ROOT GEOMETRY OF POLYNOMIAL SEQUENCES I: TYPE $(0,1)$

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Abstract. This paper is concerned with the distribution in the complex plane of the roots of a polynomial sequence $\{W_n(x)\}_{n\ge 0}$ given by a recursion $W_n(x) = aW_{n-1}(x) + (bx + c)W_{n-2}(x)$, with $W_0(x) = 1$ and $W_1(x) = t(x - r)$, where $a > 0$, $b > 0$, and $c, t, r \in \mathbb{R}$. Our results include proof of the distinct-real-rootedness of every such polynomial $W_n(x)$, derivation of the best bound for the zero-set $\{x \mid W_n(x) = 0 \text{ for some } n \ge 1\}$, and determination of three precise limit points of this zero-set. Also, we give several applications from combinatorics and topological graph theory.

1. Introduction

Gian-Carlo Rota [26] has said of the ubiquity of zeros of polynomials in combinatorics

"Disparate problems in combinatorics ... do have at least one common feature: their solution can be reduced to the problem of finding the roots of some polynomial or analytic function."

One such reduction is due to Newton's inequality, which implies that every real-rooted polynomial is log-concave. As observed by Brenti [1, 2], polynomials that arise from combinatorial problems often turn out to be real-rooted.

Given a sequence of polynomials $\{W_n(x)\}_{n\ge 0}$, we refer to the distribution of the set of zeros, taken over all n, as the *root geometry* of that sequence. General information for the root geometry of polynomials, especially the geometry of non-real roots, is given by Marden [19]; see also [21, 24].

This research arose during efforts by the present authors to affirm a quarter-century old conjecture (abbr. the LCGD conjecture) that the genus distribution (or genus polynomial, equivalently) of every graph is log-concave [8]. Although it was conjectured by Stahl [28] that genus polynomials are real-rooted, Chen and Liu [4] proved otherwise. Subsequently, various genus polynomials have been shown to have complex roots. Of course, this separates the problem of determining which graphs have real-rooted genus polynomials from trying to prove the LCGD conjecture.

After unexpected success [12] in proving the real-rootedness of the genus polynomials of iterated claws, we attempted the real-rootedness of genus polynomials for iterated 3-wheels [23]. The *iterated 3-wheel* W_3^n is the graph obtained from the cartesian product $C_3 \square P_{n+1}$, where P_k is a path graph with k vertices, by contracting a 3-cycle C_3 at one end of the product to a single vertex. By a preprocess of normalization, we transformed the problem equivalently into the following conjecture:

Conjecture 1.1. Let $W_0(x) = 1/27$, $W_1(x) = 1 + 7x$, $W_2(x) = 1 + 139x + 1120x^2 + 468x^3$, and

$$W_n(x) = (1 + 144x)W_{n-1}(x) + 54x(2 - 29x + 306x^2)W_{n-2}(x) - 5832x^3(1 - 11x)W_{n-3}(x),$$

Key words and phrases. genus distribution; real-rooted polynomial; recurrence; root geometry.

for $n \geq 3$. Then each of the polynomials $W_n(x)$ is real-rooted.

Real-rootedness of the genus polynomials of iterated 3-wheels W_3^n was confirmed by brute force computation for all $n \leq 220$. The complications encountered led us to consider the more general problem for polynomial sequences defined by a general linear recurrence of degree 3, with polynomial coefficients. As one may imagine, the difficulty did not decrease. This led us to some recurrences of degree 2. In particular, let $W_n(x)$ be a sequence of polynomials satisfying the recursion

$$W_n(x) \;=\; A(x)W_{n-1}(x) + B(x)W_{n-2}(x), \tag{1.1}$$

for $n \geq 2$, where $A(x)$ and $B(x)$ are polynomials, $W_0(x)$ is a constant, and $W_1(x)$ is a linear polynomial. When the polynomials $A(x)$ and $B(x)$ have degrees k and ℓ, respectively, we call the sequence $\{W_n(x)\}$ defined by (1.1) *a recursive polynomial sequence of type* (k, ℓ).

Classical bounds on the roots of a polynomial are given in terms of its coefficients. Examples include the Fujiwara bound [7], the Cauchy bound [3], and the Hirst-Macey bound [15]. More bounds and also some background are given by Rahman and Schmeisser [25], where the reader may also find, for instance, Rouché's theorem, Landau's inequality, and the Laguerre-Samuelson inequality, subject to bounding the roots of a polynomial. Conversely, the real-rooted polynomials with all roots in a prescribed interval have been characterized in terms of positive semi-definiteness of related Hankel matrices (see Lasserre [17]).

This paper is primarily concerned with the root geometry of a sequence of recursive polynomials of type $(0, 1)$.

2. Main Results and Examples

As a preliminary, we consider a recursive polynomial sequence of type $(0, 0)$, that is, one in which the polynomials $A(x)$ and $B(x)$ are constants, A and B. This serves as a bridge to considering a recursive sequence of polynomials of types in which $A(x)$ and $B(x)$ have other degree combinations.

Lemma 2.1. *Let $A, B \in \mathbb{R}$ with $A \neq 0$. Let $\{W_n\}_{n\geq 0}$ be a sequence of real numbers satisfying the initial condition $W_0 = 1$ and the recursion $W_n = AW_{n-1} + BW_{n-2}$. Writing $\Delta = A^2 + 4B$ and $g^{\pm} = (2W_1 - A \pm \sqrt{\Delta})/2$, we have*

$$W_n = \begin{cases} \left(1 + \dfrac{n(2W_1 - A)}{A}\right)\left(\dfrac{A}{2}\right)^n, & \text{if } \Delta = 0; \\[2ex] \dfrac{g^+(A + \sqrt{\Delta}\,)^n - g^-(A - \sqrt{\Delta}\,)^n}{2^n\sqrt{\Delta}}, & \text{if } \Delta \neq 0. \end{cases} \tag{2.1}$$

In particular, if $Re^{i\theta}$ is the polar representation of $A + \sqrt{\Delta}$, then we have

$$W_n = \left(\frac{R}{2}\right)^n \left(\cos n\theta + \frac{\sin n\theta}{\sqrt{-\Delta}}\right), \qquad \text{if } \Delta < 0. \tag{2.2}$$

Proof. The solution (2.1) to Recursion (1.1) can be found in elementary textbooks; for more extensive discussion, see Kocic and Ladas [16]. Note that when $A + \sqrt{\Delta} = Re^{i\theta}$, we have $A - \sqrt{\Delta} = Re^{-i\theta}$, since $\sqrt{\Delta}$ is either purely real or purely imaginary. Then, the expression (2.2) can be obtained from (2.1) directly. □

For instance, the Fibonacci sequence $\{f_n\}_{n\ge 0}$ is defined by the recursion

$$f_n \;=\; f_{n-1} + f_{n-2},$$

with $f_0 = f_1 = 1$. With $A = B = W_1 = 1$ (hence, $\Delta = 5$ and $g^{\pm} = (1 \pm \sqrt{5})/2$), Lemma 2.1 gives Binet's formula, as expected: $W_n = \frac{1}{\sqrt{5}}\left((g^+)^{n+1} - (g^-)^{n+1}\right)$. Thus, we see how Lemma 2.1 creates conditions forrecursive sequences of type $(0,0)$, under which the root geometry problem becomes easy.

2.1. **Main result.** The aim of this paper is to describe the root geometry of all polynomial sequences of type $(0,1)$. In order to formulate the main results of this paper, we use the following terminology.

Definition 2.2. The *zero-set* of a polynomial $p(x)$ is defined to be the set of all its roots. It is said to be *distinct-real-rooted* if all its roots are distinct and real.

Definition 2.3. Let s be a positive integer, and let $t \in \{s-1, s\}$. Let $X = \{x_1, x_2, \ldots, x_s\}$ and $Y = \{y_1, y_2, \ldots, y_t\}$ be ordered sets of real numbers. We say that the set X *interlaces the set* Y *from both sides*, denoted $X \bowtie Y$, if $t = s-1$ and

$$x_1 < y_1 < x_2 < y_2 < \cdots < x_{s-1} < y_t < x_s. \tag{2.3}$$

Note that the bow-tie symbol $\bowtie$ consists of a "times" symbol $\times$ in the middle and a bar at each side. The left (resp., right) bar indicates that the smallest (resp., largest) number in the chain (2.3) is from the set X. We say that the set X *interlaces* Y *from the right*, denoted $X \rtimes Y$, if either $X \bowtie Y$ or

$$t = s \qquad \text{and} \qquad y_1 < x_1 < y_2 < x_2 < \cdots < x_{s-1} < y_t < x_s. \tag{2.4}$$

Here the bar to the right of the "times" symbol $\times$ within the symbol $\rtimes$ means that the largest number in the chain (2.4) is from X. We observe that any set consisting of a single real number interlaces the empty set.

For any integers $m \le n$, we denote the set $\{m, m+1, \ldots, n\}$ by $[m, n]$. Moreover, when $m = 1$, we may denote the set $[1, n]$ by $[n]$. Lemma 2.4 presents some essential consequences of the interlacing property.

Lemma 2.4. *Let $f(x)$ and $g(x)$ be polynomials with zero-sets X and Y respectively. Let $\beta \in \mathbb{R}$, and let*

$$X' = X \cap (-\infty, \beta) = \{x_1, x_2, \ldots, x_p\} \qquad \textit{and} \qquad Y' = Y \cap (-\infty, \beta) = \{y_1, y_2, \ldots, y_q\}$$

be two ordered sets such that $X' \rtimes Y'$. Let $x_0 = y_0 = -\infty$ and $y_{q+1} = \beta$.

- *If $f(\beta) \neq 0$, then we have*

$$f(y_j) f(\beta)(-1)^{q-j} < 0 \qquad \textit{for all } j \in [q+1-p,\, q+1]; \tag{2.5}$$

- *If $g(\beta) \neq 0$, then we have*

$$g(x_i) g(\beta)(-1)^{p-i} > 0 \qquad \textit{for all } i \in [p-q,\, p]. \tag{2.6}$$

Proof. See Appendix 6.1. □

Definition 2.5. For any sequence $\{x_n\}$ of real numbers, we write $x_n \searrow x$ if x_n converges to the number x decreasingly, and we write $x_n \nearrow x$ if x_n converges to the number x increasingly.

Our main result, Theorem 2.6, concerns a polynomial sequence $\{W_n(x)\}$ of type $(0,1)$ in which $A(x)=a$ and $B(x)=bx+c$, with $ab\neq 0$ and $c\in\mathbb{R}$. The proof of Theorem 2.6 is given in Section 3.

