

Since b is in the region inside of an outer arc of size m_i, we have $n_\uparrow = \sum_{k=1}^{i} n_k$ and l_2 takes the values $0, 1, \ldots, m_i - 1$ once.

Case 2-b. By a similar argument to Case 2-a, the contribution of D''s is given by

$$(52) \qquad q^{-N + 2N_\uparrow + 2l_2 + 1}Q^{-1}\Psi_D$$

where l_2 takes $0, 1, \ldots, M' - 1$ once.

The sum of contributions from Eqns.(50) to (52) is given by

$$(53) \qquad q^{-N + N_\uparrow + M + M'}[N_\uparrow + M + M']Q^{-1}\Psi_D.$$

Secondly, we reverse an down arrow in D to obtain D'. By a similar argument to Case 1 and 2, the sum of contributions is given by

$$(54) \qquad q^{N_\uparrow + M + M'}[N_\downarrow + M + M']Q\Psi_D.$$

The sum of Eqns.(53), (54) and the diagonal contribution is $[Q; N]\Psi_D$. This completes the proof. $\qquad\square$

5. Action of Hamiltonian on Ψ

Let a be an arc and d be the number of arcs and dashed arc outside of a (including a itself). We call the number d the *depth* of the arc a.

In this section, we will show $e_i\Psi = 0$ for $1 \leq i \leq N$ for arbitrary q, Q, Q_0 and $e_0\Psi = 0$ under the integrable condition:

$$(55) \qquad q^{N-1}QQ_0 - 1 = 0.$$

Since we have an explicit action of e_i, $0 \leq i \leq N$, on a diagram D, we compute explicitly the D-component of $e_i\Psi$.

5.1. **Type A.**

Proposition 5.1. *We have*

$$(56) \qquad e_i\Psi = 0, \qquad 1 \leq i \leq N - 1.$$

Proof. Suppose that D does not have an arc connecting i-th and $(i + 1)$-th arrows. There is no diagram D' such that D appears in the expansion of $e_i(D)$. Thus the D-component of Eqn.(56) is obviously zero.

Below, we consider the case where D has an arc connecting the i-th and the $(i + 1)$-th arrows. We denote by e this small arc. There are two cases: 1) the depth of e is greater than one and 2) the depth of e is one. We will show that the left hand side of (56) is actually zero.

Case 1. Let d be the depth of e. There exists a unique arc e' of depth $d-1$ such that e is inside of e'. There may be several arcs of depth d inside of e'. The diagram D locally looks like

$$\text{i \quad i+1}$$
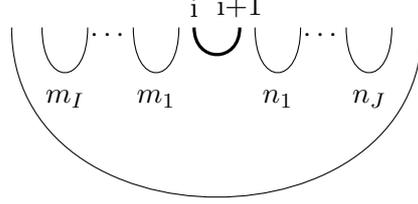
The arcs which are left to e are of size m_i, $1 \leq i \leq I$ and the arcs which are right to e are of size n_j, $1 \leq j \leq J$. Set $M' = \sum_i^I m_i$ and $N' = \sum_{j=1}^J n_j$. The arc e' is the one of size $M' + N' + 2$ and of depth $d-1$. Let D' be a diagram such that $e_i(D')$ contains the term D. Suppose that the arc of depth d and of length m_i connects the k-th and the l-th ($k < l$) arrows. We denote by D' a diagram obtained from D by connecting the l-th and the i-th arrows via an arc and also the k-th and $(i+1)$-th arrows via an arc. Since $e_i(D') = D$, the contribution of such D''s to the left hand side of Eqn.(56) is

$$(57) \qquad \sum_{i=1}^{I} \frac{[m_i]}{[1 + \sum_{j=1}^{i-1} m_i][1 + \sum_{j=1}^{i} m_i]} \Psi_D.$$

Similarly, suppose that the arc of depth d and of length n_j connects the k-th and the l-th arrows. Let D' be a diagram obtained from D by connecting the i-th and the l-th arrows via an arc and the k-th and the $(i+1)$-th arrows via an arc. The contribution of such D''s is

$$(58) \qquad \sum_{i=1}^{J} \frac{[n_i]}{[1 + \sum_{j=1}^{i-1} n_i][1 + \sum_{j=1}^{i} n_i]} \Psi_D.$$

Suppose that the arc of depth $d-1$ and of size $M + N + 2$ connects the k-th and l-th arrows. Let D' be a diagram obtained from D by connecting the k-th and i-th arrows via an arc and $(i+1)$-th and l-th arrows via an arc. The contribution of this D' is

$$(59) \qquad \frac{[M + N + 2]}{[M + 1][N + 1]} \Psi_D.$$

From Lemma A.4, the sum of Eqn.(57) to Eqn.(59) is $[2]\Psi_D$.

Since $e_i(D) = -[2]D$, the contribution of D to the left hand side of Eqn.(56) is $-[2]\Psi_D$. Thus the left hand side of Eqn.(56) is zero.

Case 2. The diagram D is locally depicted as

$$(60) \qquad \cdots \alpha \qquad \qquad \qquad \cdots \qquad \qquad \text{i \ i+1} \qquad \cdots \qquad \beta \cdots$$
$$m_{I+1} \quad m_I \qquad \qquad m_1 \qquad \qquad n_1 \qquad n_J$$

where α are β either $\uparrow$, $\downarrow$ or empty. By empty we mean that there are no arrows. The inside of the arc of size m_i or n_i is filled with arcs. Let $M' = \sum_{i=1}^I m_i$ and $N' = \sum_{i=1}^J n_i$. Since $e_i(D) = -[2]D$, the contribution of D to Eqn.(56) is $-[2]\Psi_D$. By a similar argument to Case 1, we have D''s which changes the arc e and the arc of size m_i and of depth one to two arcs of size $1 + \sum_{j=1}^{i-1} m_i$ and $1 + \sum_{j=1}^{i} m_i$. We also have similar D''s regarding the arc of size n_i. These D''s contribution to the

left hand side of Eqn.(56) is

$$(61) \qquad \sum_{i=1}^{I} \frac{[m_i]\Psi_D}{[1 + \sum_{j=1}^{i-1} m_i][1 + \sum_{j=1}^{i} m_i]} + \sum_{i=1}^{J} \frac{[n_i]\Psi_D}{[1 + \sum_{j=1}^{i-1} n_i][1 + \sum_{j=1}^{i} n_i]}$$

We have eight cases for the diagram D: a) $(\alpha, \beta) = (\uparrow, \uparrow)$, b) $(\alpha, \beta) = (\uparrow, \downarrow)$, c) $(\alpha, \beta) = (\downarrow, \downarrow)$, d) $(\alpha, \beta) = (\uparrow, \emptyset)$, e) $(\alpha, \beta) = (\downarrow, \emptyset)$ f) $(\alpha, \beta) = (\emptyset, \uparrow)$, g) $(\alpha, \beta) = (\emptyset, \downarrow)$, and h) $(\alpha, \beta) = (\emptyset, \emptyset)$ where $\alpha = \emptyset$ (resp. $\beta = \emptyset$) means that there are no arrows left to (resp. right to) α (resp. β).

Case 2-a. Let d be the sum of the numbers of arcs and up arrows left to the arrow α. Let D' be a diagram obtained from D by connecting the arrow α and the i-th arrow via an arc and putting an up arrow at the $(i+1)$-th site. Thus the contribution to Eqn.(56) is

$$(62) \qquad \frac{[d]}{[M'+1][d+M'+1]}\Psi_D.$$

Similarly, let D' be a diagram obtained from D by connecting $(i+1)$-th site and the arrow β via an arc and putting an up arrow at i-th site. The contribution of this D' is

$$(63) \qquad \frac{[d+M'+N'+2]}{[N'+1][d+M'+1]}\Psi_D.$$

The sum of contributions from Eqns.(61) to (63) is $[2]\Psi_D$ by applying Lemma A.4 to Eqn.(61). This implies that the contributions of Eqn.(56) is zero.

Case 2-b to 2-h. By a similar argument to Case 2-a, one can show that the sum of contributions is zero. In Case 2-b, 2-d and 2-h, we have a contribution from D' which is obtained from D by putting an up arrow at i-th site and a down arrow at $(i+1)$-th site. The contribution of this D' is

$$\frac{[d+M'+N'+d'+2]}{[d+M'+1][d'+N'+1]}\Psi_D,$$

where d is the number of up arrows and arcs left to the arrow α and d' is the number of down arrows and arcs right to the arrow β. This completes the proof. $\qquad \square$

Proposition 5.2. *We have $e_N\Psi = 0$.*

Proof. We have three cases for D: 1) the rightmost arrow is an up arrow, 2) the rightmost arrow forms an arc and 3) the rightmost arrow is a down arrow.

Case 1. The diagram D is depicted as

$$(64) \qquad \underbrace{\uparrow \cdots \uparrow}_{n_1} \; \smile \; \uparrow \cdots \uparrow \; \smile \; \underbrace{\uparrow \cdots \uparrow}_{n_{I+1}} \; ,$$
$$\qquad\qquad\qquad\qquad {}^{m_1} \qquad\qquad {}^{m_I}$$

where the region inside of the arc of size m_i, $1 \le i \le I$, is filled with arcs. Set $d = \sum_{i=1}^{I+1} n_i + \sum_{i=1}^{I} m_i$. We have two cases for n_{I+1}: a) $n_{I+1} \ge 2$, and b) $n_{I+1} = 1$.

We have two common contributions to the D-component of $e_N\Psi$ for both case a) and b). Since $e_N(D) = -Q^{-1}D + \cdots$, the contribution from D itself is $-Q^{-1}\Psi_D$. Let D' be a diagram obtained from D by changing the rightmost up arrow to a down arrow. We have $e_N(D') = D + \cdots$. Thus the contribution is given by $q^{-(d-1)}Q^{-1}[d]\Psi_D$.

Case 1-a. Let D' be a diagram obtained from D by connecting the $(N-1)$-th site and the N-th site via an arc. We have $e_N(D') = -q^{-1}D + \cdots$. The contribution is given by $-q^{-d}Q^{-1}[d-1]\Psi_D$. The sum of three contributions is zero, which implies $e_N\Psi = 0$.

Case 1-b. Let D' be a diagram obtained from D by connecting the $(N - 2m_I - 1)$-th site and the N-th site via an arc. We have $e_N(D') = -q^{-1}D + \cdots$. The contribution of D' is

$$(65) \qquad\qquad -q^{-d}Q^{-1}\frac{[d - m_I - 1]}{[m_I + 1]}\Psi_D.$$

Let D' be a diagram obtained from D by connecting the $(N - 1)$-th site and the N-th site via an arc and changing the $(N - 2m_I)$-th up arrow to an down arrow. We have $e_N(D') = -q^{-1}D + \cdots$. The contribution is given by

$$(66) \qquad\qquad -q^{-d}Q^{-1}\frac{[d][m_I]}{[m_I + 1]}.$$

Thus the sum of four contributions is zero, which implies $e_N\Psi = 0$.

Case 2. We have two cases for D: a) D has no down arrows, b) D has down arrows.

Case 2-a. The diagram D is depicted as Eqn.(64) with $n_{I+1} = 0$. Set $N' = \sum_{i=1}^{I} n_i$, $M = \sum_{i=1}^{I} m_i$ and $d = N' + M$. We have four types of contributions to the D-component of $e_N\Psi$. Since $e_N(D) = -Q^{-1}D + \cdots$, we have the contribution $-Q^{-1}\Psi_D$.

Let D' be a diagram obtained from D by changing the outer arc of size m_I to two down arrows. We have $e_N(D') = D + \cdots$. Thus the contribution is

$$(67) \qquad\qquad q^{-(d-1)}Q^{-1}\frac{[N' + M + 1][m_I]}{[m_I + 1]}\Psi_D.$$

Suppose that D has arcs of depth two inside of the arc of size m_I. We enumerate these arcs of depth two from right to left by $1, 2, \ldots, J$ where J is the number of arcs of depth two. We denote by $\widetilde{m_j}$, $1 \le j \le J$, the size of the j-th arc of depth two. Suppose that the i_1-th arrow and the i_2-th arrow form the arc of size $\widetilde{m_j}$. Let D' be a diagram obtained from D by connecting the i_2-th site and the N-th site via an arc and putting two down arrows at the $(N - 2m_I + 1)$-th site and the i_1-th site. We have $e_N(D') = -q^{-1}D + \cdots$. The contribution is given by

$$(68) \quad -q^{-1}\sum_{j=1}^{J}\frac{q^{-(d-1)}Q^{-1}[d + 1][m_I][\widetilde{m_j}]}{[m_I + 1][\sum_{k=1}^{j}\widetilde{m_k} + 1][\sum_{k=1}^{j-1}\widetilde{m_k} + 1]}\Psi_D = -q^{-d}Q^{-1}\frac{[d + 1][m_I - 1]}{[m_I + 1]}\Psi_D.$$

where we have used Lemma A.4 and $\sum_{j=1}^{J}\widetilde{m_j} = m_I - 1$.

