

Catalan Pairs and Fishburn Triples

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Abstract

Disanto, Ferrari, Pinzani and Rinaldi have introduced the concept of *Catalan pair*, which is a pair of partial orders (S, R) satisfying certain axioms. They have shown that Catalan pairs provide a natural description of objects belonging to several classes enumerated by Catalan numbers.

In this paper, we first introduce another axiomatic structure (T, R) , which we call the *Catalan pair of type 2*, which describes certain Catalan objects that do not seem to have an easy interpretation in terms of the original Catalan pairs.

We then introduce *Fishburn triples*, which are relational structures obtained as a direct common generalization of the two types of Catalan pairs. Fishburn triples encode, in a natural way, the structure of objects enumerated by the Fishburn numbers, such as interval orders or Fishburn matrices. This connection between Catalan objects and Fishburn objects allows us to associate known statistics on Catalan objects with analogous statistics of Fishburn objects. As our main result, we then show that several known equidistribution results on Catalan statistics can be generalized to analogous results for Fishburn statistics.

1 Introduction

The Catalan numbers $C_n = \frac{1}{n+1} \binom{2n}{n}$ are one of the most ubiquitous number sequences in enumerative combinatorics. As of May 2013, Stanley's Catalan Addendum [35] includes over 200 examples of classes of combinatorial objects enumerated by Catalan numbers, and more examples are constantly being discovered.

The Fishburn numbers F_n (sequence A022493 in OEIS [30]) are another example of a counting sequence that arises in several seemingly unrelated contexts. The first widely studied combinatorial class enumerated by Fishburn numbers is the class of interval orders, also known as $(\mathbf{2}+\mathbf{2})$ -free posets. Their study was pioneered by Fishburn [16, 17, 18]. Later, more objects counted by Fishburn numbers were identified, including non-neighbor-nesting matchings [36], ascent sequences [3], Fishburn matrices [12, 14], several classes of pattern-avoiding permutations [3, 31] or pattern-avoiding matchings [26].

It has been observed by several authors that various Fishburn classes contain subclasses enumerated by Catalan numbers [4, 7, 15, 24]. For instance, the Fishburn class of $(\mathbf{2}+\mathbf{2})$ -free posets contains the subclasses of $(\mathbf{2}+\mathbf{2}, \mathbf{3}+\mathbf{1})$ -free posets and $(\mathbf{2}+\mathbf{2}, \mathbf{N})$ -free posets, which are both enumerated by Catalan numbers. The aim of this paper is to describe a close relationship between Catalan and Fishburn classes, of which the above-mentioned inclusions are a direct consequence.

At first sight, Catalan numbers and Fishburn numbers do not seem to have much in common. The Catalan numbers can be expressed a simple formula $C_n = \frac{1}{n+1} \binom{2n}{n}$, they have a simply exponential asymptotic growth $C_n = \Theta(4^n n^{-3/2})$, and admit an algebraic generating function, namely $(1 - \sqrt{1 - 4x})/2$. In contrast, no simple formula for the Fishburn numbers F_n is known, their growth is superexponential ($F_n = \Theta(n!(6/\pi)^n \sqrt{n})$) as shown by Zagier [38]), and their generating function $\sum_{n \geq 0} \prod_{k=1}^n (1 - (1-x)^k)$, derived by Zagier [38], is not even D-finite [3].

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Nevertheless, we will show that there is a close combinatorial relationship between families of objects counted by Catalan numbers and those counted by Fishburn numbers. To describe this relationship, it is convenient to represent Catalan and Fishburn objects by relational structures satisfying certain axioms.

An example of such a structure are the so-called Catalan pairs, introduced by Disanto, Ferrari, Pinzani and Rinaldi [7, 11] (see also [2]). In this paper, we first introduce another Catalan-enumerated family of relational structures which we call Catalan pairs of type 2. Next, we introduce a common generalization of the two types of Catalan pairs, which we call Fishburn triples.

This interpretation of Catalan objects and Fishburn objects by means of relational structures allows us to detect a correspondence between known combinatorial statistics on Catalan objects and their Fishburn counterparts. This allows us to discover new equidistribution results for Fishburn statistics, inspired by analogous previously known results for statistics of Catalan objects.

In this paper, we need a lot of preparation before we can state and prove the main results. After recalling some basic notions related to posets and relational structures (Subsection 1.1), we introduce interval orders and Fishburn matrices, and characterize \mathbf{N} -free and $(\mathbf{3}+\mathbf{1})$ -free interval orders in terms of their Fishburn matrices (Subsection 1.2). In Section 2, we define Catalan pairs of type 1 and 2, which are closely related to \mathbf{N} -free and $(\mathbf{3}+\mathbf{1})$ -free interval orders, respectively. We then observe that several familiar Catalan statistics have a natural interpretation in terms of these pairs. Finally, in Section 3, we introduce Fishburn triples, which are a direct generalization of Catalan pairs. We then state and prove our main results, which, informally speaking, show that certain equidistribution results on Catalan statistics can be generalized to Fishburn statistics.

1.1 Orders and Relations

A *relation* (or, more properly, a *binary relation*) on a set X is an arbitrary subset of the Cartesian product $X \times X$. For a relation $R \subseteq X \times X$ and a pair of elements $x, y \in X$, we write xRy as a shorthand for $(x, y) \in R$. The *inverse* of R , denoted by R^{-1} , is the relation that satisfies $xR^{-1}y$ if and only if yRx . Two elements $x, y \in X$ are *comparable by R* (or *R -comparable*) if at least one of the pairs (x, y) and (y, x) belongs to R .

A relation R on X is said to be *irreflexive* if no element of X is comparable to itself. A relation R is *transitive* if xRy and yRz implies xRz for each $x, y, z \in X$. An irreflexive transitive relation is a *partial order*. A set X together with a partial order relation R on X form a *poset*.

An element $x \in X$ is *minimal* in a relation R (or *R -minimal* for short) if there is no $y \in X$ such that yRx . Maximal elements are defined analogously. This definition agrees with the standard notion of minimal elements in partial orders, but note that we will use this notion even when R is not a partial order. We let $\text{Min}(R)$ denote the set of minimal elements of R , and $\text{min}(R)$ its cardinality. Similarly, $\text{Max}(R)$ is the set of R -maximal elements and $\text{max}(R)$ its cardinality.

A *relational structure* on a set X is an ordered tuple $\mathfrak{R} = (R_1, R_2, \dots, R_k)$ where each R_i is a relation on X . The relations R_i are referred to as the *components* of \mathfrak{R} , and the cardinality of X is the *order* of \mathfrak{R} . Given \mathfrak{R} as above, and given another relational structure $\mathfrak{S} = (S_1, \dots, S_m)$ on a set Y , we say that \mathfrak{R} *contains* \mathfrak{S} (or \mathfrak{S} is a *substructure* of \mathfrak{R}) if $k = m$ and there is an injection $\phi: Y \rightarrow X$ such that for every $i = 1, \dots, k$ and every $x, y \in Y$, we have xS_iy if and only if $\phi(x)R_i\phi(y)$. In such case we call the mapping ϕ an *embedding* of \mathfrak{S} into \mathfrak{R} . If \mathfrak{R} does not contain \mathfrak{S} , we say that \mathfrak{R} *avoids* \mathfrak{S} , or that \mathfrak{R} is *\mathfrak{S} -free*. We say that \mathfrak{R} and \mathfrak{S} are *isomorphic* if there is an embedding of \mathfrak{S} into \mathfrak{R} which maps Y bijectively onto X .

By a slight abuse of terminology, we often identify a relation R of X with a single-component relational structure $\mathfrak{R} = (R)$. Therefore, e.g., saying that a relation R *avoids* a relation S means that the relational structure (R) avoids the relational structure (S) .

Throughout this paper, whenever we deal with the enumeration of relational structures, we treat them as unlabeled objects, that is, we consider an entire isomorphism class as a single object.

We assume the reader is familiar with the concept of Hasse diagram of a partial order. We shall be mainly interested in partial orders that avoid some of the three orders $\mathbf{2}+\mathbf{2}$, $\mathbf{3}+\mathbf{1}$, and \mathbf{N} , depicted on Figure 1. These classes of partial orders have been studied before. The $(\mathbf{2}+\mathbf{2})$ -free posets are known as *interval orders*, and they are the prototypical example of a class of combinatorial structures enumerated by

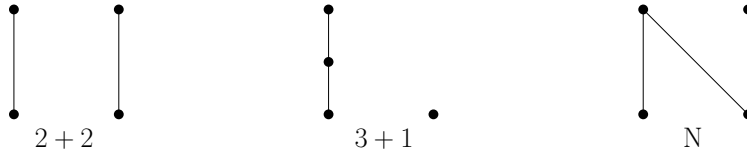


Figure 1: The posets $\mathbf{2+2}$ (left), $\mathbf{3+1}$ (center), and \mathbf{N} (right), which will frequently play the role of forbidden patterns in this paper.

Fishburn numbers. The \mathbf{N} -free posets* are also known as *series-parallel posets*. They are exactly the posets that can be constructed from a single-element poset by a repeated application of direct sums and disjoint unions. The $(\mathbf{3+1})$ -free posets do not seem to admit such a simple characterization, and their structural description is the topic of ongoing research [20, 27, 33, 34].

The $(\mathbf{2+2}, \mathbf{3+1})$ -free posets, i.e., the posets avoiding both $\mathbf{2+2}$ and $\mathbf{3+1}$, are also known as *semiorders*, and have been introduced by Luce [28] in the 1950s. They are enumerated by Catalan numbers [24]. The $(\mathbf{2+2}, \mathbf{N})$ -free posets are enumerated by Catalan numbers as well, as shown by Disanto et al. [7, 10].

1.2 Interval orders and Fishburn matrices

Let us briefly summarize several known facts about interval orders and their representations. A detailed treatment of this topic appears, e.g., in Fishburn’s book [18].

Let $R = (X, \prec)$ be a poset. An *interval representation* of R is a mapping I that associates to every element $x \in X$ a closed interval $I(x) = [l_x, r_x]$ in such a way that for any two elements $x, y \in X$, we have $x \prec y$ if and only if $r_x < l_y$. Note that we allow the intervals $I(x)$ to be degenerate, i.e., to consist of a single point.

A poset has an interval representation if and only if it is $(\mathbf{2+2})$ -free, i.e., if it is an interval order. An interval representation I is *minimal*, if it satisfies these conditions:

- for every $x \in X$, the endpoints of $I(x)$ are positive integers
- there is a positive integer $m \in \mathbb{N}$ such that for every $k \in \{1, 2, \dots, m\}$ there is an interval $I(x)$ whose right endpoint is k , as well as an interval $I(y)$ whose left endpoint is k .

Each interval order R has a unique minimal interval representation. The integer m , which corresponds to the number of distinct endpoints in the minimal representation, is known as *the magnitude* of R . Note that $x \in X$ is a minimal element of R if and only if the left endpoint of $I(x)$ is equal to 1, and x is a maximal element if and only if the right endpoint of $I(x)$ is equal to m .

