On the combinatorics of several integrable hierarchies

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Abstract. We demonstrate that statistics of certain classes of set partitions is described by generating functions related to the Burgers, Ibragimov–Shabat and Korteweg–de Vries integrable hierarchies.

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> My procedure was this: I would count the stones by eye and write down the figure. Then I would divide them into two handfuls that I would scatter separately on the table. I would count the two totals, note them down, and repeat the operation.
>
> Borges, Blue tigers (translated by Andrew Hurley)

1. Introduction

In the last decades numerous interrelations were discovered between the combinatorics and the theory of integrable systems. Mainly, these links involve *solutions*, either special ones, such as the Painlevé transcendents [5] and solitons [9], or generic ones, such as the tau-function of the Kadomtsev–Petviashvili hierarchy [1].

On the other hand, *equations* themselves exhibit a certain combinatorial nature, due to the recurrent relations which govern the higher symmetries and conservation laws of integrable hierarchies. This aspect was paid less attention so far, although a quite simple description on the language of set partitions was known for long for the Burgers hierarchy [12, 13]. We reproduce this combinatorial interpretation for the sake of completeness and as a base for further generalizations. New results obtained in the paper are related to the Ibragimov–Shabat and KdV hierarchies, see table 1. In these cases the combinatorics becomes more complicated, since the ordinary set partitions are replaced by special ones which are characterized by additional restrictions. Moreover,

Table 1. Contents of the paper.

Hierarchy	Combinatorial objects, their numbers
potential Burgers	Set partitions, Bell polynomials Y_n, Stirling numbers of the 2nd kind, Bell numbers
Burgers	Set partitions without distinguished singleton
Ibragimov–Shabat	B-type partitions, polynomials (8), B-analogs of Stirling numbers of the 2nd kind, Dowling numbers
Korteweg–de Vries	Non-overlapping partitions, polynomials (10), number triangle (12), Bessel numbers

this combinatorics comes in disguise: for instance, in the Burgers case we consider the generating function intermediately for the higher flows, but in the KdV case we have to consider a formal series for the logarithmic derivative of ψ-function which solves the Riccati equation (inversion of the Miura map, see e.g. [8]). The flows and conservation laws of the hierarchy are related with this generating function by simple algebraic relations. In the Ibragimov–Shabat case a natural choice of the generating function is dictated by the linearization procedure.

Although we are not interested in 'explicit' formulae for the coefficients of generating functions here, it should be mentioned that such formulae for the potential KdV flows actually do exist. One of them, obtained already in [8], represents the coefficient of a given monomial as a certain multiple integral. Another formula obtained in [16] is of more combinatorial nature, but it remains very complicated. Only in the case of pot-Burgers hierarchy the formula for the coefficients can be considered as truly an explicit one.

2. Potential Burgers hierarchy

The pot-Burgers hierarchy is obtained from the linear heat equation hierarchy

$$\psi_{t_n} = \psi_n \tag{1}$$

by means of the change of dependent variable $\psi = e^v$. This yields

$$v_{t_n} = e^{-v} D^n(e^v) = (D + v_1)^n(1) = Y_n(v_1, \dots, v_n), \quad n = 0, 1, 2, \dots \tag{2}$$

Here and further on we denote the derivatives as follows: $v_n = D^n(v)$, $D = \partial/\partial x$, $v_{t_n} = \partial v/\partial t_n$. Several first equations (2) are shown in the table 2. A meaningful combinatorics appears just from nothing!

It is easy to see that Y_n are polynomials with integer coefficients, homogeneous with respect to the weight $w(v_j) = j$. These polynomials play a fundamental role in combinatorics and are known under the name of *(full exponential) Bell polynomials* [4]. An equivalent definition through the exponential generating functions reads

$$\sum_{n=0}^{\infty} Y_n \frac{z^n}{n!} = e^{-v} \sum_{n=0}^{\infty} D^n(e^v) \frac{z^n}{n!} = e^{v(x+z)-v(x)} = \exp\left(\sum_{n=1}^{\infty} v_n \frac{z^n}{n!}\right)$$

Table 2. The potential Burgers hierarchy (weight $w(v_j) = j$).

$$
\begin{aligned}
v_{t_0} &= 1 \\
v_{t_1} &= v_1 \\
v_{t_2} &= v_2 + v_1^2 \\
v_{t_3} &= v_3 + 3v_1v_2 + v_1^3 \\
v_{t_4} &= v_4 + (4v_1v_3 + 3v_2^2) + 6v_1^2v_2 + v_1^4 \\
v_{t_5} &= v_5 + (5v_1v_4 + 10v_2v_3) + (10v_1^2v_3 + 15v_1v_2^2) + 10v_1^3v_2 + v_1^5
\end{aligned}
$$

and this immediately implies the explicit formula

$$
Y_n = \sum_{k_1+2k_2+\ldots+rk_r=n} \frac{n!}{(1!)^{k_1}\ldots(r!)^{k_r}k_1!\ldots k_r!}\, v_1^{k_1}\ldots v_r^{k_r}. \tag{3}
$$

Its combinatorial interpretation is obvious:

— monomials correspond to partitions of the *number* n;

— coefficients of monomials count partitions of the *set* $[n] = \{1,\ldots,n\}$ into the subsets (or blocks) of prescribed size.