Let D' be a diagram obtained from D by changing the rightmost up arrow to a down arrow. $e_N(D') = -q^{-2}D + \cdots$. The contribution is given by $-q^{-(d+1)}Q^{-1}[d - m_I][m_I + 1]^{-1}$.

By a straightforward calculation, the sum of four contributions is zero, which implies $e_N\Psi = 0$.

Case 2-b. Let D be a diagram which starts with n'_1 up arrow, followed by an outer arc of size m'_1, followed by n'_2 up arrows, followed by an outer arc of size m'_2, $\cdots$, followed by n'_{J+1} up arrows, followed by n_I down arrows, followed by an outer arc of size m_I, followed by n_{I-1} down arrows, $\cdots$, and ends with an outer arc of size m_1. We set $N' = \sum_{i=1}^{J+1} n'_i$, $M' = \sum_{i=1}^{J} m'_i$, $N_\downarrow = \sum_{i=1}^{I} n_i$, $M = \sum_{i=1}^{I} m_i$, $L = N' + M' + N_\downarrow + M$, $d = N' + M' + M$ and $v_i = \sum_{j=1}^{i-1} n_j + m_j$.

We have five types of contributions to the D-component of $e_N\Psi$. Since $e_N(D) = -Q^{-1}D + \cdots$, we have a contribution $-Q^{-1}\Psi_D$.

Let D' be a diagram obtained from D by changing the outer arc of size m_1 to two down arrows. We have $e_N(D') = D + \cdots$. The contribution is

$$(69) \qquad q^{-(d-1)}Q^{-1}\Psi_D[L+1]\frac{[m_1]}{[1+m_1]}\prod_{i=1}^{I}\frac{[1+m_i+v_i]}{[1+v_{i+1}]}.$$

Suppose that D has arcs of depth two inside of the arc of size m_1. We enumerate these arcs from right to left and denote by $\widetilde{m_j}$ its size. Suppose that the i_1-th arrow and the i_2-th arrow form the arc of size $\widetilde{m_j}$. Let D' be a diagram obtained from D by connecting the i_2-th site and the N-th site via an arc and putting two down arrows at the $(N-2m_1+1)$-th and the i_1-th sites. We have $e_N(D') = -q^{-1}D + \cdots$. The contribution is given by

$$(70) \qquad -q^{-d}Q^{-1}\Psi_D\sum_j \frac{[L+1][\widetilde{m_j}][m_1]}{[1+\sum_{k=1}^{j-1}\widetilde{m_k}][1+\sum_{k=1}^{j}\widetilde{m_k}][m_1+1]}\prod_{i=1}^{I}\frac{[1+m_i+v_i]}{[1+v_{i+1}]}$$

$$= -q^{-d}Q^{-1}\Psi_D\frac{[L+1][m_1-1]}{[m_1+1]}\prod_{i=1}^{I}\frac{[1+m_i+v_i]}{[1+v_{i+1}]}$$

where we have used Lemma A.4 and $\sum_j \widetilde{m_j} = m_1 - 1$.

Let D' be a diagram obtained from D by changing the rightmost up arrow to a down arrow. We have $e_N(D') = -q^{-\sum_{j=1}^{I}n_j-2}D + \cdots$. The contribution is given by

$$(71) \qquad -q^{-\sum_{j=1}^{I}n_j-1-d}Q^{-1}\frac{[N'+M']}{[N_\downarrow+M+1]}\Psi_D.$$

Suppose that the j_1-th arrow and the j_2-th arrow form an outer arc of size m_j for $2 \le j \le I$. Let D' be a diagram obtained from D by changing the outer arc of size m_i to two down arrows. We have $e_N(D') = -q^{-\sum_{k=1}^{j-1}n_k-2}D + \cdots$. The contribution is given by

$$(72) \qquad -q^{-(d-1)}Q^{-1}\Psi_D\sum_{j=2}^{I}q^{-\sum_{k=1}^{j-1}n_k-2}\frac{[L+1][m_j]}{[1+v_j][1+v_{j+1}]}\prod_{k=i+1}^{I}\frac{[1+m_k+v_k]}{[1+v_{k+1}]}$$

$$= -q^{-d}Q^{-1}\Psi_D[L+1]\left\{\prod_{i=1}^{I}\frac{[1+m_i+v_i]}{[1+v_{i+1}]} - \frac{q^M}{[1+v_{J+1}]} - \frac{q^{-1}[m_1]}{[m_1+1]}\prod_{i=1}^{I}\frac{[1+m_i+v_i]}{[1+v_{i+1}]}\right\}$$

where we have used Lemma A.6. By a straightforward calculation, one can show that the sum of Eqns.(69) to (72) is $Q^{-1}\Psi_D$. This cancels the contribution of D itself, which implies $e_N\Psi = 0$.

Case 3. Let D be a diagram depicted as Eqn.(16). The D-component of Ψ is explicitly given by

$$\Psi_D = q^{d(d-1)/2}Q^d\prod_{B\in S}[m_B]^{-1}\prod_{i=1}^{I}\frac{[\sum_{j=1}^{i}(n_j+m_j)]!}{[n_i+\sum_{j=1}^{i-1}(n_j+m_j)]!}\frac{[N_\uparrow+M+N'+M']!}{[N_\uparrow+M]!}$$

$$\times\prod_{i=1}^{J+1}\frac{[\sum_{j=1}^{i-1}(n'_j+m'_j)]!}{[n'_i+\sum_{j=1}^{i-1}(n'_j+m'_j)]!}$$

where $N_\uparrow = \sum_{i=1}^{I+1}n_i$, $M = \sum_{i=1}^{I}m_i$, $N' = \sum_{i=1}^{J+1}n'_i$, $M' = \sum_{i=1}^{J}m'_i$ and $d = N_\uparrow + M + M'$.

We have two cases for D: i) $n_1 \ge 2$ and ii) $n_1 = 1$. Below, we consider the case i) only since one can prove Proposition for case ii) by a similar argument.

We have ten types of contributions for the D-component of $e_N\Psi$ as follows.

a) Since $e_N(D) = -QD + \ldots$, we have a contribution from D itself, that is $-Q\Psi_D$.

b) Let D' be a diagram obtained from D by changing the outer arc of size m_i' to two down arrows. We have $e_N(D') = q^{-\sum_{j=1}^{i} n_j'} D + \ldots$.

c) Let D' be a diagram obtained from D by changing the $N_\uparrow$-th (from left) up arrow to a down arrow. We have $e_N(D') = q^{-N'} D + \ldots$.

d) Let D' be a diagram obtained from D by changing the first and the second (from right) down arrows to an arc. We have $e_N(D') = D + \ldots$.

e) Let D' be a diagram obtained from D by changing the first and the second (from right) down arrows to an arc and by changing the $N_\uparrow$-th and the $(N_\uparrow - 1)$-th (from left) up arrows to two down arrows. We have $e_N(D') = -q^{-2N'-1} D + \ldots$.

f) Let D' be a diagram obtained from D by changing the first and the second (from right) down arrows to an arc, by changing the outer arc of size m_i' to two down arrows and by changing the $N_\uparrow$-th (from left) up arrow to a down arrow. We have $e_N(D') = -(1+q^2) q^{-N'-2-\sum_{j=1}^{i} n_j} D + \ldots$.

g) Let D' be a diagram obtained from D by changing the outer arc of size m_i' to two down arrows and by changing the outer arc of size m_j' $(j < i)$ to two down arrows. We have $e_N(D') = -(1 + q^{-2}) q^{-\sum_{k=1}^{i} n_k' - \sum_{k=1}^{j} n_k'} D + \ldots$.

h) Let D'' be a diagram obtained from D by changing the first and the second (from right) down arrows to an arc and by changing the outer arc of size m_i' to two down arrows. Suppose that $m_{1,j}'$, $1 \le j \le r$, be the size of outer arcs of D'' which is inside of the outer arc of the size m_1 in D. Let D' be a diagram obtained from D'' by changing the outer arc of size $m_{1,j}'$ to two down arrows. Then, we have $e_N(D') = -q^{-1-2\sum_{k=1}^{i} n_k'} D + \ldots$.

i) Let D' be a diagram obtained from D by changing the first and the second (from right) down arrows to an arc and by changing the $N_\uparrow$-th (from left) up arrow to a down arrow. We have $e_N(D') = q^{-N'}(Q - Q^{-1}) D + \ldots$.

j) Let D' be a diagram obtained from D by changing the first and the second (from right) down arrows to an arc and by changing the outer arc of size m_i to two down arrows. We have $e_N(D') = q^{-\sum_{j=1}^{i} n_j'}(Q - Q^{-1}) D + \ldots$.

The sum of contributions from a) to j) is written as $A_1 Q\Psi_D + A_{-1} Q^{-1}\Psi_D$. We will show that $A_1 = A_{-1} = 0$. Set $v_i := \sum_{j=1}^{i-1}(n_j' + m_j')$, $w_i := 1 + n_i' + v_i$, $d_j := 1 - N_\uparrow - M - M' - \sum_{k=1}^{j} n_k'$ and $L := 1 + N_\uparrow + M + n_{I+1}' + v_{I+1}$.

The contribution to A_1 is summarized as follows. From a), we have -1. From d) we have

$$(73) \qquad \frac{q^d}{[L-1]} \prod_{i=1}^{J+1} \frac{[n_j' + v_j]}{[v_j]}.$$

From i), we have

$$(74) \qquad q^{-N'} \frac{[N_\uparrow + M]}{[L-1][M'+N']} \prod_{i=1}^{J+1} \frac{[n_j' + v_j]}{[v_j]}.$$

From j), we have

$$(75) \qquad \sum_{i=1}^{J} q^{-\sum_{j=1}^{i} n_j'} \frac{[m_i']}{[v_{i+1}][v_i]} \prod_{j=1}^{i-1} \frac{[n_j' + v_j]}{[v_j]}.$$

Applying Lemma A.5 to the sum of Eqns.(73), (74) and (75), we obtain 1 which cancels the contribution from a). Thus we have $A_1 = 0$.

We have eight types of contributions to A_{-1}. From b), we have

$$(76) \qquad \sum_{i=1}^{J} q^{d_i} \frac{[L][m_i']}{[w_i][w_{i+1}]} \prod_{j=i+2}^{J+1} \frac{[1+v_j]}{[w_j]}.$$

From c), we have

$$(77) \qquad q^{d_{J+1}} \frac{[N_\uparrow + M]}{[w_{J+1}]}.$$

From e), we have

$$(78) \qquad - q^{d'} \frac{[N_\uparrow + M][N_\uparrow + M - 1]}{[L-1][w_{J+1}][v_{J+1}]} \prod_{i=1}^{J} \frac{[n_i' + v_i]}{[v_i]},$$

where $d' := d_{I+1} - N' - 1$. From f), we have

$$(79) \qquad - (1 + q^{-2}) \sum_{i=1}^{J} q^{d_i - N'} \frac{[N_\uparrow + M][m_i']}{[v_i][v_{i+1}][w_{J+1}]} \prod_{j=1}^{i-1} \frac{[n_j' + v_j]}{[v_j]}.$$

From g), we have

$$(80) \quad - (1 + q^{-2}) \sum_{i=2}^{J} q^{d_i} \frac{[L][m_i']}{[w_i][w_{i+1}]} \prod_{k=i+2}^{J+1} \frac{[1+v_k]}{[w_k]} \sum_{j=1}^{i-1} \frac{q^{-\sum_{k=1}^{j} n_k'}[m_j']}{[v_j][v_{j+1}]} \prod_{k=1}^{j-1} \frac{[n_k' + v_k]}{[v_k]}$$

$$= - \sum_{i=1}^{J} (1 + q^{-2}) q^{d_i} \frac{[m_i'][L]}{[w_i][w_{i+1}]} \prod_{j=i+2}^{J+1} \frac{[1+v_j]}{[w_j]}$$

$$+ \sum_{i=1}^{J} (1 + q^{-2}) q^{d_i + \sum_{j=1}^{i-1} m_j'} \frac{[m_i'][L]}{[w_i][w_{i+1}]} \prod_{j=i+2}^{J+1} \frac{[1+v_j]}{[w_j]} \prod_{k=1}^{i-1} \frac{[n_k' + v_k]}{[v_{k+1}]}$$

where we have used Lemma A.5. From h), we have

$$(81) \quad \sum_{i=1}^{J} \frac{q^{d_i - \sum_{j=1}^{i} n_j' - 1}[L][m_i']}{[n_i' + v_i][1 + n_{i+1}]} \prod_{k=1}^{i} \frac{[n_k' + v_k]}{[v_k]} \prod_{k=i+1}^{J+1} \frac{[1+v_k]}{[w_k]} \left(\sum_{l} \frac{[m_{1,l}]}{[w_{i,j-1}][w_{i,j}]} \right)$$

$$= - \sum_{i=1}^{J} \frac{q^{d_i - \sum_{j=1}^{i} n_j' - 1}[L][m_i'][m_i' - 1]}{[n_i' + v_i][1 + v_{i+1}][w_i][v_{i+1}]} \prod_{k=1}^{i} \frac{[n_k' + v_k]}{[v_k]} \prod_{k=i+1}^{J+1} \frac{[1+v_k]}{[w_k]}$$

where $w_{i,j} := w_i + \sum_{k \le j} m_{1,k}$, we have used Lemma A.4 and $\sum_l m_{i,l} = m_i - 1$. From i), we have

$$(82) \qquad - q^{-N'} \frac{[N_\uparrow + M]}{[L-1][v_{I+1}]} \prod_{j=1}^{I} \frac{[n_j' + v_j]}{[v_j]}.$$

From j), we have

$$(83) \qquad - \sum_{i=1}^{J} q^{- \sum_{j=1}^{i} n_j'} \frac{[m_i']}{[v_i][v_{i+1}]} \prod_{k=1}^{i-1} \frac{[n_k' + v_k]}{[v_k]}.$$

We apply Lemma A.5 to $q^2/(1+q^2)$ times Eqn.(79) and the sum of Eqns.(78) and (82). The result cancels the contribution of Eqn.(77).