Two elements x, y of a poset $R = (X, \prec)$ are *indistinguishable* if for every element $z \in X$ we have the equivalences $x \prec z \iff y \prec z$ and $z \prec x \iff z \prec y$. In an interval order R with a minimal interval representation I , two elements x, y are indistinguishable if and only if $I(x) = I(y)$. An interval order is *primitive* if it has no two indistinguishable elements. Every interval order R can be uniquely constructed from a primitive interval order R' by replacing each element $x \in R'$ by a group of indistinguishable elements containing x .

Given a matrix M , we use the term *cell* (i, j) of M to refer to the entry in the i -th row and j -th column of M , and we let $M_{i,j}$ denote its value. We assume that the rows of a matrix are numbered top to bottom, that is, the top row has number 1. The *weight* of a matrix M is the sum of its entries[†]. Similarly, the weight of a row (or a column, or a diagonal) of a matrix is the sum of the entries in this row (or column or diagonal).

*Beware that some authors (e.g. Khamis [23]) give the term ‘ \mathbf{N} -free poset’ a meaning subtly different from ours.

[†]Some earlier papers use the term *size of M* instead of weight of M . However, in other contexts the term ‘size’ often refers to the number of rows of a matrix. We therefore prefer to use the less ambiguous term ‘weight’ in this paper.

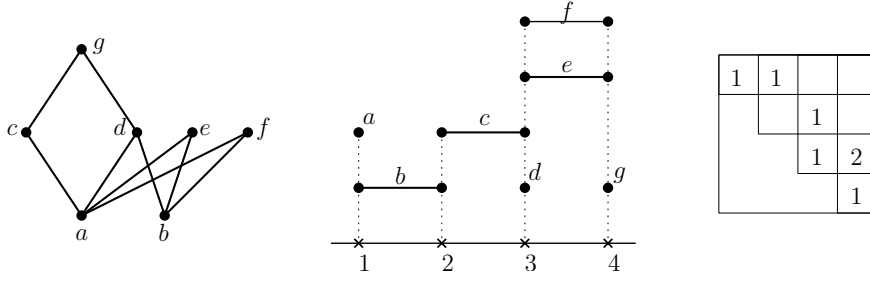


Figure 2: Three Fishburn structures: an interval order, its minimal interval representation, and its Fishburn matrix. In this paper, we use the convention that cells of value 0 in a matrix are depicted as empty boxes.

A *Fishburn matrix* is an upper-triangular square matrix of nonnegative integers whose every row and every column has nonzero weight. In other words, an $m \times m$ matrix M of nonnegative integers is a Fishburn matrix if it satisfies these conditions:

- $M_{i,j} = 0$ whenever $i > j$,
- for every $i \in \{1, \dots, m\}$ there is a j such that $M_{i,j} > 0$, and
- for every $j \in \{1, \dots, m\}$ there is an i such that $M_{i,j} > 0$.

Fishburn matrices were introduced by Fishburn [18] as a convenient way to represent interval orders. Given an interval order $R = (X, \prec)$ of magnitude m with minimal representation I , we represent it by an $m \times m$ matrix M such that $M_{i,j}$ is equal to $|\{x \in X; I(x) = [i, j]\}|$. We then say that an element $x \in X$ is *represented* by the cell (i, j) of M , if $I(x) = [i, j]$. Thus, every element of R is represented by a unique cell of M , and $M_{i,j}$ is the number of elements of R represented by cell (i, j) of M . This correspondence yields a bijection between Fishburn matrices and interval orders. For an element $x \in X$, we let c_x denote the cell of M representing x . Conversely, for a cell c of M , we let X_c be the set of elements of X represented by c . Note that the value in cell c is precisely the cardinality of X_c .

In the rest of this paper, we will use Fishburn matrices as our main family of Fishburn-enumerated object. We now show how the basic features of interval orders translate into matrix terminology.

Observation 1.1. *Let $R = (X, \prec)$ be an interval order, and let M be the corresponding Fishburn matrix. Let x and x' be two elements of R , represented respectively by cells (i, j) and (i', j') of M . The following holds:*

- *The size of R is equal to the weight of M .*
- *The number of minimal elements of R is equal to the weight of the first row of M , and the number of maximal elements of R is equal to the weight of the last column of M .*
- *The elements x and x' are indistinguishable in R if and only if they are represented by the same cell of M , i.e., $i = i'$ and $j = j'$.*
- *We have $x' \prec x$ if and only if $j' < i$.*

Let M be a Fishburn matrix, and let $c = (i, j)$ and $c' = (i', j')$ be two cells of M such that $i \leq j$ and $i' \leq j'$, i.e., the two cells are on or above the main diagonal. We shall frequently use the following terminology (see Figure 3):

- Cell c is *greater* than cell c' (and c' is *smaller* than c), if $j' < i$. The cells c and c' are *incomparable* if neither of them is greater than the other. These terms are motivated by the last part of Observation 1.1. Note that two cells c and c' of M are comparable if and only if the smallest rectangle containing both c and c' also contains at least one cell strictly below the main diagonal of M .

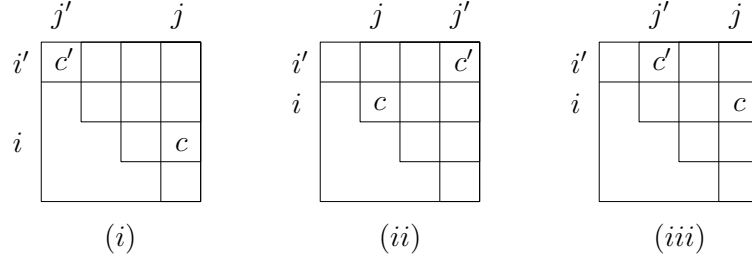


Figure 3: Mutual positions of two cells c and c' in a Fishburn matrix: (i) c is greater than c' , (ii) c is strictly South-West of c' , and (iii) c is strictly South-East of c' .

- Cell c is *South* of c' , if $i > i'$ and $j = j'$. North, West and East are defined analogously. Note that in all these cases, the two cells c and c' are incomparable.
- Cell c is *strictly South-West* (or strictly SW) from c' (and c' is strictly NE from c) if $i > i'$ and $j < j'$. This again implies that the two cells are incomparable.
- Cell c is *strictly South-East* (or strictly SE) from c' (and c' is strictly NW from c) if c and c' are incomparable, and moreover $i > i'$ and $j > j'$.
- Cell c is *weakly SW* from c' if c is South, West of strictly SW of c' . Weakly NE, weakly NW and weakly SE are defined analogously.

With this terminology, $(\mathbf{3}+\mathbf{1})$ -avoidance and \mathbf{N} -avoidance of interval orders may be characterized in terms of Fishburn matrices, as shown by the next lemma, whose first part essentially already appears in the work of Dukes et al. [12]. Refer to Figure 4.

Lemma 1.2. *Let $R = (X, \prec)$ be an interval order represented by a Fishburn matrix M . Let x and y be two elements of X , represented by cells c_x and c_y of M .*

1. *The cell c_x is strictly SW of c_y if and only if X contains two elements u and v which together with x and y induce a copy of $\mathbf{3}+\mathbf{1}$ in R such that $u \prec x \prec v$, and y is incomparable to each of u, x, v .*
2. *The cell c_x is strictly NW of c_y if and only if X contains two elements u and v which together with x and y induce a copy of \mathbf{N} in R such that $u \prec y$, $u \prec v$ and $x \prec v$, and the remaining pairs among u, v, x, y are incomparable.*

In particular, R is $(\mathbf{3}+\mathbf{1})$ -free if and only if the matrix M has no two distinct nonzero cells in a strictly SW position, and R is \mathbf{N} -free if and only if M has no two distinct nonzero cells in a strictly NW position.

Proof. We will only prove the second part of the lemma, dealing with \mathbf{N} -avoidance. The first part can be proved by a similar argument, as shown by Dukes et al. [12, proof of Proposition 16].

Suppose first that R contains a copy of \mathbf{N} induced by four elements u, v, x and y , which form exactly three comparable pairs $u \prec y$, $u \prec v$ and $x \prec v$. Let $c_u = (i_u, j_u)$, $c_v = (i_v, j_v)$, $c_x = (i_x, j_x)$ and $c_y = (i_y, j_y)$ be the cells of M representing these four elements. We claim that c_x is strictly NW from c_y . To see this, note that c_x is smaller than c_v while c_y is not smaller than c_v , hence $j_x < i_v \leq j_y$. Similarly, c_y is greater than c_u while c_x is not, implying that $i_x \leq j_u < i_y$. Since c_x and c_y are incomparable, it follows that c_x is strictly NW from c_y .

Conversely, suppose that M has two nonzero cells $c_x = (i_x, j_x)$ and $c_y = (i_y, j_y)$, with c_x being strictly NW from c_y . We thus have the inequalities $i_x < i_y \leq j_x < j_y$, where the inequality $i_y \leq j_x$ follows from the fact that c_x and c_y are incomparable. Let c_u be any nonzero cell in column i_x . This choice guarantees that c_u is incomparable to c_x and smaller than c_y . Similarly, let c_v be a nonzero cell in row j_y . Then c_v is incomparable to c_y and greater than both c_x and c_u . Any four elements x, y, u and v of R represented by the four cells c_x, c_y, c_u and c_v induce a copy of \mathbf{N} . \square

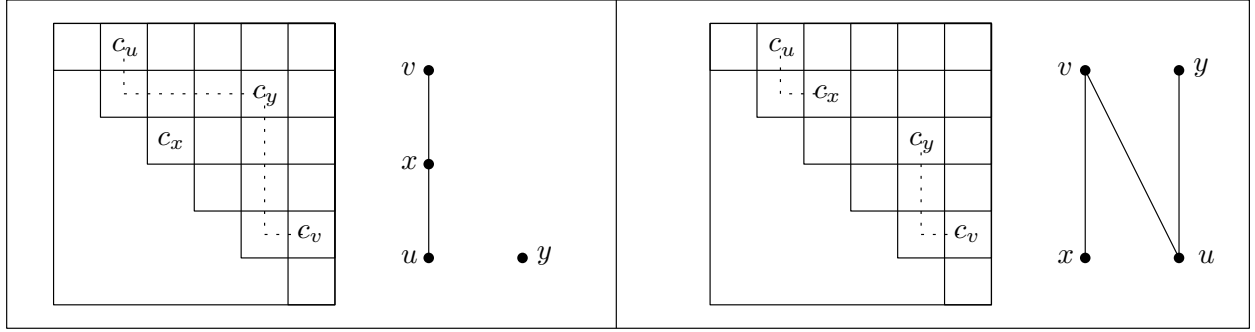


Figure 4: The two parts of Lemma 1.2. Left: if c_x is strictly SW of c_y , we get an occurrence of $\mathbf{3+1}$. Right: if c_x is strictly NW of c_y , we get an occurrence of \mathbf{N} .

2 Catalan pairs

We begin by defining the concept of Catalan pairs of type 1, originally introduced by Disanto et al. [7], who called them simply ‘Catalan pairs’.