Theorem 2.6. *Let $\{W_n(x)\}_{n\geq 0}$ be the polynomial sequence defined by the recursion*

$$W_n(x) \;=\; aW_{n-1}(x)+(bx+c)W_{n-2}(x) \tag{2.7}$$

with initial conditions $W_0(x)=1$ and $W_1(x)=t(x-r)$, where $a,b,t>0$, $c,r\in\mathbb{R}$, and $r\neq -c/b$. Also, let $R_n=\{\xi_{n,1},\ldots,\xi_{n,d_n}\}$ be the ordered zero set of $W_n(x)$. Then the polynomial $W_n(x)$ has degree $d_n=\lfloor (n+1)/2\rfloor$ and is distinct-real-rooted. Moreover, using the notations

$$x^*=-\frac{4c+a^2}{4b},\quad r^*=x^*-\frac{a}{2t},\qquad\text{and}\qquad y^*=r+\frac{(at+b)-\sqrt{(at+b)^2+4t^2(br+c)}}{2t^2}$$

we have the following conclusions:

(i) *If $r\in(-\infty,r^*]$, then $R_{n+1}\rtimes R_n$ and $R_{n+2}\bowtie R_n$ for $n\geq 1$; $\xi_{n,\,d_n-i}\nearrow x^*$ for any fixed $i\geq 0$.*
(ii) *If $r\in(r^*,-c/b)$ then $R_{n+1}\rtimes R_n$ and $R_{n+2}\bowtie R_n$ for $n\geq 1$; $\xi_{n,\,d_n-i}\nearrow x^*$ for any fixed $i\geq 1$; and $\xi_{n,d_n}\nearrow y^*$ with $x^*<y^*$.*
(iii) *If $r\in(-c/b,+\infty)$ then $R'_{n+1}\rtimes R'_n$ and $R'_{n+2}\bowtie R'_n$ for $n\geq 3$; $\xi_{n,d_n-i}\nearrow x^*$ for any fixed $i\geq 1$; $\xi_{2n,d_{2n}}\nearrow y^*$ and $\xi_{2n-1,d_{2n-1}}\searrow y^*$ with $x^*<-c/b<y^*<x_{2,d_2}$.*

The best bounds for the set $\cup_{n\geq 1}R_n$ are, in these three respective cases, $(-\infty,\,x^*)$, $(-\infty,\,y^*)$ and $(-\infty,\,r)$. Furthermore, the sequence $\xi_{n,i}$ converges to $-\infty$ for any fixed $i\geq 1$.

We observe that in the statement of Theorem 2.6, the limit point x^* does not depend on the initial polynomial $W_1(x)$, as long as the polynomial $W_1(x)$ is linear, and furthermore, no root lies in the interval $(x^*,\,-c/b)$ for case (iii).

Definition 2.7. Let $\{W_n(x)\}=\{W_n(x)\}_{n\geq 0}$ be the polynomial sequence defined recursively by

$$W_n(x)=aW_{n-1}(x)+(bx+c)W_{n-2}(x), \tag{2.8}$$

with $W_0(x)=1$ and $W_1(x)=x$, where $a,b>0$ and $c\neq 0$. In this context, we say $\{W_n(x)\}$ is a $(0,1)$-*sequence of polynomials.*

Theorem 2.8. *Let $\{W_n(x)\}_{n\geq 0}$ be a $(0,1)$-sequence of polynomials. Then the polynomial $W_n(x)$ (of degree $d_n=\lfloor (n+1)/2\rfloor$) is distinct-real-rooted. Let*

$$x^*=-\frac{4c+a^2}{4b},\quad r^*=x^*-\frac{a}{2},\qquad\text{and}\qquad y^*=\frac{a+b-\sqrt{(a+b)^2+4c}}{2}. \tag{2.9}$$

Let $R_n=\{\xi_{n,1},\ldots,\xi_{n,d_n}\}$ be the ordered zero-set of $W_n(x)$.

(i) *If $r^*\geq 0$, then $R_{n+1}\rtimes R_n$ and $R_{n+2}\bowtie R_n$ for $n\geq 1$; $\xi_{n,\,d_n-i}\nearrow x^*$ for any fixed $i\geq 0$.*
(ii) *If $0\in(r^*,-c/b)$ then $R_{n+1}\rtimes R_n$ and $R_{n+2}\bowtie R_n$ for $n\geq 1$; $\xi_{n,\,d_n-i}\nearrow x^*$ for any fixed $i\geq 1$; and $\xi_{n,d_n}\nearrow y^*$ with $x^*<y^*$.*
(iii) *If $c>0$ then $R'_{n+1}\rtimes R'_n$ and $R'_{n+2}\bowtie R'_n$ for $n\geq 3$; $\xi_{n,d_n-i}\nearrow x^*$ for any fixed $i\geq 1$; $\xi_{2n,d_{2n}}\nearrow y^*$ and $\xi_{2n-1,d_{2n-1}}\searrow y^*$ with $x^*<-c/b<y^*<x_{2,d_2}$.*

For these three cases, the respective best bounds for the set $\cup_{n\geq 1}R_n$ are $(-\infty,\,x^*)$, $(-\infty,\,y^*)$, and $(-\infty,\,r)$. Moreover, the sequence $\xi_{n,i}$ converges to $-\infty$ for any fixed $i\geq 1$.

Now we explain how Theorem 2.6 can be obtained as a corollary of Theorem 2.8. When considering the root geometry problem of general recursive polynomial sequences of type $(0,1)$, it is acceptable to assume that $\deg W_0(x) \le \deg W_1(x)$ and that the polynomial $W_0(x)$ is monic, which implies that $W_0(x) = 1$. Note that if $a < 0$, then when considering the polynomial sequence

$$\tilde{W}_n(x) = (-1)^n W_n(-x). \tag{2.10}$$

it is routine to verify that $\tilde{W}_0(x) = 1$, $\tilde{W}_1(x) = x$, and $\tilde{W}_n(x) = -a\tilde{W}_{n-1}(x) + (-bx + c)\tilde{W}_{n-2}(x)$. Consequently, supposing that $a > 0$ is without loss of generality in regard to the root geometry.

We now explain why it is enough to prove Theorem 2.8 for only the case in which

$$(i)\, W_1(x) = x, \qquad (ii)\, c \ne 0, \qquad (iii)\, b > 0. \tag{2.11}$$

(i) Here the linear polynomial $W_1(x)$ can be supposed to have the form $t(x - r)$. We can always normalize the polynomials by the linear transformation

$$\tilde{W}_n(x) \;=\; W_n(x/t + r), \tag{2.12}$$

whose root geometry differs from that of the sequence $W_n(x)$ only by magnification and translation.

(ii) If $c = 0$, then the number 0 is a root of every polynomial $W_n(x)$. In this circumstance, one may consider the polynomials $\tilde{W}_n(x)$ defined by the rule

$$\tilde{W}_n(x) \;=\; \frac{W_{n+2}(x)}{W_2(x)}.$$

It is clear that $\{\tilde{W}_n(x)\}_{n\ge 0}$ satisfies the recursion $\tilde{W}_n(x) = a\tilde{W}_{n-1}(x) + (bx + c)\tilde{W}_{n-2}(x)$. Therefore, the condition $c \ne 0$ is not really restrictive.

(iii) The case in which $b < 0$ is unexplored. In fact, when $b < 0$, both the degrees and the leading coefficients of the polynomials $W_n(x)$ may vary irregularly. We also note that dropping Condition (iii) may yield non-real-rooted polynomials $W_n(x)$. For example, when $a = 1$, $b = -1$, $c = -1$, we have $W_3(x) = -x^2 - x - 1$, which has no real roots.

We remark that in a general setting, beyond the genus polynomials of graphs, the polynomials $W_n(x)$ might have negative coefficients. In summary, this study of the root geometry of recursive polynomials of type $(0,1)$ has only two restrictions. One is that the polynomial $W_0(x)$ is a constant. The other is the assumption that the number b is positive.

2.2. **Some examples.** We now present several examples to illustrate our results.

Example 2.9. One kind of sequence of Fibonacci polynomials $W_n(x)$ is defined by the recursion

$$W_n(x) \;=\; W_{n-1}(x) + xW_{n-2}(x), \tag{2.13}$$

where $W_0(x) = 1$ and $W_1(x) = x + 1$; see [20, Table 3] and [27, A011973]. Accordingly, $a = b = 1$, $c = 0$, and $r = -1$, and we compute from Equation (2.9) that

$$x^* = -\frac{4c + a^2}{4b} = -\frac{1}{4} \qquad \text{and} \qquad r^* = x^* - \frac{a}{2} = -\frac{1}{4} - \frac{1}{2} = -\frac{3}{4} > -1 = r.$$

By Theorem 2.6(i), we know that each polynomial $W_n(x)$ is distinct-real-rooted and that all roots are less than $-1/4$. Also, for any $\epsilon > 0$, there exists a number $M' > 0$ such that every polynomial $W_n(x)$ with $n > M'$ has a root in the interval $(-1/4 - \epsilon,\, -1/4)$. Moreover, by the final conclusion of

Theorem 2.6, we know that for any $N > 0$, there exists a number $M > 0$ such that every polynomial $W_n(x)$ with $n > M$ has a root less than $-N$.

In the next two examples, we examine how the set of convergent points is affected when we change the coefficient of $W_{n-2}(x)$ in Recursion (2.13) first to $2x/5$ and then to $x + 2$.

Example 2.10. Let $W_n(x)$ be the polynomial sequence defined by the recursion

$$W_n(x) \;=\; W_{n-1}(x) + \frac{2x}{5}W_{n-2}(x),$$

with initial values $W_0(x) = 1$ and $W_1(x) = x + 1$. We see that $a = 1$, $b = 2/5$, $c = 0$, and $r = -1$. We calculate from Equation (2.9) that

$$x^* = -\frac{4c + a^2}{4b} = -\frac{5}{8} < -\frac{3}{5} = y^* \qquad \text{and} \qquad r = -1 \in \left(-\frac{9}{8},\, 0\right) = \left(r^*,\, -\frac{c}{b}\right).$$

By Theorem 2.6, the polynomial $W_n(x)$ is distinct-real-rooted, and the largest root converges to $-3/5$ increasingly. Moreover, for any positive integer i, the root sequence $x_{n,\,d_n-i}$ converges to $-5/8$ increasingly, and the root sequence $x_{n,\,i}$ converges to $-\infty$ decreasingly.

Example 2.11. Let $W_n(x)$ be the polynomial sequence defined by the recursion

$$W_n(x) \;=\; W_{n-1}(x) + (x + 2)W_{n-2}(x),$$

with initial values $W_0(x) = 1$ and $W_1(x) = x + 1$. Thus, $a = b = 1$, $c = 2$, and $r = -1$. We compute that $W_2(x) = 2x + 3$, and that

$$x^* = -\frac{9}{4}, \qquad r = -1 > -2 = -\frac{c}{b}, \qquad \text{and} \qquad y^* = -\sqrt{2}.$$

Therefore, we have $x^* < -c/b < x_{2,\,d_2}$. By Theorem 2.6, each of the polynomials $W_n(x)$ is distinct-real-rooted, and has exactly one root larger than $-9/4$. The sequence of largest roots converges to $-\sqrt{2}$ oscillatingly. Moreover, for any positive integer i, the root sequence $x_{n,\,d_n-i}$ converges to $-9/4$ increasingly, and the root sequence $x_{n,\,i}$ converges to $-\infty$ decreasingly.

