arXiv:1501.05075v4 [math.NT] 21 Nov 2015

Calculations relating to some special Harmonic numbers

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November 24, 2015

Abstract

We report on the results of a computer search for primes p which divide an Harmonic number $H_{\lfloor p/N \rfloor}$ with small $N > 1$.
Keywords: Harmonic numbers, anharmonic primes, Fermat quotient, Fermat's Last Theorem

1 Introduction

Before its complete proof by Andrew Wiles, a major result for the first case of Fermat's last theorem (FLT), that is, the assertion of the impossibility of $x^p + y^p = z^p$ in integers x, y, z, none of which is divisible by a prime $p > 2$, was the 1909 theorem of Wieferich [28] that the exponent p must satisfy the congruence

$$q_p(2) := \frac{2^{p-1} - 1}{p} \equiv 0 \pmod{p},$$

where $q_p(2)$ is known as the Fermat quotient of p, base 2. This celebrated result was to be generalized in many directions. One of these was the extension of the congruence to bases other than 2, the first such step being the proof of an analogous theorem for the base 3 by Mirimanoff [18] in 1910. Another grew from the recognition that some of these criteria could be framed in terms of certain special Harmonic numbers of the form

$$H_{\lfloor p/N \rfloor} := \sum_{j=1}^{\lfloor p/N \rfloor} \frac{1}{j} \tag{1}$$

for $N > 1$, and with $\lfloor \cdot \rfloor$ denoting the greatest-integer function. There are nice historical overviews of these developments in [22] and [21] (pp. 155–59), which go into greater detail than we attempt here.

1.1 Congruences for $H_{\lfloor p/N \rfloor}$

At the time when Wieferich's and Mirimanoff's results appeared, it was already known that three of these Harmonic numbers had close connections to the Fermat quotient, and satisfied the following congruences (all modulo p):

$$H_{\lfloor p/2 \rfloor} \equiv -2 \cdot q_p(2) \tag{2}$$

$$H_{\lfloor p/3 \rfloor} \equiv -\frac{3}{2} \cdot q_p(3) \tag{3}$$

$$H_{\lfloor p/4 \rfloor} \equiv -3 \cdot q_p(2). \tag{4}$$

All these results are due to Glaisher; those for $N = 2$ and 4 will be found in ([11], pp. 21-22, 23), and that for $N = 3$ in ([10], p. 50). Although it was not the next Harmonic number criterion published, it will be convenient to dispense next with the case $N = 6$. The apparatus needed to evaluate this Harmonic number appears in a 1905 paper of Lerch ([16], p. 476, equations 14 and 15), but the implications of Lerch's result were long overlooked, and only realized in 1938 by Emma Lehmer ([15], pp. 356ff), who gave the following congruence mod p:

$$H_{\lfloor p/6 \rfloor} \equiv -2 \cdot q_p(2) - \frac{3}{2} \cdot q_p(3). \tag{5}$$

The fact that the vanishing of $H_{\lfloor p/6 \rfloor}$ mod p is a necessary condition for the failure of the first case of FLT for the exponent p is an immediate consequence of the theorems of Wieferich and Mirimanoff. In the present study, we use Lehmer's congruence in the equivalent form

$$H_{\lfloor p/6 \rfloor} \equiv -\frac{1}{2} \cdot q_p(432), \tag{6}$$

obtained from (5) by applying in reverse the logarithmetic and factorization rules for the Fermat quotient given by Eisenstein [7]. This expression reveals that divisors p of $H_{\lfloor p/6 \rfloor}$ are instances of the vanishing of the Fermat quotient mod p for composite bases, a problem which has notably been studied in the ongoing work of Richard Fischer [9].