Definition 2.1. A *Catalan pair of type 1* (or *C1-pair* for short) is a relational structure (S, R) on a finite set X satisfying the following axioms.

C1a) S and R are both partial orders on X .

C1b) Any two distinct elements of X are comparable by exactly one of the orders S and R .

C1c) For any three distinct elements x, y, z satisfying xSy and yRz , we have xRz .

We let C_n^I denote the set of unlabeled C1-pairs on n vertices.

Note that axiom C1c might be replaced by the seemingly weaker condition that X does not contain three distinct elements x, y and z satisfying xSy, yRz and xSz (see the left part of Figure 5). Indeed, if xSy and yRz holds for some x, y and z , then both zRx and zSx would contradict axiom C1a or C1b, so the role of C1c is merely to exclude the possibility xSz , leaving xRz as the only option. This also shows that the definition of C1-pairs would not be affected if we replaced C1c by the following axiom, which we denote C1c*: “For any three distinct elements x, y, z satisfying xSy and zRy , we have zRx .”

We remark that it is easy to check that in a C1-pair (S, R) on a vertex set X , the relation $S \cup R$ is a linear order on X .

As shown by Disanto et al. [7], if (S, R) is a C1-pair, then R is a $(\mathbf{2+2}, \mathbf{N})$ -free poset, and conversely, for any $(\mathbf{2+2}, \mathbf{N})$ -free poset R , there is, up to isomorphism, a unique C1-pair (S, R) . Since there are, up to isomorphism, C_n distinct $(\mathbf{2+2}, \mathbf{N})$ -free posets on an n -elements set, this implies that C1-pairs are enumerated by Catalan numbers. In fact, the next lemma shows that the first component of a C1-pair (S, R) can be easily reconstructed from the Fishburn matrix representing the second component.

Lemma 2.2. *Let (S, R) be a C1-pair, and let M be the Fishburn matrix representing the poset R . Then S has the following properties:*

- (a) *If c is a nonzero cell of M and X_c the set of elements represented by c , then X_c is a chain in S , that is, the restriction of S to X_c is a linear order.*
- (b) *If c and d are distinct nonzero cells of M , representing sets of elements X_c and X_d respectively, and if c is weakly SW from d , then for any $x \in X_c$ and $y \in X_d$ we have xSy .*
- (c) *Apart from the situations described in parts (a) and (b), no other pair $(x, y) \in X^2$ is S -comparable.*

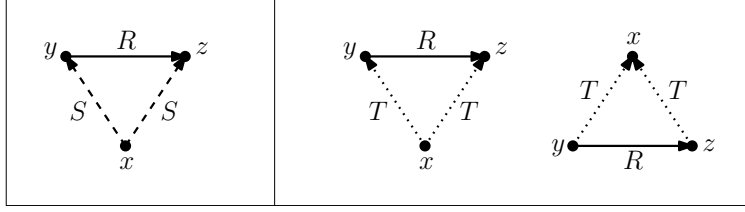


Figure 5: Left: the substructure forbidden in C1-pairs by axiom C1c. Right: the two substructures forbidden in C2-pairs by axiom C2c.

Proof. Let (S, R) be a C1-pair. Property (a) of the Lemma follows directly from the fact that the elements of X_c form an antichain in R . To prove property (b), fix distinct nonzero cells $c = (i_c, j_c)$ and $d = (i_d, j_d)$ such that c is weakly SW from d , and choose $x \in X_c$ and $y \in X_d$ arbitrarily. Since x and y are R -incomparable by Observation 1.1, we must have either xSy or ySx . Suppose for contradiction that ySx holds. As c is weakly SW from d , we know that $i_c > i_d$ or $j_c < j_d$. Suppose that $i_c > i_d$, as the case $j_c < j_d$ is analogous. Let e be any nonzero cell in column i_d of M , and let $z \in X$ be an element represented by e . Note that e is smaller than c and incomparable to d . It follows that y and z are comparable by S . However, if zSy holds, then we get a contradiction with the transitivity of S , due to ySx and zRx . On the other hand, if ySz holds, we see that x, y and z induce the substructure forbidden by axiom C1c.

To see that property (c) holds, note that by Lemma 1.2, the matrix M has no two nonzero cells in strictly NW position. Thus, if x and y are distinct elements of X represented by cells c_x and c_y , then either $c_x = c_y$ and x, y are S -comparable by property (a), or c_x and c_y are in a weakly SW position and x, y are S -comparable by (b), or one of the two cells is smaller than the other, which means that the two elements are R -comparable. \square

Various Catalan-enumerated objects, such as noncrossing matchings, Dyck paths, or 132-avoiding permutations, can be encoded in a natural way as C1-pairs (see [7]). However, there are also examples of Catalan objects, such as nonnesting matchings or $(\mathbf{2+2}, \mathbf{3+1})$ -free posets, possessing a natural underlying structure that satisfies a different set of axioms. This motivates our next definition.

Definition 2.3. A *Catalan pair of type 2* (or *C2-pair* for short) is a relational structure (T, R) on a finite set X with the following properties.

- C2a) R and $T \cup R$ are both partial orders on X .
- C2b) Any two distinct elements of X are comparable by exactly one of the relations T and R .
- C2c) There are no three distinct elements $x, y, z \in X$ satisfying xTy, xTz and yRz , and also no three elements $x, y, z \in X$ satisfying yTx, zTx and yRz .

We let \mathcal{C}_n^{II} denote the set of unlabeled C2-pairs on n vertices.

Lemma 2.4. *If (T, R) is a C2-pair on a vertex set X , then R is a $(\mathbf{2+2}, \mathbf{3+1})$ -free poset.*

Proof. Let (T, R) be a C2-pair. Suppose for contradiction that R contains a copy of $\mathbf{2+2}$ induced by four elements $\{x, y, u, v\}$, with xRy and uRv . Any two R -incomparable elements must be comparable in T , so we may assume, without loss of generality, that xTu holds. Then xTv holds by transitivity of $T \cup R$, contradicting axiom C2c. Similar reasoning shows that R is $(\mathbf{3+1})$ -free. \square

As in the case of C1-pairs, a C2-pair (T, R) is uniquely determined by its second component, and its first component can be easily reconstructed from the Fishburn matrix representing the second component.

Lemma 2.5. *Let R be $(\mathbf{2+2}, \mathbf{3+1})$ -free poset represented by a Fishburn matrix M . Let T be a relation satisfying these properties:*

- (a) If c is a nonzero cell of M and X_c the set of elements represented by c , then X_c is a chain in T , that is, the restriction of T to X_c is a linear order.
- (b) If c and d are distinct nonzero cells of M , representing sets of elements X_c and X_d respectively, and if c is weakly NW from d , then for any $x \in X_c$ and $y \in X_d$ we have xTy .
- (c) Apart from the situations described in parts (a) and (b), no other pair $(x, y) \in X^2$ is T -comparable.

Then (T, R) is a C2-pair. Moreover, if (T', R) is another C2-pair, then (T, R) and (T', R) are isomorphic relational structures. It follows that on an n -element vertex set X , there are C_n isomorphism types of Catalan pairs, where C_n is the n -th Catalan number.

Proof. Let R be a $(\mathbf{2+2, 3+1})$ -free partial order on a ground set X , represented by a Fishburn matrix M . Let T be a relation satisfying properties (a), (b) and (c) of the lemma.

We claim that (T, R) is a C2-pair. It is easy to see that (T, R) satisfies axiom C2a. Let us verify that C2b holds as well, i.e., any two distinct elements $x, y \in X$ are comparable by exactly one of T, R . Since clearly no two elements can be simultaneously T -comparable and R -comparable, it is enough to show that any two R -incomparable elements $x, y \in X$ are T -comparable. If x and y are also R -indistinguishable, then they are represented by the same cell c of M , and they are T -comparable by (a). If x and y are R -distinguishable, then they are represented by two distinct cells, say c_x and c_y . By Lemma 1.2, M has no pair of nonzero cells in strictly SW position, and hence c_x and c_y must be in weakly NW position. Hence x and y are T -comparable by (b).

To verify axiom C2c, assume first, for contradiction, that there are three elements $x, y, z \in X$ satisfying xTy , xTz and yRz . Let $c_x = (i_x, j_x)$, $c_y = (i_y, j_y)$ and $c_z = (i_z, j_z)$ be the cells of M representing x, y and z , respectively. Since c_y is smaller than c_z , we see that $j_y < i_z$. Since (x, y) belongs to T , we see that either $c_x = c_y$ or c_x is weakly NW from c_y . In any case, c_x is smaller than c_z , contradicting xTz . An analogous argument shows that there can be no three elements x, y, z satisfying yTx , zTx and yRz . We conclude that (T, R) is a C2-pair.

It remains to argue that for the $(\mathbf{2+2, 3+1})$ -free poset R , there is, up to isomorphism, at most one relation T forming a C2-pair with R . Fix a relation T' such that (T', R) is a C2-pair. We will first show that T' satisfies the properties (a), (b) and (c) of the lemma. Property (a) follows directly from axioms C2a and C2b.

To verify property (b), fix elements $x, y \in X$ represented by distinct cells $c_x = (i_x, j_x)$ and $c_y = (i_y, j_y)$ such that c_x is weakly NW from c_y . We claim that $(x, y) \in T'$. Suppose for contradiction that this is not the case. Since x and y are R -incomparable, this means that $(y, x) \in T'$ by axiom C2b. Since c_x is weakly NW from c_y , we know that $i_x < i_y$ or $j_x < j_y$. Suppose that $i_x < i_y$ (the case $j_x < j_y$ is analogous). Let c_z be any nonzero cell in column i_x and $z \in X$ an element represented by c_z . This choice of z guarantees that $(z, y) \in R$ while z and x are R -incomparable. From zRy and $yT'x$ we deduce, by the transitivity of $R \cup T'$ and the R -incomparability of x and z , that (z, x) belongs to T' , which contradicts axiom C2c. The fact that T' satisfies (c) follows from axiom C2b and the fact that no two nonzero cells of M are in strictly SW position.

We conclude that when (T, R) and (T', R) are C2-pairs, the relations T and T' may only differ by prescribing different linear orders on the classes of R -indistinguishable elements. This easily implies that (T, R) and (T', R) are isomorphic relational structures. \square

2.1 Statistics on Catalan pairs

We have seen that Catalan pairs naturally encode the structure of Fishburn matrices representing $(\mathbf{2+2, N})$ -free and $(\mathbf{2+2, 3+1})$ -free posets. However, to exploit known combinatorial properties of Catalan-enumerated objects, it is convenient to relate Catalan pairs to more familiar Catalan objects. Our Catalan objects of choice are the Dyck paths.