Example 2.12. This example illustrates how our results can be used to prove the real-rootedness of a sequence of partial genus polynomials. Let $D_n(x)$ be the polynomial sequence defined by the recursion $D_n(x) = 2D_{n-1}(x) + 8xD_{n-2}(x)$, with $D_0(x) = 1$ and $D_1(x) = 2x$, which may be recognized by those familiar with enumerative research in topological graph theory (for example, see [8,10,13]) as a partial genus distribution for the closed-end ladder L_n, which is shown in Figure 2.1.

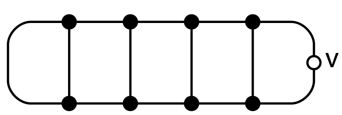

FIGURE 2.1. The closed-end ladder L_4 with a 2-valent root-vertex v.

The polynomial $D_n(x)$ is the generating function for the number of cellular imbeddings of the ladder L_n such that two different faces are incident on the root-vertex. By Theorem 2.6, each $D_n(x)$ is a distinct-real-rooted polynomial, and the root sequence $\xi_{n,\,d_n-i}$ converges to $-1/8$ for every nonnegative integer i. In particular, none of the polynomials $D_n(x)$ has a root larger than $-1/8$. Unfortunately, we do not yet know what topological information is implied by this convergent point.

3. Distinct Real-Rootedness

The proof of Theorem 2.8 begins here with an investigation of the real-rootedness of a $(0,1)$-sequence of polynomials. The remainder of the proof will be given in Section 4 and Section 5.

For any polynomial $f(x)$, we follow the usual definition that $f(\pm\infty) = \lim_{x\to\pm\infty} f(x)$. We start our analysis of $(0,1)$-sequences $\{W_n(x)\}_{n\geq 0}$ by finding a formula for the degree and the leading coefficient of each of the polynomials.

Lemma 3.1. *Let $\{W_n(x)\}_{n\geq 0}$ be a $(0,1)$-sequence of polynomials, with the constant b as in Definition 2.7, and with t_n the leading coefficient of $W_n(x)$. Then*

$$d_n = deg(W_n(x)) = \left\lfloor \frac{n+1}{2} \right\rfloor, \quad t_{2n+1} = b^n, \qquad \textit{and} \qquad t_{2n} = b^{n-1}(na+b).$$

Moreover, for all $n \geq 1$, we have

$$W_n(-\infty)(-1)^{d_n} > 0 \qquad \textit{and} \qquad W_n(+\infty) > 0$$

Proof. The formulas for the degree d_n and the leading coefficients t_n can be verified by induction on the integer n. For any polynomial $f(x)$ with positive leading coefficient, it is clear that

$$f(-\infty)(-1)^{\deg f(x)} = +\infty \qquad \text{and} \qquad f(+\infty) = +\infty.$$

Since $t_n > 0$, we infer that $W_n(-\infty)(-1)^{d_n} = W_n(+\infty) = +\infty$. The sign relations follow immediately. □

Using the intermediate value theorem for a $(0,1)$-sequence of polynomials, we derive the following criterion for their distinct-real-rootedness.

Theorem 3.2. *Let $\{W_n(x)\}_{n\geq 0}$ be a $(0,1)$-sequence of polynomials. Let $d_n = deg(W_n(x))$, and let $\beta \leq -c/b$. We denote the ordered zero-set of the polynomial $W_n(x)$ by R_n. Suppose that for some numbers $m, k \in \mathbb{N}$, we define*

$$T_m = R_m \cap (-\infty,\, \beta) \qquad \textit{and} \qquad T_{m+1} = R_{m+1} \cap (-\infty,\, \beta), \tag{3.1}$$

and suppose, further, that

$$W_m(\beta)(-1)^k \;>\; 0, \tag{3.2}$$

$$|T_m| \;=\; d_m - k, \tag{3.3}$$

$$|T_{m+1}| \;=\; d_{m+1} - k, \qquad \textit{and} \tag{3.4}$$

$$T_{m+1} \;\rtimes\; T_m. \tag{3.5}$$

Then there exists a set $T_{m+2} \subseteq R_{m+2} \cap (-\infty,\, \beta)$ such that $|T_{m+2}| = d_{m+2} - k$ and, furthermore, such that $T_{m+2} \rtimes T_{m+1}$. Moreover, if

$$T_{m+2} \;=\; R_{m+2} \cap (-\infty,\, \beta), \tag{3.6}$$

then we have $T_{m+2} \bowtie T_m$.

Proof. By Conditions (3.3) and (3.4) of the premises, we can suppose that

$$T_{m+1} \;=\; \{x_1,\, x_2,\, \ldots,\, x_p\} \qquad \text{and} \qquad T_m \;=\; \{y_1,\, y_2,\, \ldots,\, y_q\}$$

are ordered sets, where $p = d_{m+1} - k$ and $q = d_m - k$. Definition (3.1) implies that $x_p < \beta$. In view of Condition (3.5), together with the premise $\beta \le -c/b$, we have the following ordering:

$$\cdots < y_{q-2} < x_{p-2} < y_{q-1} < x_{p-1} < y_q < x_p < \beta \le -c/b. \tag{3.7}$$

Note that Condition (3.5) also implies that $p \ge 1$ and $q \in \{p-1, p\}$. For convenience, let

$$x_0 = y_0 = -\infty \qquad \text{and} \qquad x_{p+1} = y_{q+1} = \beta.$$

By applying Lemma 2.4 with $f(x) = W_{m+1}(x)$ and $g(x) = W_m(x)$, we obtain that the zero-sets X and Y become R_{m+1} and R_m respectively. Consequently, by Definition (3.1), we have

$$X' = R_{m+1} \cap (-\infty, \beta) = T_{m+1} \qquad \text{and} \qquad Y' = R_m \cap (-\infty, \beta) = T_m.$$

From Inequality (3.2) in the premises, we infer that

$$W_m(\beta) \ne 0. \tag{3.8}$$

Therefore, we can use Inequality (2.6), which gives that

$$W_m(x_i)W_m(\beta)(-1)^{p-i} > 0 \qquad \text{for all } i \in [p-q, p]. \tag{3.9}$$

Let $i \in [p]$. Since $x_i \in T_{m+1} \subseteq R_{m+1}$, we have $W_{m+1}(x_i) = 0$. Taking $n = m+2$ and $x = x_i$ in Recursion (2.7), we see that $W_{m+2}(x_i) = (bx_i + c)W_m(x_i)$. From (3.7), we see that $x_i < -c/b$. Since $b > 0$, we deduce that $bx_i + c < 0$. Thus, we can substitute $W_m(x_i) = W_{m+2}(x_i)/(bx_i + c)$ into Inequality (3.9), which gives that

$$\frac{W_{m+2}(x_i)}{bx_i + c} W_m(\beta)(-1)^{p-i} > 0.$$

Since $bx_i + c < 0$, the above inequality can be reduced to

$$W_{m+2}(x_i)W_m(\beta)(-1)^{p-i} < 0. \tag{3.10}$$

Note that Inequality (3.10) holds also for $i = p+1$. Replacing i by $i+1$ in Inequality (3.10) gives that $W_{m+2}(x_{i+1})W_m(\beta)(-1)^{p-i} > 0$. Multiplying it by Inequality (3.10), we obtain that

$$W_{m+2}(x_i)W_{m+2}(x_{i+1}) < 0.$$

By the intermediate value theorem, the polynomial $W_{m+2}(x)$ has a root in the interval (x_i, x_{i+1}). Let z_i be such a root.

When $i = 1$, Inequality (3.10) is $W_{m+2}(x_1)W_m(\beta)(-1)^{p-1} < 0$. Multiplying it by Inequality (3.2) gives that $W_{m+2}(x_1)(-1)^{p-1+k} < 0$. Since $p = d_{m+1} - k$, the above inequality is $W_{m+2}(x_1)(-1)^{d_{m+1}} > 0$. On the other hand, Lemma 3.1 gives that $W_{m+2}(-\infty)(-1)^{d_{m+2}} > 0$. Since $d_{m+1} + d_{m+2} = m+2$, we obtain that

$$W_{m+2}(-\infty)W_{m+2}(x_1)(-1)^{m+2} > 0.$$

Therefore, by the intermediate value theorem, the polynomial $W_{m+2}(x)$ has a root in the interval $(-\infty, x_1)$ when m is odd. Let z_0 be such a root.

Define

$$T_{m+2} = \begin{cases} \{z_1, z_2, \ldots, z_p\}, & \text{if } m \text{ is even;} \\ \{z_1, z_2, \ldots, z_p\} \cup \{z_0\}, & \text{if } m \text{ is odd.} \end{cases} \tag{3.11}$$

We shall now show that this set T_{m+2} has the desired properties.

- For each $j \in [0, p]$, the number z_j is chosen to be a zero of the polynomial $W_{m+2}(x)$. Therefore, $T_{m+2} \subseteq R_{m+2}$.
- For each $j \in [0, p]$, the number z_j is chosen from the interval $(x_j,\, x_{j+1})$, which is contained in the interval $(-\infty,\, \beta)$. Therefore, $T_{m+2} \subset (-\infty,\, \beta)$.
- From Definition (3.11), we see that
  - if m is even, then $|T_{m+2}| = p = d_{m+1} - k = (m+2)/2 - k = d_{m+2} - k$;
  - if m is odd, then $|T_{m+2}| = p + 1 = d_{m+1} - k + 1 = (m+3)/2 - k = d_{m+2} - k$.

  Hence, in any case, we have that $|T_{m+2}| = d_{m+2} - k$.
- Since $z_j \in (x_j,\, x_{j+1})$ for all $j \in [0, p]$, we have $T_{m+2} \rtimes T_{m+1}$ according to Definition 2.3.

