CATEGORIES GENERATED BY A TRIVALENT VERTEX

SCOTT MORRISON, EMILY PETERS, AND NOAH SNYDER

ABSTRACT. This is the first paper in a general program to automate skein theoretic arguments. In this paper, we study skein theoretic invariants of planar trivalent graphs. Equivalently, we classify trivalent categories, which are nondegenerate pivotal tensor categories over $\mathbb C$ generated by a symmetric self-dual simple object X and a rotationally invariant morphism $1 \to X \otimes X \otimes X$. Our main result is that the only trivalent categories with dim Hom $(1 \to X^{\otimes n})$ bounded by 1, 0, 1, 1, 4, 11, 40 for $0 \le n \le 6$ are quantum SO(3), quantum G_2 , a one-parameter family of free products of certain Temperley-Lieb categories (which we call ABA categories), and the H3 Haagerup fusion category. We also prove similar results where the map $1 \to X^{\otimes 3}$ is not rotationally invariant, and we give a complete classification of nondegenerate braided trivalent categories with dimensions of invariant spaces bounded by the sequence 1, 0, 1, 1, 4. Our main techniques are a new approach to finding skein relations which can be easily automated using Gröbner bases, and evaluation algorithms which use the discharging method developed in the proof of the 4-color theorem.

1. INTRODUCTION

This is the first paper in a general program to automate skein theoretic arguments in quantum algebra and quantum topology. In this paper, we study skein theoretic invariants of planar trivalent graphs following Kuperberg [Kup94]. However, the general approach will work much more broadly and in later papers we will consider other situations like those in [TW05, BJ00, BJ03, MPS11, BJL14]. One might think of this program as attempting for skein theoretic arguments what [MS12] did for principal graph arguments.

Before getting into the particulars of this paper, we will recall the basic notions of skein theory, which we illustrate using its most famous example. The Jones polynomial invariant [Jon85] of framed links can be computed by applying the Kauffman bracket skein relations

$$\bigcirc = -A^2 - A^{-2}$$

and

$$= A + A^{-1}.$$

These relations not only assign a Laurent polynomial to framed links, they also assign to each tangle a linear combination of noncrossing tangles. Unlike for ordinary tangles, the number of noncrossing tangles with n boundary points is finite (and indeed given by Catalan numbers). Our goal is to find and prove theorems modelled on the following well-known result.

Theorem. Suppose a skein theoretic invariant of framed links has the property that the span of 4-boundary point tangles modulo the relations is 2-dimensional. This invariant must be a specialization of the Jones polynomial.

The proof of this statement is straightforward. Since the span of 4-boundary point tangles is only 2-dimensional, there must be some linear relation between the three diagrams which occur in the bracket relation. Some quick calculations show that only the specific Kauffman bracket relation is compatible with the second and third Reidemeister moves. Finally it is clear that the bracket relations are enough to determine the invariant, since applying the crossing relation will turn any framed link into a linear combination of unlinked, unknotted, circles.

A number of similar results have been proved following this same outline. In each case, assumptions on the dimensions of spaces of diagrams guarantee relations of a certain form must hold, and an involved calculation determines the coefficients of these relations (possibly in terms of some parameters). Subsequently one finds an evaluation algorithm using these relations, demonstrating that they suffice to determine the invariant. Finally, one may also want to know that this invariant actually exists!

Our program has a two-fold goal. First, we are interested in generalizing this approach beyond invariants of links. We take the view that links are 'merely' a certain class of planar graphs (with vertices modeled on the under-crossing and the over-crossing), subject to some local relations. We would like to be able to prove theorems about arbitrary such classes. Second, to the extent possible we want to automate the technique — to find very general methods to derive such theorems, and where possible to implement these methods in code. This will enable us to rapidly explore many different skein theoretic settings, and moreover to explore much further out into the space of examples than is possible with by-hand calculations. (This paper restricts itself to the case of planar trivalent graphs — but we go much further than previous investigations of trivalent skein theories.)

In particular, the first step in the outline above — finding relations — is highly amenable to computer calculation, as explained in this paper (see [Thu04b] for another approach to automating skein theory). By contrast, the second step — giving an evaluation algorithm — remains an art. In this paper, our evaluation algorithms all come from the discharging technique developed in the proof of the 4-color theorem. Indeed, we hope that in the future it may be possible to systematically discover evaluation algorithms based on discharging, adapted to the available relations in a given skein theoretic setting.