The four congruences above exhaust the cases of $H_{\lfloor p/N \rfloor}$ that can be evaluated solely in terms of Fermat quotients. Historically, the case $N = 5$ also has its origins in this era, and was in fact introduced before that of $N = 6$. In 1914, Vandiver [27] proved that the vanishing of both $H_{\lfloor p/5 \rfloor}$ and $q_p(5)$ are necessary conditions for p to be an exception to the first case of FLT. Unlike the four cases already considered, here the connection of the Harmonic number with FLT was discovered before any evaluation of it (beyond the definitional one) was known. It was almost eighty years later that the connection between these results would become apparent. The ingredients needed for the evaluation of this Harmonic number were presented in a 1991 paper by Williams ([29], p. 440), and almost simultaneously by Z. H. Sun ([25], pt. 3, Theorems 3.1 and 3.2); and though they do not write out the formula explicitly, it is clearly implied to be

$$H_{\lfloor p/5\rfloor} \equiv -\frac{5}{4}\cdot q_p(5) - \frac{5}{4}\cdot F_{p-\left(\frac{5}{p}\right)}/p \pmod p, \tag{7}$$

where $F_{p-\left(\frac{5}{p}\right)}/p$ is the Fibonacci quotient (OEIS A092330), with F a Fibonacci number and $\left(\frac{5}{p}\right)$ a Jacobi symbol. In light of Vandiver's theorems on $H_{\lfloor p/5\rfloor}$ and $q_p(5)$, this result immediately established that the vanishing of the Fibonacci quotient mod p was yet another criterion for the failure of the first case of FLT for the exponent p. The fact was announced almost immediately in the celebrated paper by the Sun brothers [26], which gave fresh impetus to an already vast literature on the Fibonacci quotient; and in its honor the primes p which divide their Fibonacci quotient were named Wall-Sun-Sun-Primes. These remain hypothetical, as not a single instance has been found despite tests to high limits [19].

The remaining known formulae for the type of special Harmonic numbers in which we are interested are of much more recent origin. In some cases they were discovered simultaneously, or nearly so, by more than one researcher; and we hope we have not done injustice to any of the participants. Apart from a few published formulae which are apparently in error or underdetermined, we have the following congruences (all modulo p) which are undoubtably correct:

$$H_{\lfloor p/8\rfloor} \equiv -4\cdot q_p(2) - 2\cdot U_{p-\left(\frac{2}{p}\right)}(2,-1)/p \tag{8}$$

$$H_{\lfloor p/10\rfloor} \equiv -2\cdot q_p(2) - \frac{5}{4}\cdot q_p(5) - \frac{15}{4}\cdot F_{p-\left(\frac{5}{p}\right)}/p \tag{9}$$

$$H_{\lfloor p/12\rfloor} \equiv -3\cdot q_p(2) - \frac{3}{2}\cdot q_p(3) - 3\cdot\left(\frac{3}{p}\right)\cdot U_{p-\left(\frac{3}{p}\right)}(4,1)/p \tag{10}$$

$$\begin{aligned} H_{\lfloor p/24\rfloor} \equiv &- 4\cdot q_p(2) - \frac{3}{2}\cdot q_p(3) - 4\cdot U_{p-\left(\frac{2}{p}\right)}(2,-1)/p \\ &- 3\cdot\left(\frac{3}{p}\right)\cdot U_{p-\left(\frac{3}{p}\right)}(4,1)/p - 6\cdot\left(\frac{6}{p}\right) U_{p-\left(\frac{6}{p}\right)}(10,1)/p \end{aligned} \tag{11}$$

where the $\left(\frac{\cdot}{p}\right)$ are Jacobi symbols, and

- $F_{p-\left(\frac{5}{p}\right)}/p$ is as before the Fibonacci quotient (OEIS A092330)
- $U_{p-\left(\frac{2}{p}\right)}(2,-1)/p$ is the Pell quotient (OEIS A000129)
- $U_{p-\left(\frac{3}{p}\right)}(4,1)/p$ is a quotient derived from the Lucas sequence 1, 4, 15, 56, 209, ... (OEIS A001353)
- $U_{p-\left(\frac{6}{p}\right)}(10,1)/p$ is a quotient derived from the Lucas sequence 1, 10, 99, 980, 9701, ... (OEIS A004189)

- $\left(\frac{2}{p}\right) = (-1)^{(p^2-1)/8}$
- $\left(\frac{3}{p}\right) = (-1)^{\lfloor (p+1)/6 \rfloor}$
- $\left(\frac{6}{p}\right) = (-1)^{\lfloor (p+5)/12 \rfloor}$.

The result for $N = 8$ is derived from the 1991 paper by Williams ([29], p. 440), with an equivalent result also appearing in Sun ([25], pt. 3, Theorem 3.3). To the best of our knowledge, at the time of the discovery of this formula in 1991 the vanishing of $H_{\lfloor p/8 \rfloor}$ modulo p was not recognized as a condition for the failure of the first case of FLT for the exponent p, and this only became evident with the appearance of the 1995 paper of Dilcher and Skula [3] discussed below.