It remains to show that $T_{m+2} \bowtie T_m$. By applying Lemma 2.4 with $f(x) = W_{m+2}(x)$ and with $g(x) = W_m(x)$, we obtain that the zero-sets X and Y become R_{m+2} and R_m respectively. Consequently, by Conditions (3.6) and (3.3) of the premises, we have

$$X' = R_{m+2} \cap (-\infty,\, \beta) = T_{m+2} \qquad \text{and} \qquad Y' = R_m \cap (-\infty,\, \beta) = T_m.$$

Since $d_{m+2} = d_m + 1$, the result $|T_{m+2}| = d_{m+2} - k$ and Condition (3.3) imply that

$$|X'| = |T_{m+2}| = d_{m+2} - k = d_m + 1 - k = q + 1 = |T_m| + 1 = |Y'| + 1.$$

Therefore, the lower bound $|Y'| - |X'| + 1$ of the range of the index j in Inequality (2.5) is 0. With the aid of Inequality (3.8), we can use Inequality (2.5), which gives that

$$W_{m+2}(y_j)W_{m+2}(\beta)(-1)^{q-j} < 0 \qquad \text{for all } j \in [0,\, q+1].$$

Let $j \in [q+1]$. Replacing j by $j-1$ in the above inequality, we obtain that

$$W_{m+2}(y_{j-1})W_{m+2}(\beta)(-1)^{q-j} > 0.$$

Multiplying the above two inequalities gives that

$$W_{m+2}(y_{j-1})W_{m+2}(y_j) < 0 \qquad \text{for all } j \in [q+1]. \tag{3.12}$$

By the intermediate value theorem, we infer that the polynomial $W_{m+2}(x)$ has a root, say, w_j, in the interval $(y_{j-1},\, y_j)$, that is,

$$-\infty \;<\; w_1 \;<\; y_1 \;<\; w_2 \;<\; y_2 \;<\; \cdots \;<\; w_q \;<\; y_q \;<\; w_{q+1} \;<\; y_{q+1} = \beta.$$

Define $T = \{w_1,\, w_2,\, \ldots,\, w_{q+1}\}$. Then the above chain of inequalities implies that $T \bowtie T_m$. By the choice of the numbers w_j, we see that $w_j \in R_{m+2}$ and $w_j < \beta$. It follows that $T \subseteq R_{m+2} \cap (-\infty,\, \beta) = T_{m+2}$. Since $|T| = q + 1 = |T_{m+2}|$, we conclude that $T = T_{m+2}$. Hence, the above interlacing relation $T \bowtie T_m$ becomes $T_{m+2} \bowtie T_m$, which completes the proof. □

The usage of this method of interlacing dates back at least to Harper [14], who established the real-rootedness of the Bell polynomials in this way. We should mention that Liu and Wang [18] have further applied this method to establish several easy-to-verify criteria for the real-rootedness of polynomials in which all the coefficients are non-negative.

We make two small preparations and a lemma before the further exploration of the root geometry, which will be used in several proofs below. The polynomial $W_1(x) = x$ has the unique root $y_1 = 0$. From Recursion (2.7), it is direct to compute that the polynomial $W_2(x) = (a+b)x + c$, which has the unique root $y_2 = -c/(a+b)$. Therefore, the polynomials $W_1(x)$ and $W_2(x)$ are real-rooted, and

$$R_1 = \{0\} \qquad \text{and} \qquad R_2 = \{-c/(a+b)\}. \tag{3.13}$$

In the remainder of this section, we will use Theorem 3.2 frequently. We will always set the constant k to be either 0 or 1. The following corollary is for the particular case $k = 0$.

Lemma 3.3. *Let $\{W_n(x)\}_{n\ge 0}$ be a $(0,1)$-sequence of polynomials. Denote the ordered zero-set of $W_n(x)$ by R_n. Let $c < 0$, and*

$$-c/(a+b) < \beta \le -c/b. \tag{3.14}$$

Let N be a positive integer. If $W_m(\beta) > 0$ for all $m \in [N]$, then we have

$$\begin{aligned} R_m &\subset (-\infty,\, \beta) && \text{for all } m \in [N+2],\\ R_{m+1} &\rtimes R_m && \text{for all } m \in [N+1],\\ \text{and} \qquad R_{m+2} &\bowtie R_m && \text{for all } m \in [N]. \end{aligned}$$

In particular, if $W_m(\beta) > 0$ for all $m \ge 1$, then the above three relations hold for all $m \ge 1$.

Proof. First, we show the following relations by induction on the integer m:

$$R_m \subset (-\infty,\, \beta), \qquad R_{m+1} \subset (-\infty,\, \beta), \qquad \text{and} \qquad R_{m+1} \rtimes R_m \qquad \text{for all } m \in [N+1]. \tag{3.15}$$

Recall from Formula (3.13) that $R_1 = \{0\}$ and $R_2 = \{-c/(a+b)\}$. When $m = 1$, the relations in (3.15) become

$$R_1 \subset (-\infty,\, \beta), \qquad R_2 \subset (-\infty,\, \beta), \qquad \text{and} \qquad R_2 \rtimes R_1,$$

that is,

$$0 < \beta, \qquad -c/(a+b) < \beta, \qquad \text{and} \qquad 0 < -c/(a+b).$$

Since $a, b > 0$, the above relations hold by the negativity of the number c and Inequality (3.14) in the premises. Suppose that the relations in (3.15) hold for $m \in [N]$, and we need to show them for $m+1$.

Let $k = 0$. The upper bound $-c/b$ of the parameter β is as same as that in Theorem 3.2. We are going to verify Conditions (3.2)–(3.5). Since $k = 0$, the inequality $W_m(\beta) > 0$ in the premises is exactly Conditions (3.2). From Definition (3.1) and the induction hypothesis $R_m \subset (-\infty,\, \beta)$, we infer that $T_m = R_m \cap (-\infty,\, \beta) = R_m$. It follows that $|T_m| = |R_m| = d_m$, i.e., Condition (3.3) holds. Similarly, we have $T_{m+1} = R_{m+1}$, i.e., Condition (3.4) holds. Thus, by the induction hypothesis we have that $R_{m+1} \rtimes R_m$, which is equivalent to Condition (3.5).

Therefore, we can apply Theorem 3.2 and obtain the existence of a set $T_{m+2} \subseteq R_{m+2} \cap (-\infty,\, \beta)$ such that $T_{m+2} \rtimes T_{m+1}$ and $|T_{m+2}| = d_{m+2}$. Since the sets T_{m+2} and R_{m+2} have the same cardinality d_{m+2}, we obtain that

$$T_{m+2} = R_{m+2}. \tag{3.16}$$

Consequently, the result $T_{m+2} \subset (-\infty,\, \beta)$ becomes the desired relation

$$R_{m+2} \subset (-\infty,\, \beta); \tag{3.17}$$

and the result $T_{m+2} \rtimes T_{m+1}$ becomes the desired relation $R_{m+2} \rtimes R_{m+1}$. This completes the induction proof for the relations in (3.15).

By Equation (3.16) and Relation (3.17), we infer that $T_{m+2} = R_{m+2} \cap (-\infty,\, \beta)$, which is exactly Condition (3.6). Hence, by Theorem 3.2, we derive that $T_{m+2} \bowtie T_m$, i.e., $R_{m+2} \bowtie R_m$, which completes the proof. □

In order to continue with our discussions, we fix more parameters of $(0,1)$-sequences of polynomials. Inspired by Lemma 2.1, we introduce the following notations.

Definition 3.4. Let $\{W_n(x)\}_{n\ge 0}$ be a $(0,1)$-sequence of polynomials. We define

$$\begin{aligned}\Delta(x) &= a^2+4(bx+c) = 4bx+a^2+4c,\\ g^{\pm}(x) &= \bigl(2x-a\pm\sqrt{\Delta(x)}\,\bigr)/2 = \bigl(2x-a\pm\sqrt{4bx+a^2+4c}\,\bigr)/2,\\ g(x) &= g^-(x)g^+(x) = x^2-(a+b)x-c.\end{aligned}$$

We denote the zeros of the functions $B(x)=bx+c$, $\Delta(x)$ and $g^+(x)$ by

$$x_B = -\frac{c}{b},\quad x_\Delta = -\frac{a^2+4c}{4b},\qquad \text{and} \qquad x_g = \frac{(a+b)-\sqrt{(a+b)^2+4c}}{2}, \tag{3.18}$$

respectively. We also define

$$n_0=\frac{2ab}{a^2+2ab+4c}.$$

We observe that Lemma 2.1 implies the following:

$$W_n(x_B) = a^{n-1}W_1(x_B), \tag{3.19}$$

$$W_n(x_\Delta) = \left(1+\frac{n(2x_\Delta-a)}{a}\right)\left(\frac{a}{2}\right)^n, \text{ and} \tag{3.20}$$

$$W_n(x_g) = x_g^n. \tag{3.21}$$

The following technical lemma provides the ordering among the numbers x_Δ, x_g, x_B, and 0, for the sake of determining the sign of the value $W_n(x)$ for specific numbers x in the proofs of Theorem 3.8.

Lemma 3.5. *Let $\{W_n(x)\}_{n\ge 0}$ be a $(0,1)$-sequence of polynomials with parameters specified as in Definition 3.4.*

(i) *If $\frac{4c}{a^2+2ab}\le -1$, then $W_n(x_\Delta)>0$ for all $n\ge 1$.*

(ii) *If $\frac{4c}{a^2+2ab}>-1$, then $x_g\in\mathbb{R}$, $x_\Delta<x_g$, and*

$$W_n(x_\Delta)\ge 0 \quad\iff\quad n\le n_0=\frac{2ab}{a^2+2ab+4c}, \tag{3.22}$$

where the equality on the left hand side holds if and only if the equality on the right hand side holds. Moreover, if $c>0$, then $W_n(x_\Delta)<0$ and $x_B<x_g<0$; otherwise, we have $0<x_g<x_B$.

Proof. See Appendix 6.2. □

Below is an example illustrating the cases $-(a^2+2ab)/4<c<0$ and $c>0$, respectively.

Example 3.6. Let $\{W_n(x)\}_{n\ge 1}$ be a $(0,1)$-sequence of polynomials with parameters specified as in Definition 3.4, and with $a=b=1$ and $c=-1/2$; thus, we are in Case (ii), and we have

$$W_n(x)=W_{n-1}(x)+(x-1/2)W_{n-2}(x),$$

with initial conditions $W_0(x) = 1$ and $W_1(x) = x$. We observe that $-(a^2 + 2ab)/4 < c < 0$. By Definition 3.4, we have

$$x_\Delta = -\frac{a^2 + 4c}{4b} = -\frac{1 - 2}{4} = 1/4,$$

$x_g = 1 - \sqrt{2}/2$, $x_B = 1/2$, and $n_0 = 2$. Thus, we may confirm the inequalities $x_\Delta < x_g$ and $0 < x_g < x_B$. To illustrate that the sign of the value $W_n(x_\Delta)$ satisfies the left side of the equivalence relation (3.22), we calculate that $W_1(x_\Delta) = 1/4 > 0$, $W_2(x_\Delta) = 0$, and $W_3(x_\Delta) = -1/16 < 0$. Continuing recursively, we see that $W_n(1/4) < 0$ for $n \geq 3$.