Definition 2.6. A *Dyck path* of order n is a lattice path P joining the point $(0, 0)$ with the point (n, n) , consisting of n *up-steps* and n *right-steps*, where an up-step joins a point (i, j) to $(i, j + 1)$ and a right-step

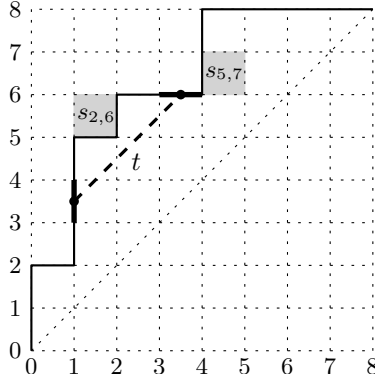


Figure 6: A Dyck path (solid) with a tunnel t (dashed). The two steps associated to t are highlighted in bold. The unit square $s_{5,7}$ is below the path while $s_{2,6}$ is above it.

joins (i, j) to $(i + 1, j)$, and moreover, every point $(i, j) \in P$ satisfies $i \leq j$. We let \mathcal{D}_n denote the set of Dyck paths of order n .

For a Dyck path P , we say that a step s of P precedes a step s' of P if s appears on P before s' when we follow P from $(0, 0)$ to (n, n) .

Given a Dyck path P , a *tunnel* of P is a segment t parallel to the diagonal line of equation $y = x$, such that the bottom-left endpoint of t is in the middle of an up-step of P , the top-right endpoint is in the middle of a right-step of P , and all the internal points of t are strictly below the path P . See Figure 6. We refer to the up-step and the right-step that contain the two endpoints of t as *the steps associated to t* .

Note that a Dyck path of order n has exactly n tunnels, and every step of the path is associated to a unique tunnel.

Let t_1 and t_2 be two distinct tunnels of a Dyck path P . Let u_i and r_i be the up-step and right-step associated to t_i , respectively. Suppose that u_1 precedes u_2 on the path P . If r_2 precedes r_1 on P , we say that t_2 is *nested* within t_1 . If r_1 precedes r_2 , then we easily see that r_1 also precedes u_2 on P . In such case, we say that t_1 *precedes* t_2 .

The following construction, due to Disanto et al. [7], shows how to represent a Dyck path by a C1-pair.

Fact 2.7 ([7]). *Let P be a Dyck path and let X be the set of tunnels of P . Define a relational structure $C^I(P) = (S, R)$ on X as follows: for two distinct tunnels t_1 and t_2 , put $(t_1, t_2) \in S$ if and only if t_1 is nested within t_2 , and $(t_1, t_2) \in R$ if and only if t_1 precedes t_2 . Then $C^I(P) = (S, R)$ is a C1-pair, and this construction yields a bijection between Dyck paths of order n and isomorphism types of C1-pairs of order n .*

There is also a simple way to encode a Dyck path by a C2-pair, which we now describe. Let P be a Dyck path of order n and let $(i, j) \in \{1, \dots, n\}^2$ be a lattice point. Let $s_{i,j}$ be the axis-aligned unit square whose top-right corner is the point (i, j) (see Figure 6). We say that $s_{i,j}$ is *above* the Dyck path P if the interior of $s_{i,j}$ is above P (with the boundary of $s_{i,j}$ possibly overlapping with P). Notice that $s_{i,j}$ is above P if and only if the i -th right-step of P is preceded by at most j up-steps. If $s_{i,j}$ is not above P , we say that it is *below* P ; see Figure 6.

Lemma 2.8. *Let P be a Dyck path of order n , and let X be the set $\{1, 2, \dots, n\}$. Define a relational structure $C^{II}(P) = (T, R)$ on X as follows: for any $i, j \in X$, put $(i, j) \in T$ if and only if $i < j$ and $s_{i,j}$ is below P , and put $(i, j) \in R$ if and only if $i < j$ and $s_{i,j}$ is above P . Then $C^{II}(P)$ is a C2-pair, and this construction is a bijection between Dyck paths of order n and isomorphism types of C2-pairs.*

Proof. It is routine to verify that $C^{II}(P)$ satisfies the axioms of C2-pairs, and that distinct Dyck paths map to distinct C2-pairs. Since both Dyck paths of order n and C2-pairs of order n are enumerated by Catalan numbers, the mapping is indeed a bijection. \square

We will now focus on combinatorial statistics on Catalan objects. For our purposes, a *statistic* on a set \mathcal{A} is any nonnegative integer function $f: \mathcal{A} \rightarrow \mathbb{N}_0$. For an integer k and a statistic f , we use the notation $\mathcal{A}[f = k]$ as shorthand for $\{x \in \mathcal{A} : f(x) = k\}$. This notation extends naturally to two or more statistics, e.g., if g is another statistic on \mathcal{A} , then $\mathcal{A}[f = k, g = \ell]$ denotes the set $\{x \in \mathcal{A} : f(x) = k, g(x) = \ell\}$. Two statistics f and g are *equidistributed* (or *have the same distribution*) on \mathcal{A} , if $\mathcal{A}[f = k]$ and $\mathcal{A}[g = k]$ have the same cardinality for every k . The statistics f and g have *symmetric joint distribution* on \mathcal{A} , if $\mathcal{A}[f = k, g = \ell]$ and $\mathcal{A}[f = \ell, g = k]$ have the same cardinality for every k and ℓ . Clearly, if f and g have symmetric joint distribution, they are equidistributed.

Recall that for a binary relation R on a set X , an element $x \in X$ is *minimal*, if there is no $y \in X \setminus \{x\}$ such that (y, x) is in R . Recall also that $\text{Min}(R)$ is the set of minimal elements of R , and $\text{min}(R)$ is the cardinality of $\text{Min}(R)$. Similarly, $\text{Max}(R)$ is the set of maximal elements of R , and $\text{max}(R)$ its cardinality.

Let P be a Dyck path of order n , and let $C^I(P) = (S, R)$ be the corresponding C1-pair, as defined in Fact 2.7. We will be interested in the four statistics $\text{min}(S)$, $\text{max}(S)$, $\text{min}(R)$ and $\text{max}(R)$. It turns out that these statistics correspond to well known statistics of Dyck paths. To describe the correspondence, we need more terminology.

For a Dyck path P , the *initial ascent* is the maximal sequence of up-steps preceding the first right-step, and the *final descent* is the maximal sequence of right-steps following the last up-step. Let $\text{asc}(P)$ and $\text{des}(P)$ denote the length of the initial ascent and final descent of P , respectively. A *return* of P is a right-step whose right endpoint touches the diagonal line $y = x$. Let $\text{ret}(P)$ denote the number of returns of P . Finally, a *peak* of P is an up-step of P that is immediately followed by a right-step, and $\text{pea}(P)$ denotes the number of peaks of P .

Observation 2.9. *Let P be a Dyck path and let $(S, R) = C^I(P)$ be the corresponding C1-pair.*

- *A tunnel t of P is minimal in R if and only if the up-step associated to t precedes the first right-step of P . In particular $\text{min}(R) = \text{asc}(P)$. Symmetrically, t is maximal in R if and only if its associated right-step succeeds the last up-step of P . Hence, $\text{max}(R) = \text{des}(P)$.*
- *A tunnel t of P is maximal in S if and only if its associated right-step is a return of P . Hence, $\text{max}(S) = \text{ret}(P)$.*
- *A tunnel t of P is minimal in S if and only if its associated up-step is immediately succeeded by its associated right-step. Hence, $\text{min}(S) = \text{pea}(P)$.*

Suppose now that $(T, R) = C^{II}(P)$ is a C2-pair representing a Dyck path P by the bijection of Lemma 2.8. We again focus on the statistics $\text{min}(T)$, $\text{max}(T)$, $\text{min}(R)$ and $\text{max}(R)$. It turns out that they again correspond to the Dyck path statistics introduced above.

Observation 2.10. *Let P be a Dyck path of order n and let $(T, R) = C^{II}(P)$ be the corresponding C2-pair.*

- *An element $i \in \{1, 2, \dots, n\}$ is minimal in R if and only if the unit square $s_{1,i}$ is below P , which happens if and only if $\text{asc}(P) \geq i$. In particular, $\text{asc}(P) = \text{min}(R)$. Symmetrically, $\text{des}(P) = \text{max}(R)$.*
- *An element $i \in \{1, 2, \dots, n\}$ is minimal in T if and only if for every $j < i$, the unit square $s_{j,i}$ is above P , which is if and only if the point $(i-1, i-1)$ belongs to P . It follows that $\text{min}(T) = \text{ret}(P)$. Symmetrically, $i \in \{1, 2, \dots, n\}$ is maximal in T if and only if for every $j > i$ the square $s_{i,j}$ is above P , which is if and only if the point (i, i) belongs to P . It follows that $\text{max}(T) = \text{ret}(P)$.*

The statistics asc , des , ret and pea are all very well studied. We now collect some known facts about them.

Fact 2.11. *Let \mathcal{D}_n be the set of Dyck paths of order n .*

- *The sets $\mathcal{D}_n[\text{asc} = k]$, $\mathcal{D}_n[\text{des} = k]$ and $\mathcal{D}_n[\text{ret} = k]$ all have cardinality $\frac{k}{2n-k} \binom{2n-k}{n}$. See [30, A033184]. Among the three statistics asc , des and ret , any two have symmetric joint distribution on \mathcal{D}_n .*

- The set $\mathcal{D}_n[\text{pea} = k]$ has cardinality $N(n, k) = \frac{1}{k} \binom{n-1}{k-1} \binom{n}{k-1}$. The numbers $N(n, k)$ are known as Narayana numbers [30, A001263]. The Narayana numbers satisfy $N(n, k) = N(n, n - k + 1)$, and hence $\mathcal{D}_n[\text{pea} = k]$ has the same cardinality as $\mathcal{D}_n[\text{pea} = n - k + 1]$.
- In fact, for any n, k, ℓ , and m , the set $\mathcal{D}_n[\text{asc} = k, \text{ret} = \ell, \text{pea} = m]$ has the same cardinality as the set $\mathcal{D}_n[\text{asc} = \ell, \text{ret} = k, \text{pea} = n - m + 1]$. This follows, e.g., from an involution on Dyck paths constructed by Deutsch [6].

For future reference, we rephrase some of these facts in the terminology of Catalan pairs. Recall that \mathcal{C}_n^I and \mathcal{C}_n^{II} denote respectively the set of C1-pairs and the set of C2-pairs of order n .

Proposition 2.12. *For any $n \geq 0$, there is a bijection $\psi: \mathcal{C}_n^I \rightarrow \mathcal{C}_n^{II}$ with the following properties. Let $(S, R) \in \mathcal{C}_n^I$ be a C1-pair, and let $(T', R') \in \mathcal{C}_n^{II}$ be its image under ψ . Then*

- $\max(S) = \max(T')$, and
- $\max(R) = \max(R')$.

