Some remarks and conjectures related to lattice paths in strips along the x-axis

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Abstract

I give an overview about some typical number sequences and polynomials which are related to lattice paths in strips along the $x-$axis and compute their generating functions in terms of Fibonacci and Lucas polynomials. In the course of this work I have been led to many conjectures and curious number triangles.

0. Introduction

Consider lattice paths in $\mathbb{Z}^2$ of length n which start at the origin $(0,0)$ and have only up-steps $U:(i,j)\to(i+1,j+1)$ and down-steps $D:(i,j)\to(i+1,j-1)$.

Let $A_{n,k}$ be the set of all lattice paths of length n which start at $(0,0)$, stop on heights 0 or -1 and are contained in the strip $-\left\lfloor\frac{k+1}{2}\right\rfloor\le y\le\left\lfloor\frac{k}{2}\right\rfloor$ of width $\left\lfloor\frac{k}{2}\right\rfloor+\left\lfloor\frac{k+1}{2}\right\rfloor=k.$ Note that such a path has $\left\lfloor\frac{n}{2}\right\rfloor$ up- steps and $\left\lfloor\frac{n+1}{2}\right\rfloor$ down- steps.

The enumeration of such paths with suitable weights depending on 2 parameters q and t leads to Rogers-Ramanujan type identities (cf. e.g. [7]). In the present paper I consider the polynomials which occur in the special case $q=1$ in more detail.

In the first part I recall some results about the numbers $a(n,k)=\left|A_{n,k}\right|$. It turns out that these numbers also count the walks of length n on the path graph P_{k+1} with vertices $\{1,2,\cdots,k+1\}$ which start at the origin 1. Most of these results are known but perhaps my point of view gives a novel approach.

For the general case I did not find anything in the literature. As for $t=1$ the generating functions can be written as quotients of Fibonacci and Lucas polynomials. The detailed study of some special cases led to many curious conjectures. Some have been found with the help of the Mathematica package Guess [14] by Manuel Kauers. Of great use has also been The On-Line Encyclopedia of Integer Sequences OEIS [18]. As far as I know the results about the polynomials $a(n,k,t)$ seem to be new, but if some of them are already known I would be very grateful to receive references in order that I can give due credit in the next version of this preliminary paper.

1. Background material

1.1. Lattice paths in strips along the x-axis

My starting point has been the set $A_{n,k}$ of all lattice paths of length n which start at $(0,0)$, stop on heights 0 or -1 and are contained in the strip $-\left\lfloor \frac{k+1}{2} \right\rfloor \le y \le \left\lfloor \frac{k}{2} \right\rfloor$ of width $\left\lfloor \frac{k}{2} \right\rfloor + \left\lfloor \frac{k+1}{2} \right\rfloor = k.$

For $n \le k$ all $\binom{n}{\left\lfloor \frac{n}{2} \right\rfloor}$ paths of length n belong to $A_{n,k}$. Note that for odd k the strips are not symmetric about the $x-$ axis.

In general we get by inclusion – exclusion (see e.g. [7],[8] or [9]) that

$$a(n,k) := \left|A_{n,k}\right| = \sum_{j\in\mathbb{Z}} (-1)^j \binom{n}{\left\lfloor \frac{n+(k+2)j}{2} \right\rfloor}. \tag{1.1}$$

The set $A_{n,0}$ is empty for $n > 0$ which gives

$$\left|A_{n,0}\right| = \sum_{j\in\mathbb{Z}} (-1)^j \binom{n}{\left\lfloor \frac{n}{2} \right\rfloor + j} = [n=0]. \tag{1.2}$$

For $k=1$ the sets $A_{n,1}$ consist only of one path. If we denote a path by the sequence of its successive heights this unique path is $(0,-1,0,-1,\cdots)$. We can write it also as $DUDU\cdots DUD.$

Therefore we have

$$\left|A_{n,1}\right| = \sum_{j\in\mathbb{Z}} (-1)^j \binom{n}{\left\lfloor \frac{n+3j}{2} \right\rfloor} = 1. \tag{1.3}$$

The sets $A_{n,2}$ are $\{(0)\},\{(0,-1)\},\{(0,1,0),(0,-1,0)\},\{(0,1,0,-1),(0,-1,0,-1)\},\cdots$ or $\{\varepsilon,\{D\},\{UD,DU\},\{UDD,DUD\},\{UDUD,UDDU,DUUD,DUDU\},\cdots\}$ if we denote by ε the trivial path, which gives by induction

$$\left|A_{n,2}\right| = \sum_{j\in\mathbb{Z}} (-1)^j \binom{n}{\left\lfloor \frac{n+4j}{2} \right\rfloor} = 2^{\left\lfloor \frac{n}{2} \right\rfloor}. \tag{1.4}$$

A very interesting case occurs for $k = 3$. In this case we have $\left|A_{n,3}\right| = F_{n+1}$, where F_n is a Fibonacci number.

The Fibonacci numbers $F_n = \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n-1-k}{k}$ satisfy $F_n = F_{n-1} + F_{n-2}$ with initial values $F_0 = 0$ and $F_1 = 1$. The first terms are $0,1,1,2,3,5,8,13,21,34,\cdots$ (cf. OEIS [18], A000045).

Thus

$$\left|A_{n,3}\right| = \sum_{j\in\mathbb{Z}} (-1)^j \binom{n}{\left\lfloor \frac{n+5j}{2} \right\rfloor} = F_{n+1} = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n-k}{k}. \tag{1.5}$$

Since $A_{0,3} = \{0\}$ and $A_{1,3} = \{(0,-1)\}$ we see that the initial values are $F_1 = F_2 = 1$.

Consider now a path in $A_{n,3}$. Let the path be given by the sequence of its $y-$coordinates. If the path ends with $(-1,0)$ or $(0,-1)$ then the path is the unique continuation of a path in $A_{n-1,3}$. If the path ends with $(0,1,0)$ or $(-1,-2,-1)$ then it is the unique continuation of a path in $A_{n-2,3}$. Therefore we have $\left|A_{n,3}\right| = \left|A_{n-1,3}\right| + \left|A_{n-2,3}\right|$. This together with the initial values gives $\left|A_{n,3}\right| = F_{n+1}$.

Remark 1

Formula (1.5) has been obtained by G.E. Andrews [1], but already in 1917 I. Schur [20] has studied the right-hand side of the identity

$$\sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} q^{k^2} \begin{bmatrix} n-k \\ k \end{bmatrix}_q = \sum_{j\in\mathbb{Z}} (-1)^j q^{\frac{j(5j-1)}{2}} \begin{bmatrix} n \\ \left\lfloor \frac{n+5j}{2} \right\rfloor \end{bmatrix}_q \tag{1.6}$$

for $|q| < 1$. Here $\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}_q$ denotes a $q-$ binomial coefficient defined by

$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(1-q)(1-q^2)\cdots(1-q^{n-k})}{(1-q)(1-q^2)\cdots(1-q^k)}$ for $0 \le k \le n$ and 0 else.

It is clear that (1.6) converges to (1.5) for $q \to 1.$ Therefore (1.6) is called a $q-$ analogue of (1.5).

If we let $n \to \infty$ in (1.6) we get the famous first Rogers-Ramanujan identity

$$\sum_{k\geq 0} \frac{q^{k^2}}{\left(1-q\right)\left(1-q^2\right)\cdots\left(1-q^k\right)} = \frac{1}{\prod_{j=1}^{\infty}\left(1-q^j\right)} \sum_{j\in\mathbb{Z}} (-1)^j q^{\frac{j(5-j)}{2}}. \tag{1.7}$$

Some of the sequences $\left(a(n,k)\right)_{n\geq 0}$ occur in the literature in other contexts.

For small values of k useful information can be found in OEIS [18]:

$\left(a(n,2)\right)_{n\geq 0}$ is A016116,

$\left(a(n,3)\right)_{n\geq 0}$ is A000045,

$\left(a(n,4)\right)_{n\geq 0} = \left(1,1,2,3,6,9,18,27,\cdots\right)$ is A182522,

$\left(a(n,5)\right)_{n\geq 0} = \left(1,1,2,3,6,10,19,33,\cdots\right)$ is A028495,

$\left(a(n,6)\right)_{n\geq 0} = \left(1,1,2,3,6,10,20,34,68,\cdots\right)$is A030436,
$\left(a(n,7)\right)_{n\geq 0} = \left(1,1,2,3,6,10,20,35,69,\cdots\right)$ is A061551 and
$\left(a(n,8)\right)_{n\geq 0} = \left(1,1,2,3,6,10,20,35,70,125,\cdots\right)$ is A178381.

For some of these sequences in OEIS combinatorial interpretations in terms of graphs are mentioned which I tried to generalize. Later I saw that these results were already well known. But perhaps my approach will give a new point of view.

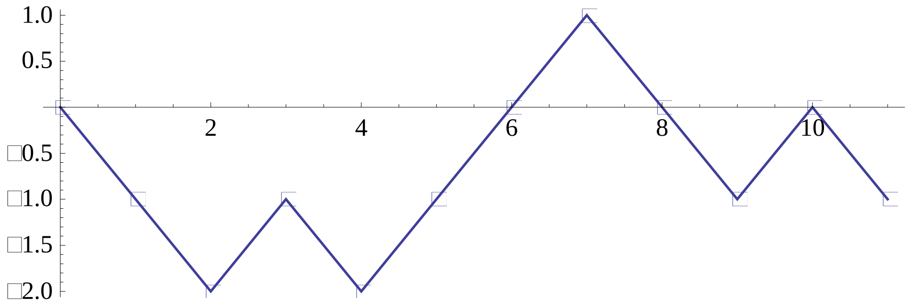
1.2. Some other combinatorial models

Proposition 1

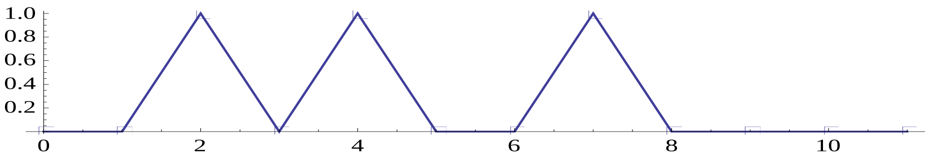
The number $a(n,2k+1)$ counts all non-negative lattice paths starting from $(0,0)$ and ending in $(n,0),$ where besides up-steps and down-steps also horizontal moves $(i,0) \to (i+1,0)$ on height 0 are allowed and the maximal height of a path is $k.$

It is easy to find a bijection between these two lattice path models. Starting from the first model we map each up-step $(i,-1) \to (i+1,0)$ and each down-step $(i,0) \to (i+1,-1)$ into a horizontal move $(i,0) \to (i+1,0).$ The non-negative parts of the path remain unaltered and the negative paths $(i,-1) \to (j,-1)$ are reflected on the line $y = -\frac{1}{2}$ into a non-negative path $(i,0) \to (j,0).$ This map obviously has a unique inverse.

For example for $k=1$ the lattice path



will be transformed to



Corollary

The number $a(n, 2k+1)$ *counts all walks of length* n *on the graph which arises by adjoining a loop at an extremity of the path graph* P_{k+1} *which start and end on this extremity.*

Remark 2

If we let $k \to \infty$ in Proposition 1, i.e. consider non-negative lattice paths starting from $(0,0)$ where besides up-steps and down-steps also horizontal steps $(i,0) \to (i+1,0)$ on height 0 are allowed then the numbers $c(n,j)$ of all such paths ending in (n,j) satisfy

$c(n,0) = c(n-1,0) + c(n-1,1)$ and $c(n,j) = c(n,j-1) + c(n,j+1)$ for $j > 0$.

We get the following table (cf. OEIS A061554)

$$\left(c(n,j)\right)_{n,j\geq 0} = \begin{pmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 2 & 1 & 1 & & & \\ 3 & 3 & 1 & 1 & & \\ 6 & 4 & 4 & 1 & 1 & \\ 10 & 10 & 5 & 5 & 1 & 1 \end{pmatrix} \tag{1.8}$$

.

It is easily verified that $c(n,j)=\binom{n}{\left\lfloor \frac{n-j}{2} \right\rfloor}$. Of course we have $c(n,0)=\binom{n}{\left\lfloor \frac{n}{2} \right\rfloor}$.

If we make the further assumption that $c(n,k+1)=0$ then we get by Proposition 1 that $c(n,0)=a(n,2k+1)$.

For example for $k=1$ we get the table

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 0 \\ 5 & 3 & 0 \\ 8 & 5 & 0 \end{pmatrix}$$

with $c(n,0)=a(n,3)=F_{n+1}$.

Another combinatorial model suggested by the entries in OEIS is

Proposition 2

The number $a(n,k)$ counts all walks of length n on the path graph P_{k+1} with vertices $\{1,2,\cdots,k+1\}$ starting at 1.

The following proof uses an idea by S.V. Ault and Ch. Kicey [2].

Proof

Let $v(n,m,k)$, $1\le m\le k+1$, be the number of all walks of length n on P_{k+1} which start at 1 and end in the point m. Then the sequences $v(n,m,k)$ are uniquely determined by the recursion $v(n,m,k)=v(n-1,m-1,k)+v(n-1,m+1,k)$ for all $m\in\{1,\cdots,k+1\}$ with initial values $v(0,m,k)=[m=1]$ if we set $v(n,0,k)=v(n,k+2,k)=0$.

As shown in [2] for $0\le m\le k+2$

$$v(n,m,k)=\sum_{j\in\mathbb{Z}}\binom{n}{\left\lfloor \frac{m+n}{2} \right\rfloor+(k+2)j}-\sum_{j\in\mathbb{Z}}\binom{n}{\left\lfloor \frac{m+n+1}{2} \right\rfloor+(k+2)j}. \tag{1.9}$$

To show this formula it suffices to check the recursion and the initial and boundary values.

The initial values are trivial.

Since

$$\binom{n}{\left\lfloor \frac{n+m+1}{2} \right\rfloor + (k+2)j} = \binom{n}{n - \left\lfloor \frac{n+m+1}{2} \right\rfloor - (k+2)j} = \binom{n}{\left\lfloor \frac{n-m}{2} \right\rfloor - (k+2)j}$$

we get

$$v(n,0,k) = \sum_{j\in\mathbb{Z}} \binom{n}{\left\lfloor \frac{n}{2} \right\rfloor + (k+2)j} - \sum_{j\in\mathbb{Z}} \binom{n}{\left\lfloor \frac{n+1}{2} \right\rfloor + (k+2)j} = 0$$

and

$$v(n,k+2,k) = \sum_{j\in\mathbb{Z}} \binom{n}{\left\lfloor \frac{k+2+n}{2} \right\rfloor + (k+2)j} - \sum_{j\in\mathbb{Z}} \binom{n}{\left\lfloor \frac{k+2+n+1}{2} \right\rfloor + (k+2)j}$$

$$= \sum_{j\in\mathbb{Z}} \binom{n}{\left\lfloor \frac{k+2+n+1}{2} \right\rfloor - (k+2)(j+1)} - \sum_{j\in\mathbb{Z}} \binom{n}{\left\lfloor \frac{k+2+n}{2} \right\rfloor - (k+2)(j+1)} = -v(n,k+2,k)$$

and thus $v(n,k+2,k) = 0.$

The recurrence follows from

$$\binom{n}{\left\lfloor \frac{m+n}{2} \right\rfloor + (k+2)j} = \binom{n-1}{\left\lfloor \frac{m-1+n-1}{2} \right\rfloor + (k+2)j} + \binom{n-1}{\left\lfloor \frac{m+1+n-1}{2} \right\rfloor + (k+2)j}.$$

Therefore

$$\sum_{m=0}^{k+1} v(n,m,k) = \sum_{j\in\mathbb{Z}} \binom{n}{\left\lfloor \frac{n}{2} \right\rfloor + (k+2)j} - \sum_{j\in\mathbb{Z}} \binom{n}{\left\lfloor \frac{n+k+2}{2} \right\rfloor + (k+2)j}$$

$$= \sum_{j\in\mathbb{Z}} (-1)^j \binom{n}{\left\lfloor \frac{n+(k+2)j}{2} \right\rfloor} = a(n,k).$$

For example for $k=3$ we have $v(2n,1,3) = F_{2n-1}, v(2n,3,3) = v(2n+1,4,3) = F_{2n}, v(2n+1,2,3) = F_{2n+1}$ and all other terms vanish.