Example 3.7. Let $\{W_n(x)\}_{n\geq 1}$ be a $(0, 1)$-sequence of polynomials with parameters specified as in Definition 3.4, and with $a = b = 1$ and $c = 1$, which puts us in Case (i); here we have

$$W_n(x) = W_{n-1}(x) + (x + 1)W_{n-2}(x),$$

with $W_0(x) = 1$ and $W_1(x) = x$. This time, we have $c > 0$. By Definition 3.4, we have $x_\Delta = -5/4$, $x_g = 1 - \sqrt{2}$, $x_B = -1$, and $n_0 = 2/7$. These data correspond to the inequalities $x_B < x_g$ and $x_\Delta < x_g$ in the conclusion of Lemma 3.5. In fact, when $c > 0$, we always have $n_0 < 1$, which implies that $W_n(x_\Delta) < 0$.

We are now ready to establish the real-rootedness of every polynomial $W_n(x)$.

Theorem 3.8. *Let $\{W_n(x)\}_{n\geq 1}$ be a $(0, 1)$-sequence of polynomials with parameters specified as in Definition 3.4. Then every polynomial $W_n(x)$ is distinct-real-rooted. Moreover, let us denote the ordered zero-set of $W_n(x)$ by R_n, and let $y_n = \max R_n$ be the largest real root of the polynomial $W_n(x)$. For all $n \geq 1$, we may conclude the following:*

(i) *if $c < 0$, then $y_n < x_B$, $R_{n+1} \rtimes R_n$, and $R_{n+2} \bowtie R_n$.*

(ii) *if $c > 0$, then $y_n > x_B$, $R'_{n+1} \subset (-\infty, x_\Delta)$, $R'_{n+2} \rtimes R'_{n+1}$, and $R'_{n+2} \bowtie R'_n$, where $R'_n = R_n \setminus \{y_n\}$.*

Proof. From Equation (3.19), we see that

$$cW_n(x_B) < 0. \tag{3.23}$$

Below we will show (i) and (ii) individually.

(i) Let $c < 0$. Then Inequality (3.23) reduces to $W_n(x_B) > 0$ for all $n \geq 1$. Take $\beta = -c/b$. Then Condition (3.14) holds trivially. By Lemma 3.3, we deduce that $R_n \subset (-\infty, x_B)$, $R_{n+1} \rtimes R_n$ and $R_{n+2} \bowtie R_n$ for all $n \geq 1$.

(ii) Let $c > 0$. Then Inequality (3.23) implies that $W_n(x_B) < 0$. By Lemma 3.1, we have $W_n(+\infty) > 0$. Therefore, by the intermediate value theorem, the polynomial $W_n(x)$ has a real root in this interval $(x_B, +\infty)$. In particular, the largest root y_n is larger than x_B. Note that $x_\Delta = -(a^2 + 4c)/(4b) < -c/b = x_B$. Thus, we have

$$x_\Delta < x_B < y_n \qquad \text{for all } n \geq 1. \tag{3.24}$$

For the remaining desired relations, it suffices to show the following:

$$R'_n \subset (-\infty, x_\Delta), \quad R'_{n+1} \subset (-\infty, x_\Delta), \quad R'_{n+1} \rtimes R'_n, \quad R'_{n+1} \bowtie R'_{n-1}, \qquad \text{for all } n \geq 2. \tag{3.25}$$

We proceed by induction on n. Consider $n = 2$. Since $d_1 = d_2 = 1$, we have $R'_1 = R'_2 = \emptyset$. Since $a, b, c > 0$, from Definition (3.18), we have

$$x_\Delta = -(a^2 + 4c)/(4b) < 0.$$

In view of Formula (3.20), we deduce that

$$W_n(x_\Delta) \;=\; \left(1 + \frac{n(2x_\Delta - a)}{a}\right)\left(\frac{a}{2}\right)^n < 0, \qquad \text{for all } n \geq 1. \tag{3.26}$$

In particular, we have $W_3(x_\Delta) < 0$. On the other hand, Lemma 3.1 gives that $W_3(-\infty)(-1)^{d_3} > 0$. Since $d_3 = 2$, it reduces to $W_3(-\infty) > 0$. Therefore, by the intermediate value theorem, we infer that the polynomial $W_3(x)$ has a root, say, r_3, in the interval $(-\infty, x_\Delta)$. From Inequality (3.24), we see that $y_3 > x_\Delta$, and thus, $r_3 < x_\Delta < y_3$. It follows that $R_3 = \{r_3, y_3\}$, and thus, $R'_3 = \{r_3\}$. Therefore, the relations in (3.25) for $n = 2$ are respectively

$$\emptyset \subset (-\infty, x_\Delta), \qquad \{r_3\} \subset (-\infty, x_\Delta), \qquad \{r_3\} \rtimes \emptyset, \qquad \{r_3\} \bowtie \emptyset,$$

all of which hold trivially, except the second one holds since $r_3 < x_\Delta$.

Suppose that the 4 relations in (3.25) hold for some $n \geq 2$, by induction, it suffices to show that

$$R'_{n+2} \subset (-\infty, x_\Delta), \qquad R'_{n+2} \rtimes R'_{n+1}, \qquad \text{and} \qquad R'_{n+2} \bowtie R'_n. \tag{3.27}$$

In applying Theorem 3.2, we set $k = 1$, $\beta = x_\Delta$ and $m = n$. We shall verify Conditions (3.2)–(3.5).

- Inequality (3.26) checks the truth for Condition (3.2).
- From Definition (3.1), we have $T_n = R_n \cap (-\infty, x_\Delta)$. Note that in the zero-set R_n, except the largest root y_n, which is not in the interval $(-\infty, x_\Delta)$ by (3.24), all the other roots (whose union is the set R'_n) are in the interval $(-\infty, x_\Delta)$ by Hypothesis (3.25). Therefore, we infer that $R_n \cap (-\infty, x_\Delta) = R'_n$, and thus, $T_n = R'_n$. It follows that $|T_n| = |R'_n| = d_n - 1$, which verifies Condition (3.3).
- Similarly, we have $T_{n+1} = R'_{n+1}$, and Condition (3.4) holds true.
- Consequently, the hypothesis $T_{n+1} \rtimes T_n$ in (3.25) can be rewritten as $R'_{n+1} \rtimes R'_n$, which verifies Condition (3.5).

By Theorem 3.2, there exists a set $T_{n+2} \subseteq R_{n+2} \cap (-\infty, x_\Delta)$ such that $|T_{n+2}| = d_{n+2} - 1$ and $T_{n+2} \rtimes T_{n+1}$. From Inequality (3.24), we see that $y_{n+2} > x_\Delta$. It follows that

$$R_{n+2} \cap (-\infty, x_\Delta) = (R'_{n+2} \cup \{y_{n+2}\}) \cap (-\infty, x_\Delta) \subseteq R'_{n+2}. \tag{3.28}$$

Thus, we have $T_{n+2} \subseteq R'_{n+2}$. Since the sets T_{n+2} and R'_{n+2} have the same cardinality $d_{n+2} - 1$, we infer that $T_{n+2} = R'_{n+2}$. Now, the result $T_{n+2} \subset (-\infty, x_\Delta)$ is one of the desired relations:

$$R'_{n+2} \subset (-\infty, x_\Delta); \tag{3.29}$$

the result $T_{n+2} \rtimes T_{n+1}$ is another one of the desired relations:

$$R'_{n+2} \rtimes R'_{n+1}.$$

In view of our goal (3.27), it suffices to show that $R'_{n+2} \bowtie R'_n$, i.e., $T_{n+2} \bowtie T_n$. By Theorem 3.2, it suffices to verify Condition (3.6), i.e.,

$$R'_{n+2} = R_{n+2} \cap (-\infty, x_\Delta).$$

In view of Relation (3.29), we deduce that $R'_{n+2} \subseteq R_{n+2} \cap (-\infty, x_\Delta)$. Together with Relation (3.28), we find the above equation, which completes the proof. □

Continuing Example 3.6 and Example 3.7, we present the approximate values of zeros in the ordered set $R_n = \{\xi_{n,1}, \ldots, \xi_{n,d_n}\}$.

Example 3.9. This example continues Example 3.6. Table 3.1 illustrates that for $n \le 8$, we have

$$y_n = \max R_n \ < \ x_B \ = \ 1/2, \qquad R_{n+1} \ \rtimes \ R_n \qquad \text{and} \qquad R_{n+2} \ \bowtie \ R_n.$$

A more careful observation suggests that the second largest root $\xi_{n,\,d_n-1}$ is bounded by the number

TABLE 3.1. The approximate zeros of $W_n(x)$ $(1 \le n \le 8)$ in Example 3.6.

	$\xi_{n,\,d_n-3}$	$\xi_{n,\,d_n-2}$	$\xi_{n,\,d_n-1}$	$\xi_{n,\,d_n} = y_n$
$n = 1$				0
$n = 2$				0.2500
$n = 3$			−1.7807	0.2807
$n = 4$			−0.2886	0.2886
$n = 5$		−4.2912	0	0.2912
$n = 6$		−1.0218	0.1046	0.2922
$n = 7$	−7.5833	−0.3639	0.1547	0.2926
$n = 8$	−1.9561	−0.1194	0.1827	0.2927

$x_\Delta = 1/4$. In fact, this is true in general; it motivates Theorem 4.1 below.

Example 3.10. This example continues Example 3.7. Table 3.2 illustrates that for $n \le 8$,

$$\xi_{n,\,d_n-1} \ < \ x_\Delta \ = \ -5/4 \qquad \text{and} \qquad y_n \ > \ x_B \ = \ -1.$$

A more careful observation suggests that the largest root y_n converges to the point x_g in an oscillating

TABLE 3.2. The approximate zeros of $W_n(x)$ $(1 \le n \le 8)$ in Example 3.7.

	$\xi_{n,\,d_n-3}$	$\xi_{n,\,d_n-2}$	$\xi_{n,\,d_n-1}$	$\xi_{n,\,d_n}$
$n = 1$				0
$n = 2$				−0.5000
$n = 3$			−2.6180	−0.3819
$n = 4$			−1.5773	−0.4226
$n = 5$		−5.1819	−1.4064	−0.4116
$n = 6$		−2.2405	−1.3444	−0.4149
$n = 7$	−8.5525	−1.7194	−1.3140	−0.4139
$n = 8$	−3.1548	−1.5342	−1.2966	−0.4142

manner, which is approximately −0.4142. In fact, this convergence is true in general; see Theorem 4.1 and Theorem 5.5.

4. Bound on the Zero-Set R_n

As consequence of the real-rootedness of the $(0,1)$-sequence polynomials $\{W_n(x)\}_{n\ge 0}$, we improve the bound of the zero-set R_n of $W_n(x)$.