For example, let us list all set partitions for $n = 2, 3, 4$:

$n=2:$	v_2	v_1^2
	2	$1+1$
	12	$1\|2$

$n=3:$	v_3	$3v_1v_2$	v_1^3
	3	$1+2$	$1+1+1$
	123	$1\|23$	$1\|2\|3$
		$2\|13$	
		$3\|12$	

$n=4:$	v_4	$4v_1v_3$	$3v_2^2$	$6v_1^2v_2$	v_1^4
	4	$1+3$	$2+2$	$1+1+2$	$1+1+1+1$
	1234	$1\|234$	$12\|34$	$1\|2\|34$	$1\|2\|3\|4$
		$2\|134$	$13\|24$	$1\|3\|24$	
		$3\|124$	$14\|23$	$1\|4\|23$	
		$4\|123$		$2\|3\|14$	
				$2\|4\|13$	
				$3\|4\|12$	

Recall, that each set partition is considered as unordered set (with blocks as the elements), that is, ordering of the blocks does not matter. However, it is often useful to enumerate the blocks somehow. For the sake of definiteness, we will adopt the enumeration corresponding to the ordering of the minimal elements in the blocks.

We see that the combinatorics behind the hierarchy (2) is quite simple. The following statement is well known, see e.g. [12, 13].

Theorem 1. *In the potential Burgers hierarchy, the coefficient of the monomial* $v_1^{k_1}\ldots v_r^{k_r}$ *is equal to the number of partitions of the set of* $n = k_1 + 2k_2 + \ldots + rk_r$

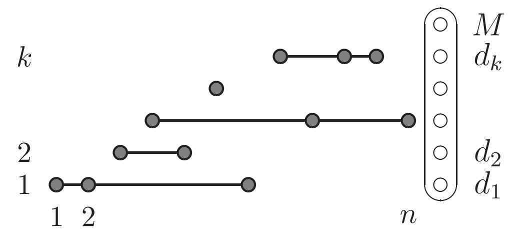
elements into k_1 blocks with 1 element, k_2 blocks with 2 elements, ..., k_r blocks with r elements.

Proof. One proof follows intermediately from the explicit formula (3) for the coefficients. However, we will not always have such a formula at hand. The following reasoning provides a more conceptual proof.

Let $\Pi_{n,k}$ denotes the set of all partitions of the set $[n]$ into k blocks and Π_n denotes the set of all partitions of $[n]$. Let us consider operations

$$d_j : \Pi_{n,k} \to \Pi_{n+1,k}, \quad j = 1, \ldots, k, \qquad M : \Pi_{n,k} \to \Pi_{n+1,k+1}$$

defined, respectively, as appending of the element $n+1$ to j-th block or adding it to the partition as a new singleton. This can be visualized by the following diagram:



Starting from the partition $\{\varnothing\}$ of the set $[0] = \varnothing$ and applying operations d_j, M, one can generate, in a unique way, any partition of $[n]$. Indeed, the required sequence of operations is uniquely recovered by deleting elements in the inverse order from n to 1.

In the theorem, a set partition π with k_1 1-blocks, ..., k_r r-blocks corresponds to the monomial $p(\pi) = v_1^{k_1} \ldots v_r^{k_r}$. The differentiation $D(p(\pi))$ by the Leibnitz rule amounts to replacing of v_i with v_{i+1} for each factor in turn, taking the multiplicity into account. In the partition language, this means that we add the new element to each block in turn. As the result, we obtain the sum of monomials $p(d_j(\pi))$ for all admissible values of j. Multiplication of the monomial $p(\pi)$ by v_1 gives the monomial $p(M(\pi))$. Thus, the polynomials

$$P_n = \sum_{\pi \in \Pi_n} p(\pi)$$

are related by the recurrent relation $P_{n+1} = (D + v_1)(P_n)$ and since $P_1 = v_1$, hence $P_n = Y_n(v_1, \ldots, v_n)$. □

A less detailed statistics is obtained if we forget about sizes of blocks and consider just their number in a given partition. Obviously, this correspond to summing up the coefficients of terms of the same degree, which gives us the *Bell polynomials* of one variable

$$B_n(u) = Y_n(u, \ldots, u) = (u\partial_u + u)^n(1) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} u^k.$$

The coefficient $\left\{ {n \atop k} \right\}$ of u^k, that is, the number of partitions of $[n]$ into k blocks, is called the *Stirling number of the second kind* [15, A048993]:

1									1
0	1								1
0	1	1							2
0	1	3	1						5
0	1	7	6	1					15
0	1	15	25	10	1				52
0	1	31	90	65	15	1			203
0	1	63	301	350	140	21	1		877

By definition, $\left\{ {n \atop 0} \right\} = 0$ at $n > 0$ and $\left\{ {0 \atop 0} \right\} = \#\{\varnothing\} = 1$. The total numbers of set partitions with n elements, the Bell or the exponential numbers [15, A000110], are given by the sums of the rows:

$$B_n = B_n(1) = Y_n(1, \dots, 1) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}, \qquad \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} = e^{e^z - 1}.$$

3. Burgers hierarchy

The right hand sides of equations (2) do not contain v and this makes the substitution $u = v_1$ possible. This brings to the Burgers hierarchy

$$u_{t_n} = D(Y_n(u, \dots, u_{n-1})), \quad n = 1, 2, \dots \tag{4}$$

which is homogeneous with respect to the weight $w(u_j) = j + 1$. Several first equations are written down in the table 3. What is the combinatorial interpretation in this case? This can be easily understood by the following example, for $n = 3$:

<table>
<tr><td>v_3</td><td>$3v_1v_2$</td><td>v_1^3</td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>u_2</td><td>$3uu_1$</td><td>u^3</td><td>$\xrightarrow{D}$</td><td>u_3</td><td>$3uu_2$</td><td>$3u_1^2$</td><td>$3u^2u_1$</td><td>$0u^4$</td></tr>
<tr><td>123</td><td>1|23</td><td>1|2|3</td><td></td><td>1234</td><td>1|234</td><td>12|34</td><td>1|2|34</td><td>~~1|2|3|4~~</td></tr>
<tr><td></td><td>2|13</td><td></td><td></td><td></td><td>2|134</td><td>13|24</td><td>1|3|24</td><td></td></tr>
<tr><td></td><td>3|12</td><td></td><td></td><td></td><td>3|124</td><td>14|23</td><td>~~1|4|23~~</td><td></td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td>~~4|123~~</td><td></td><td>2|3|14</td><td></td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td>~~2|4|13~~</td><td></td></tr>
<tr><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td>~~3|4|12~~</td><td></td></tr>
</table>

Certainly, renaming $v_j \to u_{j-1}$ does not change the combinatorics. The differentiation amounts to appending the new element to all blocks in turn, however, now we do not add it as a new block. Therefore, the partitions under consideration are constructed as in theorem 1, but we do not apply the operation M at the last step. As a result, all partitions Π_n are mapped onto some subset of partitions Π_{n+1}, namely, those partitions where the element $n + 1$ does not appear as a singleton. We arrive to the following combinatorial interpretation of equations (4).

Table 3. Burgers hierarchy (weight $w(u_j) = j + 1$).

$$u_{t_1} = u_1$$
$$u_{t_2} = u_2 + 2uu_1$$
$$u_{t_3} = u_3 + (3uu_2 + 3u_1^2) + 3u^2u_1$$
$$u_{t_4} = u_4 + (4uu_3 + 10u_1u_2) + (6u^2u_2 + 12uu_1^2) + 4u^3u_1$$
$$u_{t_5} = u_5 + (5uu_4 + 15u_1u_3 + 10u_2^2) + (10u^2u_3 + 50uu_1u_2 + 15u_1^3) + (10u^3u_2 + 30u^2u_1^2) + 5u^4u_1$$

Theorem 2. *In the Burgers hierarchy, the coefficient of the monomial $u^{k_0}u_1^{k_1}\dots u_r^{k_r}$ is equal to the number of partitions of the set with one distinguished element into k_0 blocks with 1 element, ..., k_r blocks with $(r+1)$ element and such that the distinguished element does not constitute 1-block.*

As before, one can consider more rough statistics. For instance, setting $u = 1$ gives us the total number of partitions under consideration of the set $[n+1]$:

$$D(Y_n(u,\dots,u_{n-1}))|_{u_j=1} = B'_n(1) = \sum_{k=1}^{n} k\left\{ {n \atop k} \right\}, \quad n \geq 1.$$

The sequence of these numbers (*2-Bell numbers*) starts

$$1,\ 3,\ 10,\ 37,\ 151,\ 674,\ 3263,\ 17007,\ 94828,\ 562595,\ \dots$$

According to [15, A005493], it can be characterized also in many other ways, in particular, as the number of partitions of $[n]$ with distinguished block or as the total number of blocks in all set partitions of $[n]$. These interpretations are obvious as well, since the distinguished blocks can be identified with the blocks enlarged by the operations d_j, and these operations are applied exactly as many times as there are blocks in all partitions.

4. Ibragimov–Shabat hierarchy

4.1. Recurrent relations

The table 4 displays the sequence of point changes and substitutions between equation $\psi_{t_3} = \psi_3$ and the Ibragimov–Shabat equation [10]

$$u_{t_3} = u_3 + 3u^2u_2 + 9uu_1^2 + 3u^4u_1.$$

Although this transformation looks quite harmless, it partially destroys the symmetry algebra: in the variables ψ, it consists of equations (1) of arbitrary order, while only odd order equations survive in the variables u. Indeed, the change $\psi^2 = s$ brings to equation $s_{t_n} = \dots$ where the right hand side is a full derivative only if n is odd:

$$s_{t_n} = 2\psi\psi_n = D(2\psi\psi_{n-1} - 2\psi_1\psi_{n-2} + 2\psi_2\psi_{n-3} + \dots \pm \psi^2_{(n-1)/2}). \qquad (5)$$

Table 4. Linearization of the Ibragimov–Shabat equation.

$\psi_{t_3} = \psi_3$		$u_{t_3} = u_3 + 3u^2u_2 + 9uu_1^2 + 3u^4u_1$
$\updownarrow \psi^2 = s$		$\updownarrow u^2 = v$
$s_{t_3} = D\left(s_2 - \frac{3s_1^2}{4s}\right)$		$v_{t_3} = D\left(v_2 - \frac{3v_1^2}{4v} + 3vv_1 + v^3\right)$
$\uparrow s = q_1$		$\uparrow v = w_1$
$q_{t_3} = q_3 - \frac{3q_2^2}{4q_1}$	$\overset{q=e^{2w}}{\longleftrightarrow}$	$w_{t_3} = w_3 - \frac{3w_2^2}{4w_1} + 3w_1w_2 + w_1^3$

In the analogous equation for even n the term $\psi^2_{n/2}$ remains outside the parentheses, that is, $s_{t_n} \notin \operatorname{Im} D$, and therefore the further substitution $s = q_1$ leads out of the class of evolutionary equations. The structure of odd flows is described by the following statement.