Remark 3

Proposition 2 can also be deduced from general results about lattice paths in corridors. E.g. [15], formula (9) implies that the number of walks on P_{k+1} from 1 to m is 0 if $n-m\equiv 0 \bmod 2$ and else

$$\sum_{j\in\mathbb{Z}}\binom{n}{\frac{m+n-1}{2}+(k+2)j}-\sum_{j\in\mathbb{Z}}\binom{n}{\frac{m+n+1}{2}+(k+2)j}.$$

Both results can be combined to give (1.9).

Remark 4

The number $v(2n,1,k)$ counts the walks of length $2n$ on P_{k+1} which start and end in 1 or equivalently the set of all non-negative lattice-paths of length $2n$ and height $\le k$ with up-steps U and down-steps D which start at the origin and end in $(2n,0)$ (Dyck-paths).

Helmut Prodinger has kindly brought my attention to the paper [3] by N.G. de Bruijn, D.E. Knuth and S.O. Rice which gives some more information about these numbers. I prefer to state their results in terms of Dyck paths:

Such a path P is either the trivial path $(0,0)\to(0,0)$ of length 0 or has a uniquely determined decomposition $P=UP_1DUP_2D\cdots UP_jD$ where each P_i is a Dyck path with height $\le k-1$.

Therefore

$$v_k(z)=\sum_{n\ge 0}v(n,1,k)z^n=\sum_{n\ge 0}v(2n,1,k)z^{2n} \tag{1.10}$$

satisfies

$$v_k(z)=1+z^2v_{k-1}(z)+\left(z^2v_{k-1}(z)\right)^2+\left(z^2v_{k-1}(z)\right)^3+\cdots=\frac{1}{1-z^2v_{k-1}(z)}. \tag{1.11}$$

Note that $v_0(z)=1$.

From $\left(1-z^2v_{k-1}(z)\right)v_k(z)=1$ we get by comparing coefficients

$$v(2n,1,k)=\sum_{j=0}^{n-1}v(2j,1,k)v(2n-2-2j,1,k-1) \tag{1.12}$$

for $n\ge 1$. Since $v(2n,1,0)=[n=0]$ and $v(0,1,k)=1$ all values $v(2n,1,k)$ can recursively be computed.

Formula (1.11) leads to a continued fraction.

For example from $v(2n,1,3)=F_{2n-1}$ we get (setting $F_{-1}=1$)

$$\sum_{n\geq 0}F_{2n-1}z^{2n}=\cfrac{1}{1-\cfrac{z^2}{1-\cfrac{z^2}{1-z^2}}}.$$

For arbitrary Dyck paths (1.11) gives the well-known fact that

$v(z)=v_{\infty}(z)=\frac{z}{1-v(z)}$ or $v(z)=\frac{1-\sqrt{1-4z}}{2}=\sum_{n\geq 0}C_n z^n$ where $C_n=\frac{1}{n+1}\binom{2n}{n}$ is a Catalan number.

This implies that $v(2n,1,k)=C_n$ for $n\leq k.$

Problem

We have seen that the numbers $a(n,k)=\sum_{j\in\mathbb{Z}}(-1)^j\binom{n}{\left\lfloor\frac{n+(k+2)j}{2}\right\rfloor}$ count both the set of lattice paths $A_{n,k}$ and all walks of length n on P_{k+1}. It would therefore be interesting to find a simple bijection between walks on P_{k+1} and the set of lattice paths $A_{n,k}$.

1.3. Generating functions

The generating functions of these number sequences turn out to be quotients of Fibonacci and Lucas polynomials or equivalently quotients of Chebyshev polynomials.

Recall that the Fibonacci polynomials $F_n(x,s)=\sum_{k=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}\binom{n-1-k}{k}x^{n-1-2k}s^k$ satisfy the recurrence relation $F_n(x,s)=xF_{n-1}(x,s)+sF_{n-2}(x,s)$ with initial values $F_0(x,s)=0$ and $F_1(x,s)=1$ and the Lucas polynomials $L_n(x,s)=\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\binom{n-k}{k}\frac{n}{n-k}s^k x^{n-2k}$ satisfy $L_n(x,s)=xL_{n-1}(x,s)+sL_{n-2}(x,s)$ with initial values $L_0(x,s)=2$ and $L_1(x,s)=x.$

Most identities about these polynomials can easily be proved by using the well-known Binet formulae

$F_n(x,s)=\dfrac{\alpha^n-\beta^n}{\alpha-\beta}$ and $L_n(x,s)=\alpha^n+\beta^n$ if $\alpha=\alpha(x,s)=\dfrac{x+\sqrt{x^2+4s}}{2}$ and $\beta=\beta(x,s)=\dfrac{x-\sqrt{x^2+4s}}{2}$ are the roots of the equation $z^2-xz-s=0$.

Let us do this for some formulae which will be needed in the sequel:

The identity $L_n(x,s)=F_{n+1}(x,s)+sF_{n-1}(x,s)$ follows from $\left(\alpha^n+\beta^n\right)\left(\alpha-\beta\right)=\alpha^{n+1}-\beta^{n+1}-\alpha\beta\left(\alpha^{n-1}-\beta^{n-1}\right)$, the identity

$F_{2n}(x,s)=F_n(x,s)L_n(x,s)$ from $\alpha^{2n}-\beta^{2n}=\left(\alpha^n-\beta^n\right)\left(\alpha^n+\beta^n\right)$ and $F_{k+1}(x,s)^2+sF_k(x,s)^2=F_{2k+1}(x,s)$ from $\left(\alpha^{k+1}-\beta^{k+1}\right)^2-\alpha\beta\left(\alpha^k-\beta^k\right)^2=\left(\alpha-\beta\right)\left(\alpha^{2k+1}-\beta^{2k+1}\right)$.

Since $\alpha\left(x+y,-xy\right)=x$ and $\beta\left(x+y,-xy\right)=y$ we get the well-known identities

$$\begin{aligned} L_n\left(x+y,-xy\right)&=x^n+y^n,\\ F_n(x+y,-xy)&=\frac{x^n-y^n}{x-y}. \end{aligned} \tag{1.13}$$

If we choose $x=e^{\frac{j\pi}{k+1}i}$, $y=e^{-\frac{j\pi}{k+1}i}$ for $1\le j\le k$ we get

$F_{k+1}\left(2\cos\dfrac{j\pi}{k+1},-1\right)=0$ or since $F_{k+1}(x,-1)$ is a monic polynomial of degree k

$$F_{k+1}\left(x,-1\right)=\prod_{j=1}^{k}\left(x-2\cos\frac{j\pi}{k+1}\right). \tag{1.14}$$

From (1.11) we deduce that

$$v_k(x)=\frac{F_{k+1}(1,-x^2)}{F_{k+2}(1,-x^2)}. \tag{1.15}$$

For this holds for $k=0$. If it is true for $k-1$ then

$$v_k(x)=\frac{1}{1-x^2v_{k-1}(x)}=\frac{1}{1-x^2v_{k-1}(x)}=\frac{1}{1-\dfrac{x^2F_k(1,-x^2)}{F_{k+1}(1,-x^2)}}$$
$$=\frac{F_{k+1}(1,-x^2)}{F_{k+1}(1,-x^2)-x^2F_k(1,-x^2)}=\frac{F_{k+1}(1,-x^2)}{F_{k+2}(1,-x^2)}.$$

Remark 5

Another way to obtain such results is by using the adjacency matrix $M_k = \left(m(i,j,k)\right)_{i,j=1}^{k}$ of the path graph P_k. Note that $m(i,j,k)=1$ if $\{i,j\}$ is an edge and $m(i,j,k)=0$ else, i.e. $m(1,2,k)=1,$ $m(k,k-1,k)=1,$ and $m\left(i,i\pm 1,k\right)=1$ for $1<i<k.$ All other entries are $0.$ If we set $M_{k+1}^{n} = \left(m(i,j,k+1)^{(n)}\right)_{i,j=1}^{k}$ then it is obvious that $m(1,j,k+1)^{(n)} = v(n,j,k).$

One of the reasons why Fibonacci polynomials occur in the study of path graphs is the well-known formula

$$F_{k+1}\left(x,-1\right)=\det\begin{pmatrix} x & -1 & 0 & 0 & \cdots & 0 \\ -1 & x & -1 & 0 & \cdots & 0 \\ 0 & -1 & x & -1 & \cdots & 0 \\ 0 & 0 & -1 & x & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & x \end{pmatrix} \tag{1.16}$$

which follows immediately from the recurrence relation for the Fibonacci polynomials. For the right-hand side can be interpreted as the characteristic polynomial of the adjacency matrix $M_k = \left(m(i,j,k)\right)_{i,j=1}^{k}$ of the path graph P_k.

I became aware of this fact through the blog post [22] by Qiaochu Yuan. Of course this is an old result. References may be found in the recent paper [13] by Stefan Felsner and Daniel Heldt, where similar results are obtained and in the survey article [16] by Christian Krattenthaler.

Let us recall some results from this point of view:

By (1.14) the eigenvalues of M_k are given by $\lambda_j = 2\cos\dfrac{j\pi}{k+1}$ for $1\le j\le k.$

Then $v_j = \left(F_1(\lambda_j,-1), F_2(\lambda_j,-1), \cdots, F_k(\lambda_j,-1)\right)^t$ is an eigenvector corresponding to $\lambda_j.$

For $M_k v_j = \lambda_j v_j$ is equivalent with $F_{\ell-1}\left(\lambda_j,-1\right)+F_{\ell+1}\left(\lambda_j,-1\right)=\lambda_j F_\ell\left(\lambda_j,-1\right)$ for $1\le \ell\le k.$

Note that $F_0\left(\lambda_j,-1\right)=F_{k+1}\left(\lambda_j,-1\right)=0.$

Since by (1.14) $F_\ell(\lambda_j,-1)=F_\ell(2\cos\frac{j\pi}{k+1},-1)=\dfrac{\sin\frac{\ell j\pi}{k+1}}{\sin\frac{j\pi}{k+1}}$

the eigenvectors are (up to scaling) given by $v_j=\left(\sin\frac{j\pi}{k+1},\sin\frac{2j\pi}{k+1},\cdots,,\sin\frac{kj\pi}{k+1}\right)^t$.

The normalized eigenvectors are $\sqrt{\frac{2}{k+1}}v_j$ since

$$\sum_{\ell=1}^{k}\left(\sin\frac{\ell j\pi}{k+1}\right)^2=-\frac{1}{4}\sum_{\ell=1}^{k}\left(e^{\frac{2\ell j\pi}{k+1}i}-2+e^{-\frac{2\ell j\pi}{k+1}i}\right)=-\frac{1}{4}\left(-2-2k\right)=\frac{k+1}{2}.$$

Since M_k is obviously symmetric we see that the matrix

$$U=\left(\sqrt{\frac{2}{k+1}}v_1,\sqrt{\frac{2}{k+1}}v_2,\cdots,\sqrt{\frac{2}{k+1}}v_k\right)$$

is orthogonal. Let $\Lambda_k=\left(\lambda_j[i=j]\right)_{i,j=1}^{k}$ be the diagonal matrix whose entries are the eigenvalues. Then $M_k=U\Lambda_kU^{-1}=U\Lambda_kU^t$.

Therefore from $M_{k+1}^n=U\Lambda_{k+1}^nU^t$ we get the known trigonometric representation

$$v(n,j,k)=\frac{2}{k+2}\sum_{\ell=1}^{k+1}\sin\frac{\ell\pi}{k+2}\sin\frac{\ell j\pi}{k+2}\left(2\cos\frac{j\pi}{k+2}\right)^n. \tag{1.17}$$

In the same way as above we see that $\det\left(I_k-M_kx\right)=F_{k+1}(1,-x^2)$.

From $\left(I_k-M_kx\right)^{-1}=\sum_{n\geq 0}M_k^nx^n$ and Cramer's Rule $\left(I_k-M_kx\right)^{-1}=\dfrac{adj(I_k-M_kx)}{\det\left(I_k-M_kx\right)}$

we find again (1.15) by considering the top-left entry of these matrices.

As shown in [6] and [10] the generating functions of the sequences $\left(a(n,k)\right)_{n\geq 0}$ are given by

$$\sum_{n\geq 0}a(n,2k+1)x^n=\frac{F_{k+1}(1,-x^2)}{F_{k+2}(1,-x^2)-xF_{k+1}(1,-x^2)} \tag{1.18}$$

and

$$\sum_{n\geq 0} a(n,2k)x^n = \frac{F_{k+1}(1,-x^2) + xF_k(1,-x^2)}{L_{k+1}(1,-x^2)}. \tag{1.19}$$

Observe that $\deg\left(F_{k+2}(1,-x^2) - xF_{k+1}(1,-x^2)\right) = k+1$ and $\deg L_{k+1}(1,-x^2) = 2\left\lfloor \frac{k+1}{2} \right\rfloor$.

Let $\left(a(n)\right)_{n\geq 0}$ be a sequence of real numbers with $a(0)=1$ and $\sum_{n\geq 0} a(n)x^n = \frac{c(x)}{d(x)}$ with $\deg c(x) < \deg d(x) = m$ and $d(x) = \sum_{\ell=0}^{m} d_\ell x^\ell$ with $d_0 = 1$. Then the sequence $\left(a(n)\right)$ satisfies the recursion $a_{n+m} + a_{n+m-1}d_1 + \cdots + a_n d_m = 0$ for $n \geq 0$.

We call $d(x)$ the characteristic polynomial of the recursion. If we introduce the shift operator E defined by $Ea(n) = a(n+1)$ then the recursion can also be formulated as $D(E)a(n) = 0$ where $D(x) = \sum_{\ell=0}^{m} d_\ell x^{m-\ell} = x^m d\left(\frac{1}{x}\right)$ is the reciprocal polynomial of $d(x)$.

The recursion can also be formulated as $\lambda\left(x^n D(x)\right) = 0$ if λ denotes the linear functional on the polynomials defined by $\lambda\left(x^n\right) = a(n)$.

Since $x\alpha\left(\frac{1}{x}, -1\right) = \alpha\left(1,-x^2\right)$ and $x\beta\left(\frac{1}{x},-1\right) = \beta\left(1,-x^2\right)$ we have

$x^{k-1}F_k\left(1, -\frac{1}{x^2}\right) = F_k\left(x,-1\right)$, $\quad x^k\left(F_{k+1}\left(1,-\frac{1}{x^2}\right) - \frac{1}{x}F_k\left(1,-\frac{1}{x^2}\right)\right) = F_{k+1}\left(x,-1\right) - F_k(x,-1)$

and $x^k L_k\left(1,-\frac{1}{x^2}\right) = L_k(x,-1)$.

