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GENERAL SOLUTIONS OF SUMS OF CONSECUTIVE CUBED INTEGERS EQUAL TO SQUARED INTEGERS

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ABSTRACT. All integer solutions (M, a, c) to the problem of the sums of M consecutive cubed integers $(a+i)^3$ $(a > 1,\ 0 \leq i \leq M-1)$ equaling squared integers c^2 are found by decomposing the product of the difference and sum of the triangular numbers of $(a+M-1)$ and $(a-1)$ in the product of their greatest common divisor g and remaining square factors δ^2 and σ^2, yielding $c = g\delta\sigma$. Further, the condition that g must be integer for several particular and general cases yield generalized Pell equations whose solutions allow to find all integer solutions (M, a, c) showing that these solutions appear recurrently. In particular, it is found that there always exist at least one solution for the cases of all odd values of M, of all odd integer square values of a, and of all even values of M equal to twice an integer square.
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1. Introduction

It is known since long that the sum of M consecutive cubed positive integers starting from 1 equals the square of the sum of the M consecutive integers, which itself equals the triangular number $\triangle_M$ of the number of terms M,

$$\sum_{i=1}^{M} i^3 = \left(\sum_{i=1}^{M} i\right)^2 = \left(\frac{M(M+1)}{2}\right)^2 = \triangle_M^2 \tag{1.1}$$

for $\forall M \in \mathbb{Z}^+$. The question whether this remarkable result can be extended to other integer values of the starting point, i.e. whether the sum of consecutive cubed positive integers starting from $a \neq 1$ is also a perfect square

$$\sum_{i=0}^{M-1} (a+i)^3 = c^2 \tag{1.2}$$

has been addressed by several authors but has received so far only partial answers. With the notation of this paper, Lucas stated [15] that the only solutions for $M = 5$ are $a = 0, 1, 96$ and 118 (missing the solution $a = 25$, see further Table 1), and that there are no other solutions for $M = 2$ than $a = 1$. Aubry showed [1] that a solution for $M = 3$ is $a = 23$, $c = 204$, correcting Lucas' statement that there are no other solution for $M = 3$ than $a = 1$. Other historical accounts can be found in [7]. Cassels proved [3] by using the method of finding all integral points on a given curve of genus 1 $y^2 = 3x\left(x^2+2\right)$ (with $x = a+1$, $y = c$ in this paper notations), that the only solutions for $M = 3$ are $a = -1, 0, 1$ and 23. Stroeker obtained [37] complete solutions for $2 \leq M \leq 50$ and $M = 98$, using estimates of lower bound of linear forms in elliptic logarithms to solve elliptic curve equations of the form $Y^2 = X^3 + dX$ where $d = n^2\left(n^2-1\right)/4$, $X = nx + n(n-1)/2$, $Y = ny$ (with $n = M$, $x = a$, $y = c$ in this paper notations). The method reported, although powerful, appears long and difficult and caused some problems for the cases $M = 41$ and 44. Stroeker remarked also that $M = a = 33$ with $c = 2079 = 33 \times 63$. This is not the single occurrence of $M = a$, as it occurs also for $M = a = 2017, 124993, 7747521, ...$ (see [24, 25]).

One of the reasons that these previous attempts to find all solutions (M, a, c) to (1.2) were only partially successful was most likely due to the approach taken to start the search for solutions for single values of M, one by one and in an increasing order of M values. The method proposed in this paper is to tackle the problem in a different way and instead of looking at each individual values of M one by one, to consider the problem in a more global approach by comparing different sets of known solutions, and instead of listing solutions in increasing order of M values, to look at two other parameters, δ and σ, defined further. This new beginning then leads to a more classical approach using general solutions of Pell equations, that allows to find all solutions in (M, a, c) of (1.2) for all possible cases. Note that Pell equations were already used previously by various authors (e.g. Catalan [4, 5], Cantor [2], Richaud [34]) in the 19th century in attempts to solve the present problem, albeit without reaching a complete resolution of the problem.

The approach proposed in this paper includes three steps.

Step 1 in Section 2 is based on the decomposition of the product of the difference Δ and the sum Σ of the two triangular numbers of $(a + M - 1)$ and $(a - 1)$ in simple factors, g, δ, σ where $g = \gcd(\Delta, \Sigma)$, allowing to find general expressions of a and c in function of M, δ and σ that are always solutions of (1.2).

In step 2 in Section 3, some conditions on δ and σ are explored to obtain three particular cases of solutions yielding specific expressions of M in function of k, $\forall k \in \mathbb{Z}^+$, including the case of M taking all odd positive integer values. Section 4 recalls some basics on Pell equation solutions to introduce the third step.

In step 3 in Section 5, a general solution in (M, a, c) is found for all values of δ and σ, based on solutions of simple and generalized Pell equations involving Chebyshev polynomials, allowing to find all the solutions in (M, a, c) to (1.2). As an alternative to step 3, recurrence relations are deduced in Section 6. Section 7 summarizes all findings.

Trivial solutions are not considered here. For instance, $M \leq 0$ is meaningless; for $M = 1$, the only solution is obviously $a = \alpha^2$, yielding $(M, a, c) = \left(1, \alpha^2, \alpha^3\right)$; therefore, we consider only $M > 1$. If $a < 0$, there are no solutions if $M < (1 - 2a)$; if $M = (1 - 2a)$, the only solution is $(M, a, c) = ((1 - 2a), a, 0)$; if $M > 1 - 2a$, the solutions are $(M, a, c) = \left(M, (1 - a), \left(\triangle^2_{M+a-1} - \triangle^2_{(-a)}\right)\right)$, i.e. a shift of the first term from a negative value a to a positive value $(1 - a)$ with a reduction of the number of terms $(M + 2a - 1)$. If $a = 0$ or 1, the classical solutions (1.1) are $(M, a, c) = \left(M, 0, \triangle^2_{M-1}\right)$ or $\left(M, 1, \triangle^2_{M}\right)$. Therefore, we limit our search to solutions for $M > 1$ and $a > 1$.

2. Step 1: A General theorem

Theorem 1. *For* $\forall\delta \in \mathbb{Z}^+$, $\exists\sigma, a, M, c \in \mathbb{Z}^+$, $\kappa \in \mathbb{Q}^+$, *such as* $\delta < \sigma$, $\gcd(\delta, \sigma) = 1$, $\kappa = (\sigma/\delta) > 1$,

$$\sum_{i=0}^{M-1} (a+i)^3 = c^2 \tag{2.1}$$

holds if

$$a = \frac{M\left(\kappa^2-1\right)+1+\sqrt{M^2\left(\kappa^4-1\right)+1}}{2} \tag{2.2}$$

$$c = \frac{\kappa M}{2}\left(\kappa^2 M+\sqrt{M^2\left(\kappa^4-1\right)+1}\right) \tag{2.3}$$