The result for $N = 10$ is due to a 1992 paper by Z. H. Sun ([25], pt. 3, Theorem 3.1). Since the vanishing of all the individual components was by then known to be a necessary condition for the failure of the first case of FLT for the exponent p, the same thing was immediately seen to be true for $H_{\lfloor p/10 \rfloor}$.

The result for $N = 12$ is also derived from the 1991 paper of Williams ([29], p. 440), and that for $N = 24$ is from a 2011 paper of Kuzumaki & Urbanowicz ([14], p. 139).

In their landmark joint paper of 1995, Dilcher and Skula [3] proved among other things that the vanishing modulo p of $H_{\lfloor p/N \rfloor}$ was a necessary criterion for the failure of the first case of FLT for the exponent p, for all N from 2 to 46. This result gives retrospective interest to the evaluations of $H_{\lfloor p/8 \rfloor}$ and $H_{\lfloor p/12 \rfloor}$, and furnishes the main motivation for the present study.

1.2 The case of $N = 9$

As discussed in section 3 below, when calculating the sort of special Harmonic number with which we are concerned, it is important to avoid by any means possible the calculation of the long runs of modular inverses suggested by the definition (1). In the early days of this study $H_{\lfloor p/9 \rfloor}$ was obtained from $H_{\lfloor p/10 \rfloor}$, requiring the calculation of inverses for $\frac{1}{9} - \frac{1}{10} = \frac{1}{90}$ of the numbers in the range $\{1, p\}$. This is more efficient than obtaining $H_{\lfloor p/18 \rfloor}$ from $H_{\lfloor p/24 \rfloor}$, the nearest neighbor that can be calculated by formula, which requires the calculation of inverses for $\frac{1}{18} - \frac{1}{24} = \frac{1}{72}$ of the numbers in the range $\{1, p\}$. However, assuming $H_{\lfloor p/18 \rfloor}$ will be calculated anyway, $H_{\lfloor p/9 \rfloor}$ can be derived from it without calculation of any additional inverses, and with negligible cost.

The device proposed is somewhat convoluted, but worthwhile. In 1991, Zhi-Hong Sun ([25], Theorem 2.7) gave an evaluation of

$$\sum_{k=1}^{\lfloor p/9 \rfloor} \frac{(-1)^{k-1}}{k},$$

which is in fact trivially equivalent to

$$s(1,18) = \sum_{k=\lfloor p/18\rfloor+1}^{\lfloor p/9\rfloor} \frac{1}{k},$$

to use a well-known notation for this sum. Consider the recurrence defined by

$$\begin{aligned} f(0) =& 1, f(1) = 0, f(2) = 2, \\ f(n) =& 3f(n-2) - f(n-3) \quad (n \geq 3). \end{aligned} \tag{12}$$

Sun showed that there is a composition, which we shall designate $Z(p)$, defined by

$$Z(p) \equiv s(1,18) - 2q_p(2) \pmod{p}, \tag{13}$$

such that

$$Z(p) \equiv \begin{cases} 3 \cdot \dfrac{f(p+1)-2}{p} & \text{if } p \equiv \pm 1 \pmod 9 \\ 3 \cdot \dfrac{f(p)-f(p-1)-2}{p} & \text{if } p \equiv \pm 2 \pmod 9 \\ 3 \cdot \dfrac{-f(p+1)-f(p)+f(p-1)-2}{p} & \text{if } p \equiv \pm 4 \pmod 9. \end{cases} \tag{14}$$

The evaluation of the sequence $f(n)$ can be performed efficiently in PARI using the matrix exponentiation `([0,3,-1;1,-1,1;1,0,1]^n)[1,1]`, with the calculations being performed modulo p^2 throughout. Thus, noting that by definition,

$$H_{\lfloor p/9\rfloor} \equiv H_{\lfloor p/18\rfloor} + s(1,18),$$

we have from (13) that

$$H_{\lfloor p/9\rfloor} \equiv H_{\lfloor p/18\rfloor} + Z(p) + 2q_p(2) \pmod{p}. \tag{15}$$