Therefore the sequence $\left(a(n,2k)\right)$ satisfies the recurrence

$$L_k\left(E,-1\right)a(n,2k) = 0 \tag{1.20}$$

and the sequence $\left(a(n,2k-1)\right)$ satisfies

$$\left(F_{k+1}(E,-1) - F_k(E,-1)\right)a(n,2k-1) = 0. \tag{1.21}$$

Note that $L_k\left(x,-1\right)$ and $F_{k+1}(x,-1) - F_k(x,-1)$ have analogous factorizations

$$L_k\left(x,-1\right) = \prod_{j=0}^{k-1}\left(x - 2\cos\left(\frac{2j+1}{2k}\pi\right)\right) \tag{1.22}$$

because

$$L_k\left(2\cos\left(\frac{2j+1}{2k}\pi\right),-1\right)=L_k\left(e^{\frac{2j+1}{2k}i\pi}+e^{-\frac{2j+1}{2k}i\pi},-e^{\frac{2j+1}{2k}i\pi}e^{-\frac{2j+1}{2k}i\pi}\right)=e^{\frac{2j+1}{2}i\pi}+e^{-\frac{2j+1}{2}i\pi}=0$$

and

$$F_{k+1}(x,-1)-F_k(x,-1)=\prod_{j=0}^{k-1}\left(x-2\cos\frac{2j+1}{2k+1}\pi\right) \tag{1.23}$$

because for $0\le j\le k-1$

$$F_{k+1}(2\cos\frac{2j+1}{2k+1}\pi,-1)-F_k(2\cos\frac{2j+1}{2k+1}\pi,-1)=\frac{e^{\frac{(2j+1)(k+1)}{2k+1}i\pi}-e^{-\frac{(2j+1)(k+1)}{2k+1}i\pi}}{e^{\frac{(2j+1)i\pi}{2k+1}}-e^{-\frac{(2j+1)i\pi}{2k+1}}}-\frac{e^{\frac{(2j+1)k}{2k+1}i\pi}-e^{-\frac{(2j+1)k}{2k+1}i\pi}}{e^{\frac{(2j+1)i\pi}{2k+1}}-e^{-\frac{(2j+1)i\pi}{2k+1}}}$$
$$=\frac{e^{-\frac{(2j+1)k}{2k+1}i\pi}\left(e^{(2j+1)i\pi}+1\right)-e^{\frac{(2j+1)k}{2k+1}i\pi}\left(e^{-(2j+1)i\pi}+1\right)}{e^{\frac{(2j+1)i\pi}{2k+1}}-e^{-\frac{(2j+1)i\pi}{2k+1}}}=0.$$

We now give another simple proof of (1.18) and (1.19).

Let $v_k(x,m)=\sum_{n\ge 0}v(n,m,k)x^n$.

Since each path P from $(0,0)$ to $\left(2n+m-1,m-1\right)$ has a unique decomposition

$P=P_0UP_1UP_2\cdots UP_{m-1}$ we get

$v_k(x,m)=x^{m-1}v_k(x)v_{k-1}(x)\cdots v_{k-m+1}(x)$.

This implies

$$\begin{aligned}a_k(x)&=\sum_{n\ge 0}a(n,k)x^n=\sum_{m=1}^{k+1}\sum_{n\ge 0}v(n,m,k)x^n=\sum_{m=1}^{k+1}v_k(x,m)\\&=\sum_{m=1}^{k+1}x^{m-1}v_k(x)v_{k-1}(x)\cdots v_{k-m+1}(x)=v_k(x)\left(1+xa_{k-1}(x)\right).\end{aligned} \tag{1.24}$$

Now $a_0(x)=1$ agrees with (1.19)

and since $v_1(x)=\frac{1}{1-x^2}$ we get $a_1(x)=\frac{1}{1-x}$. This agrees with (1.18).

If (1.18) and (1.19) hold for $k-1$ then we get

$$a_{2k}(x)=v_{2k}(x)\left(1+xa_{2k-1}(x)\right)=\frac{F_{2k+1}(1,-x^2)}{F_{2k+2}(1,-x^2)}\left(1+x\frac{F_k(1,-x^2)}{F_{k+1}(1,-x^2)-xF_k(1,-x^2)}\right)$$
$$=\frac{\left(F_{k+1}\left(1,-x^2\right)-xF_k\left(1,-x^2\right)\right)\left(F_{k+1}\left(1,-x^2\right)+xF_k\left(1,-x^2\right)\right)}{F_{k+1}\left(1,-x^2\right)L_{k+1}\left(1,-x^2\right)}\frac{F_{k+1}(1,-x^2)}{F_{k+1}(1,-x^2)-xF_k(1,-x^2)}$$
$$=\frac{\left(F_{k+1}\left(1,-x^2\right)+xF_k\left(1,-x^2\right)\right)}{L_{k+1}(1,-x^2)}$$

and

$$a_{2k+1}(x)=v_{2k+1}(x)\left(1+xa_{2k}(x)\right)=\frac{F_{2k+2}(1,-x^2)}{F_{2k+3}(1,-x^2)}\left(1+x\frac{\left(F_{k+1}\left(1,-x^2\right)+xF_k\left(1,-x^2\right)\right)}{L_{k+1}(1,-x^2)}\right)$$
$$=\frac{F_{k+1}\left(1,-x^2\right)L_{k+1}\left(1,-x^2\right)\left(L_{k+1}(1,-x^2)+x\left(F_{k+1}\left(1,-x^2\right)+xF_k\left(1,-x^2\right)\right)\right)}{\left(F_{k+2}\left(1,-x^2\right)-xF_{k+1}\left(1,-x^2\right)\right)\left(F_{k+2}\left(1,-x^2\right)+xF_{k+1}\left(1,-x^2\right)\right)L_{k+1}(1,-x^2)}$$
$$=\frac{F_{k+1}\left(1,-x^2\right)\left(\left(F_{k+2}\left(1,-x^2\right)+xF_{k+1}\left(1,-x^2\right)\right)\right)}{\left(F_{k+2}\left(1,-x^2\right)-xF_{k+1}\left(1,-x^2\right)\right)\left(F_{k+2}\left(1,-x^2\right)+xF_{k+1}\left(1,-x^2\right)\right)}=\frac{F_{k+1}\left(1,-x^2\right)}{\left(F_{k+2}\left(1,-x^2\right)-xF_{k+1}\left(1,-x^2\right)\right)}.$$

The denominators of (1.18) and (1.19) have a simple determinant representation. To show this we need the following well-known result.

If the (Hankel-) determinants $D_m=\det\left(a(i+j)\right)_{i,j=0}^{m-1}$ are non-vanishing then

$$P(x)=\det\begin{pmatrix} a(0) & a(1) & \cdots & a(m-1) & 1\\ a(1) & a(2) & \cdots & a(m) & x\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ a(m-1) & a(m) & \cdots & a(2m-2) & x^{m-1}\\ a(m) & a(m+1) & \cdots & a(2m-1) & x^m \end{pmatrix}$$

is a polynomial of degree m.

It satisfies $\lambda\left(x^nP(x)\right)=\det\begin{pmatrix} a(0) & a(1) & \cdots & a(m-1) & a(n+m)\\ a(1) & a(2) & \cdots & a(m) & a(n+m+1)\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ a(m-1) & a(m) & \cdots & a(2m-2) & a(n+2m-1)\\ a(m) & a(m+1) & \cdots & a(2m-1) & a(n+2m) \end{pmatrix}=0$

for all $n\geq 0$ since the rows are linearly dependent.

The reciprocal polynomial is given by

$$p(x)=\det\begin{pmatrix} a(0) & a(1) & \cdots & a(m-1) & x^m \\ a(1) & a(2) & \cdots & a(m) & x^{m-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a(m-1) & a(m) & \cdots & a(2m-2) & x \\ a(m) & a(m+1) & \cdots & a(2m-1) & 1 \end{pmatrix}.$$

Proposition 3

The characteristic polynomial $F_{k+2}(1,-x^2)-xF_{k+1}(1,-x^2)$ *of the sequence* $a(n,2k+1)$ *is*

$$F_{k+2}(1,-x^2)-xF_{k+1}(1,-x^2)=\det\begin{pmatrix} a(0) & a(1) & \cdots & a(k) & x^{k+1} \\ a(1) & a(2) & \cdots & a(k+1) & x^k \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a(k) & a(k+1) & \cdots & a(2k) & x \\ a(k+1) & a(k+2) & \cdots & a(2k+1) & 1 \end{pmatrix} \qquad (1.25)$$

where $a(n)=\binom{n}{\left\lfloor \frac{n}{2}\right\rfloor}$.

Proof

By **Fehler! Verweisquelle konnte nicht gefunden werden.** we have $\det\left(a(i+j)\right)_{i,j=0}^{n-1}=1.$ Therefore the constant terms of both sides of (1.25) coincide with 1.1.

Thus the sequence $a(n,2k+1)$ is uniquely determined by the initial values $\binom{n}{\left\lfloor \frac{n}{2}\right\rfloor}$, $0\le n\le 2k+1,$ and the determinant which also depends only on these initial values.

As an example consider the sequence $a(n,3)=F_{n+1}.$ Here we have

$$F_1=1=\binom{0}{0}, F_2=1=\binom{1}{1}, F_3=2=\binom{2}{1}, F_4=3=\binom{3}{1}$$

and

$$\det\begin{pmatrix}1 & 1 & x^2\\ 1 & 2 & x\\ 2 & 3 & 1\end{pmatrix} = 1 - x - x^2 = F_3(1,-x^2) - xF_2(1,-x^2) = \left(1-x^2\right) - x\cdot 1.$$

Proposition 4

The characteristic polynomial $L_{k+1}(1,-x^2)$ *of the sequence* $a(n,2k)$ *is*

$$L_{k+1}(1,-x^2) = \det\begin{pmatrix} a(0) & a(1) & \cdots & a(k) & x^{k+1}\\ a(1) & a(2) & \cdots & a(k+1) & x^k\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ a(k) & a(k+1) & \cdots & a(2k) & x\\ a(k+1) & a(k+2) & \cdots & a(2k+1) & 1 \end{pmatrix} \tag{1.26}$$

where $a(n) = \binom{n}{\left\lfloor \frac{n}{2} \right\rfloor}$ *for* $n \le 2k$ *and* $a(2k+1) = a(2k+1,2k) = \binom{2k+1}{k} - 1.$

The identity $a(2k+1,2k) = \binom{2k+1}{k} - 1$ follows from the fact that there are only two paths which reach the boundary $|y| = k$ from which the path D^kU^{k+1} does not belong to $A_{2k+1,2k}$ because it ends on height 1.

(1.26) is clear if $\deg L_{k+1}(1,-x^2) = k+1.$ If $k+1$ is odd we have $\deg L_{k+1}(1,-x^2) = 2\left\lfloor \frac{k+1}{2} \right\rfloor = k.$ Since $L_{k+1}(1,-x^2)$ has constant term 1 we also get (1.26).

For example choose the sequence $\left(a(n,4)\right)_{n\ge 0} = \left(1,1,2,3,6,9,18,\cdots\right).$ Here we have

$\det\begin{pmatrix}1 & 1 & 2 & x^3\\ 1 & 2 & 3 & x^2\\ 2 & 3 & 6 & x\\ 3 & 6 & 9 & 1\end{pmatrix}$. Since $\det\begin{pmatrix}1 & 2 & 3\\ 2 & 3 & 6\\ 3 & 6 & 9\end{pmatrix} = 0$ the coefficient of x^3 vanishes and we get

$$\det\begin{pmatrix}1 & 1 & 2 & x^3\\ 1 & 2 & 3 & x^2\\ 2 & 3 & 6 & x\\ 3 & 6 & 9 & 1\end{pmatrix} = 1 - 3x^2 = L_3(1,-x^2).$$

Since

$$\sum_{n\geq 0} a(2n+2,2k)x^{2n+2} = \frac{F_{k+1}(1,-x^2)}{F_{k+2}(1,-x^2)-x^2F_k(1,-x^2)} - 1 = \frac{F_{k+1}(1,-x^2)}{F_{k+1}(1,-x^2)-2x^2F_k(1,-x^2)} - 1$$
$$= \frac{2x^2F_k(1,-x^2)}{L_{k+1}(1,-x^2)} = 2x\frac{xF_k(1,-x^2)}{L_{k+1}(1,-x^2)} = 2\sum_{n\geq 0} a(2n+1,2k)x^{2n+2}$$

we have

$$a(2n+2,2k) = 2a(2n+1,2k). \tag{1.27}$$

This result also follows from the original lattice path interpretation by symmetry.

For example

$$\sum_{n\geq 0} a(n,4)x^n = \frac{F_3(1,-x^2)+xF_2(1,-x^2)}{L_3(1,-x^2)} = \frac{1+x-x^2}{1-3x^2} = 1 + x\frac{1+2x}{1-3x^2} = 1+(1+2x)\left(x+3x^3+3x^5+\cdots\right)$$

Therefore $a(n,4) = \sum_{j\in\mathbb{Z}} (-1)^j \binom{n}{\left\lfloor \frac{n+6j}{2} \right\rfloor}$ satisfies $a(0,4)=1,\ \ a(2n+1,4)=3^n$ and $a(2n+2,4) = 2\cdot 3^n$.

From (1.18) it is easily verified that for $k\geq 1$

$$\sum_{n\geq 0} a(n,2k+1)x^n = \frac{1}{1-\dfrac{x}{1-x\sum_{n\geq 0} a(n,2k-1)(-x)^n}}. \tag{1.28}$$

For

$$\frac{1}{1-\dfrac{x}{1-x\sum_{n\geq 0} a(n,2k-1)(-x)^n}} = \frac{1-x\dfrac{F_k(1,-x^2)}{F_{k+1}(1,-x^2)+xF_k(1,-x^2)}}{1-x-x\dfrac{F_k(1,-x^2)}{F_{k+1}(1,-x^2)+xF_k(1,-x^2)}}$$
$$= \frac{F_{k+1}(1,-x^2)}{F_{k+1}(1,-x^2)-xF_{k+1}(1,-x^2)-x^2F_k(1,-x^2)} = \frac{F_{k+1}(1,-x^2)}{F_{k+2}(1,-x^2)-xF_{k+1}(1,-x^2)} = \sum_{n\geq 0} a(n,2k+1)x^n.$$

Since $\sum_{n\geq 0} a(n,1)x^n = \frac{1}{1-x}$ we see that $\sum_{n\geq 0} a(n,3)x^n = \cfrac{1}{1-\cfrac{x}{1-\cfrac{x}{1+x}}}$ etc.

The continued fraction for $\sum_{n\geq 0} a(n,5)x^n$ has been observed by M. Somos ([18], A028495).

2. Polynomials associated with $A_{n,k}$.

2.1. Definitions and known results

Instead of the numbers $\left|A_{n,k}\right|$ we consider the following weights. Define a peak as a vertex preceded by an up-step U and followed by a down-step D, and a valley as a vertex preceded by a down- step D and followed by an up-step U. The height of a vertex is its $y-$ coordinate. The peaks with a height at least 1 and the valleys with height at most -2 are called extremal points. Let $E(v)$ be the set of $x-$coordinates of the extremal points of the path v, $e(v) = \left|E(v)\right|$ the number of extremal points of v and $\iota(v) = \sum_{i\in E(v)} i$ the sum of the $x-$coordinates of the extremal points. In [7] and [8] we defined the weight of v by $w(v,q) = q^{\iota(v)}t^{e(v)}$ and considered the polynomials $w\left(A_{n,k}, q\right) = \sum_{v\in A_{n,k}} q^{\iota(v)}t^{e(v)}$. These polynomials are intimately connected with Rogers-Ramanujan type theorems.