The sum of Eqn.(83) and $1/(1+q^2)$ times Eqn.(79) becomes

$$(84) \quad -\sum_{i=1}^{J} \frac{q^{-(M+N_\uparrow+\sum_{j=1}^{i} n'_j)}[m'_i][L]}{[v_i][v_{i+1}][w_{J+1}]} \prod_{k=1}^{i-1} \frac{[n'_k+v_k]}{[v_k]}$$

$$= -q^{-N_\uparrow-M}\frac{[L]}{[w_{J+1}]} + q^{-(N_\uparrow+M-\sum_{j=1}^{I} m'_j)}\frac{[L]}{[w_{J+1}]} \prod_{k=1}^{J} \frac{[n'_k+v_k]}{[v_{k+1}]}.$$

where we have used Lemma A.5. The sum of Eqn.(76) and the first term of the right hand side of Eqn.(80) is

$$(85) \qquad\qquad -q^{-N_\uparrow-M-\sum_{j=1}^{J} m'_j}[L] \prod_{i=1}^{I+1} \frac{[1+v_j]}{[w_j]} + q^{-N_\uparrow-M}\frac{[L]}{[w_{J+1}]}.$$

where we have used Lemma A.8. The sum of the second term of the right hand side of Eqn.(80) and Eqn.(81) is given by

$$(86) \quad q^{-N_\uparrow-M-\sum_{i=1}^{J} m'_j+1} \sum_{i=1}^{J} q^{-\sum_{k=1}^{i} n'_k} \frac{[L][m'_i]}{[w_i][w_{i+1}]} \prod_{k=i+2}^{J+1} \frac{[1+v_k]}{[w_k]} \prod_{k=1}^{i-1} \frac{[n'_k+v_k]}{[v_{k+1}]}$$

$$\times \left\{ (1+q^{-2})q^{\sum_{j=1}^{i-1} m'_j} - q^{-\sum_{j=1}^{i} n'_j-1}\frac{[m'_i-1]}{[v_{i+1}]} \right\}$$

By Lemma A.7, the sum of Eqns.(84), (85) and (86) is zero, which implies $A_{-1}=0$. This completes the proof.   □

Proposition 5.3. *We have* $e_0\Psi = 0$ *at* $qQQ_0 = 1$.

Proof. We compute the D-component of $e_0\Psi$ at the specialization (55). We have three cases for the leftmost arrow a of a diagram D: 1) a is a down arrow, that is, $N_\uparrow = 0$ 2) a is an up arrow, and 3) a is a down arrow forming an arc.

Case 1. Let D be a diagram depicted as

$$\underbrace{\downarrow\cdots\downarrow}_{n_1} \cup \downarrow\cdots\downarrow \cup \underbrace{\downarrow\cdots\downarrow}_{n_{I+1}}\ .$$
$$ m_1 m_I $$

We have two cases for D: a) $n_1 = 1$, and b) $n_2 \geq 2$. We consider only the Case a since one can apply a similar argument to Case b.

Case 1-a. We have four types of contributions to the D-component of $e_0\Psi$. Since we have $e_0(D) = -Q_0^{-1}D + \ldots$, the contribution to the D-component of $e_0\Psi$ is $-Q_0^{-1}\Psi_D$.

Let D' be a diagram obtained from D by reversing the leftmost down arrow to an up arrow. We have $e_0(D) = D + \ldots$. The contribution is $q^{|S|}Q[|S| + N_\downarrow]\Psi_D$.

We denote by D' a diagram obtained from D by connecting the first and the second (from left) up arrows via an arc. We have $e_0(D) = -q^{-1}D + \cdots$. The contribution is

$$-q^{|S|-1}Q\frac{[|S| + N_\downarrow - m_1 - 1]}{[m_1 + 1]}\Psi_D.$$

Let D' be a diagram obtained from D by changing the outer arc of size m_1 to two up arrows and by connecting the first and the second (from left) arrows via an arc. We have $e_0(D) = -q^{-1}D + \cdots$.

The contribution is given by

$$-q^{|S|-1}Q\frac{[m_1][|S|+N_\downarrow]}{[m_1+1]}.$$

The sum of four contributions above is

$$Q_0^{-1}(q^{N-1}QQ_0-1)\Psi_D,$$

which implies the D-component of $e_0\Psi$ is zero at the specialization (55).

Case 2. Let D be a diagram depicted as Eqn.(16). Set $N_\uparrow := \sum_{i=1}^{I+1} n_i$, $N_\downarrow := \sum_{i=1}^{J+1} n_i'$, $M := \sum_{i=1}^{I} m_i$, $M' := \sum_{i=1}^{J} m_i'$, $d := |S| + N_\uparrow$ and $v_i := \sum_{j=1}^{i}(n_j + m_j)$. We will prove Proposition in the case of $n_1 \geq 2$ since one can apply a similar argument to the case of $n_1 = 1$.

We have ten types of contributions to the D-component of $e_0\Psi$ as follows:

a) Since $e_0(D) = -Q_0 D + \cdots$, we have a contribution from D itself, that is, $-Q_0\Psi_D$.

b) Let D' be a diagram obtained from D by connecting the first and the second (from left) up arrows via an arc. We have $e_0(D') = D + \cdots$. The contribution is

$$(87) \qquad q^{-(d-1)}Q^{-1}\frac{[N_\uparrow + M]}{[N_\uparrow + N_\downarrow + M + M']}\prod_{i=1}^{I}\frac{[n_i + v_{i-1}]}{[v_i]}\Psi_D.$$

c) Let D' be a diagram obtained from D by connecting the first and second (from left) up arrows to an arc and by flipping the $N_\downarrow$-th (from right) down arrow to an up arrow. We have $e_0(D') = q^{-N_\uparrow}(Q_0 - Q_0^{-1}) + \cdots$. The contribution is given by

$$(88) \qquad q^{-N_\uparrow}(Q_0 - Q_0^{-1})\frac{[N_\downarrow + M']}{[N_\uparrow + N_\downarrow + M + M']}\prod_{i=1}^{I}\frac{[n_i + v_{i-1}]}{[v_i]}\Psi_D.$$

d) Let D' be a diagram obtained from D by connecting the first and second (from left) up arrows to an arc and by flipping the $N_\downarrow$-th and the $(N_\downarrow - 1)$ (from right) down arrows to two up arrows. We have $e_0(D') = -q^{-2N_\uparrow-1}D + \cdots$. The contribution is

$$(89) \qquad -q^{d-2N_\uparrow-1}Q\frac{[N_\downarrow + M'][N_\downarrow + M' - 1]}{[N_\uparrow + M + 1][d + N_\downarrow]}\prod_{i=1}^{I}\frac{[n_i + v_{i-1}]}{[v_i]}\Psi_D.$$

e) Let D' be a diagram obtained from D by changing the outer arc of size m_i to two up arrows. We have $e_0(D') = q^{-\sum_{j=1}^{i} n_j}D + \cdots$. The contribution is

$$(90) \qquad \sum_{i=1}^{I}q^{d-\sum_{j=1}^{i} n_j}Q\frac{[m_i]}{[1+n_i+v_{i-1}]}\frac{[d+N_\downarrow+1]}{[N_\uparrow+M+1]}\prod_{i=1}^{I}\frac{[1+v_i]}{[1+n_j+v_{j-1}]}\Psi_D$$

$$= q^d Q\Psi_D\frac{[d+N_\downarrow+1]}{[N_\uparrow+M+1]}\left(q\prod_{j=1}^{I}\frac{[1+v_j]}{[1+n_j+v_{j-1}]} - q^{1+\sum_{i=1}^{I} m_i}\right),$$

where we have used Lemma A.11.

f) Let D' be a diagram obtained from D by changing the $N_\downarrow$-th (from right) down arrow to an up arrow. We have $e_0(D') = q^{-N_\uparrow}D + \cdots$. The contribution is

$$(91) \qquad q^{d-N_\uparrow}Q\frac{[N_\downarrow + M']}{[N_\uparrow + M + 1]}\Psi_D.$$

g) Let D' be a diagram obtained from D by connecting the first and the second (from left) up arrows via an arc and flipping the $N_\downarrow$-th (from right) down arrow to an up arrow and by changing the outer arc of size m_i to two up arrows. We have $e_0(D') = -(1+q^2)q^{-N_\uparrow - 2 - \sum_{j=1}^{i} n_j} D + \cdots$. The contribution is given by

$$(92) \quad -(1+q^2)q^{d-N_\uparrow-2}Q\Psi_D \sum_{i=1}^{I} q^{-\sum_{j=1}^{i} n_j} \frac{[m_i][N_\downarrow + M']}{[v_i][N_\uparrow + M + 1]} \prod_{j=1}^{i-1} \frac{[n_j + v_{j-1}]}{[v_j]}$$

$$= -(1+q^2)q^{d-N_\uparrow-2}Q\Psi_D \frac{[N_\downarrow + M']}{[N_\uparrow + M + 1]} \left(1 - q^{\sum_{i=1}^{I} m_i} \prod_{i=1}^{I} \frac{[n_i + v_{i-1}]}{[v_i]}\right),$$

where we have used Lemma A.10.

h) Let D' be a diagram obtained from D by connecting the first and the second (from left) up arrows via an arc and changing the outer arcs of size m_i and m_j $(i < j)$ to four up arrows. We have $e_0(D') = -(1+q^{-2})q^{-\sum_{k=1}^{i} n_i - \sum_{k=1}^{j} n_j} D + \cdots$. The contribution is given by

$$(93) \quad -(1+q^{-2})q^d Q\Psi_D \frac{[d+N_\downarrow+1]}{[N_\uparrow+M+1]} \sum_{j=2}^{I} q^{-\sum_{k=1}^{j} n_k} \frac{[m_j]}{[1+n_j+v_{j-1}]} \prod_{l=j+1}^{I} \frac{[1+v_l]}{[1+n_l+v_{l-1}]}$$

$$\times \sum_{i=1}^{j-1} q^{-\sum_{k=1}^{i} n_k} \frac{[m_i]}{[v_i]} \prod_{l=1}^{i-1} \frac{[n_l+v_{l-1}]}{[v_l]}$$

$$= -(1+q^{-2})q^d Q\Psi_D \frac{[d+N_\downarrow+1]}{[N_\uparrow+M+1]} \left\{ q \prod_{j=1}^{I} \frac{[1+v_j]}{[1+n_j+v_{j-1}]} - q^{1+\sum_{i=1}^{I} m_i} \right.$$

$$\left. - \sum_{j=1}^{I} q^{-\sum_{k=1}^{j} n_k + \sum_{k=1}^{j-1} m_k} \frac{[m_j]}{[1+n_j+v_{j-1}]} \prod_{l=j+1}^{I} \frac{[1+v_l]}{[1+n_l+v_{l-1}]} \prod_{l=1}^{i-1} \frac{[n_l+v_{l-1}]}{[v_l]} \right\}.$$

where we have used Lemma A.10 and Lemma A.11.