No direct evaluations of $H_{\lfloor p/18\rfloor}$ and $H_{\lfloor p/9\rfloor}$ comparable to that for $s(1,18)$ are known, but Sun's result allows efficient evaluation modulo p of either of these special Harmonic numbers with respect to the other. Compared with a scheme in which the two quantities are calculated separately, the derivation of $H_{\lfloor p/9\rfloor}$ from $H_{\lfloor p/18\rfloor}$ requires the calculation of $\frac{5}{9}$ as many modular inverses, while the derivation of $H_{\lfloor p/18\rfloor}$ from $H_{\lfloor p/9\rfloor}$ requires the calculation of only $\frac{4}{9}$ as many inverses. However, we choose the former path because we also require the values of $H_{\lfloor p/N\rfloor}$ with $18 < N \leq 24$.

2 The Divisibility of Harmonic Numbers

It may be helpful to distinguish the purpose of the present study with that of the more general problem of the divisibility of Harmonic numbers. Eswarathasan and Levine [8] note that all primes greater than 3 divide the Harmonic numbers of indices $p-1$, $p(p-1)$, and p^2-1, and Wolstenholme's theorem states that for a prime $p > 3$, $p^2 | H_{p-1}$. These results may be inverted to give three general rules for the divisibility of Harmonic numbers:

1. H_n is divisible by $(n+1)^2$ if $n+1$ is a prime > 3;
2. H_n is divisible by $\frac{1+\sqrt{4n+1}}{2}$ if the latter is a prime > 3;
3. H_n is divisible by $\sqrt{n+1}$ if the latter is a prime > 3.

Eswarathasan and Levine define *harmonic* primes as primes that divide only the three aforementioned Harmonic numbers (OEIS A092101), and *anharmonic* primes as those that divide additional Harmonic numbers (OEIS A092102). From the fact that we consider only cases (1) where $p | H_{\lfloor p/N \rfloor}$ with $N > 1$, it will be evident that any such p is anharmonic, and that we are seeking a subset of anharmonic primes that divide Harmonic numbers of relatively small index; for example, from Table 5 we see that $N = 2$ is solved by $p = 1093$, implying that $1093 | H_{546}$, while $N = 46$ is solved by $p = 11731$, implying that $11731 | H_{255}$. In contrast, the work of Boyd [1] and Rogers [23] entails, in part, finding Harmonic numbers of large index divisible by relatively small primes, such as the case $11 | H_{10118497718552149129684042172 47}$. The overlap between our results and theirs is slight.

Indeed, our results entail only a minority of the anharmonic primes, because they require that p divide an Harmonic number H_n with $n < p-1$, and thus belong to a subset of the anharmonic primes which has sometimes been called the Harmonic irregular primes (see OEIS A092194 and the Wikipedia entry for "Regular prime"). Although the standard definition of such a prime p is that it divide an Harmonic number H_n with $n < p-1$, it may be noted that any prime satisfying this criterion must in fact divide some Harmonic number H_n of smaller index $n < (p-1)/2$, since by symmetry $H_{p-1-n} \equiv H_n \bmod p$. The inequality sign in this condition is strict because if $n = (p-1)/2$, then p is a Wieferich prime and by (2) and (4) must likewise divide the smaller $H_{\lfloor p/4 \rfloor}$. But this inequality also gives a sharp upper bound on the index of the least Harmonic number divisible by an Harmonic irregular prime p, since among the first 999 primes we have $n = (p-3)/2$ for $p = 29, 37, 3373$ (see OEIS A125854). In general, there does not seem to be any way of recognizing an Harmonic irregular prime other than by exhaustively testing it as a divisor of Harmonic numbers H_n with $n \leq (p-3)/2$ (though one would of course not attempt to do this by actual division except for very small p).

In turn, our results entail only a minority of the Harmonic irregular primes, because we must have $m = \lfloor p/N \rfloor$ for some N. Nonetheless, for some small primes this relationship is satisfied by more than one value of N; for example, with $p = 137$, $p|H_5$, and $\lfloor p/23 \rfloor = \lfloor p/24 \rfloor = \lfloor p/25 \rfloor = \lfloor p/26 \rfloor = \lfloor p/27 \rfloor = 5$.