In the present paper we consider only the case $q=1$ and study the polynomials

$$a(n,k,t) = \sum_{v\in A_{n,k}} t^{e(v)} \tag{2.1}$$

in more detail. It is obvious that $\deg\left(a(n,k,t)\right) = \left\lfloor \frac{n}{2} \right\rfloor$ for $k>1$ because the maximal degree is obtained by the path $UDUD\cdots$.

If we set $\binom{n}{k} = 0$ for $n<0$ it follows from the results in [7] and [8] that for $k\geq 1$ these polynomials can be written in the following form:

$$a(n,k,t) = \sum_{j\in\mathbb{Z}} (-1)^j \sum_{\ell\geq |j|} \binom{\left\lfloor \frac{n+(k-2)j}{2} \right\rfloor}{\ell - j} \binom{\left\lfloor \frac{n+1-(k-2)j}{2} \right\rfloor}{\ell + j} t^\ell \tag{2.2}$$

For $t=1$ we have of course $a(n,k,1)=\left|A_{n,k}\right|$.

Remark 6

A direct proof that (2.2) implies

$$a(n,k,1)=\sum_{j\in\mathbb{Z}}(-1)^j\binom{n}{\left\lfloor\frac{n+(k+2)j}{2}\right\rfloor} \tag{2.3}$$

follows from the fact that $\left\lfloor\frac{n+kj}{2}\right\rfloor+\left\lfloor\frac{n+1-kj}{2}\right\rfloor=n.$

For

$$\sum_{\ell=|j|}^{\infty}\binom{\left\lfloor\frac{n+(k-2)j}{2}\right\rfloor}{\ell-j}\binom{\left\lfloor\frac{n+1-(k-2)j}{2}\right\rfloor}{\ell+j}=\sum_{\ell=|j|}^{\infty}\binom{\left\lfloor\frac{n+(k-2)j}{2}\right\rfloor}{\ell-j}\binom{n-\left\lfloor\frac{n+(k-2)j}{2}\right\rfloor}{n-\left\lfloor\frac{n+(k-2)j}{2}\right\rfloor-\ell-j}$$

$$=\sum_{i=-\infty}^{\infty}\binom{\left\lfloor\frac{n+(k-2)j}{2}\right\rfloor}{i}\binom{n-\left\lfloor\frac{n+(k-2)j}{2}\right\rfloor}{n-\left\lfloor\frac{n+(k-2)j}{2}\right\rfloor-i-2j}$$

$$=\sum_{i=-\infty}^{\infty}\binom{\left\lfloor\frac{n+(k-2)j}{2}\right\rfloor}{i}\binom{n-\left\lfloor\frac{n+(k-2)j}{2}\right\rfloor}{\left\lfloor\frac{n+1-(k+2)j}{2}\right\rfloor-i}=\binom{n}{\left\lfloor\frac{n+1-(k+2)j}{2}\right\rfloor}=\binom{n}{\left\lfloor\frac{n+(k+2)j}{2}\right\rfloor}.$$

For $n\le k$ we have $a(n,k,t)=\sum_{\ell\ge 0}\binom{\left\lfloor\frac{n}{2}\right\rfloor}{\ell}\binom{\left\lfloor\frac{n+1}{2}\right\rfloor}{\ell}t^{\ell}.$

The simplest special cases are

$a(n,1,t)=1,$

$a(n,2,t)=(1+t)^{\left\lfloor\frac{n}{2}\right\rfloor}.$

As a generalization of (1.5) we get

$$a(n,3,t)=F_{n+1}(1,t)=\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\binom{n-k}{k}t^k=\sum_{j\in\mathbb{Z}}(-1)^j\sum_{\ell\ge|j|}\binom{\left\lfloor\frac{n+j}{2}\right\rfloor}{\ell-j}\binom{\left\lfloor\frac{n+1-j}{2}\right\rfloor}{\ell+j}t^{\ell}. \tag{2.4}$$

The first terms of $a(n,3,t)$ are

$\left(1,1,1+t,1+2t,1+3t+t^2,1+4t+3t^2,1+5t+6t^2+t^3,1+6t+10t^2+4t^3,\cdots\right)$

To show that $a(n,3,t)=F_{n+1}(1,t)$ consider a path in $A_{n,1}$. If the next to the last point is not extremal then the path is the unique continuation of a path in $A_{n-1,1}$, if it is extremal then the last two steps are a peak or a valley and the rest of the path belongs to $A_{n-2,1}$. Therefore we have $a(n,1,t)=a(n-1,1,t)+ta(n-2,1,t)$. The initial values are $a(0,1,t)=a(1,1,t)=1$.

Remark 7

Proposition 1 also holds in this case as shown in [7].

It would be interesting to know if Proposition 2 also has a simple generalization.

There is a simple table $\left(c(n,j,t)\right)$ which reduces to (1.8) for $t=1$ such that

$$c(n,0,t)=\sum_{j=0}^{n}\binom{\left\lfloor\frac{n}{2}\right\rfloor}{j}\binom{\left\lfloor\frac{n+1}{2}\right\rfloor}{j}t^j.$$

Let

$$c(n,0,t)=c(n-1,0,t)+tc(n-1,1,t),$$

and let for even numbers $j>0$

$$c(n,j,t)=c(n-1,j-1,t)+tc(n-1,j+1,t)$$

and for odd numbers j

$$c(n,j,t)=c(n-1,j-1,t)+c(n-1,j+1,t).$$

The initial values are $c(0,j,t)=[j=0]$.

Then it is easily verified that

$$\begin{aligned} c(n,2k,t)&=\sum_{j\geq 0}\binom{\left\lfloor\frac{n}{2}\right\rfloor}{j+k}\binom{\left\lfloor\frac{n+1}{2}\right\rfloor}{j}t^j,\\ c(n,2k+1,t)&=\sum_{j\geq 0}\binom{\left\lfloor\frac{n}{2}\right\rfloor}{j}\binom{\left\lfloor\frac{n+1}{2}\right\rfloor}{j+k+1}t^j. \end{aligned} \tag{2.5}$$

The first terms are

$$\begin{pmatrix} 1 &&&&& \\ 1 & 1 &&&& \\ 1+t & 1 & 1 &&& \\ 1+2t & 2+t & 1 & 1 && \\ 1+4t+t^2 & 2+2t & 2+2t & 1 & 1 & \\ 1+6t+3t^2 & 3+6t+t^2 & 2+3t & 3+2t & 1 & 1 \end{pmatrix} \tag{2.6}$$

.

In analogy to the case of numbers we could now study the polynomials $c(n,j,t,2k+1)$ which satisfy the above relations together with $c(n,k+1,t,2k+1)=0.$ These polynomials satisfy $c(n,0,1,2k+1)=a(n,2k+1).$ For $k>1$ they are different from $a(n,2k+1,t).$

I do not want to go into details but only state their generating functions without proof.

Their generating function is given by

$$\sum_{n\geq 0} c(n,0,t,2k+1)x^n = \frac{c(k,x,t)}{d(k,x,t)}.$$

Here $d(n,x,t)$ satisfies

$$d(2n,x,t) = d(2n-1,x,t) - x^2 d(2n-2,x,t),$$
$$d(2n+1,x,t) = d(2n,x,t) - tx^2 d(2n-1,x,t)$$

with initial values $d(-1,x,t)=1$ and $d(0,x,t)=1-x.$

And $c(n,x,t)$ is defined by the same recurrence

$$c(2n,x,t) = c(2n-1,x,t) - x^2 c(2n-2,x,t),$$
$$c(2n+1,x,t) = c(2n,x,t) - tx^2 c(2n-1,x,t),$$

with initial values $c(0,x,t)=c(1,x,t)=1.$

Note that $d(n,x,1) = F_{n+2}(1,-x^2) - xF_{n+1}(1,-x^2)$ and $c(n,x,1) = F_{n+1}(1,-x^2).$

2.2. Generating functions for the polynomials $a(n,k,t)$.

We now want to determine the generating functions for the polynomials $a(n,k,t).$

Again we get different results for even and odd $k.$

Theorem 1

Let

$$\Phi_n(x,t) = F_n(1+(1-t)x^2, -x^2) = \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n-1-k}{k} \left(1+(1-t)x^2\right)^{n-1-2k} (-x^2)^k. \quad (2.7)$$

Then for $k \geq 0$

$$\sum_{n\geq 0} a(n,2k+1,t)x^n = \frac{\Phi_k(x,t) - x^2\Phi_{k-1}(x,t)}{\Phi_{k+1}(x,t) - x(x+1)\Phi_k(x,t) + x^3\Phi_{k-1}(x,t)}. \quad (2.8)$$

The denominator $\Phi_{k+1}(x,t) - x(x+1)\Phi_k(x,t) + x^3\Phi_{k-1}(x,t)$ is a polynomial in x of degree $2k$ and a polynomial in t of degree $k.$ As a function of t the first terms are

$$\Phi_{k+1}(x,t) - x(x+1)\Phi_k(x,t) + x^3\Phi_{k-1}(x,t) = 1 - x - ktx^2 + (1-x)x^3t\sum_{j=0}^{k-2}(k-1-j)x^{2j} + t^2x^4B(k,x,t)$$

for some polynomial $B(k,x,t)$.

Theorem 2

Let

$$\Lambda_n(x,t) = L_n(1+(1-t)x^2, -x^2) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n-k}{k} \frac{n}{n-k} \left(1+(1-t)x^2\right)^{n-2k} (-x^2)^k. \quad (2.9)$$

Then for $k \geq 1$

$$\sum_{n\geq 0} a(n,2k,t)x^n = \frac{(1+x)\Phi_k(x,t) - x^2(1+(1-t)x)\Phi_{k-1}(x,t)}{\Lambda_k(x,t) - x^2\Lambda_{k-1}(x,t)}. \quad (2.10)$$

The denominator $\Lambda_k(x,t) - x^2\Lambda_{k-1}(x,t)$ is a polynomial in x of degree $2k$ and a polynomial in t of degree $k.$ As a function in t the first terms are

$$\Lambda_k(x,t) - x^2\Lambda_{k-1}(x,t) = 1 - x^2 - ktx^2 - t\sum_{j=2}^{k} x^{2j} + t^2x^4C(k,x,t).$$

For $t = 1$ we get again (1.18) and (1.19).

This follows from

$$\Phi_k(x,1) - x^2\Phi_{k-1}(x,1) = F_k(1,-x^2) - x^2F_{k-1}(1,-x^2) = F_{k+1}(1,-x^2),$$

$$\Phi_{k+1}(x,1) - x(x+1)\Phi_k(x,1) + x^3\Phi_{k-1}(x,1) = F_{k+1}(1,-x^2) - (x+x^2)F_k(1,-x^2) + x^3F_{k-1}(1,-x^2)$$
$$= \left(F_{k+1}(1,-x^2) - x^2F_k(1,-x^2)\right) - x\left(F_k(1,-x^2) - x^2F_{k-1}(1,-x^2)\right) = F_{k+2}(1,-x^2) - xF_{k+1}(1,-x^2),$$

$$(1+x)\Phi_k(x,1) - x^2(1+(1-1)x)\Phi_{k-1}(x,1) = (1+x)F_k(1,-x^2) - x^2F_{k-1}(1,-x^2)$$
$$= F_{k+1}(1,-x^2) + xF_k(1,-x^2)$$

and

$$\Lambda_k(x,1) - x^2\Lambda_{k-1}(x,1) = L_{k+1}(1,-x^2).$$

Before we prove these theorems let us consider some special cases.

The simplest special cases are

$$\sum_{n\geq 0} a(n,2,t)x^n = \frac{1+x}{(1+(1-t)x^2) - 2x^2} = \frac{1+x}{1-(1+t)x^2} = \sum_{n\geq 0}(1+t)^{\left\lfloor \frac{n}{2} \right\rfloor}x^n$$

and

$$\sum_{n\geq 0} a(n,3,t)x^n = \frac{1}{(1+(1-t)x^2) - x(x+1)} = \frac{1}{1-x-tx^2} = \sum_{n\geq 0} F_{n+1}(1,t)x^n.$$

Let us also consider $a(n,4,t)$.

The first terms are

$$1, 1, 1+t, 1+2t, 1+4t+t^2, 1+5t+3t^2, 1+7t+9t^2+t^3, 1+8t+14t^2+4t^3, \cdots.$$

Here we have

$$\sum_{n\geq 0} a(n,4,t)x^n = \frac{1+x-tx^2}{1-x^2-2tx^2-tx^4+t^2x^4} = \frac{1-tx^2}{1-x^2-2tx^2-tx^4+t^2x^4} + x\frac{1}{1-x^2-2tx^2-tx^4+t^2x^4}$$
$$= \sum_{n\geq 0} a(2n,4,t)x^{2n} + x\sum_{n\geq 0} a(2n+1,4,t)x^{2n}.$$

Since $\dfrac{1-tx^2}{1-x^2-2tx^2-tx^4+t^2x^4} - 1 = \dfrac{x^2+tx^2(1-tx^2)+tx^4}{1-x^2-2tx^2-tx^4+t^2x^4}$

we get by comparing coefficients and setting $a(-1,4,t) = 0$

$$a(2n+3,4,t) = a(2n+2,4,t) + ta(2n+1,4,t)$$
$$a(2n+2,4,t) = a(2n+1,4,t) + ta(2n,4,t) + ta(2n-1,4,t)$$

Thus the polynomials $a(n,4,t)$ can easily be computed.

Let now $b(n,t) = a(2n,4,t^2) + ta(2n-1,4,t^2) = \sum_{k=0}^{2n} b_{n,k} t^k$. Then the table $\left(b_{n,k}\right)$ begins as follows

$$\begin{pmatrix} & & & & & 1 & & & & & \\ & & & & 1 & 1 & 1 & & & & \\ & & & 1 & 1 & 4 & 2 & 1 & & & \\ & & 1 & 1 & 7 & 5 & 9 & 3 & 1 & & \\ & 1 & 1 & 10 & 8 & 26 & 14 & 16 & 4 & 1 & \\ 1 & 1 & 13 & 11 & 52 & 34 & 70 & 30 & 25 & 5 & 1 \end{pmatrix} \tag{2.11}$$

The sum of the rows is $b(n,1) = a(2n,4,1) + a(2n-1,4,1) = 2\cdot 3^{n-1} + 3^{n-1} = 3^n$ and the alternating sum is

$$\sum_{k=0}^{2n} (-1)^k b_{n,k} = a(2n,4,1) - a(2n-1,4,1) = 2\cdot 3^{n-1} - 3^{n-1} = 3^{n-1}.$$

The table $\left(b_{n,k}\right)$ satisfies

$$\begin{aligned} b_{n,2k+1} &= b_{n-1,2k-1} + b_{n-1,2k}, \\ b_{n,2k} &= b_{n-1,2k-2} + 2b_{n-1,2k-1} + b_{n-1,2k}. \end{aligned} \tag{2.12}$$

2.3. Proofs

Proof of Theorem 1

[7] Theorem 3.1 implies that setting $d(k,x,t) = \Phi_{k+1}(x,t) - x(x+1)\Phi_k(x,t) + x^3\Phi_{k-1}(x,t)$ we have

$$d(k,x,t)\sum_{n\geq 0} a(n,2k+1,t)x^n = c(k,x,t) \tag{2.13}$$

for some polynomial $c(k,x,t)$ of degree $\deg c(k,x,t) < \deg d(k,x,t) = 2k.$

Let us recall the proof.