i) Let D'' be a diagram obtained from D by connecting the first and the second (from left) up arrows via an arc and by changing the outer arc of size m_i to two up arrows. Suppose that $\tilde{m}_{i,j}$, $1 \le j \le r$ be the size of outer arcs of D'' which are inside of the outer arc of size m_i in D. Let D' be a diagram obtained from D'' by changing the outer arc of size $\tilde{m}_{i,j}$ to two up arrows. We have $e_0(D') = -q^{-1-2\sum_{k=1}^{i} n_k} D + \cdots$. The contribution is given by

$$(94) \quad -q^{d-1}Q\Psi_D \frac{[d+N_\downarrow+1]}{[N_\uparrow+M+1]} \sum_{i=1}^{I} q^{-2\sum_{k=1}^{i} n_k}[m_i] \prod_{j=1}^{i-1} \frac{[n_j+v_{j-1}]}{[v_j]} \prod_{j=i+1}^{I} \frac{[1+v_j]}{[1+n_j+v_{j-1}]}$$

$$\times \sum_{p=1}^{r} \frac{[\tilde{m}_{i,p}]}{[1+n_p+v_{p-1}+\sum_{k=1}^{p-1} \tilde{m}_{i,k}][1+n_p+v_{p-1}+\sum_{k=1}^{p} \tilde{m}_{i,k}]}$$

$$= -q^{d-1}Q\Psi_D \frac{[d+N_\downarrow+1]}{[N_\uparrow+M+1]} \sum_{i=1}^{I} \frac{q^{-2\sum_{k=1}^{i} n_k}[m_i][m_i-1]}{[1+n_i+v_{i-1}][v_i]} \prod_{j=1}^{i-1} \frac{[n_j+v_{j-1}]}{[v_j]} \prod_{j=i+1}^{I} \frac{[1+v_j]}{[1+n_j+v_{j-1}]},$$

where we have used Lemma A.4.

j) Let D' be a diagram obtained from D by connecting the first and the second (from left) up arrows via an arc and by changing the outer arc of size m_i to two up arrows. We have $e_0(D') =

$q^{-\sum_{k=1}^{i} n_k}(Q_0 - Q_0^{-1})D + \cdots$. The contribution is given by

$$(95) \quad (Q_0 - Q_0^{-1})\Psi_D \sum_{i=1}^{I} q^{-\sum_{k=1}^{i} n_k} \frac{[m_i]}{[v_i]} \prod_{j=1}^{i-1} \frac{[n_j + v_{j-1}]}{[v_j]}$$

$$= (Q_0 - Q_0^{-1})\Psi_D \left(1 - q^{\sum_{i=1}^{I} m_i} \prod_{i=1}^{I} \frac{[n_i + v_{i-1}]}{[v_i]}\right)$$

Note that one can apply Lemma A.12 to the third term of the right hand side of Eqn.(93) and the right hand side of Eqn.(94). The sum of contributions from a) to j) is

$$(q^{N-1}QQ_0 - 1)\left(Q_0^{-1} - (q^{d'}Q_0^{-1} + q^{1-d}Q^{-1})\frac{[N_\uparrow + M]}{[d + N_\downarrow]} \prod_{i=1}^{I} \frac{[n_i + v_{i-1}]}{[v_i]}\right)\Psi_D,$$

where $d' = N_\downarrow + M + M'$ and $N = N_\uparrow + N_\downarrow + 2M + 2M'$. The sum becomes zero under the specialization (55).

Case 3. Let D be a diagram depicted as

$$(96) \qquad \underset{\text{size } m_1}{\underbrace{\smile}} \underset{n_1}{\underbrace{\uparrow \ldots \uparrow}} \ldots \uparrow \underset{\text{size } m_I}{\underbrace{\smile}} \underset{n_I}{\underbrace{\uparrow \ldots \uparrow}} \underset{n'_{J+1}}{\underbrace{\downarrow \ldots \downarrow}} \underset{\text{size } m'_J}{\underbrace{\smile}} \downarrow \ldots \downarrow \underset{\text{size } m'_1}{\underbrace{\smile}} \underset{n'_1}{\underbrace{\downarrow \ldots \downarrow}}.$$

Set $N_\uparrow := \sum_{i=1}^{I} n_i$, $M := \sum_{i=1}^{I} m_i$, $N_\downarrow := \sum_{i=1}^{J+1} n'_i$, $M' := \sum_{i=1}^{J} m'_i$, $d = N_\uparrow + M + M'$ and $v_i := \sum_{j=1}^{i}(n_j + m_j)$. We have five types of contributions to the D-component of $e_0\Psi$.

a) Since $e_0(D) = -Q_0^{-1}D + \cdots$, we have a contribution from D, which is $-Q_0^{-1}\Psi_D$.

b) Let D' be a diagram obtained from D by changing the outer arc of size m_1 to two up arrows. We have $e_0(D') = D + \cdots$. The contribution is given by

$$q^d Q\Psi_D \frac{[m_i][d + N_\downarrow + 1]}{[N_\uparrow + M + 1]} \prod_{i=2}^{I} \frac{[1 + m_i + v_{i-1}]}{[1 + v_{i-1}]}.$$

c) Let D' be a diagram obtained from D by changing the outer arc of size m_i to two up arrows. We have $e_0(D') = -q^{-\sum_{k=1}^{i-1} -2}D + \cdots$. The contribution is given by

$$(97) \quad -q^{d-2}Q\Psi_D \frac{[d + N_\downarrow + 1]}{[N_\uparrow + M + 1]} \sum_{i=2}^{I} \frac{q^{-\sum_{k=1}^{i-1} n_k}[m_i]}{[1 + v_{i-1}]} \prod_{j=i+1}^{I} \frac{[1 + m_j + v_{j-1}]}{[1 + v_{j-1}]}$$

$$= -q^{d-2}Q\Psi_D \frac{[d + N_\downarrow + 1]}{[N_\uparrow + M + 1]} \left(q^{m_1+1} \prod_{i=2}^{I} \frac{[1 + m_j + v_{j-1}]}{[1 + v_{j-1}]} - q^{1+\sum_{i=1}^{I} m_i}\right)$$

where we have used Lemma A.11.

d) Let D' be a diagram obtained from D by connecting the first and the second (from left) up arrows via an arc and by flipping the $N_\downarrow$-th (from right) down arrow to an up arrow. We have $e_0(D') = -q^{-N_\uparrow - 2}D + \cdots$. The contribution is

$$-q^{d-N_\uparrow - 2}Q\Psi_D \frac{[N_\downarrow + M']}{[N_\uparrow + M + 1]}.$$

e) Suppose that there are several arcs of depth two inside of the outer arc of size m_1. We denote by $\tilde{m}_j$, $1 \le j \le r$, their sizes from left to right. Suppose that the i_1-th and the i_2-th (from left) arrows form the arc of size $\tilde{m}_j$. Let D' be a diagram obtained by connecting the first and the

i_1-th arrows via an arc and by putting two up arrows at the i_2-th and the $2m_1$-th sites. We have $e_0(D') = -q^{-1}D + \cdots$. The contribution is

$$-q^{d-1}Q\Psi_D \frac{[d+N_\downarrow+1][m_1]}{[N_\uparrow+M+1]} \prod_{i=2}^{I} \frac{[1+m_i+v_{i-1}]}{[1+v_{i-1}]} \sum_{j=1}^{r} \frac{[\tilde{m}_j]}{[1+\sum_{k=1}^{j-1}\tilde{m}_k][1+\sum_{k=1}^{j}\tilde{m}_k]}$$

$$= -q^{d-1}Q\Psi_D \frac{[d+N_\downarrow+1][m_1-1]}{[N_\uparrow+M+1]} \prod_{i=2}^{I} \frac{[1+m_i+v_{i-1}]}{[1+v_{i-1}]}$$

where we have used Lemma A.4 and $\sum_{j=1}^{r}\tilde{m}_j = m_1 - 1$.

By a straightforward calculation, the sum of the contributions form a) to e) is given by $(-Q_0^{-1} + Qq^{N-1})\Psi_D$ which vanishes at the specialization (55). $\square$

5.2. **Type BI, BII, BIII and standard basis.**

Theorem 5.4. *For type BI, BII, BIII and standard basis, we have*

(98) $$e_i\Psi = 0, \qquad 1 \le i \le N,$$

(99) $$e_0\Psi = 0, \qquad at \ q^{N-1}QQ_0 - 1 = 0.$$

where $Q = q^M$ for type BI.

Proof. Let Ψ^Y be the eigenfunction Ψ for type Y. Let $T^{Z\leftarrow Y}$ be the transition matrix from the Kazhdan–Lusztig basis of type Y to type Z. Since we have proved that $X\Psi^Y = [Q; N]\Psi^Y$ ($Q = q^M$ for type BI) and the multiplicity is one in Section 4, we have $\Psi^Z = T^{Z\leftarrow Y}\Psi^Y$. From Proposition 5.1 and Proposition 5.2, we have $e_i\Psi^A = 0$ for $1 \le i \le N$. Multiplying $T^{Y\leftarrow A}$ from left and plugging $\Psi^A = T^{A\leftarrow Y}\Psi^Y$, we obtain $T^{Y\leftarrow A}e_i T^{A\leftarrow Y}\Psi^Y = 0$. Since $T^{Y\leftarrow A}e_i T^{A\leftarrow Y}$ is the matrix expression of e_i on the Kazhdan–Lusztig basis of type Y, we have $e_i\Psi^Y = 0$. Similarly, from Proposition 5.3, we have $e_0\Psi^Y = 0$ at the specialization (55).

In the case of the standard basis, we define the transition matrix from a standard basis to a Kazhdan–Lusztig basis of type Z by $T^{Z\leftarrow 0}$. By a similar argument to the case of Kazhdan–Lusztig bases, we obtain Eqns.(98) and (99). $\square$

Remark 5.5. *In the proof of Theorem 5.4, we do not need an explicit expression of the transition matrix. The entries of the transition matrix from the Kazhdan–Lusztig basis to the standard basis are nothing but Kazhdan–Lusztig polynomials. Therefore, the relation $\Psi^0 = T^{0\leftarrow Y}\Psi^Y$ gives highly non-trivial relations regarding Kazhdan–Lusztig polynomials.*

5.3. Ψ **as the ground state of the Hamiltonian.**

Let A be a non-negative $N \times N$ square matrix. The matrix A is called *irreducible* if for any i, j there is a $k = k(i, j)$ such that $(A^k)_{ij} > 0$. Let $\rho(A)$ denote the spectral radius of A. Then, Perron–Frobenius Theorem for a non-negative and irreducible matrix A states that the eigenspace associated with $\rho(A)$ is one-dimensional, there exists a unique eigenvector $\mathbf{x} = (x_1, \ldots, x_N)^T$ such that the entries of $\mathbf{x}$ are positive and $A\mathbf{x} = \rho(A)\mathbf{x}$. For a general non-negative matrix A, we have

Lemma 5.6 (Lemma 6.2 in [32])**.** *Suppose that $\mathbf{x}$ be a positive vector such that $A\mathbf{x} = \lambda\mathbf{x}$ with some scalar λ. Then, we have $\rho(A) = \lambda$.*

Below, we set $q, Q > 0$ and consider the Kazhdan–Lusztig bases for type BI and BIII. From the explicit expression of the action of Temperley–Lieb algebra on the Kazhdan–Lusztig bases (see Section 3), the matrix $H' = -H^{1B} + t\mathbf{1}$ is a non-negative matrix for t sufficiently large. Since Ψ is

a positive eigenvector of H^{1B} with the eigenvalue zero, Ψ is also the positive eigenvector of H' with the eigenvalue t. We apply Lemma 5.6 to H' and Ψ. Therefore, the vector Ψ is the eigenvector of H' with the largest eigenvalue, which implies that Ψ is the ground state of H.

In the case of type A and BII, the Hamiltonian $-H^{1B} + t\mathbf{1}$ can not be a non-negative matrix. However, the bases of type A and BII can be obtained from type BIII by the change of bases. Note that the spectrum of H is invariant under the change of bases. Therefore, the vector Ψ for type A or BII is also the ground state of the Hamiltonian H^{1B}.

Recall that the two-boundary Temperley–Lieb Hamiltonian is given by $H^{1B} - a_0 e_0$. At $a_0 = 0$, H^{2B} becomes H^{1B}. The eigenfunction Ψ is an eigenvector of H^{2B} with an eigenvalue zero for arbitrary a_0. We can regard the Hamiltonian H^{2B} as a perturbation of H^{1B}. Since Ψ is the ground state of H^{1B}, Ψ is also the ground state of H^{2B} for a_0 sufficiently small.