Proof. For $a, M, c, \Delta, \Sigma, g, \Delta', \Sigma', \delta, \sigma, C \in \mathbb{Z}^+$, $i, k \in \mathbb{Z}^*$, $\kappa \in \mathbb{Q}^+$, with $M > 1$, $0 \le i \le M-1$, $\kappa = (\sigma/\delta) > 1$, the sum of cubes of M consecutive integers $(a+i)$ for $i = 0$ to $M-1$ can be written successively as

$$\sum_{i=0}^{M-1} (a+i)^3 = \sum_{i=0}^{a+M-1} i^3 - \sum_{i=0}^{a-1} i^3 \tag{2.4}$$

$$= \triangle_{a+M-1}^2 - \triangle_{a-1}^2 \tag{2.5}$$

$$= \left(\triangle_{a+M-1} - \triangle_{a-1}\right)\left(\triangle_{a+M-1} + \triangle_{a-1}\right) \tag{2.6}$$

$$= (\Delta)(\Sigma) \tag{2.7}$$

where $\Delta = \triangle_{a+M-1} - \triangle_{a-1}$ and $\Sigma = \triangle_{a+M-1} + \triangle_{a-1}$, i.e. the difference and the sum of the triangular numbers of $(a+M-1)$ and $(a-1)$, with obviously $\Delta < \Sigma$, that can also be written

$$\Delta = M\left(a+\frac{M-1}{2}\right) \tag{2.8}$$

$$\Sigma = a^2 + a(M-1) + \frac{M(M-1)}{2} \tag{2.9}$$

Let $g = \gcd(\Delta, \Sigma)$, yielding $\Delta = g\Delta'$ and $\Sigma = g\Sigma'$. For (2.1) to hold, $c^2 = g^2\Delta'\Sigma'$ and as Δ' and Σ' are coprimes and their product must be square, both must be integer squares, i.e. $\Delta' = \delta^2$ and $\Sigma' = \sigma^2$, with $\gcd(\delta, \sigma) = 1$ and $\delta < \sigma$, yielding

$$c = g\delta\sigma \tag{2.10}$$

From (2.8) and (2.9), one has then respectively

$$2a + M - 1 = \frac{2g\delta^2}{M} \tag{2.11}$$

$$= \sqrt{4g\sigma^2 - (M^2-1)} \tag{2.12}$$

where the + sign is taken in front of the square root in (2.12) as $2a + M > 1$. Solving for g yields then

$$g = \frac{M}{2\delta^2}\left(\kappa^2 M+\sqrt{M^2\left(\kappa^4-1\right)+1}\right) \tag{2.13}$$

Replacing in (2.11) or (2.12) yields then directly (2.2) and in (2.10) yields directly (2.3). ∎

3. Step 2: Three Particular Solutions and a Generalization

For which values of M does (2.1) hold ? Answers can be found for at least three particular cases. Other general approaches are developed further.
Table 1 gives for $1 < M \leq 45$ and $1 < a < 10^5$, the first values of M, a and associated $\Delta, \Sigma, g, \delta, \sigma$ and c values. For M and $a < 10^5$, there are 892 solutions (M, a, c), given in [26, **?**], such that (4.1) holds. One observes very easily that:
(i) all odd values of M have at least one entry (in bold in Table 1) with $g = 2\triangle_{(M-1)/2} = \left(M^2-1\right)/4$, $\delta = 2M$ and $\sigma = \left(2M^2-1\right)$, yielding $c = M\left(M^2-1\right)\left(2M^2-1\right)/2$ with $a = M^3 - (3M-1)/2$;
(ii) those odd values of a equal to odd integer squares have at least one entry (e.g. for $M = 17$ in Table 1) with $g = (a-1)/8$, $\delta = (2a-1)$ and $\sigma = (2a+1)$, yielding $c = (a-1)\left(4a^2-1\right)/8$ with $M = (\sqrt{a}-1)(2a-1)/2$
(iii) those even values of M equal to twice an integer square have at least one entry (in italics in Table 1) with $g = M\left(M^2-1\right)/2$, $\delta = 1$, $\sigma = \sqrt{M/2}$, yielding $c = M\left(M^2-1\right)\sqrt{M/2}/2$ with $a = \triangle_{M-1}$.
These three cases can be generalized respectively to all odd values of M, to all odd integer square values of $a = (2k+1)^2$, and to all even values of M equal to twice integer squares in the following

Theorem 2. *$\forall k \in \mathbb{Z}^+$, $\exists \delta, \sigma, M, a, c \in \mathbb{Z}^+$ such that (2.1) holds :*
(i) if $\sigma = \left(\delta^2-2\right)/2$, with

$$M = (2k+1) \tag{3.1}$$
$$a = (2k+1)^3 - (3k+1) \tag{3.2}$$
$$= M^3 - \frac{(3M-1)}{2} \tag{3.3}$$
$$c = 2k(k+1)(2k+1)(8k(k+1)+1) \tag{3.4}$$
$$= \frac{M\left(M^2-1\right)\left(2M^2-1\right)}{2} \tag{3.5}$$

(ii) if $\sigma = \delta + 2$, with

$$a = (2k+1)^2 \tag{3.6}$$
$$M = k(8k(k+1)+1) \tag{3.7}$$
$$= \frac{(\sqrt{a}-1)(2a-1)}{2} \tag{3.8}$$
$$c = \frac{k(k+1)}{2}\left(4(2k+1)^4-1\right) \tag{3.9}$$
$$= \frac{(a-1)\left(4a^2-1\right)}{8} \tag{3.10}$$

Table 1. First values of $M, a, \Delta, \Sigma, g, \delta, \sigma$ and c for $1 < M \leq 45$ and $1 < a < 10^5$

M	a	Δ	Σ	g	δ	σ	c
3	**23**	**72**	**578**	**2**	**6**	**17**	**204**
5	25	135	735	15	3	7	315
5	96	490	9610	10	7	31	2170
5	**118**	**600**	**14406**	**6**	**10**	**49**	**2940**
7	**333**	**2352**	**112908**	**12**	**14**	**97**	**16296**
8	*28*	*252*	*1008*	*252*	*1*	*2*	*504*
9	**716**	**6480**	**518420**	**20**	**18**	**161**	**57960**
11	**1315**	**14520**	**1742430**	**30**	**22**	**241**	**159060**
12	14	234	416	26	3	4	312
13	144	1950	22542	78	5	17	6630
13	**2178**	**28392**	**4769898**	**42**	**26**	**337**	**368004**
15	25	480	1080	120	2	3	720
15	**3353**	**50400**	**11289656**	**56**	**30**	**449**	**754320**
15	57960	869505	3360173145	105	91	5657	54052635
17	9	289	361	1	17	19	323
17	120	2176	16456	136	4	11	5984
17	**4888**	**83232**	**23970888**	**72**	**34**	**577**	**1412496**
18	*153*	*2907*	*26163*	*2907*	*1*	*3*	*8721*
18	680	12393	474113	17	27	167	76653
19	**6831**	**129960**	**46785690**	**90**	**38**	**721**	**2465820**
21	14	504	686	14	6	7	588
21	144	3234	23826	66	7	19	8778
21	**9230**	**194040**	**85377710**	**110**	**42**	**881**	**4070220**
23	**12133**	**279312**	**147476868**	**132**	**46**	**1057**	**6418104**
25	**15588**	**390000**	**243360156**	**156**	**50**	**1249**	**9742200**
27	**19643**	**530712**	**386358518**	**182**	**54**	**1457**	**14319396**
28	81	2646	9126	54	7	13	4914
29	**24346**	**706440**	**593409810**	**210**	**58**	**1681**	**20474580**
31	**29745**	**922560**	**885657840**	**240**	**62**	**1921**	**28584480**
32	69	2704	7396	4	26	43	4472
32	133	4752	22308	132	6	13	10296
32	*496*	*16368*	*261888*	*16368*	*1*	*4*	*65472*
33	33	1617	2673	33	7	9	2079
33	**35888**	**1184832**	**1289097488**	**272**	**66**	**2177**	**39081504**
35	225	8470	58870	70	11	29	22330
35	**42823**	**1499400**	**1835265906**	**306**	**70**	**2449**	**52457580**
37	**50598**	**1872792**	**2561979798**	**342**	**74**	**2737**	**69267996**
39	111	5070	17280	30	13	24	9360
39	**59261**	**2311920**	**3514118780**	**380**	**78**	**3041**	**90135240**
40	3276	131820	10860720	780	13	118	1196520
41	**68860**	**2824080**	**4744454820**	**420**	**82**	**3361**	**115752840**
42	64	3549	7581	21	13	19	5187
43	**79443**	**3416952**	**6314527758**	**462**	**86**	**3697**	**146889204**
45	176	8910	39710	110	9	19	18810
45	**91058**	**4098600**	**8295566906**	**506**	**90**	**4049**	**184391460**