The polynomials $\Phi_n(x,t) = F_n(1+(1-t)x^2, -x^2)$ satisfy

$$\Phi_n(x,t) = (1 + (1-t)x^2)\Phi_{n-1}(x,t) - x^2\Phi_{n-2}(x,t). \tag{2.14}$$

Therefore the polynomials

$$\phi_n(x,t) = \Phi_{n+1}(x,t) - x(x+1)\Phi_n(x,t) + x^3\Phi_{n-1}(x,t) \tag{2.15}$$

satisfy the same recurrence with initial values $\phi_0(x,t) = 1 - x$ and $\phi_1(x,t) = 1 - x - tx^2$.

Since we did not define $\Phi_{-1}(x,t)$ we would have to consider the initial values $\phi_1(x,t)$ and $\phi_2(x,t)$. But it is easy to verify that by choosing $\phi_0(x,t) = 1 - x$ we get the same result.

Let now $w_n = a(n, 2k+1, t) = w\left(A_{n,2k+1}\right)$.

By symmetry it is clear that w_n is also the weight of all paths which start from $(0,-1)$ and end in 0 or -1. Therefore the weight of all paths which begin with a down- step D is w_{n-1}.

Let $w^+_{n,j}$ be the weight of all paths of $A_{n,2k+1}$ which begin with at least j up-steps U.

Then

$w^+_{n,1} + w_{n-1} = w_n$ or

$$w^+_{n,1} = w_n - w_{n-1}. \tag{2.16}$$

The paths which begin with j up-steps U begin with one of the following steps: $U^{j+1}, U^jDU, U^jD^2U, \cdots, U^jD^{j-1}U, U^jD^j$. If we remove the part $U^\ell D^\ell$ which contributes the factor t to the weight we get

$$w^+_{n,j} = w^+_{n,j+1} + t\sum_{\ell=0}^{j-2} w^+_{n-2\ell-2,j-\ell} + tw_{n-2j}. \tag{2.17}$$

Thus $w^+_{n,1} = w^+_{n,2} + tw_{n-2}$.

This gives

$$w^+_{n,2} = w^+_{n,1} - tw_{n-2} = w_n - w_{n-1} - tw_{n-2}. \tag{2.18}$$

From

$$w^+_{n,j+1} = w^+_{n,j} - t\sum_{\ell=0}^{j-2} w^+_{n-2\ell-2,j-\ell} - tw_{n-2j}$$

we get for $j > 2$

$$w^+_{n-2,j} - w^+_{n-2,j-1} = -t\sum_{\ell=0}^{j-3} w^+_{n-2\ell-4,j-1-\ell} - tw_{n-2j}$$

Therefore

$$w^+_{n,j+1} - w^+_{n,j} + tw^+_{n-2,j} = w^+_{n-2,j} - w^+_{n-2,j-1}$$

or

$$w^+_{n,j+1} = w^+_{n,j} + (1-t)w^+_{n-2,j} - w^+_{n-2,j-1}. \tag{2.19}$$

Let now A be the operator defined by $A^j w_n = w_{n-j}$ and let $w^+_{n,j+1} = h_j(A)w_n$.

Then by (2.16) and (2.18) we have

$$h_0(A) = 1 - A, \quad h_1(A) = 1 - A - tA^2$$

and by (2.19)

$$h_j(A) = \left(1 + (1-t)A^2\right)h_{j-1}(A) - A^2 h_{j-2}(A).$$

Since $w^+_{n,k+1} = 0$ we see that

$$h_k(A)a(n,2k+1,t) = w^+_{n,k+1} = 0.$$

Since $h_k(x)$ satisfies the same recurrence and initial values as $\phi_k(x,t)$ we get (2.13).

It only remains to determine $c(k,x,t)$.

From the definition of $a(n,2k+1,t)$ it is obvious that $a(n,2k+1,t) = a(n,2k+3,t)$ for $n \le 2k+1$.

Let $A(k,x,t) = \sum_{n\ge 0} a(n,2k+1,t)x^n = \dfrac{c(k,x,t)}{d(k,x,t)}$.

The denominator $d(k,x,t)$ satisfies the recurrence
$d(k,x,t) - (1+(1-t)x^2)d(k-1,x,t) + x^2 d(k-2,x,t) = 0.$

Now we have

$$\begin{aligned}
&c(k,x,t) - (1+(1-t)x^2)c(k-1,x,t) + x^2 c(k-2,x,t)\\
&= d(k,x,t)A(k,x,t) - (1+(1-t)x^2)d(k-1,x,t)A(k-1,x,t) + x^2 d(k-2,x,t)A(k-2,x,t)\\
&= \left(d(k,x,t) - (1+(1-t)x^2)d(k-1,x,t) + x^2 d(k-2,x,t)\right)A(k,x,t)\\
&+(1+(1-t)x^2)d(k-1,x,t)\left(A(k,x,t) - A(k-1,x,t)\right) - x^2 d(k-2,x,t)\left(A(k,x,t) - A(k-2,x,t)\right)\\
&= (1+(1-t)x^2)d(k-1,x,t)\left(A(k,x,t) - A(k-1,x,t)\right) - x^2 d(k-2,x,t)\left(A(k,x,t) - A(k-2,x,t)\right).
\end{aligned}$$

Since the coefficients of x^j for $0 \le j \le 2k-1$ of $A(k,x,t)$ and $A(k-1,x,t)$ are the same we see that $(1+(1-t)x^2)d(k-1,x,t)\Big(A(k,x,t)-A(k-1,x,t)\Big)$ and $x^2 d(k-2,x,t)\Big(A(k,x,t)-A(k-2,x,t)\Big)$ are polynomials where all coefficients of x^j for $0 \le j \le 2k-1$ vanish. From the first line we see that this is a polynomial of degree $< 2k$.

Therefore

$$c(k,x,t)-(1+(1-t)x^2)c(k-1,x,t)+x^2c(k-2,x,t)=0$$

for $k \ge 3$. Direct inspection shows that $c(1,x,t)=1, c(2,x,t)=1-tx^2$.

The uniquely determined polynomials satisfying this recurrence and initial values are $c(k,x,t)=\Phi_k(x,t)-x^2\Phi_{k-1}(x,t)$.

Proof of Theorem 2

[7] Theorem 4.1 implies that with $d(k,x,t)=\Lambda_k(x,t)-x^2\Lambda_{k-1}(x,t)$ we have

$d(k,x,t)\sum_{n\ge 0} a(n,2k,t)x^n = c(k,x,t)$ for some polynomial $c(k,x,t)$ of degree $\deg c(k,x,t) < \deg d(k,x,t) = 2k$.

Let us also recall this proof. It is somewhat more complicated because we now have no symmetry for the weight.

Let $w_{n,k}^{+}$ be the weight of all paths which start with k up-steps U, $w_{n,k}^{-}$ the weight of all paths which start with k down-steps D and $w_{n,k}=w_{n,k}^{+}+w_{n,k}^{-}$. Then we have $w_{n,1}=w_n$,

$$w_{n,1}^{+}=w_{n,2}^{+}+tw_{n-2},$$
$$w_{n,1}^{-}=w_{n,2}^{-}+w_{n-2}$$

and thus

$$w_{n,2}=w_n-(1+t)w_{n-2}.$$

For the paths which start with UU there are the following possibilities: they can start with $UUU, UUDU, UUDD$. Therefore we get

$$w_{n,2}^{+}=w_{n,3}^{+}+tw_{n-2,2}^{+}+tw_{n-4}=w_{n,3}^{+}+t\left(w_{n-2,1}^{+}-tw_{n-4}\right)+tw_{n-4}=w_{n,3}^{+}+tw_{n-2,1}^{+}-t^2w_{n-4}+tw_{n-4}.$$

For the negative part we get the simpler formula

$w_{n,2}^{-}=w_{n,3}^{-}+tw_{n-2,1}^{-}$ because the valleys on height -1 are not extreme points.

Combining the last two formulae we get

$$w_{n,3} = w_n - w_{n-2} - 2tw_{n-2} + t^2 w_{n-4} - tw_{n-4}.$$

For $j > 3$ we have in the same way as above

$$w_{n,j} = w_{n,j+1} + t\sum_{\ell=0}^{j-1} w_{n-2\ell-2,j-\ell}.$$

We now define polynomials $g_n(x)$ as follows:

$$g_n(x) = g_{n-1}(x) - t\sum_{\ell=1}^{n} x^{2\ell} g_{n-\ell}(x)$$

with initial values

$g_0(x) = 2$ and $g_1(x) = 1 - (1+t)x^2 = \Lambda_1(x,t) - x^2\Lambda_0(x,t).$

Then $g_n(x) = \Lambda_n(x,t) - x^2\Lambda_{n-1}(x,t)$ for $n \geq 1.$

This is true for $1 \leq n \leq 2.$ Let it be true for $\ell < n.$

Then

$$g_n(x) = \Lambda_{n-1}(x,t) - x^2\Lambda_{n-2}(x,t) - t\sum_{\ell=1}^{n-1} x^{2\ell}\left(\Lambda_{n-\ell}(x,t) - x^2\Lambda_{n-\ell-1}(x,t)\right) - 2tx^{2n}$$
$$= \Lambda_{n-1}(x,t) - x^2\Lambda_{n-2}(x,t) - tx^2\Lambda_{n-1}(x,t) = \Lambda_n(x,t) - x^2\Lambda_{n-1}(x,t).$$

We already know that

$$w_{n,1} = g_0(\mathrm{A})w_n$$
$$w_{n,2} = g_1(A)w_n$$
$$w_{n,3} = g_2(A)w_n$$

Therefore we have $0 = w_{n,k+1} = g_k(A)w_n.$

Thus we know that with $d(k,x,t) = \Lambda_k(x,t) - x^2\Lambda_{k-1}(x,t)$ we have

$d(k,x,t)\sum_{n\geq 0} a(n,2k,t)x^n = c(k,x,t)$ for some polynomial $c(k,x,t)$ of degree $\deg c(k,x,t) < \deg d(k,x,t) = 2k.$ We now must determine $c(k,x,t).$

From the definition of $a(n,2k,t)$ it is obvious that $a(n,2k,t) = a(n,2k+2,t)$ for $n \leq 2k.$

Let $B(k,x,t) = \sum_{n\geq 0} a(n,2k,t)x^n = \dfrac{c(k,x,t)}{d(k,x,t)}.$

The denominator $d(k,x,t)$ satisfies the recurrence

$$d(k,x,t) - (1 + (1-t)x^2)d(k-1,x,t) + x^2 d(k-2,x,t) = 0.$$

Now we have

$$c(k,x,t)-(1+(1-t)x^2)c(k-1,x,t)+x^2c(k-2,x,t)$$
$$=d(k,x,t)B(k,x,t)-(1+(1-t)x^2)d(k-1,x,t)B(k-1,x,t)+x^2d(k-2,x,t)B(k-2,x,t)$$
$$=(1+(1-t)x^2)d(k-1,x,t)\Big(B(k,x,t)-B(k-1,x,t)\Big)-x^2d(k-2,x,t)\Big(B(k,x,t)-B(k-2,x,t)\Big).$$

From the first line we see that this is a polynomial of degree $<2k$.

Since the coefficients of x^j for $0\le j\le 2k-2$ of $B(k,x,t)$ and $B(k-1,x,t)$ are the same we see that

$$\text{coeff}[x^j]\Big((1+(1-t)x^2)d(k-1,x,t)\Big(B(k,x,t)-B(k-1,x,t)\Big)\Big)=0$$

for $0\le j\le 2k-2$. The same argument gives

$$\text{coeff}[x^j]\Big(x^2d(k-2,x,t)\Big(B(k,x,t)-B(k-2,x,t)\Big)\Big)=0$$

for $0\le j\le 2k-2$.

Now we must determine the coefficient of x^{2k-1}. We have

$$\text{coeff}[x^{2k-1}]\Big((1+(1-t)x^2)d(k-1,x,t)\Big(B(k,x,t)-B(k-1,x,t)\Big)\Big)=a(2k-1,2k,t)-a(2k-1,2k-2,t)$$

and

$$\text{coeff}[x^{2k-1}]\Big(x^2d(k-2,x,t)\Big(B(k,x,t)-B(k-2,x,t)\Big)\Big)=a(2k-3,2k,t)-a(2k-3,2k-4,t).$$

It suffices to show that
$a(2k-1,2k,t)-a(2k-1,2k-2,t)=a(2k-3,2k,t)-a(2k-3,2k-4,t)=t.$

We know that $a(2k-1,2k,t)-a(2k-1,2k-2,t)$ is the weight of all paths from $(0,0)$ to $(2k-1,-1)$ which touch the boundary $y=\pm k$. The uniquely determined path with this property is D^kU^{k-1} with weight t. On the other hand $a(2k-3,2k,t)-a(2k-3,2k-4,t)$ is the weight of all paths from $(0,0)$ to $(2k-3,-1)$ which touch $y=\pm(k-1)$. There is again a unique such path $D^{k-1}U^{k-2}$.

Therefore we see that

$$c(k,x,t)-(1+(1-t)x^2)c(k-1,x,t)+x^2c(k-2,x,t)=0$$

for $k\ge 3$. Direct inspection shows that $c(1,x,t)=1+x$ and $c(2,x,t)=1+x-tx^2$.

The uniquely determined polynomials satisfying this recurrence and initial values are
$c(k,x,t)=(1+x)\Phi_k(x,t)-x^2(1+(1-t)x)\Phi_{k-1}(x,t).$

2.4. Generating functions of the coefficients

Let us now consider the number of paths in $A_{n,k}$ with j extremal points. For each k there is precisely one path $DUDUD\cdots$ with no extremal points.

For $j>0$ the shortest path with j extremal points is $UDUD\cdots UD=(UD)^j$ of length $2j$.

Therefore we can write the generating function of the polynomials $a(n,k+2,t)$ for $k\geq 0$ in the form

$$\sum_{n\geq 0} a(n,k+2,t)x^n = A_0(x,k)+A_1(x,k)x^2t+A_2(x,k)x^4t^2+\cdots \tag{2.20}$$

with $A_0(x,k)=\dfrac{1}{1-x}$.

Experience with small values of k led to

Conjecture 1

For $j\geq 1$ there exist polynomials $v_j(x,k)$ with $\deg v_j(x,k)=kj$ and positive coefficients such that

$$A_j(x,k)=\frac{v_j(x,k)}{(1-x)^{j+1}(1+x)^j}. \tag{2.21}$$

These polynomials have the surprising property that

$$v_j(1,k)=(k+1)^j, \tag{2.22}$$

$$v_j(-1,2k+1)=0 \tag{2.23}$$

and more precisely $v_j(x,2k+1)$ is divisible by $(1+x)^j$ and

$$v_j(-1,2k)=(2k+1)^{j-1}. \tag{2.24}$$

Let us first consider some special cases.

For $k=0$ we get

$$\sum_{n\geq 0} a(n,2,t)x^n = \frac{1+x}{1-(1+t)x^2} = \sum_{j\geq 0} t^j \frac{x^{2j}}{(1-x)^{j+1}(1+x)^j}.$$

Therefore

$$v_j(x,0)=1. \tag{2.25}$$

For $k=1$ we see that

$$\sum_{n\ge 0} a(n,3,t)x^n = \frac{1}{1-x-tx^2} = \sum_{j\ge 0} t^j \frac{x^{2j}}{(1-x)^{j+1}}$$

implies

$$v_j(x,1) = (1+x)^j. \tag{2.26}$$

The next case is more interesting.