6. INTEGRAL STRUCTURE AND CONJECTURES

6.1. **Correlation functions.** Let $\Psi = |\Psi\rangle$ be the ground state of H^{2B} and $\mathcal{O}$ be an observable which acts on $V_1^{\otimes N}$. A correlation function is defined by

$$\langle \mathcal{O} \rangle := \frac{\langle \Psi | \mathcal{O} | \Psi \rangle}{\langle \Psi | \Psi \rangle}.$$

We consider the case where $\mathcal{O}$ is a product of $\alpha_i := (\sigma_z + 1)/2$ and $\sigma_i^{\pm}$. Let I, J be the subsets of $\{1, 2, \ldots N\}$ satisfying $I \cap J = \emptyset$. In general, an observable is written as

$$\mathcal{O}_{I,J} := \prod_{i \in I} \alpha_i \prod_{j \in J} \beta_j,$$

where β_j is σ_j^+ or σ_j^-. We compute a correlation function in the standard basis. From Definition 4.27, $|\Psi^0\rangle$ is written as

$$(100) \qquad |\Psi^0\rangle = w_1 \otimes w_2 \otimes \ldots \otimes w_N,$$

where $w_i := v_{-1} + q^{N-i} Q v_1$. We have

$$\alpha_i w_i = q^{N-i} Q v_1,$$
$$\sigma_i^+ w_i = v_1, \quad \sigma_i^- w_i = q^{N-i} Q v_{-1}.$$

Therefore, we obtain

$$(101) \qquad \langle \mathcal{O}_{I,J} \rangle = \prod_{i \in I} \frac{q^{2(N-i)} Q^2}{1 + q^{2(N-i)} Q^2} \prod_{j \in J} \frac{q^{N-j} Q}{1 + q^{2(N-j)} Q^2}.$$

6.2. **Positive integral structure.**

6.2.1. *Type A.* We have

Lemma 6.1. *All components of Ψ belong to $\mathbb{N}[q, q^{-1}, Q]$.*

Proof. Let D be a diagram depicted as Eqn.(16). In the notation used in the proof of Theorem 4.6. The explicit expression (17) is rewritten as

$$\Psi_D = q^{d(d-1)/2} Q^d \prod_{i=1}^{I} \begin{bmatrix} \sum_{j=1}^{i} n_i + m_i \\ m_i \end{bmatrix} \prod_{i=1}^{J} \begin{bmatrix} \sum_{j=1}^{i} n_i' + m_i' \\ m_i' \end{bmatrix} \begin{bmatrix} N_\uparrow + M + N_\downarrow + M' \\ N_\uparrow + M \end{bmatrix}.$$

Since a quantum binomial belongs to $\mathbb{N}[q, q^{-1}]$, we obtain $\Psi_D \in \mathbb{N}[q, q^{-1}, Q]$. $\qquad \square$

6.2.2. *Type BI.* We have

Proposition 6.2. *All components of Ψ belong to $\mathbb{N}[q, q^{-1}]$ and invariant under $q \to q^{-1}$.*

Proof. Since a component Ψ_D contains only quantum integers and terms $(q^i + q^{-i})$ for some i (see Definition 4.11), it is invariant under $q \to q^{-1}$.

Recall that we have two types of parabolic Kazhdan–Lusztig polynomials according to the choice of projection map (see e.g. [13, 42]). As in Remark 5.5, the transition matrix $T^{0 \leftarrow \mathrm{BI}}$ from the Kazhdan–Lusztig basis to the standard basis is written in terms of parabolic Kazhdan–Lusztig polynomials. The inverse of $T^{0 \leftarrow \mathrm{BI}}$, that is, $T^{\mathrm{BI} \leftarrow 0}$, is also written in terms of another parabolic Kazhdan–Lusztig polynomials (see Theorem 6 in [42]). The diagonal entries of the matrix $T^{\mathrm{BI} \leftarrow 0}$ are one and it is an upper triangular matrices whose non-zero entries are in $q^{-1}\mathbb{N}[q^{-1}]$. At $Q = q^M$, we have $\Psi^0_\epsilon \in \mathbb{N}[q]$. Since $\Psi^{\mathrm{BI}} = T^{\mathrm{BI} \leftarrow 0}\Psi^0$, we obtain $\Psi^{\mathrm{BI}}_D \in \mathbb{N}[q, q^{-1}]$. $\qquad\square$

Recall that a diagram D is characterized by a binary string $b \in \{\pm\}^N$ of length N (see Section 3.1). Given a binary string b, let J_D be the set of positions of $+$ from right. We define

$$d_D := \sum_{j \in J_D} (j + M - 1).$$

For example, when $b = (+ - + -)$ with $M = 2$, we have $d_D = 8$.

Corollary 6.3. *The component Ψ_D has the leading term q^{d_D} with the leading coefficient one.*

Proof. Let $b = b_1 \ldots b_N$ and $b' = b'_1 \ldots b'_N$ be two binary strings in $\{\pm\}^N$. We introduce the reversed lexicographic order, which is $b < b'$ if and only if $b_j = b'_j$ for $1 \le j \le i - 1$, $b_i = +$ and $b'_i = -$. This lexicographic order of binary strings induces a natural order of diagrams. If $D < D'$, then we have $d_D \ge d_{D'}$. The eigenvector satisfies $\Psi^0_D = q^{d_D}$. Recall that the diagonal entries are one and other non-zero entries are in $q^{-1}\mathbb{N}[q^{-1}]$. Since $\Psi^{\mathrm{BI}} = T^{\mathrm{BI} \leftarrow 0}\Psi^0$, one easily show that the leading term is q^{d_D} with the coefficient one. $\qquad\square$

6.2.3. *Type BII and BIII.* We have

Lemma 6.4. *All components of Ψ belong to $\mathbb{N}[q, q^{-1}, Q, Q^{-1}]$.*

Proof. We prove Lemma for type BII and type BIII separately.

Type BII. Let D be a diagram starting with n_1 up arrows, followed by an outer arc of size m_1, followed by n_2 up arrows, followed by an outer arc of size m_2, $\cdots$, followed by n_{I+1} up arrows, followed by p_{J+1} e-unpaired down arrows (and $p_{J+1} - 1, p_{J+1}$ or $p_{J+1} + 1$ o-unpaired down arrows), followed by an outer arc of size m'_{J+1}, followed by p_J e-unpaired down arrows, followed by an outer arc of size m'_J, $\cdots$, and ending with p_1 e-unpaired down arrows. Set $N_\uparrow = \sum_{i=1}^{I+1} n_i$, $M = \sum_{i=1}^{I} m_i$, $P = \sum_{i=1}^{J+1} p_i$ and $M' = \sum_{i=1}^{J+1} m'_i$. From Definition 4.17, the product $N_1 N_2 N_3$ is written as

$$\prod_{i=1}^{I} \begin{bmatrix} \sum_{j=1}^{i} n_j + m_j \\ m_i \end{bmatrix} \prod_{i=1}^{J} \begin{bmatrix} \sum_{j=1}^{i} m'_j + p_j \\ m'_i \end{bmatrix} \begin{bmatrix} N_\uparrow + M + P + M' \\ N_\uparrow + M \end{bmatrix} \in \mathbb{N}[q, q^{-1}].$$

Since $N_4 \in \mathbb{N}[q, q^{-1}, Q, Q^{-1}]$, we have $\Psi_D \in \mathbb{N}[q, q^{-1}, Q, Q^{-1}]$.

Type BIII. Let D be a diagram starting with n_1 up arrows, followed by an outer arc of size m_1, followed by n_2 up arrows, followed by an outer arc of size m_2, $\cdots$, followed by n_{I+1} up arrows, followed by n'_{J+1} down arrows, followed by an outer arc of size m'_J, followed by n'_J down arrows (with a circled integer), followed by an outer arc of size m'_{J-1}, $\cdots$, ending with n'_1 down arrows. Set $N_\uparrow = \sum_{i=1}^{I+1} n_i$, $M = \sum_{i=1}^{I} m_i$, $N_\downarrow = \sum_{i=1}^{J+1} n'_i$ and $M' = \sum_{i=1}^{J} m'_i$. From Definition 4.23, the product $N_1 N_2 N_3$ is

$$\prod_{i=1}^{I} \begin{bmatrix} \sum_{j=1}^{i} n_j + m_j \\ m_i \end{bmatrix} \prod_{i=1}^{J} \begin{bmatrix} \sum_{j=1}^{i} m'_j + n'_j \\ m'_i \end{bmatrix} \begin{bmatrix} N_\uparrow + M + N_\downarrow + M' \\ N_\uparrow + M \end{bmatrix} \in \mathbb{N}[q, q^{-1}].$$

Since $N_4 \in \mathbb{N}[q, q^{-1}, Q, Q^{-1}]$, we have $\Psi_D \in \mathbb{N}[q, q^{-1}, Q, Q^{-1}]$.

$\square$

6.3. **Sum rule.** Let $S_N^X(q, Q)$ be the sum of all components of Ψ on the Kazhdan–Lusztig bases of Type X (X=A, BI, BII or BIII), *i.e.*, $S_N^X(q, Q) = \sum_D \Psi_D$.

Let A_n be the number of $n \times n$ symmetric binary matrix with no row sum greater than one. A_n satisfies the recurrence relation

$$A_n = 2A_{n-1} + (n-1)A_{n-2},$$

with $A_0 = 1$ and $A_1 = 2$. The sequence A_n is A005425 in [44] (see also [39]).

Let B_n be the number of $n \times n$ bisymmetric binary matrix with a row sum equal to one, that is, B_n is the set of permutation matrices with symmetric about two diagonals and modulo rotation by $\pi/2$ radians. B_n satisfies the recurrence relation

$$B_n = 2B_{n-1} + (2n-2)B_{n-2},$$

with $B_1 = 1$ and $B_2 = 3$. The sequence B_n is A000902 in [44].

Let C_n be the sequence A083886 in [44], which satisfies

$$C_{n+1} = 3C_n + 2(n-1)C_{n-1}$$

with $C_1 = 1$ and $C_2 = 3$. The sequence C_n is the total number of signed permutations of size $2(n-1) \times 2(n-1)$ which are invariant under both diagonal and anti-diagonal reflections and avoiding a pattern $(-2, -1)$ [19].

Conjecture 6.5. *At $q = 1$ and $Q = 1$, we have*

$$\begin{aligned} S_N^A &= A_N, \\ S_N^{BI} &= B_{N+1}, \quad \text{for } M = 1, \\ S_N^{BI} &= C_{N+1}, \quad \text{for } M = \infty, \\ S_N^{BIII} &= C_{N+1}. \end{aligned}$$

At $q = Q = 1$, we have $S_N^{BIII} = S_N^{BI}$ for $M = \infty$. This coincidence comes from the fact that the diagram D of type BI for $M = \infty$ is the same as the diagram of type BIII with $Q = q^M$ (M large enough). The first few values of S_N^X are in Table 1.

6.3.1. *Type A and Type BIII.* At $q = 1$, the sum S_N^A is uniquely written as

$$S_N^A = \sum_{i=0}^{N} S_{N,i} Q^i$$

where $S_{N,i} \in \mathbb{N}$.

	Size N								
	1	2	3	4	5	6	7	8	9
S_N^{A}	2	5	14	43	142	499	1850	7193	29186
$S_N^{\mathrm{BI}}(M=1)$	3	10	38	156	692	3256	16200	84496	460592
$S_N^{\mathrm{BI}}(M=2)$	3	11	44	192	892	4396	22752	123248	695024
$S_N^{\mathrm{BI}}(M=3)$	3	11	45	200	952	4796	25412	140720	811280
$S_N^{\mathrm{BI}}(M=\infty)$	3	11	45	201	963	4899	26253	147345	862083
S_N^{BII}	3	9	33	129	555	2529	12273	62481	333603
S_N^{BIII}	3	11	45	201	963	4899	26253	147345	862083

TABLE 1. The first few values of S_N^X

Since a component Ψ_D is invariant under the bar involution, the sum $S_N^{III} := S_N^{\mathrm{BIII}}$ is also invariant. At $q = 1$, S_N^{III} is uniquely written as

$$S_N^{III} = \sum_{i=0}^{N} S'_{N,i}(Q + Q^{-1})^i.$$

with $S'_{N,i} \in \mathbb{N}$.

Conjecture 6.6. *We have $S_{N,i} = S_{N,N-i} = S'_{N,i} = S'_{N,N-i}$ and*

$$\text{(102)} \qquad S_{N,i} \;=\; \prod_{k=0}^{i-1} \frac{N-k}{2k+2} \cdot P_i(N),$$

$$\text{(103)} \qquad P_i(N) \;=\; N^i + \sum_{j=0}^{i-1} p_{i,j} N^j$$

where $p_{i,j} \in \mathbb{Z}$.