Statement 3. *Let us denote* $D_t = \partial_{t_1} + z^2\partial_{t_3} + z^4\partial_{t_5} + \dots$, $A = A(z) = a_0 + a_1z + a_2z^2 + \dots$, $\bar{A} = A(-z)$, *then the Ibragimov–Shabat hierarchy is equivalent to equations*

$$D_t(u) = \frac{1}{2u}D(A\bar{A}) = \frac{1}{2z}(A - \bar{A}) - uA\bar{A}, \tag{6}$$

$$z(D + u^2)(A) = A - u. \tag{7}$$

Proof. Let us consider the generating function

$$\Psi = \psi + \psi_1 z + \psi_2 z^2 + \dots$$

and set $\Psi = \sqrt{2}e^wA$. Equation (7) follows from the relations

$$zD(\Psi) = \Psi - \psi, \qquad \psi = \sqrt{q_1} = \sqrt{2e^{2w}w_1} = \sqrt{2}e^wu.$$

Next, let $\bar{\Psi} = \Psi(-z)$, then (cf (5))

$$\begin{aligned} D(\Psi\bar{\Psi}) &= z^{-1}(\Psi - \psi)\bar{\Psi} - z^{-1}\Psi(\bar{\Psi} - \psi) \\ &= z^{-1}\psi(\Psi - \bar{\Psi}) = 2\psi(\psi_1 + \psi_3z^2 + \dots) = 2\psi D_t(\psi) = D_t(s). \end{aligned}$$

Applying D^{-1} yields $\Psi\bar{\Psi} = D_t(q) = 2e^{2w}D_t(w)$, wherefrom

$$2uD_t(u) = D_t(v) = DD_t(w) = \frac{1}{2}D(e^{-2w}\Psi\bar{\Psi}) = D(A\bar{A}).$$

Second equality in (6) follows after elimination of derivatives by use of (7). □

Equation (7) is equivalent to recurrent relations

$$a_0 = u, \quad a_n = a_n(u, \dots, u_n) = (D + u^2)(a_{n-1}), \quad n = 1, 2, \dots \tag{8}$$

which are our object of study. Let us try to find a combinatorial interpretation for this recursion.

Several first polynomials a_n are presented in the table 5. Given these data as a prescribed statistics, our goal is to figure out a definition of the corresponding

Table 5. Polynomials a_n (weight $w(u_j)=2j+1$).

$$
\begin{aligned}
a_0 &= u\\
a_1 &= u_1+u^3\\
a_2 &= u_2+4u^2u_1+u^5\\
a_3 &= u_3+(5u^2u_2+8uu_1^2)+9u^4u_1+u^7\\
a_4 &= u_4+(6u^2u_3+26uu_1u_2+8u_1^3)+(14u^4u_2+44u^3u_1^2)+16u^6u_1+u^9\\
a_5 &= u_5+(7u^2u_4+38uu_1u_3+26uu_2^2+50u_1^2u_2)+(20u^4u_3+170u^3u_1u_2+140u^2u_1^3)\\
&\quad +(30u^6u_2+140u^5u_1^2)+25u^8u_1+u^{11}
\end{aligned}
$$

combinatorial objects, that is, to solve a kind of inverse problem of the enumerative combinatorics. In contrast to the Burgers hierarchy case, here we do not know an explicit formula for the coefficients, but this is not too important, the main problem is to guess what are the objects which we are counting. An invaluable aid in such an ill-posed problem may be obtained by comparison with the known data collected in the Encyclopedia of integer sequences [15]. Let us pass to the less detailed statistics by gluing together terms of the same degree. Polynomials of one variable $a_n(u,\dots,u)=(u\partial_u+u^2)^n(u)$ contain only odd powers of u and their coefficients constitute the triangle

1								1
1	1							2
1	4	1						6
1	13	9	1					24
1	40	58	16	1				116
1	121	330	170	25	1			648
1	364	1771	1520	395	36	1		4088
1	1093	9219	12411	5075	791	49	1	28640

which turns out to be known: according to [15, A039755] this is the triangle of analogs of Stirling numbers of the second kind for the so-called B type set partitions. The sums of numbers in rows, that is, the total sums of the coefficients $a_n(1,\dots,1)$ form the sequence [15, A007405] of the *Dowling numbers*, or *B-analogs of the Bell numbers*. This gives us a broad hint at a possible connection between polynomials (8) and B type partitions. This guess is proved in the next section.

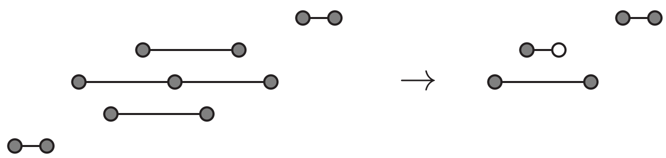
4.2. Generating operations for type B set partitions

Special classes of set partitions appear when one takes into account some additional structure of the set. Set partitions of B type (or signed set partitions, $\mathbb{Z}_2$-partitions) [6], see also e.g. [2, 17] make use of the reflection $j\to -j$.