Proof. Given a C1-pair $(S, R) \in \mathcal{C}_n^I$, fix the Dyck path $P \in \mathcal{D}_n$ satisfying $(S, R) = C^I(P)$, and define the C2-pair $(T', R') = C^{II}(P)$. The mapping $(S, R) \mapsto (T', R')$ is the bijection ψ . By Observations 2.9 and 2.10, we have $\max(S) = \text{ret}(P) = \max(T')$, and $\max(R) = \text{des}(P) = \max(R')$. \square

Proposition 2.13. *For each $n \geq 0$, the two statistics $\max(S)$ and $\max(R)$ have symmetric joint distribution on C1-pairs $(S, R) \in \mathcal{C}_n^I$.*

Proof. This follows from Observation 2.9 and from the first part of Fact 2.11. \square

3 Fishburn triples

In this section, we will show that objects from certain Fishburn-enumerated families can be represented by a triple (T, S, R) of relations, satisfying axioms which generalize the axioms of Catalan pairs. This will allow us to extend the statistics asc , des , ret and pea to Fishburn objects, where they admit a natural combinatorial interpretation. We will then show, as the main results of this paper, that some of the classical equidistribution results listed in Fact 2.11 can be extended to Fishburn objects.

Definition 3.1. A *Fishburn triple* (or *F-triple* for short) is a relational structure (T, S, R) on a set X satisfying the following axioms:

- Fa) S, R , and $T \cup R$ are partial orders on X .
- Fb) Any two distinct elements of X are comparable by exactly one of T, S , or R .
- C1c) For any three distinct elements x, y, z satisfying xSy and yRz we have xRz .
- C1c*) For any three distinct elements x, y, z satisfying xSy and zRy we have zRx .
- C2c) There are no three distinct elements $x, y, z \in X$ satisfying xTy, xTz and yRz , and also no three elements $x, y, z \in X$ satisfying yTx, zTx and zRy .

As with Catalan pairs, the Fishburn triples may equivalently be described as structures avoiding certain substructures of size at most three; see Figure 7.

Observe that a relational structure (S, R) is a C1-pair if and only if (\emptyset, S, R) is an F-triple, and a relational structure (T, R) is a C2-pair if and only if (T, \emptyset, R) is an F-triple. In this way, F-triples may be seen as a common generalization of the two types of Catalan pairs.

Note also that (T, S, R) is an F-triple if and only if (T^{-1}, S, R^{-1}) is an F-triple. We will refer to the mapping $(T, S, R) \mapsto (T^{-1}, S, R^{-1})$ as the *trivial involution* on F-triples. We may restrict the trivial involution to C1-pairs and C2-pairs, with a C1-pair (S, R) being mapped to (S, R^{-1}) and a C2-pair (T, R) being mapped to (T^{-1}, R^{-1}) . When representing Catalan pairs of either type as Dyck paths, as we did in Subsection 2.1, the trivial involution acts on Dyck paths of order n as a mirror reflection whose axis is the line $x + y = n$.

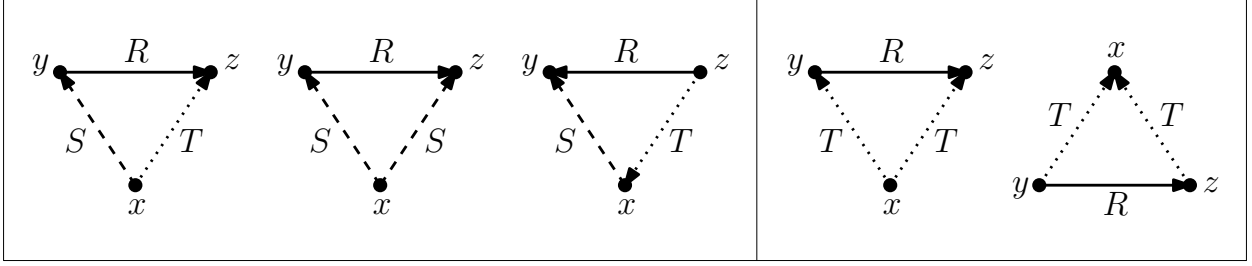


Figure 7: Left: the three minimal structures satisfying axioms Fa and Fb, but not C1c or C1c*. Right: the two structures excluded by axiom C2c.

Lemma 3.2. *If (T, S, R) is an F-triple on a vertex set X , then R is an interval order. Let M be the Fishburn matrix of R . Let x and y two elements of X , represented by cells c_x and c_y of M .*

1. *If c_x is strictly SW of c_y , then $(x, y) \in S$.*
2. *If c_x is strictly NW of c_y , then $(x, y) \in T$.*
3. *If $(x, y) \in S$, then c_x is weakly SW of c_y .*
4. *If $(x, y) \in T$, then c_x is weakly NW of c_y .*

Proof. Let (T, S, R) be an F-triple on a set X . Let us prove that R is $(\mathbf{2}+\mathbf{2})$ -free. For contradiction, assume X contains four distinct elements x, x', y, y' , such that (x, x') and (y, y') belong to R , while all other pairs among these four elements are R -incomparable. By axiom Fb, x and y are either S -comparable or T -comparable. However, if xSy holds, then C1c implies that xRy' holds as well, which is impossible. If xTy holds, then transitivity of $T \cup R$ implies xTy' or xRy' . However, xTy' is excluded by axiom C2c and xRy' contradicts the choice of x, x', y, y' . This shows that R is $(\mathbf{2}+\mathbf{2})$ -free.

We now prove the four numbered claims of the lemma. Let M be the Fishburn matrix of R , and let x and y be two elements of X represented by cells $c_x = (i_x, j_x)$ and $c_y = (i_y, j_y)$.

To prove the first claim, suppose that c_x is strictly SW of c_y . By Lemma 1.2, there are two elements $u, v \in X$ such that u, v, x, y induce a copy of $\mathbf{3}+\mathbf{1}$, where (u, x) , (x, v) , and (u, v) belong to R , and y is R -incomparable to x , u , and v . Now ySx would imply yRv by C1c, which is impossible, since y and v are R -incomparable. Next, yTx would imply yTv by transitivity of $T \cup R$, contradicting C2c. Similarly, xTy implies uTy , again contradicting C2c. This leaves xSy as the only option.

To prove the second claim, suppose that c_x is strictly NW of c_y . By Lemma 1.2, there are elements u and v such that u, v, x and y induce a copy of \mathbf{N} in R , with precisely the three pairs (x, v) , (u, v) and (u, y) belonging to R . We want to prove that (x, y) is in T . Consider the alternatives: yTx forces yTv by transitivity of $T \cup R$, contradicting C2c; on the other hand xSy forces xRu by C1c*, while ySx forces yRv by C1c, which both contradict the choice of u and v . We conclude that yTx is the only possibility.

For the third claim, proceed by contradiction, and assume that xSy holds, but c_x is not weakly SW of c_y . This means that $i_x < i_y$ or $j_x > j_y$. Suppose that $i_x < i_y$, the other case being analogous. Let c_z be a nonzero cell of M in column i_x , and let z be an element represented by c_z . By the choice of z , we know that zRy holds and that x and z are R -incomparable. This contradicts axiom C1c*.

Finally, suppose that xTy holds and c_x is not weakly NW from c_y . Since x and y are R -incomparable, this means that $i_x > i_y$ or $j_x > j_y$. Suppose that $i_x > i_y$. Let z be an element represented by a cell in column i_y , so that zRx holds, while y and z are R -incomparable. Transitivity of $T \cup R$ implies zTy , contradicting C2c. \square

Suppose we are given an interval order R on a set X , with a corresponding Fishburn matrix M , and we would like to extend R into an F-triple (T, S, R) . The four conditions in Lemma 3.2 put certain constraints on T and S , but in general they do not determine T and S uniquely. In particular, if x and y are two

elements represented by cells c_x and c_y that belong to the same row or column of M , then Lemma 3.2 does not say whether x and y should be T -comparable or S -comparable.

To obtain a unique Fishburn triple for a given R , we need to impose additional restrictions to disambiguate the relations between elements represented by cells in the same row or column of M . We will consider two ways of imposing such restrictions. In the first way, all ambiguous pairs end up S -comparable, while in the second way they will be T -comparable.

Definition 3.3. Let R be an interval order on a set X , and let M be its Fishburn matrix. The *Fishburn triple of type 1* of R (or *F1-triple of R*) is the relational structure (T_1, S_1, R) on the set X , in which T_1 and S_1 are determined by these rules:

- For any two elements $x, y \in X$, represented by distinct cells c_x and c_y in M , we have xT_1y if and only if c_x is strictly NW from c_y , and we have xS_1y if and only if c_x is weakly SW from c_y .
- If c is a cell of M and $X_c \subseteq X$ the set of elements represented by c , then S_1 induces a chain in X_c and T_1 induces an antichain in X_c .

Similarly, for R and M as above, a *Fishburn triple of type 2* of R (or *F2-triple of R*) is the relational structure (T_2, S_2, R) on X , with T_2 and S_2 defined as follows:

- For any two elements $x, y \in X$, represented by distinct cells c_x and c_y in M , we have xT_2y if and only if c_x is weakly NW from c_y , and we have xS_2y if and only if c_x is strictly SW from c_y .
- If c is a cell of M and $X_c \subseteq X$ the set of elements represented by c , then S_2 induces an antichain in X_c and T_2 induces a chain in X_c .

Note that the F1-triple and the F2-triple are up to isomorphism uniquely determined by the interval order R .

Lemma 3.4. For an interval order R , its F1-triple (T_1, S_1, R) and its F2-triple (T_2, S_2, R) are Fishburn triples.

Proof. Consider the F1-triple (T_1, S_1, R) . Clearly, it satisfies the axioms Fa and Fb of F-triples. To check axiom C1c, pick three elements $x, y, z \in X$, with xS_1y and yRz , and let $c_x = (i_x, j_x)$, $c_y = (i_y, j_y)$ and $c_z = (i_z, j_z)$ be the corresponding cells of M . Then xS_1y implies $j_x \leq j_y$, and yRz implies $j_y < i_z$. Together this proves $j_x < i_z$, and consequently xRz , and axiom C1c holds. Axiom C1c* can be proved by an analogous argument.

To prove axiom C2c, consider again three elements $x, y, z \in X$ represented by the cells $c_x = (i_x, j_x)$, $c_y = (i_y, j_y)$ and $c_z = (i_z, j_z)$. To prove the first part of the axiom, assume for contradiction that xT_1y , xT_1z and yRz holds. Since c_x is strictly NW from c_y , we have $j_x < j_y$. From yRz we get $j_y < i_z$, hence $j_x < i_z$ implying xRz , which is a contradiction. The second part of axiom C2c is proved analogously.

An analogous reasoning applies to (T_2, S_2, R) as well. □

Note that the trivial involution on F-triples maps F1-triples to F1-triples and F2-triples to F2-triples.

From Lemma 1.2, we deduce that an interval order R is \mathbf{N} -free if and only if its F1-triple has the form (\emptyset, S, R) , which means that (S, R) is a C1-pair, while R is $(\mathbf{3}+\mathbf{1})$ -free if and only if its F2-triple has the form (T, \emptyset, R) , implying that (T, R) is a C2-pair. Thus, F1-triples are a generalization of C1-pairs, while F2-triples generalize C2-pairs.