We know already that

$$\sum_{n\ge 0} a(n,4,t)x^n = \frac{1+x-tx^2}{1-x^2-2tx^2-tx^4+t^2x^4}.$$

Let us suppose that we have an expansion of the form

$$\sum_{n\ge 0} a(n,4,t)x^n = \frac{1}{1-x} + \frac{x^2 v_1(x,2)}{(1-x)^2(1+x)} t + \frac{x^4 v_2(x,2)}{(1-x)^3(1+x)^2} t^2 + \frac{x^6 v_3(x,2)}{(1-x)^4(1+x)^3} t^3 + \cdots. \tag{2.27}$$

Multiplying both sides by $1-x^2-2tx^2-tx^4+t^2x^4$ and comparing coefficients of t^j we get

$$(1-x^2)\frac{x^{2j}v_j(x,2)}{(1-x)^{j+1}(1+x)^j} - (2x^2+x^4)\frac{x^{2j-2}v_{j-1}(x,2)}{(1-x)^j(1+x)^{j-1}} + x^4\frac{x^{2j-4}v_{j-2}(x,2)}{(1-x)^{j-1}(1+x)^{j-2}} = 0$$

or equivalently

$$v_j(x,2) = (2+x^2)v_{j-1}(x,2) - (1-x^2)v_{j-2}(x,2). \tag{2.28}$$

The initial values are $v_0(x,2)=1$ and $v_1(x,2)=1+x+x^2$ by direct computation.

So if there is an expansion of the form (2.27) then $v_j(x,2)$ must be given by (2.28). But if we define $v_j(x,2)$ satisfying (2.28) and the given initial values then we get

$$\begin{aligned}
&\left(1-x^2-2tx^2-tx^4+t^2x^4\right)\sum_{n\ge 0} a(n,4,t)x^n \\
&= \left(1-x^2-2tx^2-tx^4\right)\frac{1}{1-x} + \left(1-x^2\right)\frac{x^2(1+x+x^2)}{(1-x)^2(1+x)}t \\
&= \frac{1-x^2-2tx^2-tx^4+tx^2+tx^3+tx^4}{1-x} = \frac{1-x^2-tx^2+tx^3}{1-x} = 1+x-tx^2.
\end{aligned}$$

Thus (2.27) is in fact true.

If we compute the polynomials $v_j(x,2)=\sum_{\ell=0}^{2j} c(j,\ell)x^\ell$ we get the following array of the coefficients $c(j,\ell)$

1										
1	1	1								
1	2	4	1	1						
1	3	9	5	7	1	1				
1	4	16	14	26	8	10	1	1		
1	5	25	30	70	34	52	11	13	1	1

(2.28) implies the following formulae:

$c(-1,\ell)=c(-2,\ell)=0, c(0,\ell)=[\ell=0],\ c(1,\ell)=[\ell\le 2],$

$c(j,\ell)=2c(j-1,\ell)-c(j-2,\ell)+c(j-1,\ell-2)+c(j-2,\ell-2).$

Surprisingly this is almost the same array as (2.11) . More precisely

$v_j(x,2)=x^{2j}b\left(j,\frac{1}{x}\right)$ since both sides satisfy the same recurrence (2.28).

Thus we have $v_j(1,2)=\sum_{\ell=0}^{2j} c(j,\ell)=3^j$ and $v_j(-1,2)=\sum_{\ell=0}^{2j}(-1)^\ell c(j,\ell)=3^{j-1}.$

Let us also mention that $v_j(x,3)$ satisfies

$v_j(x,3)=\left(x^3+x+2\right)v_{j-1}(x,3)+(1-x)(1+x)^2 v_{j-2}(x,3)$

with $v_0(x,1)=1$ and $v_1(x,1)=(1+x)\left(1+x^2\right).$

For $v_j(x,4)$ we get

$v_j(x,4)=\left(x^4+x^2+3\right)v_{j-1}(x,4)+\left(2x^4+x^2-3\right)v_{j-2}(x,4)+\left(1-x^2\right)^2 v_{j-3}(x,4).$

If we let $k\to\infty$ we get

$$a(n,t)=\sum_{\ell\ge 0}\binom{\left\lfloor\frac{n}{2}\right\rfloor}{\ell}\binom{\left\lfloor\frac{n+1}{2}\right\rfloor}{\ell}t^\ell. \tag{2.29}$$

Here we have

$$\sum_{n\geq 0} a(n,t)x^n = \frac{1}{1-x} + \frac{x^2 r_0(x)}{(1-x)^3(1+x)}t + \frac{x^4 r_1(x)}{(1-x)^5(1+x)^3}t^2 + \frac{x^6 r_2(x)}{(1-x)^7(1+x)^5}t^3 + \cdots \tag{2.30}$$

where

$$r_j(x) = \sum_{\ell=0}^{j} \binom{j}{\ell}^2 x^{2\ell} + \sum_{\ell=1}^{j} jN_{j,\ell} x^{2\ell-1} = \sum_{\ell=0}^{j} \binom{j}{\ell}^2 x^{2\ell} + \sum_{\ell=1}^{j} \binom{j}{\ell}\binom{j}{\ell-1} x^{2\ell-1}. \tag{2.31}$$

The numbers $N_{n,k} = \binom{n}{k-1}\binom{n}{k}\frac{1}{n}$ are the Narayana numbers (cf. OEIS A001263).

The coefficient table of $\left(r_j(x)\right)_{j\geq 0}$ is

$$\begin{array}{ccccccccc} &&&&1&&&& \\ &&&1&1&1&&& \\ &&1&2&4&2&1&& \\ &1&3&9&9&9&3&1& \\ 1&4&16&24&36&24&16&4&1 \end{array} \tag{2.32}$$

It should be noted that (2.32) consists of the even numbered rows of OEIS A088855.

Note that (2.30) is equivalent with

$$(1-x)^2\left(1-x^2\right)^{2k-1} \sum_{n\geq 0} \binom{\left\lfloor \frac{n+2k}{2} \right\rfloor}{k}\binom{\left\lfloor \frac{n+1+2k}{2} \right\rfloor}{k} x^n = r_{k-1}(x) \tag{2.33}$$

for all k.

To show this identity we make use of the identities

$$(1-x)^{2k+1} \sum_{n\geq 1} \binom{n+k-1}{k}\binom{n+k}{k} x^{n-1} = \sum_{j=1}^{k} \binom{k-1}{j-1}\binom{k+1}{j} x^{j-1} \tag{2.34}$$

and

$$(1-x)^{2k+1}\sum_{n\ge 0}\binom{n+k}{k}^2 x^n = \sum_{j=0}^{k}\binom{k}{j}^2 x^j. \tag{2.35}$$

These imply

$$\left(1-x^2\right)^{2k+1}\sum_{n\ge 0}\binom{\left\lfloor \frac{n+2k}{2}\right\rfloor}{k}\binom{\left\lfloor \frac{n+1+2k}{2}\right\rfloor}{k}x^n$$
$$=\left(1-x^2\right)^{2k+1}\sum_{n\ge 1}\binom{n+k-1}{k}\binom{n+k}{k}x^{2n-1}+\left(1-x^2\right)^{2k+1}\sum_{k\ge 0}\binom{n+k}{k}^2 x^{2n}$$

$$=\sum_{j=0}^{k}\binom{k}{j}^2 x^{2j}+\sum_{j=1}^{k}\binom{k-1}{j-1}\binom{k+1}{j}x^{2j-1}=(1+x)^2 r_{k-1}(x)$$

by observing that

$$(1+x)^2\left(\sum_{j=0}^{k-1}\binom{k-1}{j}^2 x^{2j}+\sum_{j=1}^{k-1}\binom{k-1}{j-1}\binom{k-1}{j}x^{2j-1}\right)=\sum_j\binom{k-1}{j}^2x^{2j}+2\sum_j\binom{k-1}{j-1}^2x^{2j-1}+\sum_j\binom{k-1}{j-1}^2x^{2j}$$
$$+\sum_j\binom{k-1}{j-1}\binom{k-1}{j}x^{2j-1}+2\sum_j\binom{k-1}{j-1}\binom{k-1}{j}x^{2j}+\sum_j\binom{k-1}{j-2}\binom{k-1}{j-1}x^{2j-1}$$
$$=\sum_j\left(\binom{k-1}{j-1}+\binom{k-1}{j}\right)^2x^{2j}+\sum_j\binom{k-1}{j-1}\left(2\binom{k-1}{j-1}+\binom{k-1}{j}+\binom{k-1}{j-2}\right)$$
$$=\sum_j\binom{k}{j}^2x^{2j}+\sum_j\binom{k-1}{j-1}\binom{k+1}{j}x^{2j-1}.$$

To prove (2.34) and (2.35) we show more generally that for $m \geq 0$

$$\left(1-x\right)^{2k+1}\sum_{n\ge 0}\binom{n+k}{k}\binom{n+k-m}{k}x^{n-m}=\sum_{j=0}^{k}\binom{k-m}{j}\binom{2k-j}{k}(-1)^j(1-x)^j$$
$$=\sum_{j=m}^{k}\binom{k-m}{j-m}\binom{k+m}{j}x^{j-m}.$$

Since $\sum_{n\ge 0}\binom{n+k}{k}x^n=\dfrac{1}{(1-x)^{k+1}}$

we get

$$\sum_{n\geq 0}\binom{n+k}{k}\binom{n+k-m}{k}x^{n-m}=\frac{1}{k!}D^k\frac{x^{k-m}}{(1-x)^{k+1}}=\frac{1}{k!}D^k\frac{\Big(1-(1-x)\Big)^{k-m}}{(1-x)^{k+1}}=\frac{1}{k!}D^k\sum_{j=0}^{k-m}(-1)^j\binom{k-m}{j}\Big(1-x\Big)^{j-}$$

$$=\sum_{j=0}^{k-m}(-1)^j\binom{k-m}{j}\binom{2k-j}{k}\Big(1-x\Big)^{j-2k-1}$$

It remains to show that

$$\sum_{j=0}^{k-m}(-1)^j\binom{k-m}{j}\binom{2k-j}{k}\Big(1-x\Big)^j=\sum_{j=m}^{k}\binom{k-m}{j-m}\binom{k+m}{j}x^{j-m}. \tag{2.36}$$

This follows by comparing coefficients in two different expansions of $(1+z)^{k+m}(x+z)^{k-m}$.

On the one hand we have

$$(1+z)^{k+m}(x+z)^{k-m}=(1+z)^{k+m}(x-1+1+z)^{k-m}=\sum_j\binom{k-m}{j}(x-1)^j(1+z)^{2k-j}$$

$$=\sum_{j,\ell}\binom{k-m}{j}(x-1)^j\binom{2k-j}{\ell}z^\ell$$

On the other hand we get

$$(1+z)^{k+m}(x+z)^{k-m}=\sum_{j,\ell}\binom{k+m}{j}\binom{k-m}{\ell}x^\ell z^{j+k-m-\ell}.$$

Comparing the coefficients of z^k in both sums we get (2.36).

Remark 8

For $m=0$ identity (2.36) reduces to

$$\sum_{j=0}^{k}(-1)^j\binom{k}{j}\binom{2k-j}{k}\Big(1-x\Big)^j=\sum_{j=0}^{k}\binom{k}{j}^2x^j. \tag{2.37}$$

It is mentioned in OEIS A063007 without proof. A combinatorial proof has been given by H.S. Wilf [21], p. 117. The above proof is inspired by the paper [19] by Jocelyn Quaintance, which contains tables of seven unpublished manuscript notebooks of H. W. Gould from 1945 – 1990. Similar identities can be found in [4] and the literature cited there.

A more general identity of this sort has been proved combinatorially in [4]. It can equivalently be formulated as

$$\sum_{j=0}^{n}\binom{n}{j}\binom{n+2m+x}{j+m}z^j=\sum_{j=0}^{n}\binom{n}{j}\binom{2n+2m+x-j}{n+m}(z-1)^j \tag{2.38}$$

and reduces to (2.36) for $\left(n,x\right) \to (k-m,0)$.

Since there seems to be some interest in such identities I will give another proof of (2.33).
We first show by induction that

$$b(n,j,x) := (1-x)^{k+j+1}\frac{D^j}{j!}\frac{x^n}{(1-x)^{k+1}} = \sum_{i=0}^{n}\binom{j+k-n}{k-i}\binom{n}{i}x^i. \tag{2.39}$$

Observe that $b(n,0,x) = (1-x)^{k+1}\dfrac{x^n}{(1-x)^{k+1}} = x^n = \sum_{i=0}^{n}\binom{j+k-n}{k-i}\binom{n}{i}x^i$

because $\binom{k-n}{k-i} = [i=n]$ for $i \le n$. We also have

$b(0,j,x) = (1-x)^{k+j+1}\dfrac{D^j}{j!}\dfrac{1}{(1-x)^{k+1}} = \dbinom{j+k}{k}.$

Since $D^j xf(x) = xD^j f(x) + jD^{j-1}f(x)$ the sequence $\left(b(n,j,x)\right)_{n\ge 0, j\ge 0}$ satisfies

$b(n,j,x) = xb(n-1,j,x) + (1-x)b(n-1,j-1,x).$

Comparing coefficients this is equivalent with

$$\binom{j+k-n}{k-i}\binom{n}{i} = \binom{j+k-n+1}{k-i+1}\binom{n-1}{i-1} + \binom{j+k-n}{k-i}\binom{n-1}{i} - \binom{j+k-n}{k-i+1}\binom{n-1}{i-1}.$$

This is clear because the right-hand side is

$$\binom{j+k-n+1}{k-i+1}\binom{n-1}{i-1} - \binom{j+k-n}{k-i+1}\binom{n-1}{i-1} + \binom{j+k-n}{k-i}\binom{n-1}{i}$$
$$= \binom{j+k-n}{k-i}\binom{n-1}{i-1} + \binom{j+k-n}{k-i}\binom{n-1}{i} = \binom{j+k-n}{k-i}\binom{n}{i}.$$

Thus for $m \le k$ (2.39) gives

$$(1-x)^{k+j+1}\frac{D^j}{j!}\frac{x^{k-m}}{(1-x)^{k+1}} = \sum_{i=0}^{k-m}\binom{j+m}{k-i}\binom{k-m}{i}x^i. \tag{2.40}$$

If we choose $m=0$ and $j=k$ we get

$$(1-x)^{2k+1}\sum_{n\geq 0}\binom{n+k}{k}^2 x^n = (1-x)^{2k+1}\frac{D^k}{k!}\sum_{n\geq 0}\binom{n+k}{k}x^{n+k}$$
$$= (1-x)^{2k+1}\frac{D^k}{k!}\frac{x^k}{(1-x)^{k+1}} = \sum_{j=0}^{k}\binom{k}{j}^2 x^j. \tag{2.41}$$

For $m=1$ and $j=k-2$ we get

$$(1-x)^{2k+1}\sum_{n\geq 0}\binom{n+k}{k}\binom{n+k-1}{k-2}x^{n+1} = (1-x)^{2k+1}\frac{D^{k-2}}{(k-2)!}\sum_{n\geq 0}\binom{n+k}{k}x^{n+k-1}$$
$$= (1-x)^{2k+1}\frac{D^{k-2}}{(k-2)!}\frac{x^{k-1}}{(1-x)^{k+1}} = \sum_{i=0}^{k-1}\binom{k-1}{i-1}\binom{k-1}{i}x^i. \tag{2.42}$$

Comparing coefficients gives

$$(1-x)^2\sum_{n\geq 0}\binom{\left\lfloor\frac{n+2k}{2}\right\rfloor}{k}\binom{\left\lfloor\frac{n+1+2k}{2}\right\rfloor}{k}x^n = (1-x)^2\sum_{n\geq 0}\binom{n+k}{k}^2 x^{2n} + (1-x)^2\sum_{n\geq 0}\binom{n+k}{k}\binom{n+k+1}{k}x^{2n+1}$$
$$= \sum_{n\geq 0}\binom{n+k-1}{k-1}^2 x^{2n} + \sum_{n\geq 0}\binom{n+k}{k}\binom{n+k-1}{k-2}x^{2n+1}.$$

Combining these identities we finally get the desired result (2.33).