Theorem 4.1. *Let $\{W_n(x)\}_{n\ge 0}$ be a $(0,1)$-sequence of polynomials. Let us denote the ordered zero-set of $W_n(x)$ by R_n, and we let $R_n' = R_n \setminus \{y_n\}$, where $y_n = \max R_n$ is the largest real root of the polynomial $W_n(x)$.*

(i) *If $c \le -(a^2+2ab)/4$, then $R_n \subset (-\infty, x_\Delta)$ for all $n \ge 1$.*

(ii) *If $-(a^2+2ab)/4 < c < 0$, then we have*
- *$R_n \subset (-\infty, x_\Delta)$, for $n < n_0$;*
- *$R_n' \subset (-\infty, x_\Delta)$ and $y_n = x_\Delta$, for $n = n_0$;*
- *$R_n' \subset (-\infty, x_\Delta)$ and $y_n \in (x_\Delta,\, x_g)$, for $n > n_0$;*

(iii) *If $c > 0$, then we have $R_n' \subset (-\infty, x_\Delta)$, and*

$$x_B < y_2 < y_4 < y_6 < \cdots < y_{2n} < \cdots < x_g < \cdots < y_{2n-1} < \cdots < y_5 < y_3 < y_1 = 0. \tag{4.1}$$

Proof. We treat the three cases individually.

(i) Let $c \le -(a^2+2ab)/4$. Since $a, b > 0$, it is routine to check that

$$-c/(a+b) \;<\; -(a^2+4c)/(4b) \;<\; -c/b,$$

which verifies Condition (3.14) for $\beta = x_\Delta$. By Lemma 3.5, we have

$$W_n(x_\Delta) > 0 \qquad \text{for all } n \ge 1. \tag{4.2}$$

Now, by Lemma 3.3, we deduce that $R_n \subset (-\infty,\, x_\Delta)$ for all $n \ge 1$.

(ii) Let $-(a^2+2ab)/4 < c < 0$.

Case $n < n_0$**.** Recall from Formula (3.13) that $R_1 = \{0\}$ and $R_2 = \{-c/(a+b)\}$. If $n_0 \le 1$, then nothing needs to be shown in this case. Next suppose that $n_0 > 1$, i.e., $a^2+4c < 0$. Together with $b > 0$, this implies that $0 < -(a^2+4c)/(4b) = x_\Delta$, i.e., $R_1 \subset (-\infty,\, x_\Delta)$. If $n_0 \le 2$, then nothing else needs to be shown. And then suppose that $n_0 > 2$, i.e., $a^2+ab+4c < 0$. Together with $a, b > 0$, it is routine to check that

$$-\,c/(a+b) < -(a^2+4c)/(4b), \tag{4.3}$$

i.e., $R_2 \subset (-\infty,\, x_\Delta)$. If $n_0 \le 3$, nothing else needs to be shown. So we may suppose that $n_0 > 3$.

Let $N = \lceil n_0 \rceil - 3$. Since $n_0 > 3$, the integer N is positive. Take $\beta = x_\Delta$. From $x_\Delta < -c/b$, together with Inequality (4.3), we see that Condition (3.14) holds true. By Lemma 3.5, we have $W_n(x_\Delta) > 0$ for all $n \in [N]$. By Lemma 3.3, we have $R_n \subset (-\infty,\, x_\Delta)$ for all $n \in [N+2] = [\lceil n_0 \rceil - 1]$, i.e., for all $n < n_0$.

Case $n = n_0$. It follows that the number n_0 is an integer. By Lemma 3.5, we have $W_{n_0}(x_\Delta) = 0$. It suffices to show that the polynomial $W_{n_0}(x)$ has no roots larger than the number x_Δ. If $n_0 = 1$, then the polynomial $W_{n_0}(x) = W_1(x) = x$ has only one root. So we are done. Suppose that $n_0 \ge 2$. By the interlacing property $R_{n_0} \rtimes R_{n_0-1}$ obtained in Theorem 3.8, we see that the second largest root of the polynomial $W_{n_0}(x)$ is less than the largest root of the polynomial $W_{n_0-1}(x)$, which is less than the number x_Δ, in view of the case $n < n_0$. This completes the proof for the case $n = n_0$.

Case $n > n_0$**.** First, we show that $y_n < x_g$, i.e., $R_n \subset (-\infty, x_g)$. We do this by applying Lemma 2.4 for $\beta = x_g$. Recall from Definition (3.18) that $x_g = (a+b-\sqrt{(a+b)^2+4c})/2$. Since $a, b > 0$ and $-(a^2+2ab)/4 < c < 0$, it is routine to check that

$$-c/(a+b) \;<\; (a+b-\sqrt{(a+b)^2+4c})/2. \tag{4.4}$$

By Lemma 3.5 (ii), we have

$$\max(0,\, x_\Delta) < x_g < x_B. \tag{4.5}$$

The particular inequality $x_g < x_B$, together with Inequality (4.4), verifies Condition (3.14). On the other hand, since $x_g > 0$, Formula (3.21) implies that

$$W_n(x_g) \;>\; 0, \qquad \text{for all } n \geq 1. \tag{4.6}$$

From Lemma 3.3, we deduce that $R_n \subset (-\infty, x_g)$ for all $n \geq 1$.

By Lemma 3.5, we have $W_n(x_\Delta) < 0$. In view of Inequality (4.6), the polynomial $W_n(x)$ has different signs at the ends of the interval $(x_\Delta,\, x_g)$. Therefore, the polynomial $W_n(x)$ has an odd number, say p_n, of roots in the interval $(x_\Delta,\, x_g)$. In particular, we have

$$p_n \geq 1 \qquad \text{for all } n > n_0. \tag{4.7}$$

It suffices to show that $p_n = 1$, for all $n > n_0$. We proceed the proof by induction on n. Note that the largest root of the polynomial $W_{\lfloor n_0 \rfloor}(x)$ is less than or equal to the number x_Δ. By the interlacing property $R_{\lfloor n_0 \rfloor+1} \bowtie R_{\lfloor n_0 \rfloor}$, the polynomial $W_{\lfloor n_0 \rfloor+1}(x)$ has at most one root larger than the number x_Δ, i.e., $p_{\lfloor n_0 \rfloor+1} \leq 1$. In view of Inequality (4.7), we deduce that $p_{\lfloor n_0 \rfloor+1} = 1$. Thus, we can suppose that $p_n = 1$ for some $n > n_0$. If $n \leq 2$, then the degree $d_n \leq 1$. It follows immediately that $p_n = 1$. Suppose that $n \geq 3$. By the interlacing property $R_{n+1} \bowtie R_n$, the third largest root of the polynomial $W_{n+1}(x)$ is less than the second largest root of the polynomial $W_n(x)$, which is at most x_Δ since $p_n = 1$. Therefore, the polynomial $W_{n+1}(x)$ has at most two roots larger than the number x_Δ, i.e., $p_n \leq 2$. Since the integer p_n is odd, in view of Inequality (4.7), we infer that $p_n = 1$. This completes and the induction and hence the proof of (ii).

(iii) Let $c > 0$. The bound for the set R'_n has been confirmed in Theorem 3.8. It suffices to show Inequality (4.1). By Theorem 3.8, we have $y_n > x_B$ for all $n \geq 1$. It suffices to show that

$$y_{2n} \;<\; y_{2n+2} \;<\; x_g \qquad \text{and} \tag{4.8}$$

$$x_g \;<\; y_{2n+1} \;<\; y_{2n-1} \tag{4.9}$$

for all $n \geq 0$, where $y_0 = x_B$ and $y_{-1} = +\infty$. We proceed by induction on the integer n. When $n = 0$, the desired inequalities (4.8) and (4.9) become $y_2 < x_g < y_1$, i.e.,

$$-c/(a+b) < (a+b-\sqrt{(a+b)^2+4c})/2 < 0.$$

Since $a, b, c > 0$, it is routine to check the truth of the above inequalities. Now, based on the induction hypothesis that

$$y_{2n} \;<\; x_g \;<\; y_{2n-1}, \tag{4.10}$$

we are going to show the inequalities (4.8) and (4.9).

Since the number y_{2n} is largest real root of the polynomial $W_{2n}(x)$, and $y_{2n-1} > y_{2n}$ by the hypothesis (4.10), we infer that the value $W_{2n}(y_{2n-1})$ has the same sign as the limit $W_{2n}(+\infty)$, which is positive by Lemma 3.1. Therefore, we find $W_{2n}(y_{2n-1}) > 0$. Replacing n by $2n-1$ in Recursion (2.8), and taking $x = y_{2n-1}$, we obtain that

$$W_{2n+1}(y_{2n-1}) = aW_{2n}(y_{2n-1}) > 0. \tag{4.11}$$

On the other hand, by Lemma 3.5, we have $x_g < 0$. Thus from Equation (3.21), we infer that

$$W_n(x_g)(-1)^n \ > \ 0, \qquad \text{for all } n \geq 1. \tag{4.12}$$

In particular, we have $W_{2n+1}(x_g) < 0$. Together with (4.11), we see that the polynomial $W_{2n+1}(x)$ attains different signs at the ends of the interval $(x_g,\, y_{2n-1})$. By the intermediate value theorem, the polynomial $W_{2n+1}(x)$ has a root in the interval $(x_g,\, y_{2n-1})$. By Theorem 3.8, only the largest root y_{2n+1} of the polynomial $W_{2n+1}(x)$ is larger than the number x_B. Since $x_B < x_g$, we conclude that $y_{2n+1} \in (x_g,\, x_{2n-1})$. This proves Inequality (4.9).

Denote by z_{2n+1} the second largest root of the polynomial $W_{2n+1}(x)$. From the interlacing property $R_{2n+1} \rtimes R_{2n}$, we infer that

$$W_{2n+1}(x)W_{2n+1}(+\infty) < 0 \qquad \text{for all } x \in (z_{2n+1},\, y_{2n+1}).$$

By Lemma 3.1, we see that the limit $W_{2n+1}(+\infty) = +\infty$. It follows that

$$W_{2n+1}(x) < 0 \qquad \text{for all } x \in (z_{2n+1},\, y_{2n+1}). \tag{4.13}$$

Now, from Inequality (4.9) and the hypothesis (4.10), we see that $y_{2n} < x_g < y_{2n+1}$. From Theorem 3.8, we see that $z_{2n+1} < x_B < y_{2n}$. By Inequality (4.13), we infer that $W_{2n+1}(y_{2n}) < 0$.

Replacing n by $2n+2$ in Recursion (2.8), and taking $x = y_{2n}$, we obtain that

$$W_{2n+2}(y_{2n}) = aW_{2n+1}(y_{2n}) < 0. \tag{4.14}$$

By Inequality (4.12), we have $W_{2n+2}(x_g) > 0$. By the intermediate value theorem, the polynomial $W_{2n+2}(x)$ has a root in the interval $(y_{2n},\, x_g)$. Since only its largest root is larger than the number x_B, and since $y_{2n} > x_B$, we conclude that $y_{2n+2} \in (y_{2n},\, x_g)$. This proves Inequality (4.8), which completed the induction. □

In summary, we see that "almost all" zeros lie in the open interval $(-\infty, x_\Delta)$. Precisely speaking, when $c \leq -(a^2 + 2ab)/4$, all roots lie in $(-\infty, x_\Delta)$; when $c > -(a^2 + 2ab)/4$, only the largest root of the polynomial $W_n(x)$ is possibly but "eventually" larger than x_Δ, with maximum value $\max(x_g,\, 0)$.