Our main goal is to use F1-triples and F2-triples to identify combinatorial statistics on Fishburn objects that satisfy nontrivial equidistribution properties. Inspired by the Catalan statistics explored in Subsection 2.1, we focus on the statistics that can be expressed as the number of minimal or maximal elements of a component of an F1-triple or an F2-triple.

For an interval order R , let (T_1, S_1, R) be its F1-triple and (T_2, S_2, R) its F2-triple. We may then consider the number of minimal and maximal elements in each of the five relations T_1 , S_1 , T_2 , S_2 , and R , for a total of ten possible statistics.

In fact, we have no nontrivial result for the two statistics $\min(S_1)$ and $\min(S_2)$. Furthermore, the trivial involution on F-triples maps the minimal elements of T_1 to maximal elements of T_1 and vice versa, and the same is true for T_2 and R as well. Therefore we will not treat the statistics $\min(T_1)$, $\min(T_2)$ and $\min(R)$ separately, and we focus on the five statistics $\max(S_1)$, $\max(S_2)$, $\max(T_1)$, $\max(T_2)$, and $\max(R)$.

To gain an intuition for these five statistics, let us describe them in terms of Fishburn matrices. Let (T_1, S_1, R) and (T_2, S_2, R) be as above, and let M be the Fishburn matrix representing the interval order R .

Recall from Observation 1.1 that $\max(R)$ equals the weight of the last column of M . To get a similar description for the remaining four statistics of interest, we need some terminology. Let us say that a cell c of the matrix M is *strong-NE extreme cell* (or *sNE-cell* for short), if c is a nonzero cell of M , and any other cell strongly NE from c is a zero cell. Similarly, c is a *weak-NE extreme cell* (or *wNE-cell*) if it is a nonzero cell and any other cell weakly NE from c is a zero cell. Note that every weak-NE extreme cell is also a strong-NE extreme cell; in particular, being strong-NE extreme is actually a weaker property than being weak-NE extreme. In an obvious analogy, we will also refer to sSE-cells, wSE-cells, etc.

Observation 3.5. *Let R be an interval order represented by a Fishburn matrix M , let (T_1, S_1, R) be its F1-triple and (T_2, S_2, R) be its F2-triple. Then*

- $\max(S_1)$ is equal to the number of wNE-cells of M ,
- $\max(S_2)$ is equal to the total weight of the sNE-cells of M ,
- $\max(T_1)$ is equal to the total weight of the sSE-cells of M , and
- $\max(T_2)$ is equal to the number of wSE-cells of M .

We are finally ready to state our main results. Let \mathcal{F}_n denote the set of interval orders on n elements.

Theorem 3.6. *Fix $n \geq 0$. There is an involution $\phi: \mathcal{F}_n \rightarrow \mathcal{F}_n$ with these properties. Suppose $R \in \mathcal{F}_n$ is an interval order with F1-triple (T_1, S_1, R) and F2-triple (T_2, S_2, R) . Let $R' = \phi(R)$ be its image under ϕ , with F1-triple (T'_1, S'_1, R') and F2-triple (T'_2, S'_2, R') . Then the following holds:*

- $\max(S_1) = \max(T'_2)$, and hence $\max(T_2) = \max(S'_1)$, since ϕ is an involution,
- $\max(S_2) = \max(T'_1)$, and hence $\max(T_1) = \max(S'_2)$, and
- $\max(R) = \max(R')$.

In other words, the pair of statistics $(\max(S_1), \max(T_2))$ and the pair of statistics $(\max(S_2), \max(T_1))$ both have symmetric joint distribution on \mathcal{F}_n , and the symmetry of both these pairs is witnessed by the same involution ϕ which additionally preserves the value of $\max(R)$.

Theorem 3.7. *Let (T_1, S_1, R) be the F1-triple of an interval order R . For any $n \geq 0$, the pair of statistics $(\max(S_1), \max(R))$ has a symmetric joint distribution over \mathcal{F}_n .*

In matrix terminology, Theorem 3.7 states that the statistics ‘number of wNE-cells’ and ‘weight of the last column’ have symmetric joint distribution over Fishburn matrices of weight n ; see Figure 8.

Note that by combining Theorems 3.6 and 3.7, we may additionally deduce that $\max(R)$ and $\max(T_2)$ also have symmetric joint distribution over \mathcal{F}_n .

Before we present the proofs of the two theorems, let us point out how they relate to the results on Catalan statistics discussed previously. Theorem 3.6 is a generalization of Proposition 2.12. To see this, consider the situation when R is $(\mathbf{2}+\mathbf{2}, \mathbf{N})$ -free. With the notation of Theorem 3.6, this implies that $T_1 = \emptyset$ and (S_1, R) is a C1-pair. Consider then the F2-triple (T'_2, S'_2, R') . Theorem 3.6 states that $\max(T_1) = \max(S'_2)$, but since $T_1 = \emptyset$, it follows that $S'_2 = \emptyset$ as well, since \emptyset is the only relation with n maximal elements. Consequently, R' has an F2-triple of the form (T'_2, \emptyset, R') , hence (T'_2, R') is a C2-pair and R' is $(\mathbf{2}+\mathbf{2}, \mathbf{3}+\mathbf{1})$ -free. We conclude that by restricting the mapping ϕ from Theorem 3.6 to $(\mathbf{2}+\mathbf{2}, \mathbf{N})$ -free posets R , we get a bijection from C1-pairs to C2-pairs with the same statistic-preserving properties as in Proposition 2.12.

Theorem 3.7 is inspired by Proposition 2.13, and can be seen as extending the statement of this proposition from C1-pairs to F1-triples.

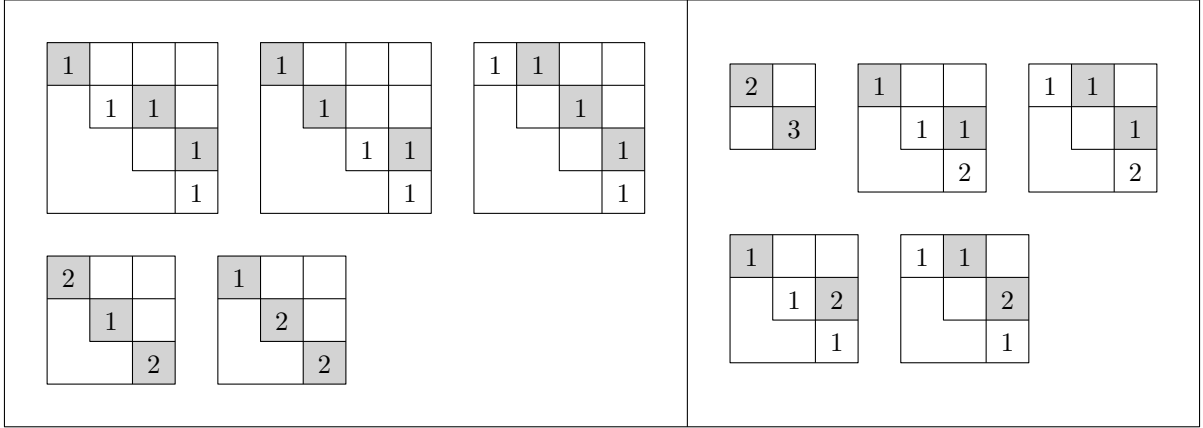


Figure 8: Illustration of the symmetric joint distribution of the number of wNE-cells and the weight of last column over Fishburn matrices. Left: the Fishburn matrices of weight 5, with 3 wNE-cells, and last column of weight 2. Right: the Fishburn matrices of weight 5, with 2 wNE-cells, and last column of weight 3. The wNE-cells are shaded.

3.1 Proofs

To prove Theorems 3.6 and 3.7, it is more convenient to work with Fishburn matrices rather than relational structures, and to interpret the relevant statistics using Observation 3.5.

Recall that an interval order is primitive if it has no two indistinguishable elements. Primitive interval orders correspond to Fishburn matrices whose entries are equal to 0 or 1; we call such matrices *primitive Fishburn matrices*.

An *inflation* of a primitive Fishburn matrix M is an operation which replaces the value of each 1-cell of M (i.e., a cell of value 1) by a positive integer, while the 0-cells are left unchanged. Clearly, by inflating a primitive Fishburn matrix we again obtain a Fishburn matrix, and any Fishburn matrix can be uniquely obtained by inflating a primitive Fishburn matrix.

Another useful operation on primitive Fishburn matrices is the *extension*. Informally speaking, it creates a primitive Fishburn matrix P' with $k + 1$ columns from a primitive Fishburn matrix P with k -columns, by splitting the last column of P into two new columns. Formally, suppose that $P = (P_{i,j})_{i,j=1}^k$ is a k -by- k primitive Fishburn matrix. We say that a $(k + 1)$ -by- $(k + 1)$ matrix $P' = (P'_{i,j})_{i,j=1}^{k+1}$ is an extension of P , if P' has the following properties (see Figure 9):

- The last row of P' consists of k 0-cells followed by a 1-cell. In other words, for $j \leq k$ we have $P'_{k+1,j} = 0$, while $P'_{k+1,k+1} = 1$.
- For every $j < k$ and for every $i \leq j$, we have $P_{i,j} = P'_{i,j}$. That is, the first $k - 1$ columns of P' are identical to the first $k - 1$ columns of P , except for an extra 0-cell in the last row.
- If $P_{i,k} = 0$ for some i , then $P'_{i,k} = P'_{i,k+1} = 0$. That is, each 0-cell in the last column of P gives rise to two 0-cells in the same row and in the last two columns of P' .
- If $P_{i,k} = 1$, then there are three options for the values of $P'_{i,k}$ and $P'_{i,k+1}$:
 1. $P'_{i,k} = P'_{i,k+1} = 1$. In such case we say that the 1-cell $P_{i,k}$ is *duplicated* into $P'_{i,k}$ and $P'_{i,k+1}$.
 2. $P'_{i,k} = 0$ and $P'_{i,k+1} = 1$. We then say that $P_{i,k}$ is *shifted* into $P'_{i,k+1}$.
 3. $P'_{i,k} = 1$ and $P'_{i,k+1} = 0$. We then say that $P_{i,k}$ is *ignored* by the extension.

We say that an extension of P into P' is *valid*, if there is at least one 1-cell in the penultimate column of P' , or equivalently, if at least one 1-cell in the last column of P has been duplicated or ignored. It is easy

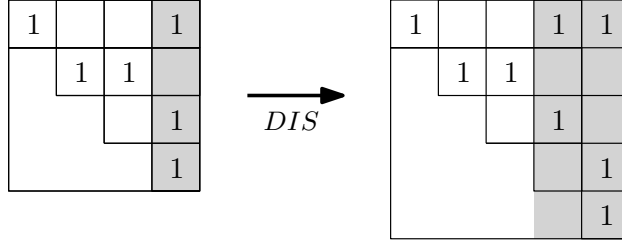


Figure 9: Example of an extension of a primitive Fishburn matrix. The word DIS is the code of the extension.

to see that if P' is a valid extension of a primitive Fishburn matrix P , then P' is itself a primitive Fishburn matrix, and conversely, any primitive Fishburn matrix P' with at least two columns is a valid extension of a unique primitive Fishburn matrix P .