Taking into account the stringency of the conditions on p, it is perhaps unsurprising that few instances have been found where a given p divides distinct Harmonic numbers of the special type under consideration. By (2) and (4) all Wieferich primes p divide both $H_{\lfloor p/4 \rfloor}$ and $H_{\lfloor p/2 \rfloor}$, but otherwise we have found only two instances where a non-Wieferich prime $p < 72,000,000$ divides two distinct Harmonic numbers of index m in such a way that $m = \lfloor p/N \rfloor$ is satisfied for some $N \leq 1000$ in each case: for $p = 761$, p divides both H_8 (with $N = 85$ through 95) and H_{23} (with $N = 32$, 33), and for $p = 845921$, p divides both H_{1011} (with $N = 836$) and H_{1524} (with $N = 555$). Perhaps further solutions exist with larger N, but we expect such primes to be rare.

There is, however, a sense in which our results do relate to the general problem of the divisibility of Harmonic numbers. While the formulae (2) through (11) are framed in terms of p, they may conversely be seen as divisibility conditions on H_n. For example, with $N = 2$ (the Wieferich primes), we seek cases where the numerator of H_n is divisible by a prime $2n+1$, $4n+1$, or $4n+3$. The still unresolved case $N = 5$ asks whether it is possible for H_n to be divisible by a prime of one of the forms $5n + 1$, $5n + 2$, $5n + 3$, or $5n + 4$, and the still unresolved case $N = 12$ whether it is possible for H_n to be divisible by a prime of one of the forms $12n + 1$, $12n + 5$, $12n + 7$, or $12n + 11$. Incidentally, we also tested the numerators of H_n for divisibility by any of these linear forms without the restriction that the divisors be prime, and found only one additional solution for $n \leq 10,000$, for a divisor of the form $12n + 1$: H_{10} is divisible by 121, which is of course 11^2. For the same ranges of p as covered by Table 5 below, the following table shows, for small k, all linear forms for which there exists no known $p = kn + r$ $(r < k)$ dividing an Harmonic number H_n.

Table 1: Linear forms for which no known prime $p = kn + r$ divides an Harmonic number H_n ($n \leq 24$); starred rows contain all possible forms of p

k	r
$\star$ 5	1, 2, 3, 4
7	1, 3, 4, 5, 6
8	3, 7
9	1, 5, 7, 8
10	1, 3
11	1, 2, 3, 4, 6, 7, 8, 10
$\star$ 12	1, 5, 7, 11
13	1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 12
14	1, 3, 5, 9
15	1, 2, 4, 8, 11, 13, 14
16	5, 7, 9, 11, 13
$\star$ 17	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16
$\star$ 18	1, 5, 7, 11, 13, 17
19	1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 14, 15, 16, 17
$\star$ 20	1, 3, 7, 9, 11, 13, 17, 19
21	1, 4, 5, 8, 10, 13, 17, 19, 20
22	1, 3, 7, 9, 13, 15, 17, 19, 21
23	1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22
24	11, 13, 23

The example noted above of 121 dividing H_{10} underscores the fact that if $k-1$ is a prime, then $(k-1)^2 | H_{k-2}$ and $(k-1)^2 \equiv 1 \bmod k$. Thus, although there are no known *prime* divisors of H_n of such forms as $12n+1$, $14n+1$, $18n+1$, and $20n+1$, it is not because such divisors are *algebraically* impossible.

3 Computational Considerations

In the cases where such formulae as (2) through (11) exist for the kinds of special Harmonic numbers with which we are concerned, it would be difficult to overemphasize their superiority for computational purposes over naíive application of the definition of the Harmonic number as a sum of reciprocals (1). As we noted in [4], the limits attainable by the two methods differ by several orders of magnitude. A striking example is furnished by the cases of $H_{\lfloor p/6 \rfloor}$ (calculated by a particular simple formula) versus $H_{\lfloor p/7 \rfloor}$ (a smaller sum by definition, but necessarily calculated by brute force). Of the two, the calculations for $H_{\lfloor p/6 \rfloor}$ were begun more than a year later, since (as will be explained below) they overlapped previous work which there initially seemed no purpose in repeating. However, the calculations for $H_{\lfloor p/6 \rfloor}$ now extend more than 50,000 times as far as those for $H_{\lfloor p/7 \rfloor}$.