Definition 1. A partition π of the set $\{-n,\dots,n\}$ is called the B_n type partition if:
1) $\pi=-\pi$, that is for each block $\beta\in\pi$ also $-\beta\in\pi$;
2) π contains only one block $\pi_0\in\pi$ such that $\pi_0=-\pi_0$.

We will denote Π_n^B the set of all such partitions and $\Pi_{n,k}^B$ those partitions which contain k block pairs.

In a brief notation for B type partitions, the negative elements of the 0-block are omitted, and only that block of each pair is displayed for which the element with minimal absolute value is positive; the minus signs are denoted by over bars. For instance, in this notation the partition $-5,-4|-3,0,3|-2,1|-1,2|4,5$ is represented as $03|1\bar{2}|45$. A graphical representation is clear from the diagram



Now let us define the map p from Π_n^B into the set of monomials on the variables u_j. Let $|\beta|$ denote the number of positive elements in the block β:

$$|\beta| = \#\{i \in \beta : i > 0\}.$$

It is clear that the number of negative elements in the block is $|\bar{\beta}|$. Let a set partition $\pi \in \Pi_{n,k}^B$ consists of 0-block π_0 and block pairs $\pi_1, \bar{\pi}_1, \ldots, \pi_k, \bar{\pi}_k$, such that the element of π_j with minimal absolute value is positive. For such a partition, let

$$p(\pi) = u_{|\pi_0|} \cdot u_{|\pi_1|-1} u_{|\bar{\pi}_1|} \cdots u_{|\pi_k|-1} u_{|\bar{\pi}_k|}.$$

As an example, let us write down Π_3^B partitions, collecting together all partitions corresponding to the same monomial:

u_3	$5u^2u_2$	$8uu_1^2$	$9u^4u_1$	u^7
0123	$0\vert 123$	$0\vert 12\bar{3}$	$0\vert 12\vert 3$	$1\vert 2\vert 3\vert 4$
	$0\vert 1\bar{2}\bar{3}$	$0\vert 1\bar{2}3$	$0\vert 1\bar{2}\vert 3$	
	$012\vert 3$	$01\vert 23$	$0\vert 13\vert 2$	
	$013\vert 2$	$01\vert 2\bar{3}$	$0\vert 1\bar{3}\vert 2$	
	$023\vert 1$	$02\vert 13$	$0\vert 23\vert 1$	
		$02\vert 1\bar{3}$	$0\vert 2\bar{3}\vert 1$	
		$03\vert 12$	$01\vert 2\vert 3$	
		$03\vert 1\bar{2}$	$02\vert 1\vert 3$	
			$03\vert 1\vert 2$	

The resulting polynomial is exactly a_3. The following theorem demonstrates that this is not just a coincidence and the polynomials a_n are, indeed, the $\mathbb{Z}_2$-analogs of the full exponential Bell polynomials Y_n.

Theorem 4. *The polynomials (8) are equal to*

$$a_n = \sum_{\pi \in \Pi_n^B} p(\pi).$$

Proof. Let us denote the sum in the right hand side p_n. Obviously, $p_0 = u = a_0$, so we only have to prove that p_n satisfy the same recurrent relations as a_n, that is, $p_n = (D + u^2)(p_{n-1})$.

Notice, that deleting of elements $\pm n$ from any B_n type set partition gives us a B_{n-1} type set partition. Therefore, Π^B_n is constructed from Π^B_{n-1} by adding $\pm n$ in all possible ways. It is easy to see that this is done by the following operations:

$d_0 : \Pi^B_{n-1,k} \to \Pi^B_{n,k}$, insertion of both elements $\pm n$ into 0-block;

$d_j : \Pi^B_{n-1,k} \to \Pi^B_{n,k}$, $j = 1, \ldots, k$, insertion of $\pm n$ into blocks $\pm\pi_j$;

$\bar{d}_j : \Pi^B_{n-1,k} \to \Pi^B_{n,k}$, $j = 1, \ldots, k$, insertion of $\pm n$ into blocks $\mp\pi_j$;

$M : \Pi^B_{n-1,k} \to \Pi^B_{n,k+1}$, adding of the new block pair $\{-n\}$, $\{n\}$.

Starting from the trivial partition of the set $\{0\}$ and applying these operations, one can obtain, in a unique way, any B type set partition. Let us keep track of the monomial $p(\pi)$, $\pi \in \Pi^B_{n-1,k}$ under these operations:

d_0 : the factor $u_{|\pi_0|}$ is replaced with $u_{|\pi_0|+1}$;

d_j : the factor $u_{|\pi_j|-1}$ is replaced with $u_{|\pi_j|}$;

$\bar{d}_j$: the factor $u_{|\bar{\pi}_j|}$ is replaced with $u_{|\bar{\pi}_j|+1}$;

M : two new factors u are added.