The first few polynomials $P_i(N)$'s are

$$\begin{aligned}
P_1(N) &= N + 1, \\
P_2(N) &= N^2 - N + 2, \\
P_3(N) &= N^3 - 6N^2 + 17N - 16, \\
P_4(N) &= N^4 - 14N^3 + 83N^2 - 230N + 248.
\end{aligned}$$

Let $\mathcal{A}_n$ be a set of symmetric binary matrices of size $n \times n$ with no row sum greater than one. For $a = (a_{i,j})_{1 \le i,j \le n} \in \mathcal{A}_n$, we define the weight of a by $\mathrm{wt}(a) := \#\{a_{i,j} = 1 | i \le j\}$. Then,

Conjecture 6.7. *We have*

$$S_{N,i} = \#\{a \in \mathcal{A}_N | \mathrm{wt}(a) = i\}.$$

Let $\mathcal{C}_n$ be a set of signed permutation matrices of size $2(n-1) \times 2(n-1)$ which are invariant under the diagonal and anti-diagonal reflections and avoids the pattern $(-2, -1)$. For $c = (c_{i,j})_{1 \le i,j \le 2n} \in \mathcal{C}_n$, we define the weight of c by

$$\mathrm{wt}(c) := n_+ - n_-,$$

where

$$n_+ \quad := \quad \#\{c_{i,j} = 1 | 1 \le i \le n, i \le j \le n\},$$
$$n_- \quad := \quad \#\{c_{i,j} = 1 | 1 \le i \le n, n+1 \le j \le 2n+1-i\}.$$

At $q = 1$, the sum S_N is rewritten as

$$S_N = \sum_{i=0}^{N} \widetilde{S}_{N,N-2i} Q^{N-2i}.$$

where $\widetilde{S}_{N,i} \in \mathbb{N}$. Then,

Conjecture 6.8. *we have*

$$\widetilde{S}_{N,i} = \#\{c \in \mathcal{C} | \mathrm{wt}(c) = i\}.$$

6.3.2. *Type BII.* At $q = 1$, the sum $S_N := S_N^{BII}$ is uniquely written as

$$S_N = \sum_{j=0}^{N} S_{N,j} (Q + Q^{-1})^j.$$

where $S_{N,j} \in \mathbb{N}$. Then, we have two conjectures:

Conjecture 6.9. *We have*

$$S_{N,1} = \frac{1}{8}(2N^2 + 4N + 1 - (-1)^N).$$

We have checked the conjecture up to $N = 20$.

Conjecture 6.10. *We have*

$$S_{N,N-i} \quad = \quad \prod_{j=1}^{i} (2j)^{-1} \cdot P_i(N),$$

$$P_i(N) \quad = \quad N^{2i} - iN^{2i-1} + \sum_{k=0}^{2i-2} p_{i,k} N^k,$$

where $p_{i,k} \in \mathbb{Z}$.

We have checked the conjecture up to $j = 7$ and $N = 20$. The first few polynomials $P_j(N)$'s are

$$P_1(N) \quad = \quad N^2 - N + 2,$$
$$P_2(N) \quad = \quad N^4 - 2N^3 + 3N^2 + 14N - 8,$$
$$P_3(N) \quad = \quad N^6 - 3N^5 + N^4 + 51N^3 - 2N^2 - 96N + 96.$$

6.4. Components of Ψ and enumerations of binary/permutation matrices.

6.4.1. *Type A.* In the case of Type A, some components are related to an enumeration of symmetric binary matrices.

Let $\mathcal{S}$ be the set of symmetric binary matrices with no row sum greater than one. Let $s = (s_{ij})_{1 \le i,j \le N} \in \mathcal{S}$. Since s is symmetric, we consider only s_{ij} with $i \le j$. We denote by N_1 the number of one and define

$$A(s) := \{(i,j) | s_{ij} = 1, i \le j\}.$$

For $l = (i,j) \in A(s)$, we define $n_l := N - 2i + j$. We define a map $F : \mathcal{S} \to \mathbb{N}[q, q^{-1}, Q]$ by

$$F(s) := q^{N(N-1)/2} Q^{N-N_1} \prod_{l \in A(s)} q^{-n_l}.$$

Let $l = (i,j)$ and $l' = (i',j')$ be elements in $A(s)$. We call s admissible if there exists no pair (l, l') such that $i < i' < j < j'$. For an admissible symmetric binary matrix s, we define a map g from $s \in \mathcal{S}$ to a binary string $b = (b_1 \ldots b_N)$ of length N: for each $l = (i,j) \in A(s)$, we set $b_j = -$ and otherwise we set $b_j = +$. Then we consider a set

$$\mathrm{Adm}(b) := \{s \in \mathcal{S} | s \text{ is admissible}, g(s) = b\}.$$

Example 6.11. *Let s be*

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ & 1 & 0 & 0 & 0 \\ & & 0 & 0 & 0 \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}.$$

Then s is admissible, $F(s) = q^{-1}Q^2$ and $g(s) = + - - + -$.

Let $b = \underbrace{- \cdots -}_{i} \underbrace{+ \cdots +}_{j} \underbrace{- \cdots -}_{k} \underbrace{+ \cdots +}_{N-i-j-k}$ with $i, j, k \geq 0$ and $D(b)$ be a diagram of type A associated with b. Then we have

Conjecture 6.12.

$$\Psi_{D(b)} = \sum_{s \in \mathrm{Adm}(v)} F(s)$$

6.4.2. *Type BIII.* Let $\mathcal{C}_N$ be the set of signed permutation matrices of size $2(N-1) \times 2(N-1)$ which are symmetric both diagonal and anti-diagonal reflections and avoid the pattern $(-2, -1)$.

Let $c = (c_{i,j})_{1 \leq i,j \leq N} \in \mathcal{C}_{N+1}$. Since c is bisymmetric, we consider only $c_{i,j}$ with $1 \leq i \leq N$ and $i \leq j \leq 2N + 1 - i$. Define

$$A(c) := \{(i,j) | c_{i,j} = 1, 1 \leq i \leq N, i \leq j \leq 2N\}.$$

We call an element of $A(c)$ a *link*. A link $(i,j) \in A(c)$ is said to be *fundamental* if and only if $1 \leq i \leq N$ and $1 \leq j \leq 2N + 1 - i$. We denote by $A^+(c)$ the set of fundamental links. Given two links $l = (i_1, j_1)$ and $l' = (i_2, l_2)$, we define a *cross point* $p(l, l') = (i_2, j_1)$ if and only if $i_1 < i_2 < j_1 < j_2$, $i_1 + j_1 \neq 2N + 1$ and $i_2 + j_2 \neq 2N + 1$. We denote by $B(c)$ the set of cross points $p(l, l')$ for $l, l' \in A(c)$. We consider the coordinate system where x-direction is rightward and y-direction is downward. We define a down-left path from a link $l = (i_1, j_1) \in A^+(c)$ to a diagonal point (i_2, i_2) as follows. Note that a link l does not belong to $B(c)$. First, we go down from l until it reaches to a cross point or a diagonal point. If a path reaches a cross point form up (resp. right), we make a turn on the cross point and go left (resp. down). We continue this procedure until a path reaches to a diagonal point. We define a binary string $b = b_1 \ldots b_N \in \{\pm\}^N$ of length N by $b_i = -$ if there exists a path from $l \in A^+(c)$ to the diagonal point (i,i), and $b_i = +$ otherwise. Therefore, by composing the above procedure, we have a map $B : \mathcal{C}_N \to \{\pm\}^N$.

Suppose that $b = \underbrace{- \cdots -}_{i} \underbrace{+ \cdots +}_{j} \underbrace{- \cdots -}_{k} \underbrace{+ \cdots +}_{N-i-j-k}$ with $i, j, k \geq 0$ and $D(b)$ be a diagram of type BIII associated with b. Then,

Conjecture 6.13. *At $q = Q = 1$, we have*

$$\Psi_{D(b)} = \#\{c \in \mathcal{C}_{N+1} | B(c) = b\}.$$

APPENDIX A.

Lemma A.1. *Let $A = (a_{i,j})_{1 \leq i,j \leq N+1}$ be a tridiagonal matrix whose entries are*

$$a_{i,i} = q^{N+2-2i}\frac{Q - Q^{-1}}{q - q^{-1}}, \qquad a_{i,i-1} = [N + 1 - i], \qquad a_{i,i+1} = q^{N-2i}[i].$$

Then, the eigenvalues of A are

$$(104) \qquad \frac{Qq^{N-2\lambda} - Q^{-1}q^{-N+2\lambda}}{q - q^{-1}}, \qquad \lambda = 0, 1, \ldots, N.$$

Proof. Let x_λ be the expression (104). To show that x_λ is the eigenvalue of A, it is enough to show that the determinant of $A^{(1)} = A - x_\lambda \mathbf{1}$ is equal to zero. We diagonalize the tridiagonal matrix $A^{(1)} = (a_{i,j}^{(1)})$ from the right bottom corner. First, we subtract $a_{N+1,N}^{(1)}/a_{N+1,N+1}^{(1)}$ times the $(N + 1)$-th column vector from the N-th column and obtain a matrix $A^{(2)} = (a_{i,j}^{(2)})$. Then, we subtract $a_{N,N-1}^{(2)}/a_{N,N}^{(2)}$ times the N-th column vector from the $(N-1)$-th column. We continue this procedure until we obtain an upper triangular matrix $L = (l_{i,j})$. From a direct computation, the diagonal entries of the matrix L is written in terms of a set of Laurent polynomials $\{v_1, \ldots, v_{N+2}\}$ as $l_{i,i} = v_i/v_{i+1}$ where $v_i = a_{i,i}^{(1)}v_{i+1} - a_{i+1,i}^{(1)}a_{i,i+1}^{(1)}v_{i+2}$ with the initial conditions $v_{N+2} = 1$ and $v_{N+1} = a_{N+1,N+1}^{(1)}$. Note that $a_{i,i+1}^{(1)} = a_{i,i+1}$, $a_{i,i-1}^{(1)} = a_{i,i-1}$ and

$$a_{i,i}^{(1)} = q^{N+1-i-\lambda}[\lambda + 1 - i]Q + q^{1-i+\lambda}[\lambda - N - 1 + i]Q^{-1}.$$

We will prove that for $1 \leq n \leq N$

$$(105) \qquad v_n = \sum_{j=0}^{m(n)} Q^{m(n)-2j}\alpha(n, j),$$

where

$$\alpha(n, j) \quad := \quad q^{d(n,j)}\begin{bmatrix} m(n) \\ j \end{bmatrix}\prod_{i=0}^{m(n)-j-1}[\lambda - N + i]\prod_{i=0}^{j-1}[\lambda - i],$$

$$m(n) \quad := \quad N + 2 - n,$$

$$d(n, j) \quad := \quad m(n)(m(n) - 1)/2 - \lambda m(n) + j(-N + 2\lambda).$$

For $n = N+1, N+2$, Eqn.(105) holds true. We assume that Eqn.(105) is true up to some $n \leq N+1$. We have

$$v_{n-1} = a_{n-1,n-1}^{(1)}v_n - a_{n,n-1}a_{n-1,n}v_{n+1}$$

$$= \sum_{j=0}^{m(n)+1} Q^{m(n-1)-2j}\left(q^{N+2-n-\lambda}[\lambda + 2 - n]\alpha(n, j) + q^{2-n+\lambda}[\lambda - N - 2 + n]\alpha(n, j - 1)\right)$$

$$- \sum_{j=0}^{m(n)+1} Q^{m(n-1)-2j}q^{N-2n+3}[n - 1][N + 2 - n]\alpha(n + 1, j - 1),$$

where $\alpha(n, m(n) + 1) = \alpha(n, -1) = 0$. By a direct computation, the above expression is equal to Eqn.(105).

The determinant of $A^{(1)}$ is equal to the one of L, that is, $\prod_{i=1}^{N+1} l_{i,i} = v_1$. The explicit expression of v_1 is

$$v_1 = \prod_{i=1}^{N}[\lambda - i] \sum_{j=0}^{N+1} Q^{N+1-2j} q^{d(1,j)} \begin{bmatrix} N+1 \\ j \end{bmatrix}.$$

Note that $0 \le \lambda \le N$. We have $v_1 = 0$, *i.e.*, the eigenvalues of A is x_λ for $0 \le \lambda \le N$. □

Lemma A.2 (Lemma A.1 in [41])**.** *Set $M := \sum_{i=1}^{I} m_i$. We have*

$$(106) \qquad \sum_{i=1}^{I}[m_i]\left[1 + \sum_{j=1}^{i} n_j\right] \frac{\prod_{j=i+1}^{I}[1 + \sum_{k=1}^{j}(n_k + m_k)]}{\prod_{j=i}^{I}[1 + n_j + \sum_{k=1}^{j-1}(n_k + m_k)]} = [M]$$

Proof. We prove Lemma by induction. Let $f(I)$ be the left hand side of Eqn.(106) When $I = 1$, we have $f(I) = m_1$ by a straightforward calculation. Set $N = \sum_{i=1}^{I+1} n_i$. We assume that Lemma holds true up to I. We have

$$\begin{aligned}
f(I+1) &= f(I)\frac{[1 + \sum_{k=1}^{I+1}(n_k + m_k)]}{[1 + n_{I+1} + \sum_{k=1}^{I}(n_k + m_k)]} + \frac{[m_{I+1}][1 + N']}{[1 + N + M]} \\
&= \frac{[M][1 + M + N + m_{I+1}]}{[1 + M + N]} + \frac{[m_{I+1}][1 + N']}{[1 + N + M]} \\
&= [M + m_{I+1}].
\end{aligned}$$