Computing $b(n,j,x)$ in another way we get

$$(1-x)^{k+j+1}\frac{D^j}{j!}\frac{x^{k-m}}{(1-x)^{k+1}} = (1-x)^{k+j+1}\frac{D^j}{j!}\frac{\sum_{\ell=0}^{k-m}\binom{k-m}{\ell}(-1)^\ell(1-x)^\ell}{(1-x)^{k+1}}$$
$$= (1-x)^{k+j+1}\frac{D^j}{j!}\sum_{\ell=0}^{k-m}\binom{k-m}{\ell}(-1)^\ell(1-x)^{\ell-k-1} = \sum_{\ell=0}^{k-m}(-1)^\ell\binom{k-m}{\ell}\binom{k+j-\ell}{j}(1-x)^\ell.$$
(2.43)

Comparison of (2.40) and (2.43) gives

$$\sum_{i=0}^{k-m}\binom{j+m}{k-i}\binom{k-m}{i}x^i = \sum_{\ell=0}^{k-m}(-1)^\ell\binom{k-m}{\ell}\binom{k+j-\ell}{j}(1-x)^\ell. \tag{2.44}$$

For $m=0$ and $j=k$ this reduces again to (2.37).

For $m=1$ and $j=k-2$ we get

$$\sum_{i=0}^{k-1}\binom{k-1}{k-i}\binom{k-1}{i}x^i=\sum_{\ell=0}^{k-1}(-1)^\ell\binom{k-1}{\ell}\binom{2k-2-\ell}{k-2}(1-x)^\ell,$$

which by $k\to k+1$ can be written as

$$\begin{aligned}\sum_{i=0}^{k}N_{k,i}x^i&=\sum_{i=0}^{k}\binom{k}{i-1}\binom{k}{i}\frac{1}{k}x^i=\sum_{\ell=0}^{k}(-1)^\ell\frac{1}{k+1}\binom{k+1}{\ell}\binom{2k-\ell}{k}(1-x)^\ell\\&=\sum_{\ell=0}^{k}(-1)^\ell\frac{1}{n-\ell+1}\binom{2n-2\ell}{n-\ell}\binom{2k-\ell}{k}(1-x)^\ell.\end{aligned}\tag{2.45}$$

In this form it has been proved in [5], (2.2) and [17], (1.3).

Remark 9

A slight modification of the above proof gives the following $q-$analogue of (2.33):

$$\left(x;q^k\right)_2\left(qx^2;q\right)_{2k-1}\sum_{n\geq 0}\begin{bmatrix}\left\lfloor\frac{n+2k}{2}\right\rfloor\\k\end{bmatrix}\begin{bmatrix}\left\lfloor\frac{n+1+2k}{2}\right\rfloor\\k\end{bmatrix}x^n=r_{k-1}(x,q)\tag{2.46}$$

with

$$r_n(x,q)=\sum_{j=0}^{2n}q^{\left\lfloor\frac{(j+1)^2}{4}\right\rfloor}\begin{bmatrix}n\\\left\lfloor\frac{j}{2}\right\rfloor\end{bmatrix}\begin{bmatrix}n\\\left\lfloor\frac{j+1}{2}\right\rfloor\end{bmatrix}x^j=\sum_{j=0}^{n}q^{2\binom{j+1}{2}}\begin{bmatrix}n\\j\end{bmatrix}^2x^{2j}+\sum_{j=1}^{n}q^{j^2}\begin{bmatrix}n\\j\end{bmatrix}\begin{bmatrix}n\\j-1\end{bmatrix}x^{2j-1}.\tag{2.47}$$

Here $\begin{bmatrix}n\\k\end{bmatrix}$ is $q-$binomial coefficient and $\left(x;q\right)_n=\prod_{j=0}^{n-1}\left(1-q^jx\right)$.

Comparing coefficients gives

$$\left(x;q^k\right)_2\sum_{n\geq 0}\begin{bmatrix}\left\lfloor\frac{n+2k}{2}\right\rfloor\\k\end{bmatrix}\begin{bmatrix}\left\lfloor\frac{n+1+2k}{2}\right\rfloor\\k\end{bmatrix}x^n=\sum_{n\geq 0}q^n\begin{bmatrix}n+k-1\\k-1\end{bmatrix}^2x^{2n}+\sum_{n\geq 0}q^{n+1}\begin{bmatrix}n+k\\k\end{bmatrix}\begin{bmatrix}n+k-1\\k-2\end{bmatrix}x^{2n+1}.$$

Let D_q be the $q-$differential operator defined by $D_q f(x)=\frac{f(x)-f(qx)}{(1-q)x}$ and define polynomials $b(n,j,x,q)$ by $b(n,j,x,q)=\left(x;q\right)_{k+j+1}\frac{D_q^j}{[j]!}\frac{x^n}{\left(x;q\right)_{k+1}}$.

Then

$$b(n,j,x,q)=\sum_{i=0}^{n}q^{i(j+i-n)}\begin{bmatrix}j+k-n\\k-i\end{bmatrix}\begin{bmatrix}n\\i\end{bmatrix}x^i. \tag{2.48}$$

Since $D_q^j xf(x)=q^j xD_q^j f(x)+[j]D_q^{j-1}f(x)$ the sequence $b(n,j,x,q)$ satisfies

$$b(n,j,x,q)=q^j xb(n-1,j,x,q)+(1-q^{k+j}x)b(n-1,j-1,x,q). \tag{2.49}$$

Comparing coefficients this is equivalent with

$$\begin{bmatrix}j+k-n\\k-i\end{bmatrix}\begin{bmatrix}n\\i\end{bmatrix}=q^{n-i}\begin{bmatrix}j+k-n+1\\k-i+1\end{bmatrix}\begin{bmatrix}n-1\\i-1\end{bmatrix}+\begin{bmatrix}j+k-n\\k-i\end{bmatrix}\begin{bmatrix}n-1\\i\end{bmatrix}-q^{k-2i+n+1}\begin{bmatrix}j+k-n\\k-i+1\end{bmatrix}\begin{bmatrix}n-1\\i-1\end{bmatrix}$$

.

The right-hand side is

$$q^{n-i}\begin{bmatrix}j+k-n+1\\k-i+1\end{bmatrix}\begin{bmatrix}n-1\\i-1\end{bmatrix}-q^{k-i+1}\begin{bmatrix}j+k-n\\k-i+1\end{bmatrix}\begin{bmatrix}n-1\\i-1\end{bmatrix}+\begin{bmatrix}j+k-n\\k-i\end{bmatrix}\begin{bmatrix}n-1\\i\end{bmatrix}$$
$$=q^{n-i}\begin{bmatrix}n-1\\i-1\end{bmatrix}\begin{bmatrix}j+k-n\\k-i\end{bmatrix}+\begin{bmatrix}j+k-n\\k-i\end{bmatrix}\begin{bmatrix}n-1\\i\end{bmatrix}=\begin{bmatrix}n\\i\end{bmatrix}\begin{bmatrix}j+k-n\\k-i\end{bmatrix}.$$

Now observe that

$$\frac{1}{\left(x;q\right)_{k+1}}=\sum_{n\geq 0}\begin{bmatrix}n+k\\k\end{bmatrix}x^n.$$

Therefore

$$\left(x;q\right)_{2k+1}\sum_{n\geq 0}\begin{bmatrix}n+k\\k\end{bmatrix}^2 x^n=\left(x;q\right)_{2k+1}\frac{D^k}{[k]!}\frac{x^k}{\left(x;q\right)_{k+1}}=b(k,k,x,q)=\sum_{i=0}^{k}q^{i^2}\begin{bmatrix}k\\i\end{bmatrix}^2 x^i$$

and

$$\left(qx^2;q\right)_{2k-1}\sum_{n\geq 0}\begin{bmatrix}n+k-1\\k-1\end{bmatrix}^2\left(qx^2\right)^n=\sum_{i=0}^{k-1}q^{i^2}\begin{bmatrix}k-1\\i\end{bmatrix}^2\left(qx^2\right)^i=\sum_{j=0}^{k-1}q^{j^2+j}\begin{bmatrix}k-1\\j\end{bmatrix}^2 x^{2j}.$$

In the same way we get

$$\left(x;q\right)_{2k-1}\sum_{n\geq 0}\begin{bmatrix}n+k\\k\end{bmatrix}\begin{bmatrix}n+k-1\\k-2\end{bmatrix}x^{n+1}=\left(x;q\right)_{2k-1}\frac{D^{k-2}}{[k-2]!}\frac{x^{k-1}}{\left(x;q\right)_{k+1}}$$
$$=b(k-1,k-2,x,q)=\sum_{i=0}^{k-1}q^{i(i-1)}\begin{bmatrix}k-1\\k-i\end{bmatrix}\begin{bmatrix}k-1\\i\end{bmatrix}x^{i}$$

and therefore

$$\left(qx^2;q\right)_{2k-1}\sum_{n\geq 0}q^{n+1}\begin{bmatrix}n+k\\k\end{bmatrix}\begin{bmatrix}n+k-1\\k-2\end{bmatrix}x^{2n+1}=\sum_{i=1}^{k-1}q^{i^2}\begin{bmatrix}k-1\\k-i\end{bmatrix}\begin{bmatrix}k-1\\i\end{bmatrix}x^{2i-1}.$$

From the easily verified formulae

$$x^{n}=\sum_{\ell=0}^{n}(-1)^{\ell}\begin{bmatrix}n\\\ell\end{bmatrix}q^{\binom{\ell+1}{2}-n\ell}\left(x;q\right)_{\ell} \tag{2.50}$$

and

$$\frac{D_q^j}{[j]!}\frac{\left(x;q\right)_{\ell}}{\left(x;q\right)_{k+1}}=q^{\ell j}\begin{bmatrix}k+j-\ell\\j\end{bmatrix}\frac{\left(x;q\right)_{\ell}}{\left(x;q\right)_{k+j+1}} \tag{2.51}$$

we get

$$\frac{D_q^j}{[j]!}\frac{x^n}{\left(x;q\right)_{k+1}}=\frac{D_q^j}{[j]!}\frac{\sum_{\ell=0}^{n}(-1)^{\ell}\begin{bmatrix}n\\\ell\end{bmatrix}q^{\binom{\ell+1}{2}-n\ell}\left(x;q\right)_{\ell}}{\left(x;q\right)_{k+1}}=\frac{\sum_{\ell=0}^{n}(-1)^{\ell}q^{\binom{\ell+1}{2}+\ell(j-n)}\begin{bmatrix}n\\\ell\end{bmatrix}\begin{bmatrix}k+j-\ell\\j\end{bmatrix}\left(x;q\right)_{\ell}}{\left(x;q\right)_{k+j+1}}. \tag{2.52}$$

As special cases we get

$$\left(x;q\right)_{2k+1}\frac{D_q^k}{[k]!}\frac{x^k}{\left(x;q\right)_{k+1}}=\sum_{\ell=0}^{k}(-1)^{j}\begin{bmatrix}k\\\ell\end{bmatrix}\begin{bmatrix}2k-\ell\\k\end{bmatrix}q^{\binom{\ell+1}{2}}\left(x;q\right)_{\ell}$$

and

$$\left(x;q\right)_{2k-1}\frac{D^{k-2}}{[k-2]!}\frac{x^{k-1}}{\left(x;q\right)_{k+1}}=\sum_{\ell=0}^{k-1}(-1)^{\ell}\begin{bmatrix}k-1\\\ell\end{bmatrix}\begin{bmatrix}2k-2-\ell\\k-2\end{bmatrix}q^{\binom{\ell}{2}}\left(x;q\right)_{\ell}.$$

Comparing (2.52) with (2.48) we get as $q-$analogue of (2.44)

$$\sum_{i=0}^{n} q^{i(j+i-n)} \begin{bmatrix} j+k-n \\ k-i \end{bmatrix} \begin{bmatrix} n \\ i \end{bmatrix} x^i = \sum_{\ell=0}^{n} (-1)^\ell q^{\binom{\ell+1}{2}+\ell(j-n)} \begin{bmatrix} n \\ \ell \end{bmatrix} \begin{bmatrix} k+j-\ell \\ j \end{bmatrix} \left(x;q\right)_\ell . \tag{2.53}$$

The most interesting special cases are

$$\sum_{i=0}^{n} q^{i^2} \begin{bmatrix} n \\ i \end{bmatrix}^2 x^i = \sum_{\ell=0}^{n} (-1)^\ell q^{\binom{\ell+1}{2}} \begin{bmatrix} n \\ \ell \end{bmatrix} \begin{bmatrix} 2n-\ell \\ j \end{bmatrix} \left(x;q\right)_\ell \tag{2.54}$$

for $j = n = k$ and

$$\sum_{i=0}^{k-1} q^{i(i-1)} \begin{bmatrix} k-1 \\ k-i \end{bmatrix} \begin{bmatrix} k-1 \\ i \end{bmatrix} x^i = \sum_{\ell=0}^{k-1} (-1)^\ell q^{\binom{\ell+1}{2}-\ell} \begin{bmatrix} k-1 \\ \ell \end{bmatrix} \begin{bmatrix} 2k-2-\ell \\ k-2 \end{bmatrix} \left(x;q\right)_\ell$$

for $j = k-2$ and $n = k-1.$ By substituting $k \to k+1$ this can be written as

$$\sum_{i=0}^{k} q^{i(i-1)} \frac{1}{[k]} \begin{bmatrix} k \\ i \end{bmatrix} \begin{bmatrix} k \\ i-1 \end{bmatrix} x^i = \sum_{\ell=0}^{k} (-1)^\ell q^{\binom{\ell}{2}} \frac{1}{[k+1]} \begin{bmatrix} k+1 \\ \ell \end{bmatrix} \begin{bmatrix} 2k-\ell \\ k \end{bmatrix} \left(x;q\right)_\ell , \tag{2.55}$$

where $\frac{1}{[k]} \begin{bmatrix} k \\ i \end{bmatrix} \begin{bmatrix} k \\ i-1 \end{bmatrix}$ is a $q-$analogue of a Narayana number.

Conjecture 2

For each $j \geq 1$ *there exists a polynomial* $p_j(z) = p_j(x,z)$ *with degree* $\deg_z p_j(z) = \frac{(j-1)(j+2)}{2}$ *and* $\deg_x p_j(x,z) = \binom{j+2}{3} - 1$ *such that*

$$\sum_{k \geq 0} v_j(x,k) z^k = \frac{p_j(x,z)}{(1-z)\prod_{\ell=1}^{j} (1-x^\ell z)^{j+1-\ell}}. \tag{2.56}$$

Moreover it seems that the polynomials have a certain symmetry:

$$(-1)^{\binom{j}{2}} x^{\binom{j+2}{3}-1} z^{\frac{(j+2)(j-1)}{2}} p_j\left(\frac{1}{x}, \frac{1}{z}\right) = p_j(x,z).$$

Let us consider some special cases.

First we have

$$v_1(x,k) = \sum_{i=0}^{k} x^i \tag{2.57}$$

Let the coefficient of t in $a(n,k+2,t)$ be $[t](a(n,k+2,t) = c(n)$. Then $c(n+2) - c(n) = \min(n+1, k+1)$. This implies that $c(n+2) - c(n) - \big(c(n+1) - c(n-1)\big) = 1$ for $0 \le n \le k$ and vanishes for $n > k$, which is equivalent with (2.57).