(iii) if $\delta = 1$ and $\sigma = k$ with $k > 1$ and

$$M = 2k^2 \tag{3.11}$$

$$a = k^2\left(2k^2-1\right) \tag{3.12}$$

$$= \frac{M(M-1)}{2} \tag{3.13}$$

$$c = k^3\left(4k^4-1\right) \tag{3.14}$$

$$= \frac{M\left(M^2-1\right)}{2}\sqrt{\frac{M}{2}} \tag{3.15}$$

Proof. For $k, \delta, \sigma, g, a, M > 1, c \in \mathbb{Z}^+$, $q \in \mathbb{Q}^+$, for (2.1) to hold:
(i) let $\sigma = \left(\delta^2-2\right)/2$. Replacing in (2.13) yields

$$g = \frac{M^2}{8\delta^4}\left(\delta^4 - 4\delta^2 + 4 + \sqrt{\delta^8 - 8\delta^6 + 8\delta^4 + 16\delta^2\left(\frac{\delta^2}{M^2} - 2\right) + 16}\right) \tag{3.16}$$

For the polynomial in δ under the square root sign to be a square, let $(\delta/M)^2 - 2 = 2$, i.e. $\delta = 2M$, yielding $\delta^8 - 8\delta^6 + 8\delta^4 + 32\delta^2 + 16 = \left(\delta^4 - 4\delta^2 - 4\right)^2$, giving $g = M^2\left(\delta^2-4\right)/4\delta^2 = \left(M^2-1\right)/4$ where δ was replaced by $2M$. As $g \in \mathbb{Z}^+$, M cannot be even and must be odd, i.e. $M = 2k+1$ $\forall k \in \mathbb{Z}^+$, yielding $g = k(k+1)$, $\delta = 2M = 2(2k+1)$, $\sigma = 2M^2 - 1 = 8k(k+1) + 1$. Replacing in (2.2) and (2.3) yields directly (3.2) to (3.5).
(ii) Let $\sigma = \delta + 2$. Replacing in (2.13) yields

$$g = \frac{M}{2\delta^4}\left(M(\delta+2)^2 + \sqrt{M^2\left((\delta+2)^4 - \delta^4\right) + \delta^4}\right) \tag{3.17}$$

For the polynomial in δ under the square root sign to be a square, let $M = q\delta$ and $\delta = 8q(q+1)+1$ with $q \in \mathbb{Q}^+$. It yields then $\sqrt{q^2\left((\delta+2)^4 - \delta^4\right) + \delta^2} = \delta^2 - 4q(\delta+1)$, giving $g = q(q+1)/2$. As $g \in \mathbb{Z}^+$, $q \in \mathbb{Z}^+$ and let $q = k$, yielding $g = \triangle_k$, $\delta = 16\triangle_k + 1$, $\sigma = 16\triangle_k + 3$, $M = k\delta = k(16\triangle_k + 1) =$(3.7). Replacing in (2.11) and (2.10) yields (3.6) and (3.9). Replacing k in function of a from (3.6) yields also (3.8) and (3.10).
(iii) Let $\delta = 1$ and $\sigma = k$ with $k > 1$. Then (2.13) reads

$$g = \frac{M\left(k^2M + \sqrt{M^2\left(k^4-1\right)+1}\right)}{2} \tag{3.18}$$

which takes integer values if $M = 2k^2$, yielding $g = k^2\left(4k^4-1\right) = M\left(M^2-1\right)/2$. Replacing in (2.11) and (2.10) yields directly (3.12) to (3.15). □

The case (i) of Theorem 2 confirms the statement of Stroeker ([37], p. 297) about all odd values of M (in bold in Table 1) having a solution to (2.1). The first 50 000 values of (M, a, c) for this case (i) are given in [28, 29]
For the case (ii) of Theorem 2, Table 2 gives the first five values of M, a, g, δ, σ and c. The first 50 000 values of (M, a, c) for this case (ii) are given in [30, 31].
As there exist other triplets of values of (M, a, c) such that (2.1) holds with $M = k\delta$ for the same value of $g = \Delta_k$, the case (ii) of Theorem 2 can be generalized to other values of a as follows. Table 3 shows some of these values for $1 \leq k \leq 2$ and $1 < M_n, a_n < 10^5$ and indexed by n for increasing values of M_n.

TABLE 2. Values of $M, a, g, \delta, \sigma, c$ for $1 \leq k \leq 5$ for case (ii) of Theorem 2

k	M	a	g	δ	σ	c
1	17	9	1	17	19	323
2	98	25	3	49	51	7497
3	291	49	6	97	99	57618
4	644	81	10	161	163	262430
5	1205	121	15	241	241	878445

TABLE 3. Values of $M_n, a_n, g, \delta_n, \sigma_n, c_n$ for $1 \leq k \leq 2$ and $1 < M_n, a_n < 10^5$

k	n	M_n	a_n	g	δ_n	σ_n	c_n
1	1	17	9	1	17	19	323
1	2	305	153	1	305	341	104005
1	3	5473	2737	1	5473	6119	33489287
1	4	98209	49105	1	98209	109801	10783446409
2	1	98	25	3	49	51	7497
2	2	4898	1225	3	2449	2549	18727503

It is easily seen from Table 3 that, for each value of k,

$$\sigma_n - \delta_n = \sigma_{n-1} + \delta_{n-1} \tag{3.19}$$

$$\sigma_n + \delta_n = \left(2\,(2k+1)^2 - 1\right)\sigma_{n-1} + \left(2\,(2k+1)^2 + 1\right)\delta_{n-1} \tag{3.20}$$

with $\sigma_0 = \delta_0 = 1$, yielding the recurrence relations

$$\delta_n = 2\,(2k+1)^2\,\delta_{n-1} - \delta_{n-2} \tag{3.21}$$

$$\sigma_n = 2\,(2k+1)^2\,\sigma_{n-1} - \sigma_{n-2} \tag{3.22}$$

i.e. δ_n and σ_n fulfill the Diophantine equation $(2k\,(k+1)+1)\,\delta_n^2 - 2k\,(k+1)\,\sigma_n^2 = 1$. The general solutions of this Diophantine equation can be expressed in function of Chebyshev polynomials of the second kind $U_n\left((2k+1)^2\right)$ as

$$\delta_n = U_n\left((2k+1)^2\right) - U_{n-1}\left((2k+1)^2\right) \tag{3.23}$$

$$\sigma_n = U_n\left((2k+1)^2\right) + U_{n-1}\left((2k+1)^2\right) \tag{3.24}$$

These results are generalized for all k and $n \in \mathbb{Z}^+$ in the next

Theorem 3. *$\forall k,n\in\mathbb{Z}^{+}$, $\exists\delta_n,\sigma_n,M_n,a_n,c_n\in\mathbb{Z}^{+}$, $M_n>1$, such that (2.1) holds if $\sigma_n=\sqrt{\delta_n^2+\left(\delta_n^2-1\right)/2k\left(k+1\right)}$, with*

$$M_n = k\delta_n \tag{3.25}$$

$$= k\left[U_n\left(\left(2k+1\right)^2\right)-U_{n-1}\left(\left(2k+1\right)^2\right)\right] \tag{3.26}$$

$$a_n = \frac{\delta_n+1}{2} \tag{3.27}$$

$$= \frac{U_n\left(\left(2k+1\right)^2\right)-U_{n-1}\left(\left(2k+1\right)^2\right)+1}{2} \tag{3.28}$$

$$c_n = \delta_n\sqrt{\frac{k\left(k+1\right)\left(\left(2k\left(k+1\right)+1\right)\delta_n^2-1\right)}{8}} \tag{3.29}$$

$$= \frac{k\left(k+1\right)U_{2n}\left(\left(2k+1\right)^2\right)}{2} \tag{3.30}$$