□

Lemma A.3. *Set $M = \sum_{i=1}^{J} m_i$ and $N = \sum_{j=1}^{J+1} n_j$. We have*

$$(107) \qquad \sum_{i=1}^{J}\left[1 + \sum_{j=1}^{i} n_j\right][m_i]\frac{\prod_{j\ge i+2}^{J+1}[1 + \sum_{k=1}^{j-1}(n_k + m_k)]}{\prod_{j\ge i}^{J+1}[1 + n_j + \sum_{k=1}^{j-1}(n_k + m_k)]} = \frac{[M]}{[M + N + 1]}$$

Proof. We prove Lemma by induction. When $J = 1$, Lemma holds true by a direct computation. We assume that Lemma is true up to some $J - 1 \ge 1$. Let $f(J)$ be the left hand side of Eqn.(107) and $w_i = 1 + n_i + \sum_{i=1}^{i-1}(n_i + m_i)$. We have

$$\begin{aligned}
f(J) &= f(J-1)\frac{[1 + \sum_{i=1}^{J}(n_i + m_i)]}{[w_{J+1}]} + \frac{[1 + \sum_{i=1}^{J} n_i][m_J]}{[w_J][w_{J+1}]} \\
&= \frac{[M - m_J][1 + M + N - n_{J+1}] + [N + 1 - n_{J+1}][m_J]}{[M + N - m_J - n_{J+1} + 1][1 + M + N]} \\
&= \frac{[M]}{[M + N + 1]}.
\end{aligned}$$

This completes the proof. □

Lemma A.4. *Set $M_i = \sum_{j=1}^{i} m_j$.*

$$(108) \qquad \sum_{i=1}^{I} \frac{[m_i]}{[x + M_{i-1}][x + M_i]} = \frac{[M_I]}{[x][x + M_I]}$$

Proof. We prove Lemma by induction. Let $f(I)$ be the left hand side of Eqn.(108). Lemma is true when $I = 1$. We assume that Lemma holds true up to I. We have

$$
\begin{aligned}
f(I+1) &= f(I) + \frac{[m_{I+1}]}{[x + M_{I+1}][x + M_I]} \\
&= \frac{[M_{I+1}]}{[x][x + M_{I+1}]}
\end{aligned}
$$

$\square$

Lemma A.5. *Set $v_i := \sum_{j=1}^{i-1} n_j + m_j$. We have*

$$
(109) \qquad \frac{q^{\sum_{i=1}^{I} m_i}}{[v_{I+1}]} \prod_{j=1}^{I} \frac{[n_j + v_j]}{[v_j]} + \sum_{i=1}^{I} \frac{q^{-\sum_{j=1}^{i} n_j}[m_i]}{[v_i][v_{i+1}]} \prod_{j=1}^{i-1} \frac{[n_j + v_j]}{[v_j]} = 1.
$$

Proof. We prove Lemma by induction. Let $f(I)$ be the left hand side of Eqn.(109). By a straightforward calculation, we have $f(1) = 1$. We assume Lemma holds true up to I. We have

$$
\begin{aligned}
f(I+1) &= f(I) - \frac{q^{\sum_{i=1}^{I} m_i}}{[v_{I+1}]} \prod_{j=1}^{I} \frac{[n_j + v_j]}{[v_j]} + \frac{q^{-\sum_{i=1}^{I+1} n_i}[m_{I+1}]}{[v_{I+1}][v_{I+2}]} \prod_{i=1}^{I} \frac{[n_i + v_i]}{[v_i]} \\
&\quad + \frac{q^{\sum_{i=1}^{I+1} m_i}}{[v_{I+2}]} \prod_{j=1}^{I+1} \frac{[n_j + v_j]}{[v_j]} \\
&= 1.
\end{aligned}
$$

$\square$

Lemma A.6. *Set $v_i := \sum_{j=1}^{i-1} m_j + n_j$. We have*

$$
(110) \qquad \sum_{i=1}^{I} \frac{q^{-\sum_{j=1}^{i-1} n_j - 1}[m_i]}{[1 + v_i][1 + v_{i+1}]} \prod_{j=i+1}^{I} \frac{[1 + m_j + v_j]}{[1 + v_{j+1}]} = \prod_{j=1}^{I} \frac{[1 + m_j + v_j]}{[1 + v_{j+1}]} - \frac{q^{\sum_{i=1}^{I} m_i}}{[1 + v_{J+1}]}.
$$

Proof. We prove Lemma by induction. Let $f(I)$ be the left hand side of Eqn.(110). By a straightforward calculation, Lemma is true for $I = 1$. We assume that Lemma holds true up to I. We have

$$
\begin{aligned}
f(I+1) &= f(I)\frac{[1 + m_{I+1} + v_{I+1}]}{[1 + v_{I+2}]} + \frac{q^{-\sum_{i=1}^{I} n_i - 1}[m_{I+1}]}{[1 + v_{I+1}][1 + v_{I+2}]} \\
&= \prod_{i=1}^{I+1} \frac{[1 + m_i + v_i]}{[1 + v_{i+1}]} - \frac{q^{\sum_{j=1}^{I} m_j}[1 + m_{I+1} + v_{I+1}]}{[1 + v_{I+1}][1 + v_{I+2}]} + \frac{q^{-\sum_{i=1}^{I} n_i - 1}[m_{I+1}]}{[1 + v_{I+1}][1 + v_{I+2}]} \\
&= \prod_{i=1}^{I+1} \frac{[1 + m_i + v_i]}{[1 + v_{i+1}]} - \frac{q^{\sum_{i=1}^{I+1} m_i}}{[1 + v_{J+2}]}.
\end{aligned}
$$

$\square$

Lemma A.7. *Set $v_i := \sum_{j=1}^{i-1} n_j + m_j$ and $w_i := 1 + n_i + v_i$. We have*

$$(111) \quad q^{-\sum_{j=1}^I m_j + 1} \sum_{i=1}^I \frac{q^{-\sum_{j=1}^i n_j}[m_i]}{[w_i][w_{i+1}]} \prod_{k=i+2}^{I+1} \frac{[1+v_k]}{[w_k]} \prod_{k=1}^{i-1} \frac{[n_k+v_k]}{[v_{k+1}]}$$

$$\times \left\{ (1+q^{-2}) q^{\sum_{j=1}^{i-1} m_j} - \frac{q^{-\sum_{j=1}^i n_j - 1}[m_i - 1]}{[v_{i+1}]} \right\}$$

$$= q^{-\sum_{i=1}^I m_i} \prod_{j=1}^{I+1} \frac{[1+v_j]}{[w_j]} - \frac{q^{\sum_{i=1}^I m_i}}{[w_{I+1}]} \prod_{j=1}^I \frac{[n_j+v_j]}{[v_{j+1}]}.$$

Proof. We prove Lemma by induction. Let $f(I)$ be the left hand side of Eqn.(111). By a straight-forward calculation, Lemma is true for $I = 1$. We assume that Lemma holds true up to I. We have

$$f(I+1) = q^{-m_{I+1}} \frac{[1+v_{I+2}]}{[w_{I+2}]} f(I) + \frac{q^{1-v_{I+2}}[m_{I+1}]}{[w_{I+1}][w_{I+2}]} \prod_{j=1}^I \frac{[n_j+v_j]}{[v_{j+1}]}$$

$$\times \left\{ (1+q^{-2}) q^{\sum_{j=1}^I m_j} - \frac{q^{-\sum_{j=1}^{I+1} n_j - 1}[m_{I+1} - 1]}{[v_{I+2}]} \right\}$$

$$= q^{-\sum_{i=1}^{I+1} m_j} \prod_{i=1}^{I+2} \frac{[1+v_i]}{[w_i]} - \frac{q^{\sum_{i=1}^I m_i - m_{I+1}}[1+v_{I+2}]}{[w_{I+1}][w_{I+2}]} \prod_{i=1}^I \frac{[n_i+v_i]}{[v_{i+1}]}$$

$$+ \frac{q^{1-v_{I+2}}[m_{I+1}]}{[w_{I+1}][w_{I+2}]} \prod_{i=1}^I \frac{[n_i+v_i]}{[v_{i+1}]} \left\{ (1+q^{-2}) q^{\sum_{j=1}^I m_j} - \frac{q^{-\sum_{j=1}^{I+1} n_j - 1}[m_{I+1} - 1]}{[v_{I+2}]} \right\}$$

$$= q^{-\sum_{i=1}^{I+1} m_i} \prod_{j=1}^{I+2} \frac{[1+v_j]}{[w_j]} - \frac{q^{\sum_{i=1}^{I+1} m_i}}{[w_{I+2}]} \prod_{j=1}^{I+1} \frac{[n_j+v_j]}{[v_{j+1}]}.$$

$\square$

Lemma A.8. *Set $v_i := \sum_{j=1}^i (n_j + m_j)$ and $w_i = 1 + n_i + v_i$. We have*

$$(112) \quad \sum_{i=1}^I q^{-1-\sum_{j=1}^i n_j} \frac{[m_i]}{[w_i][w_{i+1}]} \prod_{j=i+2}^{I+1} \frac{[1+v_j]}{[w_j]} = \prod_{i=1}^{I+1} \frac{[1+v_i]}{[w_i]} - \frac{q^{\sum_{i=1}^I m_i}}{[w_{J+1}]}$$

Proof. We prove Lemma by induction. Let $f(I)$ be the left hand side of Eqn.(112). By a straight-forward calculation, Lemma holds true when $I = 1$. We have

$$f(I+1) = f(I) \frac{[1+v_{I+2}]}{[w_{I+2}]} + \frac{q^{-1-\sum_{j=1}^{I+1} n_j}[m_{I+1}]}{[w_{I+1}][w_{I+2}]}$$

$$= \prod_{i=1}^{I+2} \frac{[1+v_i]}{[w_i]} - \frac{1}{[w_{I+1}][w_{I+2}]} \left(q^{\sum_{i=1}^I m_i}[1+v_{I+2}] - q^{-1-\sum_{i=1}^{I+1} n_i}[m_{I+1}] \right)$$

$$= \prod_{i=1}^{I+2} \frac{[1+v_i]}{[w_i]} - \frac{q^{\sum_{i=1}^{I+1} m_i}}{[w_{J+2}]}.$$

$\square$

Lemma A.9 ([41, Lemma A.2]). *We have*

$$\sum_{i=1}^{K} I_i \cdot J_i + \frac{[2z+2]}{[1+x+\sum_{i=1}^{K} m_i]} I_K = [1+x]^{-1} \left[2 + 2z + 2 \sum_{i=1}^{K} m_i \right]$$

where

$$I_i := \prod_{j=1}^{i} \frac{[2+2z+2\sum_{k=j}^{K} m_k]}{[2+2z+m_j+2\sum_{k=j+1}^{K} m_k]},$$

$$J_i := \frac{[m_i][x+3+2z+\sum_{j=1}^{i} m_j + 2\sum_{j=i+1}^{K} m_j]}{[1+x+\sum_{j=1}^{i-1} m_j][1+x+\sum_{j=1}^{i} m_j]}.$$

Lemma A.10. *Set* $v_i := \sum_{j=1}^{i} (n_j + m_j)$. *We have*

(113) $$\sum_{i=1}^{I} q^{-\sum_{j=1}^{i} n_j} \frac{[m_i]}{[v_i]} \prod_{l=1}^{i-1} \frac{[n_l + v_{l-1}]}{[v_l]} = 1 - q^{\sum_{i=1}^{I} m_i} \prod_{l=1}^{I} \frac{[n_l + v_{l-1}]}{[v_l]}$$

Proof. We prove Lemma by induction on I. Let $f(I)$ be the left hand side of Eqn.(113). By a straightforward calculation, Lemma holds true for $I = 1$. We have

$$f(I+1) = f(I) + q^{-\sum_{j=1}^{I+1} n_j} \frac{[m_{I+1}]}{[v_{I+1}]} \prod_{l=1}^{I} \frac{[n_l + v_{l-1}]}{[v_l]}$$

$$= 1 - \prod_{l=1}^{I} \frac{[n_l + v_{l-1}]}{[v_l]} \left\{ q^{\sum_{i=1}^{I} m_i} - q^{-\sum_{j=1}^{I+1} n_j} \frac{[m_{I+1}]}{[v_{I+1}]} \right\}$$

$$= 1 - q^{\sum_{i=1}^{I+1} m_i} \prod_{l=1}^{I+1} \frac{[n_l + v_{l-1}]}{[v_l]}$$

$\square$

Lemma A.11. *Set* $v_i := \sum_{j=1}^{i} (n_j + m_j)$. *We have*

(114) $$\sum_{i=1}^{I} \frac{q^{-\sum_{j=1}^{i} n_j} [m_i]}{[1+n_i+v_{i-1}]} \prod_{j=i+1}^{I} \frac{[1+v_j]}{[1+n_j+v_{j-1}]} = q \prod_{j=1}^{I} \frac{[1+v_j]}{[1+n_j+v_{j-1}]} - q^{1+\sum_{i=1}^{I} m_i}$$