Therefore, application of all possible operations maps the monomial $p(\pi)$ to the sum of monomials $(D + u^2)(p(\pi))$. □

5. Korteweg–de Vries hierarchy

5.1. Recurrent relations

Let us recall (for a proof, see e.g. [8]) a computation method of the KdV conservation laws and flows, based on solving of the Riccati equation

$$D(f) + f^2 = \lambda - u, \quad \lambda = z^2/4 \tag{9}$$

by the formal power series

$$f(z) = -\frac{z}{2} + \frac{f_1(u)}{z} + \frac{f_2(u, u_1)}{z^2} + \ldots + \frac{f_n(u, \ldots, u_{n-1})}{z^n} + \cdots$$

Equation (9) is equivalent to the recurrent relations

$$f_1 = u, \quad f_{n+1} = D(f_n) + \sum_{s=1}^{n-1} f_s f_{n-s}, \quad n = 1, 2, \ldots \tag{10}$$

which will be the main object of our study. Several polynomials f_n are written down in the table 6. The flows are computed from the polynomials with odd subscripts: let

$$g(z) = -\frac{1}{2z} - \frac{g_1}{z^3} - \frac{g_3}{z^5} - \ldots - \frac{g_{2m-1}}{z^{2m+1}} - \cdots = \frac{1}{2(f(z) - f(-z))}$$

Table 6. Polynomials f_n (weight $w(u_j) = j + 2$).

$$f_1 = u$$
$$f_2 = u_1$$
$$f_3 = u_2 + u^2$$
$$f_4 = u_3 + 4uu_1$$
$$f_5 = u_4 + (6uu_2 + 5u_1^2) + 2u^3$$
$$f_6 = u_5 + (8uu_3 + 18u_1u_2) + 16u^2u_1$$
$$f_7 = u_6 + (10uu_4 + 28u_1u_3 + 19u_2^2) + (30u^2u_2 + 50uu_1^2) + 5u^4$$

which is equivalent to recurrent relations

$$g_1 = u, \quad g_{2m+1} = f_{2m+1} + 2\sum_{s=1}^{m} g_{2s-1}f_{2m-2s+1}, \quad m = 1, 2, \dots$$

then the KdV hierarchy reads

$$u_{t_{2m+1}} = D(g_{2m+1}) = u_{2m+1} + \dots, \quad m = 0, 1, 2, \dots$$

Moreover, polynomials (10) with odd subscripts serve as common conserved densities for all these flows.

One interpretation of the polynomials f_n can be seen intermediately from the recurrent relations (10). Let us consider expressions φ builded from the variable u and operations $M(a, b)$, $d_j(a)$, $1 \le j \le \deg a$ where $\deg a$ is equal to the number of instances of u in a. Such expressions can be called 'unexpanded monomials'. For any expression φ its value expand(φ) is computed according to the following rules:

- independently on the order of operations, all d_j are applied before M;
- the action of $d_j(a)$ amounts to replacing of j-th instance of u_i in a with u_{i+1} (u is identified with u_0, as usual);
- $M(a, b)$ is replaced by the product ab.

Let Φ_n denote the set of all expressions with the total number of symbols u, d, M equal to n. For instance:

	unexpanded monomials	expanded monomials
$n = 1$	u	u
$n = 2$	$d_1(u)$	u_1
$n = 3$	$d_1(d_1(u)),\quad M(u, u)$	$u_2,\quad u^2$
$n = 4$	$d_1(d_1(d_1(u))),$	$u_3,$
	$d_1(M(u, u)),\quad d_2(M(u, u))$	$uu_1,\ uu_1$
	$M(d_1(u), u),\quad M(u, d_1(u))$	$uu_1,\ uu_1$

Theorem 5. *The number of different expressions builded from symbols M, d_j, u with the same monomial as their value is equal to the coefficient of this monomial in polynomials f_n. In other words,*

$$f_n = \sum_{\varphi \in \Phi_n} \operatorname{expand}(\varphi). \tag{11}$$

Proof. Any expression from Φ_{n+1}, $n > 0$ is either of the form $d_j(a)$ where $a \in \Phi_n$, $1 \le j \le \deg a$ or of the from $M(a, b)$ where $a \in \Phi_s$, $b \in \Phi_{n-s}$. Taking into account the obvious properties

$$\sum_{j=1}^{\deg a} \operatorname{expand}(d_j(a)) = D(\operatorname{expand}(a)),$$
$$\operatorname{expand}(M(a, b)) = \operatorname{expand}(a) \operatorname{expand}(b),$$

this implies that polynomials (11) satisfy the recurrent relation (10). □

This interpretation is fairly intuitive, but it is desirable to compare it with something more standard. As before, let us pass to polynomials of one variable by collecting together terms of the same degrees. This brings us to a number triangle which apparently is not in the OEIS:

1						1	
1						1	
1	1					2	
1	4					5	
1	11	2				14	
1	26	16				43	(12)
1	57	80	5			143	
1	120	324	64			509	
1	247	1170	490	14		1922	
1	502	3948	2944	256		7651	
1	1013	12776	15403	2730	42	31965	

Nevertheless, the sequence of coefficients sum totals turns out to be known: $f_{n+1}[1]$ is equal to the number of *non-overlapping partitions* of the set $[n]$, or the *Bessel number* B^*_n [15, A006789]. Notice, that identifying of all u_j results in the Riccati equation $u\partial_u(f) + f^2 = \lambda - u$ which is, indeed, equivalent to the Bessel equation. Moreover, one can see in the triangle the Euler numbers [15, A000295], the Catalan numbers [15, A000108] and powers of 4.