Proof. For $k,n,\delta_n,\sigma_n,g,M_n,a_n,c_n\in\mathbb{Z}^{+}$, $M_n>1$, let $\sigma_n=\sqrt{\delta_n^2+\left(\delta_n^2-1\right)/2k\left(k+1\right)}$. Replacing in (2.13) yields

$$g = \frac{M_n}{2\delta_n^4}\left(\left(\frac{\left(2k\left(k+1\right)+1\right)\delta_n^2-1}{2k\left(k+1\right)}\right)M_n+\sqrt{M_n^2\left(\left(\frac{\left(2k\left(k+1\right)+1\right)\delta_n^2-1}{2k\left(k+1\right)}\right)^2-\delta_n^4\right)+\delta_n^4}\right) \tag{3.31}$$

For the polynomial in δ_n under the square root sign to be a square, let $M_n=k\delta_n$, yielding after simplification $g=k\left(k+1\right)/2=\triangle_k$. Replacing further in (2.2) and (2.10) yields (3.27) and (3.29). Replacing further δ_n by (3.23) yields (3.26) and (3.28). Replacing δ_n and σ_n in (2.10) yields (3.30), noting that $\delta_n\sigma_n=U_n^2\left(\left(2k+1\right)^2\right)-U_{n-1}^2\left(\left(2k+1\right)^2\right)=U_{2n}\left(\left(2k+1\right)^2\right)$ as can be shown by replacing $U_n^2\left(x\right)$ and $U_{n-1}^2\left(x\right)$ in function of Chebyshev polynomials of the first kind, respectively $T_{2n+2}\left(x\right)$ and $T_{2n}\left(x\right)$ (see e.g. [36]) and simplifying appropriately. □

For the case (iii) of Theorem 2, the first 50 000 values of (M,a,c) are given in [32]. There exist also other triplets of values of (M,a,c) such that (2.1) holds with $\delta=1$ and $\sigma=k$. Table 4 shows some of these values for $1<k\le 5$ and $1<M_n,a_n<10^5$ and indexed by n for increasing values of M_n. The generalization of the case (iii) of Theorem 2 to other values of M is included in a following more general theorem. This theorem will use the solutions of simple and generalized Pell equations, that are recalled in the next section.

4. Pell equations: A Reminder

Pell equations of the general form

$$X^2-DY^2=N \tag{4.1}$$

with $X,Y,N\in\mathbb{Z}$ and square free $D\in\mathbb{Z}^{+}$, i.e. $\sqrt{D}\notin\mathbb{Z}^{+}$, have been investigated in various forms since long (see historical accounts in [7, 14, 38, 13]) and are treated in several classical text books (see e.g. [20, 21, 39, 11] and references therein). A simple reminder is given here and further details can be found in the references.

TABLE 4. Values of M_n, a_n, g_n, c_n with $\delta = 1$ and $\sigma = k$ for $1 < k \leq 5$ and $1 < M_n, a_n < 10^5$

k	n	M_n	a_n	g_n	c_n
2	1	8	28	252	504
2	2	63	217	15624	31248
2	3	496	1705	968440	1936880
2	4	3905	13420	60027660	120055320
3	1	18	153	2907	8721
3	2	323	2737	936054	2808162
3	3	5796	49105	301406490	904219470
4	1	32	496	16368	65472
4	2	1023	15841	16728096	66912384
5	1	50	1225	62475	312375
5	2	2499	61201	156062550	780312750

For $N = 1$, the simple Pell equation reads classically

$$X^2 - DY^2 = 1 \tag{4.2}$$

which has, beside the trivial solution $(X_0, Y_0) = (1, 0)$, a whole infinite branch of solutions $\forall n \in \mathbb{Z}^+$ given by

$$X_n = \frac{\left(X_1 + \sqrt{D}Y_1\right)^n + \left(X_1 - \sqrt{D}Y_1\right)^n}{2} \tag{4.3}$$

$$Y_n = \frac{\left(X_1 + \sqrt{D}Y_1\right)^n - \left(X_1 - \sqrt{D}Y_1\right)^n}{2\sqrt{D}} \tag{4.4}$$

where (X_1, Y_1) is the fundamental solution to (4.2), i.e. the smallest integer solution different from the trivial solution $(X_1 > 1, Y_1 > 0, \in \mathbb{Z}^+)$. Among the five methods listed by Robertson [35] to find the fundamental solution (X_1, Y_1), the classical method based on the continued fraction expansion of the quadratic irrational $\sqrt{D}$ introduced by Lagrange [12] is at the core of several other methods. It can be summarized as follows. One computes the r^{th} convergent (p_r/q_r) of the continued fraction $[\alpha_0; \alpha_1, ..., \alpha_r, \alpha_{r+1}, ...]$ of $\sqrt{D}$ which becomes periodic after the following term $\alpha_{r+1} = 2\alpha_0$ if $\sqrt{D}$ is a quadratic surd or quadratic irrational (i.e. $\sqrt{D} \notin \mathbb{Z}^+$) and with $\alpha_0 = \left\lfloor \sqrt{D} \right\rfloor$ i.e. the greatest integer $\leq \sqrt{D}$. The terms p_i and q_i can be found by the recurrence relations

$$p_i = \alpha_i p_{i-1} + p_{i-2}\ , \quad q_i = \alpha_i q_{i-1} + q_{i-2} \tag{4.5}$$

with $p_{-2} = 0, p_{-1} = 1, q_{-2} = 1, q_{-1} = 0$. The fundamental solution is then $(X_1, Y_1) = (p_r, q_r)$ if $r \equiv 1\,(mod\,2)$ or $(X_1, Y_1) = (p_{2r+1}, q_{2r+1})$ if $r \equiv 0\,(mod\,2)$.
For the general case of $N \neq 1$, the generalized Pell equation (4.1) can have either no solution at all, or one or several fundamental solutions (X_1, Y_1), and all integer solutions, if they exist, come on double infinite branches that can be expressed in function of the fundamental solution(s) (X_1, Y_1) and $(X_1, -Y_1)$. Several authors (see e.g. [12, 6, 20, 19, 35, 16, 17, 11] and references therein) discussed how to find the fundamental solution(s) of the generalized Pell equation, based on Lagrange's

method of continued fractions with various modifications (see e.g. [22]), and further how to find additional solutions from the fundamental solution(s). The method indicated by Matthews [17] based on an algorithm by Frattini [8, 9, 10] using Nagell's bounds [20, 18], will be used further.
Noting now (x_f, y_f) the fundamental solutions of the related simple Pell equation (4.3), the other solutions (X_n, Y_n) can be found from the fundamental solution(s) by

$$X_n + \sqrt{D}Y_n = \pm\left(X_1 + \sqrt{D}Y_1\right)\left(x_f + \sqrt{D}y_f\right)^n \tag{4.6}$$

for a proper choice of sign $\pm$ [35], or by the recurrence relations

$$X_n = x_f X_{n-1} + D y_f Y_{n-1} \tag{4.7}$$

$$Y_n = x_f Y_{n-1} + y_f X_{m-1} \tag{4.8}$$

that can also be written as

$$X_n = 2x_f X_{n-1} - X_{n-2} \tag{4.9}$$

$$Y_n = 2x_f Y_{n-1} - Y_{n-2} \tag{4.10}$$

or in function of Chebyshev's polynomials of the first kind $T_{n-1}(x_f)$ and of the second kind $U_{n-2}(x_f)$ evaluated at x_f (see [23])

$$X_n = X_1 T_{n-1}(x_f) + D Y_1 y_f U_{n-2}(x_f) \tag{4.11}$$

$$Y_n = X_1 y_f U_{n-2}(x_f) + Y_1 T_{n-1}(x_f) \tag{4.12}$$

For $N = \eta^2$ an integer square, the generalized Pell equation (4.1) admits always integer solutions. The variable change $(X', Y') = ((X/\eta), (Y/\eta))$ transforms the generalized Pell equation in a simple Pell equation $X'^2 - DY'^2 = 1$ which has integer solutions (X'_n, Y'_n). The integer solutions to the generalized Pell equation can then be found as $(X_n, Y_n) = (\eta X'_n, \eta Y'_n)$, or from ((4.9),(4.10)) with $(X_0, Y_0) = (1, 0)$ and $(X_1, Y_1) = (\eta x_f, \eta y_f)$, yielding simply

$$X_n = \eta T_n(x_f) \tag{4.13}$$

$$Y_n = \eta y_f U_{n-1}(x_f) \tag{4.14}$$

(which is also valid for the simple Pell equation (4.2) with $\eta = 1$). Note however that not all solutions in (X, Y) may be found in this way (see e.g.[39]) and, depending on the value of D, other fundamental solutions may exist.