Note that a primitive Fishburn matrix P whose last column has weight m has exactly 3^m extensions; one of them is invalid and $3^m - 1$ are valid. For convenience, we will represent each extension P' of P by a word $w = w_1 \cdots w_m$ of length m over the alphabet $\{D, S, I\}$, defined as follows: suppose that c_1, c_2, \dots, c_m are the 1-cells in the last column of P , listed in top-to-bottom order. Then w_i is equal to D (or S or I), if the cell c_i is duplicated (or shifted, or ignored, respectively) in the extension P' . We will call w the *code* of the extension from P to P' . Notice that the 1-cell in the bottom-right corner of P' is not represented by any symbol of w .

Given a word $w = w_1 w_2 \cdots w_m$ of length m , the *reverse* of w , denoted by \bar{w} , is the word $w_m w_{m-1} \cdots w_1$.

Observation 3.8. *Let P be a k -by- k primitive Fishburn matrix with m 1-cells in the last column, and let P' be its valid extension, with code $w = w_1 \cdots w_m$. Let c be a 1-cell in the j -th column of P' .*

- *Suppose that $j < k$, which implies, in particular, that c is also a 1-cell in P . Then c is an sNE-cell of P' (or wNE-cell of P' , or sSE-cell of P' , or wNE-cell of P') if and only if it is a sNE-cell of P (or wNE-cell of P , or sSE-cell of P , or wNE-cell of P , respectively).*
- *Suppose that $j = k$, which means that c is also a 1-cell in P , and this 1-cell was duplicated or ignored by the extension from P to P' . Suppose c is the i -th 1-cell in the last column of P , counted from the top (i.e., there are $i - 1$ 1-cells above c and $m - i$ 1-cells below c in the last column of P). Then c is a wNE-cell of P' if and only if $i = 1$ and $w_1 = I$, while c is a wSE-cell of P' if and only if $i = m$ and $w_m = I$. Furthermore, c is an sNE-cell of P' if and only if all the 1-cells of P above it were ignored (i.e., $w_1 = w_2 = \cdots = w_{i-1} = I$), while c is an sSE-cell of P' if and only if all the 1-cells of P below it were ignored (i.e., $w_{i+1} = w_{i+2} = \cdots = w_m = I$).*
- *If $j = k + 1$, i.e., c is in the last column of P' , then c is an sNE-cell and also an sSE-cell. Moreover, c is a wNE-cell if and only if it is the topmost 1-cell of the last column of P' , and it is a wSE-cell if and only if it is the bottommost 1-cell of the last column of P' .*

3.1.1 Proof of Theorem 3.6

To prove Theorem 3.6, we first describe the involution ϕ , and then verify that it has the required properties.

Let M be a k -by- k Fishburn matrix. As explained above, M can be constructed in a unique way from the 1-by-1 matrix $\boxed{1}$ by a sequence of $k - 1$ valid extensions followed by an inflation. In particular, there is a sequence of matrices P_1, P_2, \dots, P_k, M , where $P_1 = \boxed{1}$, for $i > 1$ the matrix P_i is an extension of P_{i-1} , and M is an inflation of P_k .

Define a new sequence $\bar{P}_1, \dots, \bar{P}_k$ as follows:

- $\bar{P}_1 = P_1 = \boxed{1}$.
- For each $i > 1$, \bar{P}_i is an extension of \bar{P}_{i-1} , and the code of \bar{P}_i is the reverse of the code of P_i .

Observe that for any $1 \leq j \leq i \leq k$, the j -th column of \overline{P}_i has the same weight as the j -th column of P_i . Furthermore, from Observation 3.8, we immediately deduce that for every i and j such that $1 \leq j \leq i \leq k$, the following relationships hold:

- The number of sNE-cells in the j -th column of P_i equals the number of sSE-cells in the j -th column of \overline{P}_i .
- The number of sSE-cells in the j -th column of P_i equals the number of sNE-cells in the j -th column of \overline{P}_i .
- The number of wNE-cells in the j -th column of P_i equals the number of wSE-cells in the j -th column of \overline{P}_i .
- The number of wSE-cells in the j -th column of P_i equals the number of wNE-cells in the j -th column of \overline{P}_i .

As the last step in the definition of ϕ , we describe how to inflate \overline{P}_k into a matrix \overline{M} . Fix a column index $j \leq k$. Let m be the number of 1-cells in the j -th column of P_k . Since M is an inflation of P_k , it has m nonzero cells in its j -th column; let x_1, x_2, \dots, x_m be the weights of these non-zero cells, ordered from top to bottom. As we know, \overline{P}_k also has m 1-cells in its j -th column. We inflate these cells by using values x_m, x_{m-1}, \dots, x_1 , ordered from top to bottom. Doing this for each j , we obtain an inflation \overline{M} of \overline{P}_k . We then define ϕ by $\phi(M) = \overline{M}$.

Let us check that ϕ has all the required properties. Clearly, ϕ is an involution, and it preserves the weight of the last column (indeed, of any column). Moreover, the number of wNE-cells of M is equal to the number of wSE-cells of \overline{M} , since these numbers are not affected by inflations. It remains to see that the total weight of the sNE-cells of M equals the total weight of the sSE-cells of \overline{M} . Fix a column $j \leq k$, and suppose that M has exactly ℓ sNE-cells in its j -th column. These must be the ℓ topmost nonzero cells of the j -th column of M , and in the notation of the previous paragraph, their total weight is $x_1 + x_2 + \dots + x_\ell$. It follows that P_k also has ℓ sNE-cells in its j -th column, therefore \overline{P}_k has ℓ sSE-cells in its j -th column, and these are the bottommost ℓ 1-cells of the j -th column of \overline{P}_k . In \overline{M} , these ℓ cells have total weight $x_1 + \dots + x_\ell$. We see that the sNE-cells of M have the same weight as the sSE-cells of \overline{M} , and Theorem 3.6 is proved.

We remark that the involution ϕ actually witnesses more equidistribution results than those stated in Theorem 3.6. E.g., the total weight of wNE-cells of M equals the total weight of wSE-cells of $\phi(M)$, and the number of sNE-cells of M equal the number of sSE-cells of $\phi(M)$. These facts do not seem to be easy to express in terms of F1-triples or F2-triples.

Moreover, from the construction of ϕ it is clear that the conclusions of Theorem 3.6 remain valid even when restricted to primitive Fishburn matrices, or to Fishburn matrices of a given number of rows. No such restriction is possible in Theorem 3.7, as seen from the examples in Figure 8.

3.1.2 Proof of Theorem 3.7

As with Theorem 3.6, our proof will be based on the concepts of extension and inflation. However, in this case we are not able to give an explicit bijection. Instead, we will proceed by deriving a formula for the corresponding refined generating function.

For a matrix M , let $w(M)$ denote its weight, $lc(M)$ the weight of its last column, $pc(M)$ the total weight of the columns preceding the last one (so $w(M) = lc(M) + pc(M)$), and $ne(M)$ its number of wNE-cells. Our goal is to show that the statistics lc and ne have symmetric joint distribution over the set of Fishburn matrices of a given weight.

We will use the standard notation $(a; q)_n$ for the product $(1 - a)(1 - aq)(1 - aq^2) \cdots (1 - aq^{n-1})$.

Let \mathcal{M} be the set of nonempty Fishburn matrices. Our main object of interest will be the generating function

$$F(x, y, z) = \sum_{M \in \mathcal{M}} x^{w(M)} y^{lc(M)} z^{ne(M)}.$$

Theorem 3.7 is equivalent to the identity $F(x, y, z) = F(x, z, y)$. Therefore, the theorem follows immediately from the following proposition.

Proposition 3.9. *The generating function $F(x, y, z)$ satisfies*

$$F(x, y, z) = xyz \sum_{n \geq 0} ((1 - xy)(1 - xz); 1 - x)_n.$$

Let us remark that the formula above is a refinement of a previously known formula for the generating function

$$G(x, y) = F(x, y, 1) = \sum_{M \in \mathcal{M}} x^{w(M)} y^{lc(M)},$$

which also corresponds to refined enumeration of interval orders with respect to their number of maximal elements (see [30, sequence A175579]). Formulas for $G(x, y)$ have been obtained in different contexts by several authors (namely Zagier [38], Kitaev and Remmel [25], Yan [37], Levande [26], Andrews and Jelínek [1]), and there are now three known expressions for this generating function:

$$G(x, y) = \sum_{n \geq 0} \frac{xy}{1 - xy} (1 - x; 1 - x)_n \quad [25]$$

$$= \sum_{n \geq 1} (1 - xy; 1 - x)_n \quad [26, 37, 38]$$

$$= -1 + \sum_{n \geq 0} pq^n (p; q)_n (q; q)_n \text{ with } p = \frac{1}{1 - xy}, q = \frac{1}{1 - x}. \quad [1]$$

The second of these three expressions can be deduced from Proposition 3.9 by the substitution $z = 1$ and a simple manipulation of the summands. Other authors have derived formulas for generating functions of Fishburn matrices (or equivalently, interval orders) refined with respect to various statistics [3, 13, 22], but to our knowledge, none of them has considered the statistic $ne(M)$.

Proof of Proposition 3.9. Our proof is an adaptation of the approach from our previous paper [22]. We first focus on primitive Fishburn matrices. Let \mathcal{P}_k be the set of primitive k -by- k Fishburn matrices, and let $\mathcal{P} = \bigcup_{k \geq 1} \mathcal{P}_k$. Define auxiliary generating functions

$$P_k(x, y, z) = \sum_{M \in \mathcal{P}_k} x^{pc(M)} y^{lc(M)-1} z^{ne(M)}, \text{ and}$$

$$P(x, y, z) = \sum_{k \geq 1} P_k(x, y, z).$$

Since $lc(M)$ is positive for every $M \in \mathcal{P}$, the exponent of the factor $y^{lc(M)-1}$ in P_k is nonnegative. The idea behind subtracting 1 from the exponent is that y now only counts the 1-cells in the last column that are not wNE-cells, while the unique wNE-cell of the last column is counted by the variable z only. Note that the wNE-cells of the previous columns contribute to x as well as to z .

Consider a matrix $M \in \mathcal{P}_k$, with $w(M) = n$, $lc(M) = m$ and $ne(M) = \ell$. This matrix contributes with the summand $x^{n-m} y^{m-1} z^\ell$ into $P_k(x, y, z)$; let us call $x^{n-m} y^{m-1} z^\ell$ the *value* of M . Let us determine the total value of the matrices obtained by extending M . Let c be the topmost 1-cell in the last column of M . If M' is an extension of M , then $ne(M') = ne(M) + 1$ if c was ignored by the extension, and $ne(M') = ne(M)$ otherwise. Any other 1-cell in the last column may be ignored, duplicated or shifted, contributing respectively a factor x , or xy , or y into the value of M' . It follows that the total value of all the matrices obtained by a valid extension of M is equal to

$$xz(x^{n-m}(x + y + xy)^{m-1} z^\ell) + xy(x^{n-m}(x + y + xy)^{m-1} z^\ell) + y(x^{n-m}(x + y + xy)^{m-1} z^\ell) - yx^{n-m} y^{m-1} z^\ell,$$

where the first three summands are respectively the values of the matrices in which the cell c has been ignored, duplicated and shifted, and the final subtracted term is the value of the matrix obtained by the invalid extension. Note that an extension M' also includes a 1-cell in the bottom-right corner, which contributes a factor of y into the value of M' , but the effect of this extra factor is cancelled by the fact that the topmost 1-cell in the last column of M' does not contribute any factor of y into the value of M' .