Before ending this section, we mention that the recurrence system defined by Recursion (2.7) can be solved always by transforming the polynomials $W_n(x)$ into Chebyshev polynomials. More precisely, by induction and by the fact that Chebyshev polynomials of the second kind satisfy the recursion $U_n(t) = 2tU_{n-1}(t) - U_{n-2}(t)$ with initial conditions $U_0(t) = 1$ and $U_1(t) = t$, we obtain that

$$W_n(x) = \sqrt{-bx-c}^{\,n}\left(\frac{x}{\sqrt{-bx-c}}U_{n-1}\left(\frac{a}{2\sqrt{-bx-c}}\right) - U_{n-2}\left(\frac{a}{2\sqrt{-bx-c}}\right)\right).$$

By this, it is now clear that all roots of $W_n(x)$ are real and bounded, as described in Theorems 3.8 and 4.1.

5. Limit Points of the Zero-Set R_n

In this section, we show that one of the intervals $(-\infty, x_\Delta)$, $(-\infty, x_g)$, and $(-\infty, y_2)$ is the best bound of all zeros, depending on the range of the constant term c of the linear polynomial coefficient $B(x) = bx + c$. More precisely, we will demonstrate the aforementioned three limit points of the zero-set $\cup_{n\ge1} R_n$ over the course of several subsections. We say that a proposition *holds for large n*, if there exists a number N such that the proposition holds whenever $n > N$.

5.1. **The number x_g can be a limit point.** The following lemma will help determine all limit points of the zero-set $\cup_{n\ge1} R_n$, which are larger than the number x_Δ.

Lemma 5.1. *Let $\{W_n(x)\}_{n\ge0}$ be a $(0,1)$-sequence of polynomials. Let $x_0 \ne x_g$ and $\Delta(x_0) > 0$. Then $(x_0 - x_g)W_n(x_0) > 0$, for large n.*

Proof. See Appendix 6.3. □

Using Lemma 5.1, we can confirm that the roots outside the interval $(-\infty, x_\Delta)$ converges to the number x_g when $n \to \infty$ as follows.

Theorem 5.2. *Let $\{W_n(x)\}_{n\ge0}$ be a $(0,1)$-sequence of polynomials. Let us denote the ordered zero-set of $W_n(x)$ by R_n, and we let $y_n = \max R_n$ be the largest real root of the polynomial $W_n(x)$.*

(i) *If $-(a^2 + 2ab)/4 < c < 0$, then we have $y_n \nearrow x_g$.*
(ii) *If $c > 0$, then we have $y_{2n} \nearrow x_g$ and $y_{2n+1} \searrow x_g$.*

Proof. We treat the two cases individually.

(i) Suppose that $-(a^2 + 2ab)/4 < c < 0$. Since $R_{n+1} \rtimes R_n$, the sequence y_n increases. In virtue of Theorem 4.1, we have $y_n < x_g$ for all $n \ge 1$. Therefore, the sequence y_n converges to a finite number y^* as $n \to \infty$. If $y^* < x_g$, then there exists $x_0 \in (x_\Delta,\, x_g)$ such that the values $W_n(x_0)$ and $W_n(x_g)$ have the same sign for large n, i.e., $W_n(x_0) > 0$ for large n; see Inequality (4.6). This contradicts Theorem 5.1. Hence, we have that $y_n \nearrow x_g$.

(ii) Suppose that $c > 0$. From Theorem 4.1, we see that the sequence y_{2n} converges to a finite number y^*. Then we have $x_B < y^* \le x_g$. Suppose to the contrary that $y^* < x_g$, so there exists $x_0 \in (\ell_e,\, x_g)$ such that the numbers $W_{2n}(x_0)$ and $W_{2n}(x_g)$ have the same sign for large n, i.e., $W_{2n}(x_0) > 0$ for large n; see Inequality (4.12). This contradicts Theorem 5.1. Along the same line, we can show the convergence $y_{2n+1} \searrow x_g$, which completes the proof. □

An illustration for the convergences above can be found in Tables 3.1 and 3.2.

5.2. **The number x_Δ is a limit point.** In an analog with Lemma 5.1, we give a characterization of the sign of the value $W_n(x_0)$ for the case $\Delta(x_0) < 0$. This time the criterion for the sign is for all positive integers n. We define l_{x_0} to be the straight line $\sqrt{-\Delta(x_0)}\, x + (2x_0 - a)y = 0$, and the radian $\theta(x_0)$ to be $\arctan \frac{\sqrt{-\Delta(x_0)}}{a}$.

Lemma 5.3. *Let $\{W_n(x)\}_{n\ge0}$ be a $(0,1)$-sequence of polynomials, and let $\Delta(x_0)<0$.*

- *If the radian $n\theta(x_0)$ lies to the left of the line l_{x_0}, then $W_n(x_0)<0$;*
- *If the radian $n\theta(x_0)$ lies on the line l_{x_0}, then $W_n(x_0)=0$;*
- *If the radian $n\theta(x_0)$ lies to the right of the line l_{x_0}, then $W_n(x_0)>0$.*

Proof. See Appendix 6.4. □

Let us get some illustration of this characterization from the example below.

Example 5.4. This example continues Example 3.6. Take $x_0=-1$, we have $\Delta(x_0)=-5<0$ and $\theta(x_0)=\arctan(\sqrt{5})$. The line l_{x_0} becomes $\sqrt{5}x-3y=0$. Thus a radian ϕ lies to the left of the line l_{x_0} if and only if

$$\phi\in\big(\arctan(\sqrt{5}/3)+2\ell\pi,\ \arctan(\sqrt{5}/3)+(2\ell+1)\pi\big)\quad\text{for some integer }\ell. \tag{5.1}$$

By approximating $\arctan(\sqrt{5}/3)\approx0.6405$, we have that $\theta(x_0)\approx1.1502$, $2\theta(x_0)\approx2.3005$ and $3\theta(x_0)\approx3.4507$. By Relation (5.1), we deduce that θ_{x_0}, $2\theta_{x_0}$ and $3\theta_{x_0}$ lie to the left of the line l_{x_0}. In the same way we can deduce that the radians $4\theta(x_0)\approx4.6010$, $5\theta(x_0)\approx5.7513$ and $6\theta(x_0)\approx6.9015$ lie to the right of the line l_{x_0}. The truth is, as one may compute directly, that

$$\begin{aligned} W_1(-1)&=-1, & W_2(-1)&=-5/2, & W_3(-1)&=-1,\\ W_4(-1)&=11/4, & W_5(-1)&=17/4, & W_6(-1)&=1/8. \end{aligned}$$

The above data verifies the fact that $W_n(x_0)<0$ for $n\in\{1,2,3\}$, and that $W_n(x_0)>0$ for $n\in\{4,5,6\}$, coinciding with the characterization.

Now we are ready to justify that the number x_Δ is a limit point.

Theorem 5.5. *Let $\{W_n(x)\}_{n\ge0}$ be a $(0,1)$-sequence of polynomials, and let us denote the ordered zero-set of $W_n(x)$ by $R_n=\{\xi_{n,1},\ldots,\xi_{n,d_n}\}$. Then*

$$\lim_{n\to\infty}\xi_{n,\,d_n-i}\;=\;x_\Delta \tag{5.2}$$

for all $i\ge0$ if $c\le-(a^2+2ab)/4$; and for all $i\ge1$ otherwise.

Proof. Let $c\le-(a^2+2ab)/4$. We will show the limit (5.2) for all $i\ge0$. As will be seen, the other case can be done in the same vein.

From the interlacing property obtained in Theorem 3.8, we see that the sequence $\{\xi_{n,\,d_n-i}\}_{n\ge1}$ increases and all its members are less than the number x_Δ, which implies that it converges to a number which is at most x_Δ. Suppose, by way of contradiction, that the limit point of the sequence $\{\xi_{n,\,d_n-i}\}_{n\ge1}$ is not the point x_Δ.

When $i=0$, there exists a point $x_0<x_\Delta$ such that the numbers $W_n(x_0)$ and $W_n(x_\Delta)$ have the same sign, i.e., we have $W_n(x_0)>0$ for large n. Therefore, by Lemma 5.3, the radian $n\theta(x_0)$ resides in certain one side of the line l_{x_0} forever for large n. This is impossible because $\theta(x_0)<\pi/2$. Hence we deduce that $\lim_{n\to\infty}\xi_{n,d_n}=x_\Delta$.

Now for $i=1$, the sequence $\{\xi_{n,\,d_n-1}\}_{n\ge1}$ converges to some point less than x_Δ. Thus, there exists a number $x_1<x_\Delta$ such that the numbers $W_n(x_1)$ and $W_n(x_\Delta)$ have distinct signs, i.e., we have $W_n(x_1)<0$ for large n. Here again, the radian $\theta(x_1)$ resides in certain one side of the line l_{x_1} for large n, a contradiction. This confirms the truth of the limit (5.2) for $i = 1$. Continuing in this way, we can deduce that for a general $i \geq 2$, there exists a number $x_i < x_\Delta$, such that

$$W_n(x_i)(-1)^i \ > \ 0 \qquad \text{for large } n,$$

which contradicts Lemma 5.3. Hence, we conclude that the limit (5.2) holds true for all $i \geq 0$.

Now we consider the other possibility that $c > -(a^2 + 2ab)/4$. In fact, the above contradiction idea still works. This is because that, whatever sign does the value $W_n(x_\Delta)$ have, it is a fixed sign. However, the sign of the value $W_n(x_0)$ for any point $x_0 < x_\Delta$ can not be invariant for large n. This completes the proof. □

5.3. **Negative infinity is a limit point.** Now we are ready to study the negative infinity as a limit point.

Theorem 5.6. *Let* $\{W_n(x)\}_{n\geq 0}$ *be a* $(0,1)$*-sequence of polynomials, and let us denote the ordered zero-set of* $W_n(x)$ *by* $R_n = \{\xi_{n,1}, \ldots, \xi_{n,d_n}\}$*. Then* $\lim_{n\to\infty} \xi_{n,i} \ = \ -\infty$*, for all positive integers* i*.*

Proof. From the interlacing property $R_{n+2} \bowtie R_n$ obtained in Theorem 3.8, we see that the sequences $\{\xi_{2n,\,i}\}_{n\geq 1}$ decreases, and so does the sequence $\{\xi_{2n-1,\,i}\}_{n\geq 1}$. Therefore, these two sequences converge respectively. We shall show that both of these sequences converge to the negative infinity.