The nine cases of $H_{\lfloor p/N \rfloor}$ represented by formulae have all been tested to high limits (Table 2) without finding solutions for $H_{\lfloor p/5 \rfloor}$ and $H_{\lfloor p/12 \rfloor}$, and this suggests the question of whether the latter two cases ever vanish modulo p. These cases have resisted tests to limits that are at least four orders of magnitude greater than the least zeros (wherever one is known) of $H_{\lfloor p/N \rfloor}$ with $N \leq 46$. These limits are at least two orders of magnitude greater than the least zeros (whenever one is known) of all the Fermat quotients $q_p(b)$ with base $b \leq 100$, than all three known zeros of the Pell quotient (see OEIS A238736 for its zeros), and than the lesser of the known zeros of the Lucas quotient $U_{p-(\frac{3}{p})}(4,1)/p$ (see OEIS A238490 for its zeros). But if there are mathematical reasons for the observed non-vanishing of $H_{\lfloor p/5 \rfloor}$ and $H_{\lfloor p/12 \rfloor}$, they remain elusive.

On the other hand, we are not inclined to attach as much significance to the fact that there are still no known zeros of $H_{\lfloor p/N \rfloor}$ for $N = 17, 18, 20, 29, 31, 43$, as these cases require brute-force processing and have still not been particularly well tested. Explicit computation of Harmonic numbers, even using modular arithmetic, is an unappealing scenario; as noted by Schwindt, for fixed N it has algorithmetic complexity of order $p \log p$. Where such computation is unavoidable, it can be somewhat accelerated by employing the elementary rearrangement

$$H_n = H_{\lfloor n/4 \rfloor} \quad + \quad \frac{3}{2} \cdot \sum_{j=\lfloor n/4 \rfloor+1}^{\lfloor n/2 \rfloor} \frac{1}{j} \quad + \quad \sum_{\substack{k=\lfloor n/2 \rfloor+1 \\ k \text{ odd}}}^{n} \frac{1}{k}, \tag{16}$$

which reduces the number of terms by about $\frac{1}{4}$. The term $H_{\lfloor n/4 \rfloor}$ could likewise be decomposed in a similar manner, though for the sake of simplicity this was not done in the present study; and more sophisticated rearrangements are possible, where all the even terms are successively generated from the odd terms. But such devices can bring about only modest improvements in runtime.

As to the solution of the remaining cases with $N = 17, 18, 20, 29, 31, 43$, there are indeed heuristic reasons for doubting the efficacy of the present means of testing. If one considers the sequence of least divisors p of Harmonic numbers $H_{\lfloor p/N \rfloor}$ for $N = 2$ through 46 (see Table 6), arranged in ascending order of p and displayed on a logarithmetic scale (see Figure 1), it is readily seen that the points tend to describe a line of positive curvature. Thus, within the limits studied, it has been found that the sequence of least divisors p grows at a superexponential rate. This portion of our data, accumulating since January 2014, now represents about 2 years of processing time on a typical desktop computer with a 3.20 GHz processor. Perhaps a few more such p can be found using the present methods and equipment, but it does not seem reasonable to expect that all six will be (assuming they exist).

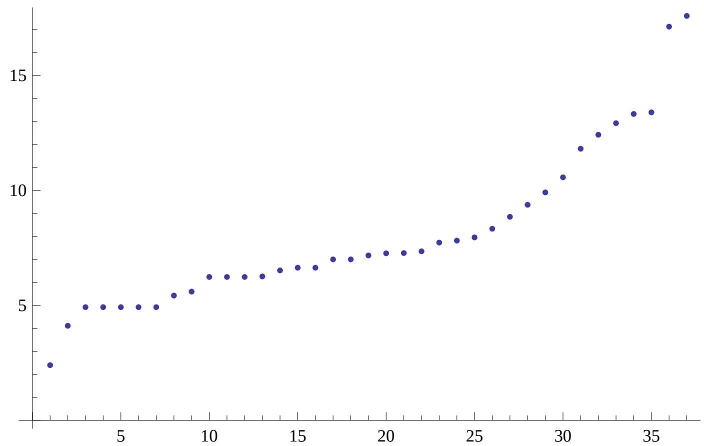

Figure 1: Least divisors $p < 319,900,000$ of Harmonic numbers $H_{\lfloor p/N \rfloor}$ for $N = 2$ through 46, arranged in ascending order of p; horizontal axis = order of discovery; vertical axis = $\log p$.