5. Step 3: A More General Approach

The three cases of Theorem 2 can now be generalized as shown in the next theorem, that includes also the general method to find all solutions (M, a, c) such that (2.1) holds.

Theorem 4. *For* $\forall \delta, n \in \mathbb{Z}^+$*,* $\exists \sigma, a, M, M_n, c, x_f, y_f, X_1, Y_1, D, N \in \mathbb{Z}^+$*,* $\kappa \in \mathbb{Q}^+$*, with* $\gcd(\delta, \sigma) = 1$*,*$\kappa = (\sigma/\delta) > 1$*, such as (2.1) holds with*

$$\begin{aligned}
(5.1)\quad M_n &= X_1 y_f U_{n-1}(x_f) + Y_1 T_n(x_f) \\
(5.2)\quad a_n &= \frac{1}{2\delta^2}\left[\left(\sigma^2-\delta^2\right)\left(X_1+\left(\sigma^2+\delta^2\right)Y_1\right)y_f U_{n-1}(x_f) + \right. \\
&\quad \left. \left(X_1+\left(\sigma^2-\delta^2\right)Y_1\right)T_n(x_f)+\delta^2\right] \\
(5.3)\quad c_n &= \frac{\sigma}{2\delta^3}\left(X_1 y_f U_{n-1}(x_f) + Y_1 T_n(x_f)\right) \\
&\quad \left[\left(\sigma^2 X_1+\left(\sigma^4-\delta^4\right)Y_1\right)y_f U_{n-1}(x_f)+\left(X_1+\sigma^2 Y_1\right)T_n(x_f)\right]
\end{aligned}$$

where (x_f, y_f) *and* (X_1, Y_1) *are the fundamental solutions of respectively the simple (4.2) and generalized Pell equations (4.1) with* $D = \left(\sigma^4 - \delta^4\right)$ *and* $N = \delta^4$*, and* $T_n(x_f)$ *and* $U_n(x_f)$ *Chebyshev's Polynomials of the first and second kind evaluated at* x_f*.*

Proof. Let $n, M, M_n, a, a_n, c, c_n, \sigma, \delta, x_f, y_f, X_1, Y_1 \in \mathbb{Z}^+$, $k \in \mathbb{Z}^*$, $\kappa, C, C_n \in \mathbb{Q}^+$, with $M > 1$, $\gcd(\delta, \sigma) = 1$, $\kappa = (\sigma/\delta) > 1$. As g (2.13), a (2.2) and c (2.3) in Theorem 1 must be integers, the condition for the polynomial$\left(M^2\left(\kappa^4-1\right)+1\right)$ under the square root sign in (2.13), (2.2) and (2.3) to be a squared integer or a squared rational allows to find for which values of M (2.1) holds. Let

$$M^2\left(\kappa^4-1\right)+1 = C^2 \tag{5.4}$$

which can be rewritten as a Pell equation as

$$C^2 - \left(\kappa^4-1\right)M^2 = 1 \tag{5.5}$$

or as[1]

$$\left(\delta^2 C\right)^2 - \left(\sigma^4-\delta^4\right)M^2 = \delta^4 \tag{5.6}$$

which is a generalized Pell equation that always admits at least one fundamental solution as the right hand term is a squared integer. Noting (x_f, y_f) the fundamental solutions of the simple Pell equation (4.2) and (X_1, Y_1) the fundamental solution(s) of the generalized Pell equation (4.1) with $X = \left(\delta^2 C\right)$, $Y = M$, $D = \left(\sigma^4 - \delta^4\right)$ and $N = \delta^4$, all solutions can be found by (4.6) or ((4.11),(4.12)) $\forall n \in \mathbb{Z}^+$ yielding

$$C_n = \frac{\left(\left(\sigma^4-\delta^4\right)Y_1 y_f U_{n-1}(x_f) + X_1 T_n(x_f)\right)}{\delta^2} \tag{5.7}$$

and (5.1). Then replacing M by M_n (5.1) in (2.2) and (2.3) yields directly (5.2) and (5.3). □

Note that, although (5.1) yields all integer solutions in M to (5.6), some of them do not yield integer solutions to a_n (5.2) and c_n (5.3) and must be rejected.
For $\delta = 1$, (5.6) is a simple Pell equation similar to (4.2). It is easy to see that its fundamental solution is $(x_f, y_f) = \left(\sigma^2, 1\right)$ (see e.g. [22]). (5.6) admits then an infinitude of solutions $\forall n \in \mathbb{Z}^+$ for each integer value of σ, that can be written as

[1] Note that (5.5) could also be written as $C^2 - \left(\sigma^4-\delta^4\right)\left(M/\delta^2\right)^2 = 1$, but not all solutions may be obtained in this way.

((4.3), (4.4)) or ((4.13), (4.14)), yielding $C_n = T_n\left(\sigma^2\right)$ and

$$M_n = U_{n-1}\left(\sigma^2\right) \tag{5.8}$$

$$a_n = \frac{U_n\left(\sigma^2\right) - U_{n-1}\left(\sigma^2\right) + 1}{2} \tag{5.9}$$

$$c_n = \frac{\sigma U_n\left(\sigma^2\right) U_{n-1}\left(\sigma^2\right)}{2} \tag{5.10}$$

where the relation $U_n\left(\sigma^2\right) = T_n\left(\sigma^2\right) + \sigma^2 U_{n-1}\left(\sigma^2\right)$ (see e.g. [36]) was used in (5.9) and (5.10). Note as well that $g_n = U_n\left(\sigma^2\right) U_{n-1}\left(\sigma^2\right)/2$ as can be found from (2.10) or (2.11).

This generalizes the case (iii) of Theorem 2 and gives an infinitude of solutions (M_n, a_n, c_n) $\forall n \in \mathbb{Z}^+$ for $\delta = 1$ and for each value of σ, $\forall \sigma \in \mathbb{Z}^+$.

For $\delta > 1$, three of the fundamental solutions are always $(X_1, Y_1) = \left(\delta^2, 0\right)$ and $\left(\sigma^2, \pm 1\right)$, corresponding to respectively $(C, M) = (1, 0)$ and $\left(\left(\sigma^2/\delta^2\right), \pm 1\right)$. Depending on the value of $D = \left(\sigma^4 - \delta^4\right)$, other fundamental solutions may exist. All solutions in M can then be found by (5.1) $\forall n \in \mathbb{Z}^+$ for $(X_1, Y_1) = \left(\delta^2, 0\right)$ or $\left(\sigma^2, \pm 1\right)$, and $\forall n \in \mathbb{Z}^*$ for other fundamental solutions. Furthermore, solutions found for $(X_1, Y_1) = \left(\sigma^2, -1\right)$ and $\left(\delta^2, 0\right)$ yield integer values for M_n, a_n and c_n, while the solutions found for $(X_1, Y_1) = \left(\sigma^2, 1\right)$, although yielding integer values of M_n, do not yield integer values for a_n and c_n, as can be seen easily from (5.2) and (5.3), and must be rejected, although these non-integer values satisfy (2.1).