Suppose, by way of contradiction, that the sequence $\{\xi_{2n,\,1}\}_{n\geq 1}$ converges to some real number x^*. Then for any number $x_0 < x^*$, the number $W_n(x_0)$ has the sign of $W_n(-\infty)$. It follows that the sign of the number $W_n(x_0)$ would not change for large n, which contradicts Theorem 5.3. This proves that $\lim_{n\to\infty} \xi_{2n,i} \ = \ -\infty$ for $i = 1$. Its truth for general i, in fact, along the same lines, if it does not hold for some $i \geq 2$, then we can deduce the existence of a number x_i such that $x_i < x_\Delta$ and that the sign of the number $W_n(x_i)$ keeps invariant for large n, which leads to a contradiction.

Along the same lines, we can prove that $\lim_{n\to\infty} \xi_{2n-1,\,i} \ = \ -\infty$, for all $i \geq 1$.

Now, for any fixed $i \geq 1$, the subsequences $\{\xi_{2n,\,i}\}_{n\geq 1}$ and $\{\xi_{2n-1,\,i}\}_{n\geq 1}$ converge to the same point $-\infty$. Hence, the joint sequence $\{\xi_{n,i}\}_{n\geq 1}$ converges to the negative infinity as well, which completes the proof. □

For an illustration for the convergences in Theorem 5.5 and Theorem 5.6, the reader can refer to Tables 3.1 and 3.2.

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6. Appendix: Technical proofs

6.1. **Proof of Lemma 2.4.** Let $x_0 = y_0 = -\infty$ and $y_{q+1} = \beta$. The interlacing property $X' \rtimes Y'$ in the premises implies that $p \ge 1$ and $q \in \{p-1, p\}$. Since $X' \subset (-\infty, \beta)$, we infer that $x_p < \beta$. Therefore, we have that

$$\cdots < y_{q-2} < x_{p-2} < y_{q-1} < x_{p-1} < y_q < x_p < \beta.$$

We shall show Inequality (2.5) and Inequality (2.6) respectively.

Let $i \in [p]$. From the definition $X' = X \cap (-\infty, \beta)$ and the interlacing property $X' \rtimes Y'$ in the premises, we see that the number x_{p+1-i} is the unique root of the polynomial $f(x)$ in the interval (y_{q+1-i}, y_{q+2-i}). Suppose that $f(\beta) \neq 0$. By the intermediate value theorem, we infer that

$f(y_{q+1-i})f(y_{q+2-i}) < 0$, that is,

$$\begin{aligned} f(y_q)f(\beta) &< 0 \qquad (i=1),\\ f(y_{q-1})f(y_q) &< 0 \qquad (i=2),\\ &\vdots\\ f(y_{q-p+1})f(y_{q-p+2}) &< 0 \qquad (i=p). \end{aligned}$$

Multiplying the first i inequalities in the above list results in that

$$f(y_{q+1-i})f(\beta)(-1)^{i-1} < 0.$$

Replacing i by $q+1-j$ in it yields Inequality (2.5) for $j \in [q+1-p,\, q]$. When $j = q+1$, since $y_{q+1} = \beta$ stands as a premise, Inequality (2.5) holds true trivially.

From the definition $Y' = Y \cap (-\infty,\, \beta)$, we deduce that the polynomial $g(x)$ has no roots in the interval (y_q, β). Suppose that $g(\beta) \neq 0$. By the intermediate value theorem, we infer that $g(x)g(\beta) > 0$ for all $x \in (y_q, \beta)$. In particular, we have

$$g(x_p)g(\beta) \;>\; 0, \tag{6.1}$$

which is Inequality (2.6) for $j = p$. Below we can suppose that $p \geq 2$, and thus, $q \geq 1$.

Let $j \in [q]$. Similar to the previous proof, we have $g(x_{p-j})g(x_{p+1-j}) < 0$, that is,

$$\begin{aligned} g(x_{p-1})g(x_p) &< 0 \qquad (j=1),\\ g(x_{p-2})g(x_{p-1}) &< 0 \qquad (j=2),\\ &\vdots\\ g(x_{p-q})g(x_{p-q+1}) &< 0 \qquad (j=q). \end{aligned}$$

Multiplying the first j inequalities in the above list, we find that

$$g(x_{p-j})g(x_p)(-1)^{j-1} \;<\; 0.$$

Multiplying it by Inequality (6.1) results in that $g(x_{p-j})g(\beta)(-1)^{j-1} \;<\; 0$. Replacing j by $p-i$ in it yields that

$$g(x_i)g(\beta)(-1)^{p-i} \;>\; 0.$$

Together with Inequality (6.1), we obtain Inequality (2.6). This completes the proof. □

6.2. **Proof of Lemma 3.5.** From Equation (3.20), we have that

$$\text{the numbers } W_n(x_\Delta) \text{ and } a + n(2x_\Delta - a) \text{ have the same sign.} \tag{6.2}$$

(i) If $c \leq -(a^2+2ab)/4$, then we have

$$x_\Delta = -\frac{a^2+4c}{4b} \geq -\frac{a^2-(a^2+2ab)}{4b} = \frac{a}{2}, \tag{6.3}$$

that is, $2x_\Delta - a \geq 0$. It follows that $a + n(2x_\Delta - a \geq 0) > 0$ for all $n \geq 1$. By Relation (6.2), we obtain that $W_n(x_\Delta) > 0$.

(ii) Below we suppose that $c > -(a^2+2ab)/4$. From the deduction (6.3), we see that $2x_\Delta - a < 0$. If $n < n_0$, then we have $a + n(2x_\Delta - a) > a + \frac{2ab}{a^2+2ab+4c} \cdot \bigl(2(-\frac{a^2+4c}{4b}) - a\bigr) = 0$, which, by (6.2), implies that $W_n(x_\Delta) > 0$. Similarly, if $n = n_0$ then we have that $a + n(2x_\Delta - a) = 0$, and thus $W_n(x_\Delta) = 0$

by the relation (6.2); and if $n > n_0$ then we have that $a + n(2x_\Delta - a) < 0$, and thus $W_n(x_\Delta) < 0$ by the relation (6.2).

When $c > 0$, we have that $n_0 = \frac{2ab}{a^2+2ab+4c} < 1$. Therefore, the case $n > n_0$ happens for all $n \geq 1$, that is, $W_n(x_\Delta) < 0$. Thus, by Definition 3.4 we obtain that $x_g < 0$. Moreover, we have

$$x_g - x_B = \frac{(a+b) - \sqrt{(a+b)^2 + 4c}}{2} + \frac{c}{b} = \frac{ab + b^2 + 2c - b\sqrt{a^2 + 2ab + b^2 + 4c}}{2b}.$$

Thus, to show that $x_g > x_B$, it suffices to show that $(ab+b^2+2c)^2 > b^2(a^2+2ab+b^2+4c)$. By direct calculation, this inequality is equivalent to $4c(ab + c) > 0$, which is true since $a, b, c > 0$.

When $c < 0$, we have $x_g > 0$ from Definition (3.4) straightforwardly. Suppose to the contrary that $x_g \geq x_B$. It follows that $2c + b(a + b) \geq b\sqrt{(a + b)^2 + 4c} > 0$. Solving $(2c + b(a + b))^2 \geq (b\sqrt{(a + b)^2 + 4c})^2$ with $c < 0$, we see that $c \leq -ab$. On the one hand, by solving $2c+b(a+b) > 0$, we get $-b(a + b)/2 < c \leq -ab$, which implies that $a < b$. On the other hand, we have $-(a^2 + 2ab)/4 < c \leq -ab$, which implies that $a > 2b$. Hence, $2b < a < b$, a contradiction. This proves $x_g < x_B$ when $-(a^2 + 2ab)/4 < c < 0$. □

6.3. **Proof of Lemma 5.1.** By Lemma 2.1, the value $W_n(x_0)$ can be recast as the following form

$$W_n(x_0) \;=\; \frac{\left(A(x_0) + \sqrt{\Delta(x_0)}\,\right)^n}{2^n\sqrt{\Delta(x_0)}} \left[g^+(x_0) - g^-(x_0)\left(\frac{A(x_0) - \sqrt{\Delta(x_0)}}{A(x_0) + \sqrt{\Delta(x_0)}}\right)^n\right].$$

Since $A(x_0) = a > 0$ and $\sqrt{\Delta(x_0)} > 0$, we deduce that

$$\left|\frac{A(x_0) - \sqrt{\Delta(x_0)}}{A(x_0) + \sqrt{\Delta(x_0)}}\right| \;<\; 1.$$

Thus we obtain that

$$W_n(x_0)g^+(x_0) > 0 \qquad \text{for large } n. \tag{6.4}$$

Note that the function $2g^+(x) = 2x - a + \sqrt{4(bx + c) + a^2}$ is increasing. Since $g^+(x_g) = 0$, we infer that $(x_0 - x_g)g^+(x_0) > 0$. In view of (6.4), we conclude that $(x_0 - x_g)W_n(x_0) > 0$ for large n, which completes the proof. □

6.4. **Proof of Lemma 5.3.** By Lemma 2.1, the sign of the value $W_n(x_0)$ is equal to the sign of the value $F = \cos\theta + \ell\sin\theta$, where $\theta = n\theta(x_0)$, and $\ell = (2x_0 - a)/\sqrt{-\Delta(x_0)}$.

If $x_0 = a/2$, then the line l_{x_0} becomes the imagine axis $x = 0$. In this case, the sign of the value $W_n(x_0)$ is determined by the sign of the value $\cos\theta$. In other words, we have $W_n(x_0) > 0$ if and only if the radian $n\theta_0$ lies in the right open half-plane, and $W_n(x_0) < 0$ if and only if the radian $n\theta_0$ lies in the left open half-plane.

Below we can suppose that $x_0 \neq a/2$. It follows that $\ell \neq 0$.

- Assume that $\ell > 0$. It is elementary to find the following equivalence relation

$$F > 0 \quad\iff\quad \begin{cases} \tan\theta > -1/\ell, & \text{if } \cos\theta > 0; \\ \sin\theta > 0, & \text{if } \cos\theta = 0; \\ \tan\theta < -1/\ell, & \text{if } \cos\theta < 0. \end{cases}$$

In this case, we have $F > 0$ if and only if the radian θ lies to the right of the line $y = -x/\ell$, that is, of the line l_{x_0}. By symmetry, we have $F < 0$ if and only if the radian θ lies to the left of the line l_{x_0}. It follows immediately that $F = 0$ if and only if the radian θ lies on the line l_{x_0}.

- Now suppose that $\ell < 0$. Then we have the following equivalence relation in the same vein:

$$F > 0 \quad \Longleftrightarrow \quad \begin{cases} \tan\theta < -1/\ell, & \text{if } \cos\theta > 0; \\ \sin\theta < 0, & \text{if } \cos\theta = 0; \\ \tan\theta > -1/\ell, & \text{if } \cos\theta < 0. \end{cases}$$

In this case, we have the same desired characterization.

This completes the proof. □

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