4 Previous Calculations

The following account does not aim to be complete, but we hope that it includes all significant contributions to the problem. In referring to earlier work, for the sake of brevity we focus mainly on the results of successful searches. In [4], we noted that a statement in pp. 389–390 of [3], implying that $H_{\lfloor p/N \rfloor} \equiv 0 \mod p$ has a solution $p < 2,000$ for every N between 2 and 46 other than 5, is incorrect (at least if we require $p > N$ to ensure that $H_{\lfloor p/N \rfloor}$ is not a vacuous sum). Indeed, as reported below, for $N = 17, 18, 20, 29, 31, 43$ there are no solutions with $p < 319,900,000$, and for $N = 5, 12$ there are no solutions with $p < 9,520,000,000,000$.

The divisors p of $H_{\lfloor p/2 \rfloor}$ and $H_{\lfloor p/4 \rfloor}$ are the Wieferich primes (OEIS A001220). The first of these, 1093, was found in 1913 by Meissner, and the second, 3511, in 1922 by Beeger. The Wieferich primes have inspired some of the most intensive numerical searches ever conducted, but no further instances have been found. The current search record doubtless belongs to the ongoing test by PrimeGrid [20], which had reached $p < 484,357,700,000,000,000$ as of 6 October 2015.

The divisors p of $H_{\lfloor p/3 \rfloor}$ are the Mirimanoff primes (OEIS A014127). The first of these, 11, was announced by Jacobi in 1828 [12], the second, 1006003, by Kloss in 1965 [13]. No further instances have been found. The current search record appears to be $p < 970,453,984,500,000$, held by Dorais and Klyve [6].

We have found no computational literature on $H_{\lfloor p/5 \rfloor}$ apart from the negative result in [3] that it does not vanish modulo p for $p < 2,000$, though inevitably, reports of negative results are more difficult to locate. One wonders why such results were not reported by Schwindt [24], since he could have obtained them as a byproduct of his study of the interval $H_{\lfloor p/5 \rfloor} - H_{\lfloor p/6 \rfloor}$. In any case, we have tested $H_{\lfloor p/5 \rfloor}$ to a much greater limit than that attainable by Schwindt's method, without finding any instance where it vanishes modulo p.

The divisors of $H_{\lfloor p/6 \rfloor}$ (OEIS A238201, but ignoring solutions with $p < 6$) were investigated in 1983 by Schwindt [24], whose study employed a rather naïve method and only reached $p < 600,000$, finding the single nontrivial case $p = 61$. We know of no further computations relating to these numbers until they were obtained in another guise as the zeros of $q_p(432)$ by Richard Fischer [9], whose ongoing test, which had reached $p < 65,442,059,485,993$ as of 30 September 2015, has found two further solutions. The vast range studied by Fischer has not been completely retested, and initially we carried our calculations just far enough to rediscover his second solution. The three solutions are included in Tables 4 and 5 below. At a rather late stage in the present study we decided to continue the replication of his calculations, for the sole reason that (to the best of our knowledge) they have been performed only once. (A similar study by Ležák [17], conducted in apparent unawareness of Fischer's work, extends only to $p < 35,000,000,000$.)

5 The Present Calculations

The present study, which supersedes the corresponding section of [5], does not attempt to extend existing calculations in the cases $N = 2, 3, 4, 6$, and relies on known values for those cases in the tables below. Although Cikánek [2] also gives similar conditions for the first case of FLT for N up to 94 (with certain qualifications), it was decided to focus computing resources on the original range considered by Dilcher and Skula, with N ranging from 2 to 46.

The search for solutions of $p | H_{\lfloor p/N \rfloor}$ has now been carried to $319,900,000$ for all values of N in the range, and solutions have been found for all but eight cases out of 45 (see Table 5 below). For values obtainable by the formulae (2) through (11), the work was performed in PARI, using matrix exponentiation with modular arithmetic to evaluate the required recurrence sequences, and it has now been run almost continuously on a typical desktop computer since January 2014, except in the case of $N = 6$ which was only begun in May 2015. The following search limits have been attained:

Table 2: Search limits on divisors p of Harmonic numbers $H_{\lfloor p/N \rfloor}$ for which a formula exists