Relations (5.7) and (5.1) to (5.3) read

- for $(X_1, Y_1) = \left(\sigma^2, -1\right)$, $C_n = \kappa^2 T_n\left(x_f\right) - \delta^2\left(\kappa^4 - 1\right) y_f U_{n-1}\left(x_f\right)$ and

$$M_n = \sigma^2 y_f U_{n-1}\left(x_f\right) - T_n\left(x_f\right) \tag{5.11}$$

$$a_n = \frac{T_n\left(x_f\right) - \left(\sigma^2 - \delta^2\right) y_f U_{n-1}\left(x_f\right) + 1}{2} \tag{5.12}$$

$$c_n = \frac{\sigma\delta y_f U_{n-1}\left(x_f\right)\left(\sigma^2 y_f U_{n-1}\left(x_f\right) - T_n\left(x_f\right)\right)}{2} \tag{5.13}$$

- for $(X_1, Y_1) = \left(\delta^2, 0\right)$, $C_n = T_n\left(x_f\right)$ and

$$M_n = \delta^2 y_f U_{n-1}\left(x_f\right) \tag{5.14}$$

$$a_n = \frac{T_n\left(x_f\right) + \left(\sigma^2 - \delta^2\right) y_f U_{n-1}\left(x_f\right) + 1}{2} \tag{5.15}$$

$$c_n = \frac{\sigma\delta y_f U_{n-1}\left(x_f\right)\left(\sigma^2 y_f U_{n-1}\left(x_f\right) + T_n\left(x_f\right)\right)}{2} \tag{5.16}$$

Note that the solutions (5.11) to (5.13) found for $(X_1, Y_1) = \left(\sigma^2, -1\right)$ are smaller than those (5.14) to (5.16) found for $(X_1, Y_1) = \left(\delta^2, 0\right)$ for a same value of $n \geq 1$. Furthermore, if $2g$ is a multiple of M, $2g = \mu M$, $\mu \in \mathbb{Z}^+$, (like for $M = 33, 35, 42...$), the fundamental solutions of the simple Pell equation are $(x_f, y_f) = \left(\left(\mu\sigma^2 - M\right), \mu\right)$, as can be easily verified.

If $g = \triangle_M$ (like for $M = 5, 15, ...$) or $\triangle_{M-1}$ (like for $M = 5, 13, 15, 17, 40, ...$), then

$$(5.17) \qquad (x_f, y_f) = \left(\left((M \pm 1)\,\sigma^2 - M\right), (M \pm 1)\right)$$

$$= \left(\left(2\left|\delta^4 - 2\sigma^2 + 1\right|\left(\frac{\sigma^2 - 1}{\left|\delta^4 - 2\sigma^2 + 1\right|}\right)^2 - 1\right),\right.$$

$$(5.18) \qquad \left.\left(\frac{2\left(\sigma^2 - 1\right)}{\left|\delta^4 - 2\sigma^2 + 1\right|}\right)\right)$$

with $M = \left(\delta^4 - 1\right) / \left(\left|2\sigma^2 - \delta^4 - 1\right|\right)$ (see [33]) with the $+$ (respectively $-$) sign in (5.17) for $g = \triangle_M$ (resp. $\triangle_{M-1}$) and vertical bars denote the absolute value.
Values of (x_f, y_f) and (X_1, Y_1) from [17] yielding solutions (M_n, a_n, c_n) for the first values of M, $1 < M \leq 45$ and $1 < a < 10^5$ of Table 1 can be found in [33].
For the case (i) of Theorem 2 with $M = (2k+1)$, $g = \left(M^2 - 1\right)/4$, $\delta = 2M$ and $\sigma = \left(2M^2 - 1\right)$, one of the other fundamental solutions is always

$$(5.19) \qquad (X_1, Y_1) = \left(\left(\frac{\delta\left(\sigma^2 - 2\right)}{2}\right), \frac{\delta}{2}\right)$$

$$(5.20) \qquad = \left(M\left(\left(2M^2 - 1\right)^2 - 2\right), M\right)$$

as can be easily verified. More generally, all solutions for this case can be found by (3.1) to (3.5).
For the case (ii) of Theorem 2 with $a = (2k+1)^2$, $g = \triangle_k$, $\delta = 16\triangle_k + 1 = 2a - 1$, $\sigma = 16\triangle_k + 3 = 2a + 1$, $M = k\delta = k\left(16\triangle_k + 1\right)$, one of the other fundamental solutions is always known and can be expressed in function of δ, k or a as

$$(5.21) \qquad (X_1, Y_1) = \left(\left(4\delta\left(\sqrt{\frac{\delta + 1}{2}}\right)^3\left(\sqrt{\frac{\delta + 1}{2}} - 1\right)\right), 2\delta\right)$$

$$(5.22) \qquad = \left(\left(4\left(16\triangle_k + 1\right)\left(8k\left(2k+1\right)^3 + 1\right)\right), 2\left(16\triangle_k + 1\right)\right)$$

$$(5.23) \qquad = \left((2a - 1)\,a^2\left(1 - a^{-1/2}\right), 2\,(2a - 1)\right)$$

6. Recurrent Relations

The sets of solutions (M_n, a_n, c_n) are obviously not independent. As (5.1) to (5.3) are linear combinations of Chebyshev polynomials, one has also the general recurrence relations

$$(6.1) \qquad M_n = 2x_f M_{n-1} - M_{n-2}$$

$$(6.2) \qquad a_n = 2x_f a_{n-1} - a_{n-2} - (x_f - 1)$$

$$(6.3) \qquad c_n = 2\left(2x_f^2 - 1\right) c_{n-1} - c_{n-2} + \delta\sigma^3 y_f^2$$

among values of M_n, a_n, c_n calculated for the same fundamental solutions (X_1, Y_1). These relations are immediate from (5.1) to (5.3) and the recurrence and other formulas for Chebyshev polynomials (see e.g [36]). For the sake of the recurrence, initial values of (M_n, a_n, c_n) for $n = 0, 1$ are shown in Table 5 for the cases (i) $M_n = k\delta_n$ of Theorem 3, (ii) $\delta = 1$ and $(x_f, y_f) = \left(\sigma^2, 1\right)$, (iii) $\delta > 1$ and $(X_1, Y_1) = \left(\sigma^2, -1\right)$, (iv) $\delta > 1$ and $(X_1, Y_1) = \left(\delta^2, 0\right)$, and (v) the general case for $\delta > 1$ and with other fundamental solutions (X_1^*, Y_1^*) of the generalized Pell equation.

TABLE 5. Initial values of (M_n, a_n, c_n) for $n = 0, 1$ for recurrence relations

X_1, Y_1	n	(M_n, a_n, c_n) for $M_n = k\delta_n$, $(x_f, y_f) = \left((2k+1)^2, 0\right)$
$[1, 0]$	0	$(1, 1, 1)$
	1	$\left(k(8k(k+1)+1), (2k+1)^2, (k(k+1)/2)\left(4(2k+1)^4 - 1\right)\right)$
X_1, Y_1	n	(M_n, a_n, c_n) for $\delta = 1$, $(x_f, y_f) = (\sigma^2, 1)$
$[1, 0]$	0	$(0, 1, 0)$
	1	$\left(1, \sigma^2, \sigma^3\right)$
X_1, Y_1	n	(M_n, a_n, c_n) for $\delta > 1$
$\sigma^2, -1$	0	$(-1, 1, 0)$
	1	$\left((\sigma^2 y_f - x_f), (x_f - (\sigma^2 - \delta^2) y_f + 1)/2,\ \delta\sigma y_f (\sigma^2 y_f - x_f)/2\right)$
$\delta^2, 0$	0	$(0, 1, 0)$
	1	$\left(\delta^2 y_f,\ (x_f + (\sigma^2 - \delta^2) y_f + 1)/2,\ \delta\sigma y_f (\sigma^2 y_f + x_f)/2\right)$
X_1^*, Y_1^*	0	$\left(Y_1^*, (X_1^* + (\sigma^2 - \delta^2) Y_1^* + \delta^2)/2\delta^2,\ \sigma Y_1^* (X_1^* + \sigma^2 Y_1^*)/2\delta^3\right)$
	1	$\left(X_1^* y_f + Y_1^* x_f,\right.$ $\left((x_f + (\sigma^2 - \delta^2) y_f) X_1^* + (\sigma^2 - \delta^2)(x_f + (\sigma^2 + \delta^2) y_f) Y_1^* + \delta^2\right)/2\delta^2,$ $\left.\sigma (y_f X_1^* + x_f Y_1^*)\left((x_f + \sigma^2 y_f) X_1^* + (\sigma^2 x_f + (\sigma^4 - \delta^4) y_f) Y_1^*\right)/2\delta^3\right)$