To see this consider all paths with $n+2$ steps which begin with DU. Their weight is $c(n)$. The remaining paths with weight t start with $UD, U^2D^2, \cdots$ and $D^2U^2, D^3U^3, \cdots$. There number is $\left\lfloor \frac{n+2}{2} \right\rfloor + \left\lfloor \frac{n+3}{2} \right\rfloor - 1 = n+1$ if $n \le k$. For $n > k$ the heights are bounded by $\left\lfloor \frac{k+2}{2} \right\rfloor$ and $\left\lfloor \frac{k+3}{2} \right\rfloor - 1$. Therefore their number is $k+1$.

The coefficient table is

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 1 & & 1 & \\ 1 & & 1 & & 1 & & 1 \end{array}$$

with generating function for the row sums

$$\frac{1}{(1-z)(1-xz)} = 1 + (1+x)z + (1+x+x^2)tz^2 + \cdots.$$

Thus $p_1(x,z) = 1$.

For $j = 2$ we conjecture that

$$v_2(x,k) = 1 + \sum_{i=1}^{k} x^i (i + 1 + x + x^2 + \cdots + x^i) \tag{2.58}$$

The coefficient table is

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & 2 & 1 & & & & \\ & & & 1 & 2 & 4 & 1 & 1 & & & \\ & & 1 & 2 & 4 & 5 & 2 & 1 & 1 & & \\ & 1 & 2 & 4 & 5 & 7 & 2 & 2 & 1 & 1 & \\ 1 & 2 & 4 & 5 & 7 & 8 & 3 & 2 & 2 & 1 & 1 \end{array} \tag{2.59}$$

This table consists of two sequences $c = \left(1,2,4,5,7,8,\cdots\right)$ with generating function $\dfrac{1+z+z^2}{(1-z)^2(1+z)}$ and

$d = \left(1,1,2,2,3,3,\cdots\right)$ with generating function $\dfrac{1}{(1+z)(1-z)^2}$.

The sequence c is the sequence of all positive integers which are not multiples of 3. Thus $c_{2n} = 3n+1$ and $c_{2n+1} = 3n+2$.

The generating function of the rows is therefore

$$\sum_{k\geq 0} v_2(x,k)z^k = \frac{1-z^2x^3}{(1-z)(1-zx)^2(1-zx^2)}. \tag{2.60}$$

This implies $p_2(x,z) = 1 - x^3z^2$.

For $x = 1$ this reduces to $\dfrac{1+z}{(1-z)^3} = \sum_{k\geq 0}(k+1)^2 z^k$.

For $x = -1$ we get $\dfrac{1+z^2}{(1-z^2)^2} = \sum_{n\geq 0}(2k+1)z^{2k}$.

Thus $v_2(1,k) = (k+1)^2$ and $v_2(-1,2k) = 2k+1$ and $v_2(-1,2k+1) = 0$.

It is perhaps interesting that the coefficient table (2.59) has the form

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & c_1 & 1 & & & & \\ & & & 1 & c_1 & c_2 & d_1 & 1 & & & \\ & & 1 & c_1 & c_2 & c_3 & d_2 & d_1 & 1 & & \\ & 1 & c_1 & c_2 & c_3 & c_4 & d_3 & d_2 & d_1 & 1 & \\ 1 & c_1 & c_2 & c_3 & c_4 & c_5 & d_4 & d_3 & d_2 & d_1 & 1 \end{array}$$

The sequences c and d are uniquely determined by $v_2(1,k) = (k+1)^2$, $v_2(-1,2k) = 2k+1$ and $v_2(-1,2k+1) = 0$.

Observe also that

$$c_0 + \left(c_1 + d_0\right) + (c_2 + d_1) + \cdots + \left(c_n + d_{n-1}\right) = 1 + 3 + 5 + \cdots + (2n+1) = (n+1)^2,$$
$$c_0 - \left(c_1 + d_0\right) + (c_2 + d_1) + \cdots + \left(c_{2n} + d_{2n-1}\right) = 1 - 3 + \cdots + (4n+1) = 2n+1$$

and $\sum_{k=0}^{2n+1} (-1)^k c_k = -(n+1)$ and $\sum_{k=0}^{2n} (-1)^k d(k) = n+1.$

For $j \geq 3$ the situation becomes more complicated.

The coefficient table of $v_3(x,k) = \left(u(k,j)\right)_{j=0}^{3k}$ starts with

1												
1	3	3	1									
1	3	9	5	7	1	1						
1	3	9	15	12	10	9	3	1	1			
1	3	9	15	27	16	20	12	14	3	3	1	1

This table has some unusual properties.

Each column $[x^{k-j}]v_3(x,k)$ for $k \geq j$ is given by $\left(c_n\right)_{n\geq 0} = \left(1,3,9,15,27,55,69,93,111,\cdots\right)$

with generating function

$$\frac{1+2x+4x^2+2x^3+x^4}{(1-x)^3(1+x)^2}.$$

This implies that

$c_{2n} = 1 + 3n + 5n^2$ and $c_{2n-1} = 5n^2 - 3n + 1.$

On the right-hand side of the table each column $[x^{2k+j}]v_3(x,k)$ for $k \geq j$ equals

$\left(1,1,3,3,6,6,\cdots\right) = \left(\binom{\left\lfloor \frac{n+4}{2} \right\rfloor}{2}\right)_{n\geq 0}$. Its generating function is $\frac{1}{(1-x)^3(1+x)^2}$.

If we write the table in the form

$$\left(u(n,k)\right)_{n,k=0}^{\infty}=\begin{pmatrix} 1 &&&&&&&&&&&& \\ 1 & 3 & 3 & 1 &&&&&&&&& \\ 1 & 3 & 9 & 5 & 7 & 1 & 1 &&&&&& \\ 1 & 3 & 9 & 15 & 12 & 10 & 9 & 3 & 1 & 1 &&& \\ 1 & 3 & 9 & 15 & 27 & 16 & 20 & 12 & 14 & 3 & 3 & 1 & 1 \end{pmatrix}$$

then the table between the green and the red column looks like

$$\begin{pmatrix} &&& 3 &&& \\ && 5 && 7 && \\ & 12 && 10 && 9 & \\ 16 && 20 && 12 && 14 \end{pmatrix}=\begin{pmatrix} &&& u(1,2) &&& \\ && u(2,3) && u(2,4) && \\ & u(3,4) && u(3,5) && u(3,6) & \\ u(4,5) && u(4,6) && u(4,7) && u(4,8) \end{pmatrix}.$$

It has some curious properties:

Let $R_k=\left(u\left(n+1,2n+2-k\right)-u\left(n,2n-k\right)\right)_{n\geq k+1}$ be the sequence of successive differences of the south-east columns $\left(u(n,2n-k)\right)_{n\geq k+1}$.

Then

$$R_{2k}=\left(4k+4,2k+2,4k+5,2k+2,4k+6,2k+2,\cdots\right)$$

and

$$R_{2k+1}=\left(4k+5,2k+2,4k+6,2k+2,4k+7,2k+2,\cdots\right).$$

Let $L_k=\left(u\left(n+1,n+2+k\right)-u\left(n,n+k+1\right)\right)_{n\geq k+1}$ be the sequence of successive differences of the south-west columns $\left(u(n,n+k+1)\right)_{n\geq k+1}$.

Then

$$L_{2k}=\left(k+2,5k+7,k+4,5k+11,k+6,5k+15,\cdots\right)$$

and

$$L_{2k-1}=\left(k+2,5k+5,k+4,5k+9,k+6,5k+13,\cdots\right).$$

The exact values are

$$u(2k-1,2k+2j)=3k^2-\frac{j(5j+1)}{2}\text{ for } 0\leq j\leq k-1$$

and

$$u(2k-1,2k-1+2j)=3k^2-\frac{j(5j-1)}{2}\text{ for } 1\leq j\leq k-1.$$

Furthermore for $0 \le j \le k-1$

$$u(2k, 2k+2j) = 3k^2 + 2k - \frac{j(5j+3)}{2}$$

and

$$u(2k, 2k+2j+1) = 3k^2 + 4k - \frac{j(5j+7)}{2}.$$

The generating function is given by

$$\sum_{k \ge 0} v_3(x,k) z^k = \frac{1 + x^2 z - \left(5 + x + 2x^2\right) x^3 z^2 + \left(5x^2 + x + 2\right) x^4 z^3 - x^7 z^4 - x^9 z^5}{(1-z)(1-xz)^3 (1-x^2 z)^2 (1-x^3 z)} \tag{2.61}$$

For $x = 1$ this reduces to $\frac{1+4z+z^2}{(1-z)^4} = \sum_{k \ge 0} (k+1)^3 z^k$

and for $x = -1$ we get $\frac{1+6z^2+z^4}{(1-z^2)^3} = \sum_{n \ge 0} (2n+1)^2 z^{2n}.$

More generally by choosing $x = 1$ we get

$$\sum_{k \ge 0} v_j(1,k) z^k = \sum_{k \ge 0} (k+1)^j z^k = \frac{p_j(1,z)}{(1-z)^{1+\binom{j+1}{2}}}. \tag{2.62}$$

Recall that the Eulerian numbers $\left\langle {n \atop k} \right\rangle$ defined by the recurrence

$$\left\langle {n \atop k} \right\rangle = (k+1) \left\langle {n-1 \atop k} \right\rangle + (n-k) \left\langle {n-1 \atop k-1} \right\rangle$$

with initial values $\left\langle {0 \atop k} \right\rangle = \left[k = 0\right]$ and boundary values $\left\langle {n \atop -1} \right\rangle = 0$ satisfy

$$\sum_{k \ge 0} (k+1)^j z^k = \frac{\sum_{\ell=0}^{j} \left\langle {j \atop \ell} \right\rangle z^\ell}{(1-z)^{j+1}}. \tag{2.63}$$

Comparing (2.62) with (2.63) we see that

$$p_j(1,z) = (1-z)^{\binom{j}{2}} \sum_{\ell=0}^{j} \left\langle {j \atop \ell} \right\rangle z^{\ell}. \tag{2.64}$$

Conjecture 3

Let $r_j(x)$ *be the polynomial defined in* (2.31). *Then*

$$p_j(x,1) = \left(1-x\right)^{j-1} \prod_{\ell=3}^{j} \left(1-x^{\ell}\right)^{\max(j+1-\ell,0)} r_{j-1}(x). \tag{2.65}$$

For example

$p_2(x,1) = 1 - x^3 = (1-x)r_1(x) = (1-x)(1+x+x^2)$

and

$p_3(x,1) = 1 + x^2 - 5x^3 + x^4 - x^5 + 5x^6 - x^7 - x^9 = (1-x)^2(1-x^3)(1+2x+4x^2+2x^3+x^4).$

3. Related results

At last let us consider the polynomials

$$a(n,k,t,z) = \sum_{j \in \mathbb{Z}} z^j \sum_{\ell \geq |j|} \binom{\left\lfloor \frac{n+(k-2)j}{2} \right\rfloor}{\ell - j} \binom{\left\lfloor \frac{n+1-(k-2)j}{2} \right\rfloor}{\ell + j} t^{\ell}. \tag{3.1}$$

Computations for small values of k suggest

Conjecture 4

For $k \geq 1$

$$\sum_{n \geq 0} a(n,k,t,z) x^n = \frac{c(k,x,t,z)}{d(k,x,t,z)} \tag{3.2}$$

with

$$c(2k+1,x,t,z) = \left((1+x)^2 - tx^2\right)\left(\Phi_{k+1}(x,t) - x(x+1)\Phi_k(x,t) + x^3\Phi_{k-1}(x,t)\right) \left(\Phi_k(x,t) - x^2\Phi_{k-1}(x,t)\right) + t(1+z)x^{2k+2}, \tag{3.3}$$

$$d(2k+1,x,t,z) = \left((1+x)^2 - tx^2\right)\left(\Phi_{k+1}(x,t) - x(x+1)\Phi_k(x,t) + x^3\Phi_{k-1}(x,t)\right)^2 - t\frac{(1+z)^2}{z} x^{2k+3}, \tag{3.4}$$

and

$$c(2k,x,t,z) \\ = \Big(\Lambda_k(x,t) - x^2\Lambda_{k-1}(x,t)\Big)\Big((1+x)\Phi_k(x,t) - x^2(1+(1-t)x)\Phi_{k-1}(x,t)\Big) + t(1+z)x^{2k+1} \tag{3.5}$$

and

$$d(2k,x,t,z) = \Big(\Lambda_k(x,t) - x^2\Lambda_{k-1}(x,t)\Big)^2 - t\frac{(1+z)^2}{z}x^{2k+2}. \tag{3.6}$$

Closely related is

Conjecture 5

$$\sum_{n\geq 0} a(n,2k+1,t,1)x^n = \frac{\Lambda_{k+1}(x,t) - (1-t)x^2\Lambda_k(x,t)}{(1-x)\Lambda_{k+1}(x,t) - x^2(1-(1-t)x)\Lambda_k(x,t)} \tag{3.7}$$

and

$$\sum_{n\geq 0} a(n,2k,t,1)x^n = \frac{(1-x)\Phi_k(x,t) - x^2(1-(1-t)x)\Phi_{k-1}(x,t)}{\Big((1-x)^2 - tx^2\Big)\Big(\Phi_k(x,t) - x^2\Phi_{k-1}(x,t)\Big)}. \tag{3.8}$$

The following proposition has been proved with other methods in [6] and [10] and can also be deduced from Conjecture 4.

Proposition 5

The generating function of the sequence

$$\Big(a(n,k,1,z)\Big)_{n\geq 0} = \left(\sum_{j\in\mathbb{Z}} z^j \binom{n}{\left\lfloor \frac{n+(k+2)j}{2} \right\rfloor}\right)_{n\geq 0} \tag{3.9}$$

is given by

$$\sum_{n\geq 0} a(n,k,1,z)x^n = \frac{F_{k+2}(1,-x^2) + xF_{k+1}(1,-x^2) + zx^{k+1}}{L_{k+2}(1,-x^2) - x^{k+2}\left(z + \frac{1}{z}\right)}. \tag{3.10}$$

For $z = 1$ further simplifications occur (cf. [6] or [10]).

$$\sum_{n\geq 0} a(n,2k,1,1)x^n = \frac{F_{k+1}(1,-x^2) - xF_k(1,-x^2)}{(1-2x)F_{k+1}(1,-x^2)} \tag{3.11}$$

and

$$\sum_{n\geq 0} a(n,2k+1,1,1)x^n = \frac{L_{k+1}(1,-x^2)}{L_{k+2}(1,-x^2) - xL_{k+1}(1,-x^2)}. \tag{3.12}$$

For small values of k these sequences occur several times in the literature.

The sequence $\left(a(n,1,1,1)\right)_{n\geq 0} = \left(1,1,3,5,11,21,\cdots\right)$ is the so-called Jacobsthal sequence A001045 with generating function

$$\sum_{n\geq 0} a(n,1,1,1)x^n = \frac{L_1(1,-x^2)}{L_2(1,-x^2) - xL_1(1,-x^2)} = \frac{1}{1-x-2x^2}.$$

The sequence $\left(a(n,2,1,1)\right)_{n\geq 0} = \left(1,1,2,4,8,16,\cdots\right)$ is A011782, the sequence $\left(a(n,3,1,1)\right)_{n\geq 0} = \left(1,1,2,3,7,12,\cdots\right)$ is A099163, and the sequence $\left(a(n,4,1,1)\right)_{n\geq 0} = \left(1,1,2,3,6,11,22,43,\cdots\right)$ is A005578.

Finally let us mention that by (3.9)

$$a(n,k) = a(n,k,1,-1) = 2a(n,2k+2,1,1) - a(n,k,1,1). \tag{3.13}$$

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