Summing over all $M \in \mathcal{P}_k$, and recalling that every matrix in \mathcal{P}_{k+1} may be uniquely obtained as a valid extension of a matrix in \mathcal{P}_k , we deduce that

$$P_{k+1}(x, y, z) = (xz + xy + y)P_k(x, x + y + xy, z) - yP_k(x, y, z).$$

Summing this identity over all $k \geq 1$ and noting that $P_1(x, y, z) = z$, we see that

$$\begin{aligned} P(x, y, z) - z &= (xz + xy + y)P(x, x + y + xy, z) - yP(x, y, z), \text{ or equivalently,} \\ P(x, y, z) &= \frac{z}{1+y} + \frac{xz + xy + y}{1+y}P(x, x + y + xy, z). \end{aligned}$$

From this functional equation, by a simple calculation (analogous to [22, proof of Theorem 2.1]), we obtain the formula

$$\begin{aligned} P(x, y, z) &= \sum_{n \geq 0} \frac{z}{(1+x)^n(1+y)} \prod_{i=0}^{n-1} \frac{(1+x)^{i+1}(1+y) - 1 - x + xz}{(1+x)^i(1+y)} \\ &= \frac{z}{1+y} \sum_{n \geq 0} \left(\frac{1+x-xz}{(1+x)(1+y)}; \frac{1}{1+x} \right)_n. \end{aligned} \quad (1)$$

Any Fishburn matrix may be uniquely obtained by inflating a primitive Fishburn matrix, which corresponds to the identity

$$F(x, y, z) = \frac{xy}{1-xy} P \left(\frac{x}{1-x}, \frac{xy}{1-xy}, z \right),$$

where the factor $\frac{xy}{1-xy}$ on the right-hand side corresponds to the contribution of the topmost 1-cell in the last column. Substituting into (1) gives

$$\begin{aligned} F(x, y, z) &= \frac{xy}{1-xy} \cdot \frac{z}{1+\frac{xy}{1-xy}} \sum_{n \geq 0} \left(\frac{1 + \frac{x}{1-x} - z \frac{x}{1-x}}{\left(1 + \frac{x}{1-x}\right) \left(1 + \frac{xy}{1-xy}\right)}; \frac{1}{1 + \frac{x}{1-x}} \right)_n \\ &= xyz \sum_{n \geq 0} \left((1-xy)(1-xz); 1-x \right)_n. \end{aligned}$$

This completes the proof of Proposition 3.9, and of Theorem 3.7. \square

4 Further Remarks and Open Problems

Diagonal-free fillings of polyominoes. We have seen in Lemma 1.2 that Fishburn matrices with no two positive cells in strictly SW position correspond to $(\mathbf{2}+\mathbf{2}, \mathbf{3}+\mathbf{1})$ -free posets, while those with no two positive cells in strictly SE position correspond to $(\mathbf{2}+\mathbf{2}, \mathbf{N})$ -free posets. Both classes are known to be enumerated by Catalan numbers, and in particular, the two types of matrices are equinumerous. This can be seen as a very special case of symmetry between fillings of polyomino shapes avoiding increasing or decreasing chains.

To be more specific, a k -by- k Fishburn matrix (or indeed, any upper-triangular k -by- k matrix) can also be represented as a right-justified array of boxes with k rows of lengths $k, k-1, k-2, \dots, 1$, where each box is filled by a nonnegative integer. Such arrays of integers are a special case of the so-called *fillings of polyominoes*, which are a rich area of study.

Given a Fishburn matrix M , we say that a m -tuple of nonzero cells c_1, \dots, c_m forms an *increasing chain* (or *decreasing chain*) if for each $i < j$, the cell c_j is strictly NE from c_i (strictly SE from c_i , respectively). It follows from general results on polyomino fillings (e.g. from a theorem of Rubey [32, Theorem 5.3]) that there is a bijection which maps Fishburn matrices avoiding an increasing chain of length m to those that avoid a decreasing chain of length m , and the bijection preserves the number of rows, as well as the weight of every row and column.

We have seen that certain equidistribution results related to Fishburn matrices without increasing chains of length 2 can be extended to general Fishburn matrices. It might be worthwhile to look at Fishburn matrices avoiding chains of a given fixed length $m > 2$, and see if these matrices exhibit similar kinds of equidistribution.

F-triples for other objects. Disanto et al. [7] have shown that many Catalan-enumerated objects, such as Dyck paths, 312-avoiding permutations, non-crossing partitions or non-crossing matchings, admit a natural bijective encoding into C1-pairs. For other Catalan objects, e.g. non-nesting matchings, an encoding into C2-pairs is easier to obtain.

Since F1-triples and F2-triples are a direct generalization of C1-pairs and C2-pairs, it is natural to ask which Fishburn-enumerated objects admit an easy encoding into such triples. We have seen that such an encoding exists for interval orders, as well as for Fishburn matrices. It is also easy to describe an F1-triple structure on non-neighbor-nesting matchings [5, 36], since those are closely related to Fishburn matrices.

A more challenging problem is to find an F-triple structure on ascent sequences. Unlike other Fishburn objects, ascent sequences do not exhibit any obvious ‘trivial involution’ that would be analogous to duality of interval orders or transposition of Fishburn matrices.

Problem 1. Find a natural way to encode an ascent sequence into an F-triple.

There are several results showing that certain subsets of ascent sequences are enumerated by Catalan numbers (see e.g. Duncan and Steingrímsson [15], Callan et al. [4], or Mansour and Shattuck [29]). It might be a good idea to first try to encode those Catalan-enumerated classes into C1-pairs or C2-pairs.

Another Fishburn-enumerated family for which we cannot find an F-triple encoding is the family of $(\uparrow\downarrow)$ -avoiding permutations. We say that a permutation $\pi = \pi_1 \cdots \pi_n$ is $(\uparrow\downarrow)$ -avoiding, if there are no three indices i, j, k such that $i + 1 = j < k$ and $\pi_i + 1 = \pi_k < \pi_j$. Parviainen [31] has shown that $(\uparrow\downarrow)$ -avoiding permutations are enumerated by Fishburn numbers.

Numerical evidence suggests that there is a close connection between the statistics of Fishburn matrices expressible in terms of F-triples, and certain natural statistics on $(\uparrow\downarrow)$ -avoiding permutations. In a permutation $\pi = \pi_1 \pi_2 \cdots \pi_n$, an element π_i is a *left-to-right maximum* (or *LR-maximum*) if π_i is larger than any element among π_1, \dots, π_{i-1} . Let $\text{LRmax}(\pi)$ denote the number of LR-maxima of π . Analogously, we define LR-minima, RL-maxima and RL-minima of π , their number being denoted by $\text{LRmin}(\pi)$, $\text{RLmax}(\pi)$ and $\text{RLmin}(\pi)$, respectively. Let $\text{Av}_n(\uparrow\downarrow)$ be the set of $(\uparrow\downarrow)$ -avoiding permutations of size n . Observe that for a $(\uparrow\downarrow)$ -avoiding permutation π , its composition inverse π^{-1} avoids $(\uparrow\downarrow)$ as well. Let \mathcal{M}_n be the set of Fishburn matrices of weight n .

Conjecture 4.1. *For every n , there is a bijection $\phi: \text{Av}_n(\uparrow\downarrow) \rightarrow \mathcal{M}_n$ with these properties:*

- $\text{LRmax}(\pi)$ is the weight of the first row of $\phi(\pi)$,
- $\text{RLmin}(\pi)$ is the weight of the last column of $\phi(\pi)$,
- $\text{RLmax}(\pi)$ is the number of *wNE*-cells of $\phi(\pi)$,
- $\text{LRmin}(\pi)$ is the number of positive cells of $\phi(\pi)$ belonging to the main diagonal, and
- $\phi(\pi^{-1})$ is obtained from $\phi(\pi)$ by transpose along the North-East diagonal.

Together with Theorem 3.7, Conjecture 4.1 implies the following weaker conjecture.

Conjecture 4.2. *For any n , LRmax and RLmax have symmetric joint distribution on $\text{Av}_n(\uparrow\downarrow)$.*

Primitive Fishburn matrices and Motzkin numbers. Recall that an interval order is primitive if it has no two indistinguishable elements. Primitive interval orders are encoded by primitive Fishburn matrices which are Fishburn matrices whose every entry is equal to 0 or 1. Primitive interval orders (and therefore also primitive Fishburn matrices) were enumerated by Dukes et al. [13], see also [30, sequence A138265]. As we pointed out before, every Fishburn matrix can be uniquely obtained from a primitive Fishburn matrix by inflation. Thus, knowing the enumeration of primitive objects we may obtain the enumeration of general Fishburn objects and vice versa.

In particular, with the help of Lemma 1.2, we easily deduce that the number of primitive $(\mathbf{2}+\mathbf{2}, \mathbf{3}+\mathbf{1})$ -free interval orders of size n , as well as the number of primitive $(\mathbf{2}+\mathbf{2}, \mathbf{N})$ -free interval orders of size n is the $(n-1)$ -th Motzkin number (see [30, A001006]). Maybe this connection between Motzkin objects and primitive Fishburn objects will reveal further combinatorial results, in the same way the connection between Catalan objects and Fishburn objects inspired Theorems 3.6 and 3.7.

Tamari-like lattices. Disanto et al. [9] defined a lattice structure on the set of interval orders of a given size, and showed that the restriction of this lattice to the set of $(\mathbf{2}+\mathbf{2}, \mathbf{N})$ -free posets yields the well-known Tamari lattice of Catalan objects [19, 21]. They also gave a simple description of the Tamari lattice on $(\mathbf{2}+\mathbf{2}, \mathbf{N})$ -free posets [8].

It might be worth exploring whether the lattice structure introduced by Disanto et al. for interval orders admits an easy interpretation in terms of Fishburn triples or Fishburn matrices. It might also be worthwhile to look at the restriction of this lattice to $(\mathbf{2}+\mathbf{2}, \mathbf{3}+\mathbf{1})$ -free posets.

The missing bijection. For Theorem 3.7, we could provide a proof by a generating function argument, but not a direct bijective proof. By finding a bijective proof, or even better, a proof that generalizes Deutsch's [6] involution on Catalan objects, we might, e.g., gain more insight into the statistic $\min(S_1)$, which generalizes the Narayana-distributed statistic pea on Dyck paths.

Problem 2. Prove Theorem 3.7 bijectively.

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