For $\delta = 1$, one has also the remarkable recurrence relation

$$M_n = M_{n-1} + 2a_{n-1} - 1 \tag{6.4}$$

among all values of M_n $\forall n \in \mathbb{Z}^+$ (see Table 7 further in section 7). One has also a similar relation among values of M_n and a_n for $\delta > 1$ if only those solutions (5.11), (5.12) and (5.14), (5.15) calculated respectively for $(X_1, Y_1) = \left(\sigma^2, -1\right)$ and $\left(\delta^2, 0\right)$ are considered.

If all solutions(M_n, a_n, c_n) are ordered in increasing value order and indexed by a new index $j \in \mathbb{Z}^+$, one obtains simply $j = n$ for solutions for $\delta = 1$, and for $\delta > 1$, if there are no fundamental solutions other than $(X_1, Y_1) = \left(\delta^2, 0\right)$ and $\left(\sigma^2, \pm 1\right)$, one obtains $j = 2n - 1$ for the solutions (5.11) to (5.13) and $j = 2n$ for the solutions (5.14) to (5.16). If fundamental solutions other than $(X_1, Y_1) = \left(\delta^2, 0\right)$ and $\left(\sigma^2, \pm 1\right)$ exist, the solutions (M_n, a_n, c_n) (5.1) to (5.3) calculated with these other fundamental solutions (including for $n = 0$) have to be inserted accordingly (see example further in Section 7).

For $\delta > 1$, if only those value of (M_j, a_j, c_j) (5.11) and 5.13) and $(M_{j+1}, a_{j+1}, c_{j+1})$ (5.14 to 5.16) calculated respectively with $(X_1, Y_1) = \left(\sigma^2, -1\right)$ and $\left(\delta^2, 0\right)$ are considered, two such sets of solutions are called a "recurrent pair" and obviously, for $\delta = 1$, all sets of (M_j, a_j, c_j) solutions form "recurrent pairs". The following theorem give other remarkable recurrent relations.

Theorem 5. *For* $\forall \delta \in \mathbb{Z}^+$, $\exists \sigma, \kappa, j, a_j, M_j, c_j \in \mathbb{Z}^+$ *with* $\gcd(\delta, \sigma) = 1$, $\kappa = (\sigma/\delta) > 1$, *and such as (2.1) holds, if* (M_j, a_j, c_j) *and* $(M_{j+1}, a_{j+1}, c_{j+1})$ *form a* *"recurrent pair", then*

$$M_{j+1} = M_j + 2a_j - 1 \tag{6.5}$$
$$= \kappa^2 M_j + \sqrt{M_j^2\left(\kappa^4-1\right)+1} = \kappa^2 M_j + C_j \tag{6.6}$$
$$M_j = \left(2\kappa^2-1\right) M_{j+1} - 2a_{j+1} + 1 \tag{6.7}$$
$$a_{j+1} + a_j = \kappa^2 M_{j+1} - M_j + 1 \tag{6.8}$$
$$a_{j+1} - a_j = \left(\kappa^2-1\right) M_{j+1} \tag{6.9}$$
$$c_j = \frac{\kappa M_j M_{j+1}}{2} \tag{6.10}$$
$$c_{j+1} + c_j = \kappa^3 M_{j+1}^2 \tag{6.11}$$

Proof. Let $\sigma, \delta, j, n, k, a, M, c \in \mathbb{Z}^+$ with $\gcd(\delta, \sigma) = 1$, $\kappa = (\sigma/\delta) > 1$, and (M_n, a_n, c_n) solutions of (2.1), yielding (M_j, a_j, c_j) and $(M_{j+1}, a_{j+1}, c_{j+1})$ to be a "recurrent pair" with $n = j$ for $\delta = 1$, and, for $\delta > 1$, $n = j$ or $n = j+1$ respectively for (M_n, a_n, c_n) (5.11) to (5.13) for $(X_1, Y_1) = \left(\sigma^2, -1\right)$ or (5.14) to (5.16) for $(X_1, Y_1) = \left(\delta^2, 0\right)$. Then,
(i) (6.5) is immediate from (5.8) and (5.9) for $\delta = 1$, and from (5.11), (5.12) and (5.14) for $\delta > 1$.
(ii) (6.6) is immediate from (2.11), (2.13) and (5.4).
(iii) For $\delta = 1$, replacing M_{j+2} by the recurrence relation (6.1) in (6.5) written for M_{j+2} yields directly (6.7). For $\delta > 1$, in $\left(2\kappa^2-1\right) M_{j+1} - 2a_{j+1} + 1$, replacing M_{j+1} and a_{j+1} by (5.14) and (5.15) yields

$$\begin{aligned}\left(2\kappa^2-1\right) M_{j+1} - 2a_{j+1} + 1 &= \left(2\sigma^2-\delta^2\right) y_f U_{n-1}(x_f) - \\ &\quad \left(T_n(x_f) + \left(\sigma^2-\delta^2\right) y_f U_{n-1}(x_f)\right) &(6.12)\\ &= \sigma^2 y_f U_{n-1}(x_f) - T_n(x_f) &(6.13)\end{aligned}$$

which is M_n (5.11). Therefore (6.7) holds for $\delta \geq 1$.
(iv) Summing and subtracting a_{j+1} and a_j extracted respectively from (6.7) and (6.5) yield directly (6.8) and (6.9).
(v) (2.3) and (6.6) yield directly (6.10).
(vi) Replacing M_j from (6.7) in (6.10) and replacing with c_{j+1} from (6.10) yield directly (6.11). □

This theorem means that once a solution (M_n, a_n, c_n) has been found for $\delta > 1$ from (5.11) to (5.13) with $(X_1, Y_1) = \left(\sigma^2, -1\right)$, the other solution (M_n, a_n, c_n) of the "recurrent pair" can be found directly from (6.5) to (6.11) without having to be calculated from (5.14) to (5.16) for $(X_1, Y_1) = \left(\delta^2, 0\right)$.