5.2. Generating operations for non-overlapping partitions

This class of set partitions was introduced in [7], see also [3, 11]. Its definition engages the order relation on the partitioned set $[n] = \{1, \ldots, n\}$.

Definition 2. Blocks α and β of a set partition π overlap if

$$\min\alpha < \min\beta < \max\alpha < \max\beta.$$

The set partition is called non-overlapping (NOP) if any two blocks in it do not overlap. All NOPs of the set $[n]$ will be denoted Π^*_n.

The interval $[\min\alpha, \max\alpha]$ is called the *support* of the block α. The above definition of NOP is equivalent to the property that supports of any two blocks either do not intersect or lie one in another. The left diagram below shows overlapping blocks and the right diagram shows non-overlapping ones:

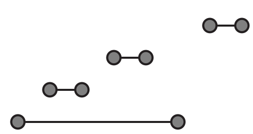
Remark 1. A neighbour class of *non-crossing* partitions is characterized by a more restrictive condition which forbids the pattern $\alpha_1 < \beta_1 < \alpha_2 < \beta_2$ for any elements of any two blocks. It is under active study in combinatorics as well, moreover, it makes sense to combine such types of restrictions with symmetries like the reflection for the B type partitions, see e.g. [14]. It is an open question, whether some integrable hierarchies may be associated with such kind of objects.

Some simple properties of NOPs are the following.

— At $n = 0, 1, 2, 3$ we have $\Pi^*_n = \Pi_n$ and there is only one overlapping partition $13|24$ in Π_4,.

— Singletons do not overlap with any block.

— NOPs containing only doublets can be easily identified with the balanced sets of parentheses:



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The last property explains where from the Catalan numbers appear in the triangle (12). The deletion of differentiation in equation (10) brings to the recursion for the 'dispersionless terms':

$$f_1 = u, \quad f_{n+1} = \sum_{s=1}^{n-1} f_s f_{n-s} \quad \to \quad u, 0, u^2, 0, 2u^3, 0, 5u^4, 0, \ \ldots$$

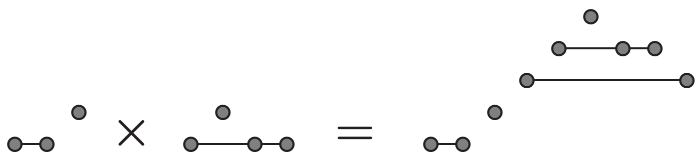
In order to establish a correspondence with the general polynomials f_n, let us identify the variable u with the set partition $\{\varnothing\}$ and define the action of the operations M and d_j on the NOPs, in such a way that expressions Φ_{n+1} be in a one-to-one correspondence with Π^*_n.

Degree. Let $\deg\pi = k$ if π contains $k-1$ multiplets (blocks with more than one element).

Operation M. Let $\rho \in \Pi_r^*$, $\sigma \in \Pi_s^*$. Denote by $(\sigma)_{r+1}$ the partition of the set $\{r+2, r+s+1\}$ obtained from σ by adding $r+1$ to each element, and define

$$M(\rho,\sigma) = \rho \cup \{\{r+1, r+s+2\}\} \cup (\sigma)_{r+1} \in \Pi_{r+s+2}^*.$$

This can be illustrated by the diagram



In particular, if $\rho = \{\varnothing\}$ then $(\sigma)_1$ is bounded by the doublet $\{1, s+2\}$, and if $\sigma = \{\varnothing\}$ then the doublet $\{r+1, r+2\}$ is appended to ρ. Notice that $\deg M(\rho,\sigma) = \deg\rho \deg\sigma$.

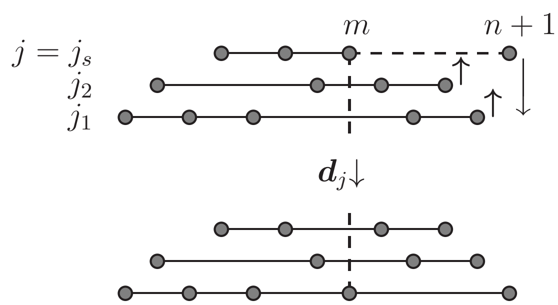
Operation d_j. It consists of adding one element $n+1$ to $\pi \in \Pi_n^*$. If $j = 1$ then the element is added just as a singleton. For $1 < j \le k = \deg\pi$, the operation requires a detailed description. Let us denote $\mu_2, \dots, \mu_k$ all multiplets in π, ordered by increase of their minimal elements. Assume that all blocks with support containing μ_j are enumerated by a sequence $j_1 < \dots < j_s = j$. Let us divide each of these blocks into the left and right parts with respect to $m = \max\mu_j$:

$$\mu_{j_r}^- = \{i \in \mu_{j_r} : i < m\}, \quad \mu_{j_r}^+ = \{i \in \mu_{j_r} : i \ge m\}$$

and form the new blocks

$$\tilde\mu_{j_1} = \mu_{j_1}^- \cup \{m, n+1\}, \quad \tilde\mu_{j_r} = \mu_{j_r}^- \cup \mu_{j_{r-1}}^+, \quad r = 2, \dots, s$$

as shown on the following diagram. The rest blocks of the partition do not change under this operation.



