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A Note on a Question of Erdős & Graham

Kellen Myers

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Abstract

Erdős & Graham ask whether the equation $x^2 + y^2 = z^2$ is partition regular, i.e. whether it has a finite Rado number. This note provides a lower bound and also states results in the affirmative for two similar quadratic equations.

1 Introduction

In [Sch16], Schur shows that for any finite coloring of the positive integers (a function $\chi : \mathbb{Z}^+ \to \{1, 2, \ldots, r\}$), there will be a triple (x, y, z) such that $x+y = z$ and $\chi(x) = \chi(y) = \chi(z)$. This triple is said to be a monochromatic solution to the equation $x + y = z$. The least N such that this statement holds for any coloring $\chi : \{1, 2, \ldots, N\} \to \{1, 2, \ldots, r\}$ is called the r-color Schur number and is denoted $S(r)$.

It is easy to see that $S(2) = 5$, and one might note trivially that $S(1) = 2$. The values of $S(3)$ and $S(4)$ are also known. We can associate such a quantity to any equation $\mathcal{E}$, not just $x+y = z$, which denote $R_r(\mathcal{E})$. In cases where no such N exists, we say $R_r(\mathcal{E}) = \infty$. An equation $\mathcal{E}$ is called r-regular if the quantity $R_r(\mathcal{E})$ is finite, and it is called regular if it is r-regular for all r.

In [Rad33], Rado provides necessary and sufficient conditions for linear, homogeneous equations to be regular. He also gives necessary and sufficient conditions (essentially non-triviality) for such equations to be 2-regular. For that reason, we call these quantities "Rado numbers." There have been many papers, starting with [BB82], giving certain Rado numbers (often parametrized families of equations). In most cases the equations are linear and there are 2 colors.

In this note, we state three results from a forthcoming paper [MP] that do not fall into these categories. The equations are quadratic, and in one case, the number of colors is 3.

In [EG80], Erdős and Graham ask whether the equation $x^2+y^2=z^2$ is 2-regular. In [Gra08], Graham notes that it is not clear which answer is correct. In [FH13] it is proved that $9x^2+16y^2=n^2$ (along with a family of similar quadratic equations) is 2-regular, but only x and y are supposed to be monochromatic (note n). We offer the following three results, which have been a part of ongoing work to settle the question:

Theorem 1. $R_2\left(x^2+y^2+z^2=w^2\right)=105$.

Theorem 2. $R_3\left(x_1^2+x_2^2+x_3^2+x_4^2=y_1^2+y_2^2+y_3^2\right)=32$.

Theorem 3. $R_2\left(x^2+y^2=z^2\right)>6500$.

In the third of these results, we do not exclude the possibility that it is infinite.

These results are all obtained computationally and will be detailed in [MP]. The first result also inspires a new sequence representing the Rado number for the equation $x_1^2+x_2^2+\cdots+x_k^2=z^2$, which is tabulated as follows:

$k=$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$N=$	?	105	37	23	18	20	20	15	16	20	23	17	21	26	17	23

This is now entry A250026 in the Online Encyclopedia of Integer Sequences.

References

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[Gra08] Ron Graham, *Old and New Problems in Ramsey Theory*, Bolyai Soc. math. Stud. **17** (2008), 105–118.

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