7. Summary and Examples

In summary, to find all solutions (M, a, c) such that the sum of M consecutive cubed integers starting from $a > 1$ equal a squared integer c^2, the following approach is proposed.
1) Calculate first all solutions for odd integer values of $M = (2k+1)$ $\forall k \in \mathbb{Z}^+$ by (3.1) to (3.5). Table 6 shows the first ten values of an infinitude of solutions.
2) Second, calculate all solutions for the cases $M_n = k\delta_n$ given by Theorem 3 $\forall k, n \in \mathbb{Z}^+$ either by (3.26), (3.28), (3.30) in function of Chebyshev polynomials,

TABLE 6. Values of (M, a, c) for $M = (2k+1)$, $1 \leq k \leq 10$

(3, 23, 204), (5, 118, 2940), (7, 333, 16296), (9, 716, 57960), (11, 1315, 159060), (13, 2178, 368004), (15, 3353, 754320), (17, 4888, 1412496), (19, 6831, 2465820), (21, 9230, 4070220)

TABLE 7. Values of (M_n, a_n, c_n) for $\delta = 1$, $2 \leq \sigma \leq 5$ and $2 \leq n \leq 6$

$\sigma = 2$	$\sigma = 3$
(8, 28, 504)	(18, 153, 8721)
(63, 217, 31248)	(323, 2737, 2808162)
(496, 1705, 1936880)	(5796, 49105, 904219470)
(3905, 13420, 120055320)	(104005, 881145, 291155861205)
(30744, 105652, 7441492968)	(1866294, 15811497, 93751283088567)
$\sigma = 4$	$\sigma = 5$
(32, 496, 65472)	(50, 1225, 312375)
(1023, 15841, 66912384)	(2499, 61201, 780312750)
(32704, 506401, 68384391040)	(124900, 3058801, 1949220937250)
(1045505, 16188976, 69888780730560)	(6242501, 152878825, 4869153120937875)
(33423456, 517540816, 71426265522241344)	(312000150, 7640882425, 12163142546881874625)

or by the recurrence relations (6.1) to (6.3) with $(x_f, y_f) = \left((2k+1)^2, 0\right)$ and the initial values of Table 5 (see Table 3).
3) Third, calculate all solutions for the case $\delta = 1$ $\forall \sigma \in \mathbb{Z}^+$ either by (5.8) to (5.10), or by the recurrence relations (6.1) to (6.3) with $(x_f, y_f) = \left(\sigma^2, 1\right)$ and the initial values of Table 5. Table 7 shows the first five values of the infinitude of solutions for $2 \leq \sigma \leq 5$.
4) Fourth, for all other cases with $\delta > 1$, find the fundamental solutions of the simple and generalized Pell equations with $D = \left(\sigma^4 - \delta^4\right)$ and $N = \delta^4$ for all values of $\sigma \in \mathbb{Z}^+$ and of $\delta \in \mathbb{Z}^+$ such that $1 < \delta < \sigma$ and $\gcd(\delta, \sigma) = 1$.
4.1) If there are no other fundamental solutions than $(X_1, Y_1) = \left(\delta^2, 0\right)$ and $\left(\sigma^2, \pm 1\right)$, then $g = \mu M/2$ ($\mu \in \mathbb{Z}^+$) and $(x_f, y_f) = \left(\left(\mu\sigma^2 - M\right), \mu\right)$; if $g = \triangle_M$ or $\triangle_{M-1}$, then $M_1 = \left(\delta^4 - 1\right) / \left(\left|\delta^4 - 2\sigma^2 + 1\right|\right)$ and (5.18) gives (x_f, y_f). Then (5.11) to (5.16) yield an infinitude of integer solutions (M_n, a_n, c_n) $\forall n \in \mathbb{Z}^+$ with $(X_1, Y_1) = \left(\sigma^2, -1\right), \left(\delta^2, 0\right)$. Alternatively, the recurrence relations (6.1) to (6.3) with the initial values of Table 5 or the recurrence relations of Theorem 5 can be used. Solutions have to be ordered then in increasing order of M_j.
For $\sigma = 3$ and $\delta = 2$, $g = 120 = \triangle_{15}$, $D = 65$ and $N = 16$ yielding $(x_f, y_f) = (129, 16)$ and only three fundamental solutions $(X_1, Y_1) = (4, 0)$ and $(9, \pm 1)$ (see [17]). The first ten solutions are shown in Table 8 for $(X_1, Y_1) = (9, -1)$ and $(4, 0)$, arranged by M_j increasing values and with respectively $j = 2n - 1$ and $j = 2n$.
4.2) If there are fundamental solutions $(X_1, Y_1) = (X_1^*, \pm Y_1^*)$ other than $\left(\delta^2, 0\right)$ and $\left(\sigma^2, \pm 1\right)$, then (x_f, y_f) has to be calculated separately (see [17]). All integer solutions (M_n, a_n, c_n) are found by (5.1) to (5.3) for $(X_1, Y_1) = (X_1^*, Y_1^*)$ $\forall n \in \mathbb{Z}^*$ and by (5.11) to (5.16) for $(X_1, Y_1) = \left(\sigma^2, -1\right), \left(\delta^2, 0\right)$ $\forall n \in \mathbb{Z}^+$. Alternatively, the recurrence relations (6.1) to (6.3) can be used with the initial values of Table 5. Solutions have to be ordered then in increasing order of M_j.

TABLE 8. Values of (M_j, a_j, c_j) for $\delta = 2$, $\sigma = 3$ with $(x_f, y_f) = (129, 16)$ and $1 \leq n \leq 5$ with $j = 2n-1$ and $j = 2n$ respectively for $(X_1, Y_1) = (9, -1)$ and $(4, 0)$

X_1, Y_1	n	j	(M_j, a_j, c_j)
9, -1	1	1	(15, 25, 720)
4, 0	1	2	(64, 105, 13104)
9, -1	2	3	(3871, 6321, 47938464)
4, 0	2	4	(16512, 26961, 872242272)
9, -1	3	5	(998703, 1630665, 3190880053872)
4, 0	3	6	(4260032, 6955705, 58058190109584)
9, -1	4	7	(257661503, 420705121, 212391358097903424)
4, 0	4	8	(1099071744, 1794544801, 3864469249201901760)
9, -1	5	9	(66475669071, 108540290425, 14137193574521767668240)
4, 0	5	10	(283556249920, 462985602825, 257226802107318794853360)

TABLE 9. Values of (M_j, a_j, c_j) for $\delta = 3$, $\sigma = 4$ with $(x_f, y_f) = (2024, 153)$ and $0 \leq n \leq 3$ with $j = 3n+1$, $j = 3n-1$ and $j = 3n$ respectively for $(X_1, Y_1) = (159, 12), (16, -1)$ and $(9, 0)$

X_1, Y_1	n	j	(M_j, a_j, c_j)
159,12	0	1	(12, 14, 312)
16,-1	1	2	(424, 477, 389232)
9,0	1	3	(1377, 1548, 4105296)
159,12	1	4	(48615, 54635, 5117020440)
16,-1	2	5	(1716353, 1928872, 6378077594592)
9,0	2	6	(5574096, 6264280, 67270624549920)
159,12	2	7	(196793508, 221160443, 83849042274507096)
16,-1	3	8	(6947796520, 7808071356, 104513105644422184080)
9,0	3	9	(22563939231, 25357801869, 1102316769603603585072)
159,12	3	10	(796620071769, 895257416606, 1373975729120835063673080)

For $\sigma = 4$ and $\delta = 3$, $D = 175$ and $N = 81$ yielding $(x_f, y_f) = (2024, 153)$ and five fundamental solutions $(X_1, Y_1) = (9, 0), (16, \pm 1)$ and $(159, \pm 12)$ (see [17]). The first ten solutions are shown in Table 9 for $(X_1, Y_1) = (159, 12), (16, -1)$ and $(9, 0)$, arranged by M_j increasing values and with respectively $j = 3n+1$, $j = 3n-1$ and $j = 3n$.

8. Conclusions

The approach proposed in this paper to find all solutions (M, a, c) to (1.2) by investigating two other parameters, δ and σ, instead of investigating each individual values of M one by one in an increasing order of M values allows to find quite simple and elegant general solutions (M, a, c) based on solutions of simple and generalized Pell equations involving Chebyshev polynomials. Alternatively, recurrence relations can be used in order to simplify the computational part. This approach allows to find all possible solutions (M, a, c) to (1.2) but with the drawback that solutions are not ordered by increasing M values.

However, it is found that there are always at least one solution for every cases of all odd values of M, of all odd integer square values of a, and of all even values of M equal to twice an integer square.

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