Proof. We prove Lemma by induction. Let $f(I)$ be the left hand side of Eqn.(114). By a straightforward calculation, Lemma holds true for $I = 1$. We have

$$f(I+1) = f(I) \frac{[1+v_{I+1}]}{[1+n_{I+1}+v_I]} + \frac{q^{-\sum_{j=1}^{I+1} n_j} [m_{I+1}]}{[1+n_{I+1}+v_I]}$$

$$= q \prod_{j=1}^{I+1} \frac{[1+v_j]}{[1+n_j+v_{j-1}]} + \frac{1}{[1+n_{I+1}+v_I]} \left\{ q^{-\sum_{j=1}^{I+1} n_j} [m_{I+1}] - q^{1+\sum_{i=1}^{I} m_i} [1+v_{I+1}] \right\}$$

$$= q \prod_{j=1}^{I+1} \frac{[1+v_j]}{[1+n_j+v_{j-1}]} - q^{1+\sum_{i=1}^{I+1} m_i}$$

$\square$

Lemma A.12. *Set $v_i := \sum_{j=1}^{i}(n_j + m_j)$. We have*

$$
(115) \quad \sum_{i=1}^{I} q^{-\sum_{k=1}^{i} n_k} \frac{[m_i]}{[1 + n_i + v_{i-1}]} \prod_{j=1}^{i-1} \frac{[n_j + v_{j-1}]}{[v_j]} \prod_{j=i+1}^{I} \frac{[1 + v_j]}{[1 + n_j + v_{j-1}]}
$$

$$
\times \left\{ (1 + q^{-2}) q^{\sum_{k=1}^{i-1} m_k} - q^{-1 - \sum_{k=1}^{i} n_k} \frac{[m_i - 1]}{[v_i]} \right\}
$$

$$
= q^{-1} \prod_{i=1}^{I} \frac{[1 + v_j]}{[1 + n_j + v_{j-1}]} - q^{2 \sum_{k=1}^{I} m_k - 1} \prod_{i=1}^{I} \frac{[n_i + v_{i-1}]}{[v_i]}
$$

Proof. We prove Lemma by induction on I. Let $f(I)$ be the left hand side of Eqn.(115). By a direct calculation, Lemma holds true for $I = 1$. We have

$$
\begin{aligned}
f(I + 1) &= f(I) \frac{[1 + v_{I+1}]}{[1 + n_{I+1} + v_I]} + q^{-\sum_{k=1}^{I+1} n_k} \frac{[m_{I+1}]}{[1 + n_{I+1} + v_I]} \prod_{j=1}^{I} \frac{[n_j + v_{j-1}]}{[v_j]} \\
&\quad \times \left\{ (1 + q^{-2}) q^{\sum_{k=1}^{I} m_k} - q^{-1 - \sum_{k=1}^{I+1} n_k} \frac{[m_{I+1} - 1]}{[v_{I+1}]} \right\} \\
&= q^{-1} \prod_{i=1}^{I+1} \frac{[1 + v_i]}{[1 + n_i + v_{i-1}]} - \prod_{i=1}^{I} \frac{[n_i + v_{i-1}]}{[v_i]} \left\{ q^{2 \sum_{k=1}^{I} m_k - 1} \frac{[1 + v_{I+1}]}{[1 + n_{I+1} + v_I]} \right. \\
&\quad \left. - q^{-\sum_{k=1}^{I+1} n_k} \frac{[m_{I+1}]}{[1 + n_{I+1} + v_I]} \left((1 + q^{-2}) q^{\sum_{k=1}^{I} m_k} - q^{-1 - \sum_{k=1}^{I+1} n_k} \frac{[m_{I+1} - 1]}{[v_{I+1}]} \right) \right\} \\
&= \text{right hand side of Eqn.(115).}
\end{aligned}
$$

□

REFERENCES

[1] M. T. Batchelor, J. de Gier, and B. Nienhuis, *The quantum symmetric XXZ chain at $\Delta = -1/2$, alternating sign matrices and plane partitions*, J. Phys. A **34** (2001), L265–L270, `arXiv:cond-mat/0101385`.

[2] R. E. Behrend, P. Di Francesco, and P. Zinn-Justin, *On the weighted enumeration of alternating sign matrices and descending plane partitions*, J. Combin. Theory Ser. A **119** (2012), 331–363, `arXiv:1103.1176`.

[3] B. D. Boe, *Kazhdan–Lusztig polynomials for Hermitian symmetric spaces*, Trans. Amer. Math. Soc. **309** (1988), 279–294.

[4] F. Brenti, *Parabolic Kazhdan–Lusztig polynomials for Hermitian symmetric spaces*, Trans. Amer. Math. Soc. **361** (2009), 1703–1729.

[5] D. M. Bressoud, *Proofs and Confirmations – The Story of the Alternating–Sign Matrix Conjecture*, Cambridge Univ. Press, 1999.

[6] L. Cantini and A. Sportiello, *Proof of the Razumov–Stroganov conjecture*, J. Combin. Theory Ser. A **118** (2011), no. 5, 1549–1574, `arXiv:1003.3376`.

[7] J. de Gier, *Loops, matchings and alternating-sign matrices*, Discr. Math. **298** (2005), no. 1-3, 365–388, `arXiv:math/0211285`.

[8] J. de Gier and A. Nichols, *The two-boundary Temperley–Lieb algebra*, J. Algebra **321** (2009), 1132–1167, `arXiv:math/0703338`.

[9] J. de Gier and P. Pyatov, *Bethe Ansatz for the Temperley–Lieb loop model with open boundaries*, J. Stat. Mech. **0403** (2004), P03002, `arXiv:hep-th/0312235`.

[10] _____, *Factorised solutions of Temperley–lieb qKZ equations on a segment*, Adv. Theor. Math. Phys. **14** (2010), 795–877, `arXiv:0710.5362`.

[11] J. de Gier, P. Pyatov, and P. Zinn-Justin, *Punctured plane partitions and the q-deformed Knizhnik–Zamolodchikov and Hirota equations*, J. Combin. Theory Ser. A **116** (2009), 772–794, `arXiv:0712.3584`.

[12] J. de Gier and V. Rittenberg, *Refined Razumov–Stroganov conjectures for open boundaries*, J. Stat. Mech. Theory Exp. (2004), P09009, `arXiv:math-ph/0408042`.

[13] V. Deodhar, *On some geometric aspects of Bruhat orderings. II. The parabolic analogue of Kazhdan–Lusztig polynomials*, J. Algebra **111** (1987), no. 2, 483–506.

[14] P. Di Francesco and P. Zinn-Justin, *Around the Razumov–Stroganov conjecture: proof of a multi-parameter sum rule*, Elect. J. Comb. **12** (2005), R6, `arXiv:math-ph/0410061`.

[15] _____, *Quantum Knizhnik–Zamolodchikov equation, generalized Razumov–Stroganov sum rules and extended Joseph polynomials*, J. Phys. A: Math. Gen. **38** (2005), L815–L822, `arXiv:math-ph/0508059`.

[16] _____, *Quantum Knizhnik–Zamolodchikov equation: reflecting boundary conditions and combinatorics*, J. Stat. Mech. (2007), P12009, `arXiv:0709.3410`.

[17] I. B. Frenkel and M. G. Khovanov, *Canonical bases in tensor products and graphical calculus for $U_q(\mathfrak{sl}_2)$*, Duke Math. J. **87** (1997), no. 3, 409–480.

[18] I. B. Frenkel, M. G. Khovanov, and Jr. A. A. Krillov, *Kazhdan–Lusztig polynomials and canonical basis*, Tansform. Groups **3** (1998), no. 4, 321–336, `arXiv:q-alg/9709042`.

[19] A. Hardt and J. M. Troyka, *Restricted symmetric signed permutations*, Pure Math. Appl. **23** (2012), no. 3, 179–217.

[20] M. Kashiwara, *Crystallizing the q-analogue of universal enveloping algebras*, Comm. Math. Phys. **133** (1990), 249–260.

[21] _____, *Global crystal bases of quantum groups*, Duke Math. J. **69** (1993), no. 2, 455–485.

[22] D. Kazhdan and G. Lusztig, *Representations of Coxeter groups and Hecke algebras*, Invent. Math. **53** (1979), no. 2, 165–184.

[23] S. Kolb, *Quantum symmetric Kac–Moody pairs*, Adv. Math. **267** (2014), 395–469, `arXiv:1207.6036`.

[24] G. Letzter, *Symmetric Pairs for Quantized Enveloping Algebras*, J. Algebra **220** (1999), 729–767.

[25] _____, *Quantum symmetric pairs and their zonal spherical functions*, Transform. Groups **8** (2003), 261–292, `arXiv:math/0204103`.

[26] G. Lusztig, *Canonical bases arising from quantized enveloping algebras*, J. Amer. Math. Soc. **3** (1990), 447–498.

[27] _____, *Canonical bases arising from quantized enveloping algebras, II*, Progr. Theor. Phys. Suppl. **102** (1990), 175–201.

[28] _____, *Hecke Algebra with Unequal Parameters*, CRM monograph series, vol. 18, American Mathematical Society, 2003.

[29] P. Martin, *Temperley–Lieb algebras and the long distance properties of statistical mechanical models*, J. Phys. A **23** (1990), no. 1, 7–30.

[30] P. P. Martin and H. Saleur, *The blob algebra and the periodic Temperley–Lieb algebra*, Lett. Math. Phys. **30** (1994), 189–206, `arXiv:hep-th/9302094`.

[31] S. Mitra, B. Nienhuis, J. de Gier, and M. T. Batchelor, *Exact expressions for correlations in the ground state of the dense $O(1)$ loop model*, J. Stat. Mech. (2004), P09010, `arXiv:cond-mat/0401245`.

[32] B. Nachtergaele, S. Ng, and S. Starr, *Ferromagnetic Ordering of Energy Levels for $U_q(\mathfrak{sl}_2)$ Symmetric Spin Chains*, Lett. Math. Phys. **100** (2012), 327–356, `arXiv:1105.5264`.

[33] A. Nichols, V. Rittenberg, and J. de Gier, *One-boundary Temperley–Lieb algebras in the XXZ and loop models*, J. Stat. Mech. Theory Exp. (2005), P03003, `arXiv:cond-mat/0411512`.

[34] A. Razumov, Yu. Stroganov, and P. Zinn-Justin, *Polynomial solutions of qKZ equation and ground state of XXZ spin chain at $\Delta = -1/2$*, J. Phys. A: Math. Theor. **40** (2007), no. 39, 11827–11847, `arXiv:0704.3542`.

[35] A. V. Razumov and Yu. G. Stroganov, *Spin chains and combinatorics*, J. Phys. A **34** (2001), 3185–3190, `arXiv:cond-mat/0012141`.

[36] _____, *Spin chains and combinatorics: twisted boundary conditions*, J. Phys. A **34** (2001), 5335–5340, `arXiv:cond-mat/0102247`.

[37] _____, *Combinatorial nature of ground state vector of $O(1)$ loop model*, Theor. Math. Phys. **138** (2004), 333–337, [russian: Teor. Mat. Fiz.**138** 395–400 (2004)], `arXiv:math/0104216`.

[38] _____, *$O(1)$ loop model with different boundary conditions and symmetry classes of alternating-sign matrices*, Theor. Math. Phys. **142** (2005), 237–243, [russian: Teor. Mat. Fiz.**142** 284–292 (2005)], `arXiv:cond-mat/0108103`.

[39] R. W. Robinson, *Counting arrangements of bishops*, Combinatorial Mathematics, IV, (Proc. Fourth Australian Conf., Univ. Adelaide, Adelaide, 1975), Lecture Notes in Math., vol. 560, Springer, Berlin, 1976, pp. 198–214.

[40] H. Saleur, *Virasoro and Temperley Lieb algebras*, Knots, topology and quantum field theories (Florence, 1989), World Sci. Publ., River Edge, NJ, 1989, pp. 485–496.

[41] K. Shigechi, *Canonical bases of a coideal subalgebra in $U_q(\mathfrak{sl}_2)$*, preprint (2014), `arXiv:1412.7285`.

[42] _____, *Kazhdan–Lusztig polynomials for the Hermitian symmetric pair (B_N, A_{N-1})*, preprint (2014), `arXiv:1412.6740`.

[43] K. Shigechi and P. Zinn-Justin, *Path representation of maximal parabolic Kazhdan-Lusztig polynomials*, J. Pure and Appl. Algebra **216** (2012), 2533–2548, arXiv:1001.1080.

[44] N. J. A. Sloane, *The On-line Encyclopedia of Integer Sequences*, https://oeis.org.

[45] J. Stokman and B. Vlaar, *Koorwinder Polynomials and the XXZ Spin Chain*, preprint (2013), To appear in J. Approx. Theory, arXiv:1310.5545.

[46] H. Temperley and E. Lieb, *Relations between the "percolation" and "colouring" problem and other graph-theoretical problems with regular lattices: some exact results for the "percolation" problem*, Proc. Roy. Soc. London Ser. A **322** (1971), no. 1549, 251–280.

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