Theorem 6. *The operations M, d_j generate any non-overlapping partition, in a unique way.*

Proof. The last operation bringing to a given partition is uniquely defined by consideration of the block β containing the maximal element of the partition. If it is a singleton, then the last operation was d_1; if it is a doublet, then it was M; if it is a multiplet, then the operation was d_j where j is the maximal number such that the support of multiplet μ_j contains the last to the end element of β. In each case, applying of inverse operation brings to NOPs with lesser numbers of elements. □

Taking the theorem 5 into account, the established bijection allows to associate a certain monomial with each NOP, although not in a quite effective way, because we first have to build an exression $\varphi \in \Phi_n$ corresponding to $\pi \in \Pi^*_{n-1}$ and then to compute expand(φ):

$$\begin{array}{ccc} \Phi_n & \leftrightarrow & \Pi^*_{n-1} \\ \text{expand} \downarrow & \swarrow & \\ f_n & & \end{array}$$

Nevertheless, it is easy to trace at the degree of monomial under this correspondence; it is one more than the number of multiplets in the partition. This gives us the following interpretation of the number triangle (12).

Corollary 7. *The number of NOPs of n elements containing k multiplets is equal to the number in the n-th row and k-th column of the number triangle (12), starting their enumeration from 0. This number is equal to the coefficient of u^{k+1} in the polynomial $F_{n+1}(u) = f_{n+1}(u, \dots, u)$ defined by the recurrent relations*

$$F_1 = u, \quad F_{n+1} = u\partial_u(F_n) + \sum_{s=1}^{n-1} F_s F_{n-s}, \quad n = 1, 2, \dots$$

6. Conclusion

We have established a relation between several classes of set partitions and generating functions for integrable hierarchies. Hopefully, this observation may turn useful for both theories. Of course, we have too few examples at the moment to make far-reaching conclusions. A conjecture is that each integrable hierarchy has an underlying generating function which may be interpreted as statistics for some kind of combinatorial objects (possibly unknown). As further steps, it would be interesting to reveal the combinatorics associated with the mKdV equation, KdV-like equations of 5-th order, nonlinear Schrödinger equation and so on.

On the other hand, the objects studied in the combinatorics are so plentiful and diverse that it seems doubtful that any one can be associated with an integrable hierarchy. In all likelihood, this is a very special property, so it would be interesting to understand what is the integrability intermediately in combinatorial terms (rather than of the level of generating functions). In particular, one can try to obtain a proof of commutativity of the flows of a hierarchy based on their combinatorial interpretation.

References

[1] Alexandrov A, Mironov A, Morozov A and Natanzon S 2012 Integrability of Hurwitz partition functions. *J. Phys. A: Math. Theor.* **45:4** 045209

[2] Benoumhani M 1996 On Whitney numbers of Dowling lattices. *Discrete Mathematics* **159** 13–33

[3] Claesson A 2001 Generalized pattern avoidance. *Europ. J. Combinatorics* **22** 961–971

[4] Comtet L 1974 *Advanced combinatorics. The art of finite and infinite expansions* (D. Reidel, Dordrecht)

[5] Deift P 2000 Integrable systems and combinatorial theory. *Notices of the AMS* **47:6** 631–640

[6] Dowling T A 1973 A class of geometric lattices based on finite groups. *J. of Combin. Theory, Ser. B* **14:1** 61–86

[7] Flajolet P and Schott R 1990 Non-overlapping partitions, continued fractions, Bessel functions and a divergent series. *Europ. J. Combinatorics* **11:5** 421–432

[8] Gel'fand I M and Dikii L A 1975 Asymptotic properties of the resolvent of Sturm–Liouville equations, and the algebra of Korteweg–de Vries equations. *Russian Math. Surveys* **30:5** 77–113

[9] Grosset M-P and Veselov A P 2005 Bernoulli numbers and solitons. *J. of Nonl. Math. Phys.* **12:4** 469–474

[10] Ibragimov N Kh and Shabat A B 1980 Infinite Lie–Bäcklund algebras. *Funct. Anal. Appl.* **14:4** 313–315

[11] Klazar M 2003 Bell numbers, their relatives, and algebraic differential equations. *J. of Combin. Theory, Ser. A* **102** 63–87.

[12] Lambert F, Loris I, Springael J and Willox R 1994 On a direct bilinearization method: Kaup's higher-order water wave equation as a modified nonlocal Boussinesq equation. *J. Phys. A: Math. Gen.* **27** 5325–5334.

[13] Lambert F and Springael J 2008 Soliton equations and simple combinatorics. *Acta Appl. Math.* **102** 147–178

[14] Reiner V 1997 Non-crossing partitions for classical reflection groups. *Discrete Mathematics* **177:1–3** 195–222

[15] Sloane N J A *The On-Line Encyclopedia of Integer Sequences,* published electronically at `http://oeis.org`, 2010 (A000108 Catalan numbers; A000110 Bell numbers; A000295 Euler numbers; A005493 2-Bell numbers; A006789 Bessel numbers; A007405 Dowling numbers; A008277, A048993 Stirling numbers of the 2nd kind; A039755 B-analogs of Stirling numbers of the 2nd kind)

[16] Schimming R 1995 An explicit expression for the Korteweg–de Vries hierarchy. *Acta Appl. Math.* **39** 489–505

[17] Suter R 2000 Two analogues of a classical sequence. *J. Integer Seq.* **3** Article 00.1.8.