N	limit of p
5	9,520,500,000,000
6	14,015,000,000,000
8	2,335,500,000,000
10	2,335,500,000,000
12	9,669,000,000,000
24	2,335,500,000,000

For those values of N necessitating a brute-force search, the work was performed concurrently on a second machine, which has likewise been running from January 2014 to the present, with minor interruptions. After testing several programming languages, the best runtimes were again found to be achieved in PARI, which is very efficient at calculating modular inverses. To expedite the computations, values of $H_{\lfloor p/N \rfloor}$ were generally obtained either by addition or by subtraction from the nearest neighbor that could be calculated by formula. Thus, $H_{\lfloor p/7 \rfloor}$ was obtained from $H_{\lfloor p/8 \rfloor}$, $H_{\lfloor p/11 \rfloor}$ from $H_{\lfloor p/12 \rfloor}$, and all cases with $N > 12$ from $H_{\lfloor p/24 \rfloor}$. Initially (as reported in an earlier version of this study) $H_{\lfloor p/9 \rfloor}$ was obtained from $H_{\lfloor p/10 \rfloor}$, but later the more efficient strategy discussed in section 1.2 above was implemented. The calculations were separated into multiple concurrent runs; hence the limits so far attained vary, as follows:

Table 3: Search limits on divisors p of Harmonic numbers $H_{\lfloor p/N \rfloor}$ for which no known formula exists

N	limit of p
7	333,000,000
11	349,000,000
9; 13–23	319,900,000
25–46	333,330,000

In all, solutions remain unknown for $N = 5, 12, 17, 18, 20, 29, 31, 43$, and so for as long as feasible the work will continue.

The results of the searches follow. First (Table 4), we report separately those results for which the search-limits are given in Table 2 above, since these have particular interest and in some cases coincide with OEIS sequences. Then (Table 5) we report all results combined, including those in Table 4. We believe that the zero $p = 31251349243$ for $N = 24$ is the largest presently known for any value of $N > 1$.

6 Acknowledgements

We should like to thank the University of Winnipeg Library, and in particular Joffrey Abainza, for providing access to computing resources. We should also like to thank T. D. Noe for creating OEIS sequence A238201 based on our [4].

Table 4: Divisors p of Harmonic numbers $H_{\lfloor p/N \rfloor}$ for which a formula exists

N	p	OEIS reference
2	1093, 3511	A001220
3	11, 1006003	A014127
4	1093, 3511	A001220
5	—	
6	61, 1680023, 7308036881	A238201
8	269, 8573, 1300709, 11740973, 241078561	
10	227, 17539, 4750159	
12	—	
24	137, 577, 247421, 307639, 366019, 5262591617, 31251349243	

Table 5: Divisors p of Harmonic numbers $H_{\lfloor p/N \rfloor}$ for all N, 2 through 46

N	p	N	p
2	1093, 3511	25	137
3	11, 1006003	26	137, 67939
4	1093, 3511	27	137, 23669
5	—	28	20101
6	61, 1680023, 7308036881	29	—
7	652913	30	27089407
8	269, 8573, 1300709, 11740973, 241078561	31	—
		32	761
9	677, 6691	33	761
10	227, 17539, 4750159	34	1553
11	246277, 1156457	35	4139, 4481, 4598569
12	—	36	1297
13	43214711	37	1439, 26833
14	2267, 6898819	38	2473, 3527, 4047089
15	134227	39	407893
16	38723, 38993, 4292543	40	509, 177553
17	—	41	509, 151883
18	—	42	509, 190657
19	521, 911	43	—
20	—	44	6967, 27361
21	1423, 5693, 5782639, 212084723	45	609221
22	2843	46	11731
23	137, 264391		
24	137, 577, 247421, 307639, 366019, 5262591617, 31251349243		

Table 6: Least divisors $p < 319,900,000$ of Harmonic numbers $H_{\lfloor p/N \rfloor}$ for $N = 2$ through 46, arranged in ascending order of p

p	N
11	3
61	6
137	23
137	24
137	25
137	26
137	27
227	10
269	8
509	40
509	41
509	42
521	19
677	9
761	32
761	33
1093	2
1093	4
1297	36
1423	21
1439	37
1553	34
2267	14
2473	38
2843	22
4139	35
6967	44
11731	46
20101	28
38723	16
134227	15
246277	11
407893	39
609221	45
652913	7
27089407	30
43214711	13

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