In this paper we concentrate on skein theoretic invariants of planar trivalent graphs. By the diagrammatic calculus for tensor categories, this paper can also be thought of as providing classifications of nondegenerate pivotal tensor categories over \mathbb{C} generated by a symmetrically self-dual simple object X and a rotationally invariant morphism $1 \to X \otimes X \otimes X$, where the dimensions of the first few invariant spaces $\operatorname{Hom}(1 \to X^{\otimes n})$ are small. We call such a tensor category a trivalent category. We note that the arguments in the paper are elementary and skein theoretic, so no knowledge of tensor categories is needed except for the existence proofs.

We now summarize this paper's results in a table. To use this table, compute the dimensions of the spaces of diagrams with n boundary points (equivalently the invariant spaces $\operatorname{Hom}(1 \to X^{\otimes n})$ for your tensor category) for the first few values of n. If an initial segment of this sequence of dimensions appears in the first column of some row of this table, then the second column explicitly identifies the category for you. If, on the other hand, you only have upper bounds for the dimensions of invariant spaces, then your category appears in the corresponding row, or some previous row.

dimension bounds	new examples	reference
1,0,1,1,2,	$SO(3)_{\zeta_5}$	Theorem A
$1,0,1,1,3,\ldots$	$SO(3)_q$ or $OSp(1 2)$	
$1,0,1,1,4,8,\ldots$	$ABA \subset TL_{\sqrt{dt^{-1}}} * TL_t$	Theorem B
$1,0,1,1,4,9,\ldots$	$(G_2)_{\zeta_{20}}$	
$1,0,1,1,4,10,\ldots$	$(G_2)_q$	
$1,0,1,1,4,11,37,\ldots$	H3	Theorem C
$1,0,1,1,4,11,40,\ldots$	nothing more	

In this table, * denotes the free product, and H3 denotes the fusion category Morita equivalent to the Haagerup fusion category constructed in [GS12b]. It is fascinating to see the Haagerup subfactor once again appearing as the first surprising example in a classification of 'small' categories.

The same classification is shown in Figure 1.

Kuperberg proved in [Kup94] that if the dimensions are exactly 1, 0, 1, 1, 4, 10 and in addition the diagrams with no internal faces give bases for these spaces, then the category must be $(G_2)_q$. In order to apply Kuperberg's result one needs to do a calculation to verify linear independence (cf. [MPS11, Lemma 3.9]). Our classification is more satisfying, as it does not include this linear independence assumption. To get an even more satisfying result, one would need to drop the condition that the trivalent vertex generates all morphisms. Dropping the generating assumption would introduce some additional examples (e.g. from subfactors, or from quantum subgroups of $(G_2)_q$).

It is worth noting that for these results up to and including the row 1, 0, 1, 1, 4, 10, we can also give proofs that do not use a computer, following Kuperberg. However, these by-hand calculations are not enlightening, and we prefer to give computer-assisted arguments uniformly in all cases, because they are easier to follow and more reliable.¹ By contrast, the 1, 0, 1, 1, 4, 11 results would be quite difficult, and probably impossible, to check by hand.

We also prove classification results when the map $1 \to X \otimes X \otimes X$ has a nontrivial rotational eigenvalue. This case turns out to be much easier than the rotationally invariant case (likely easy enough to check by hand in a tedious week), and there are correspondingly many fewer possibilities. If the dimensions

¹Indeed, N.S. initially did such calculations by hand. Due to human error this initial version missed the ABA case, but the error was easily caught by the more reliable computer.

are below $1, 0, 1, 1, 4, 11, 40, \ldots$ then the category must be a twisted version of $\operatorname{Rep}(S_3)$ or a twisted version of the Haagerup fusion category, or possibly one other new tensor category. This new candidate category is interesting as it can not come from subfactors or quantum groups. Finding such exotic tensor categories is one of our main motivations for this project.

As a corollary in the spirit of the results in [MPS11], we see that if X is a simple object in a pivotal category and $X^{\otimes 2} \cong 1 \oplus X \oplus A \oplus B$ for some simple objects A and B, and moreover dim $\operatorname{Hom}(X \to (A \oplus B)^{\otimes 2}) \leq 3$, then dim $\operatorname{Hom}(1 \to X^{\otimes 5}) \leq 10$ and so the category generated by the morphism $X^{\otimes 2} \to X$ must be either a twisted $\operatorname{Rep}(S_3)$ category, or an $SO(3)_q$, ABA, or $(G_2)_q$ category.

These results also allow a complete classification of braided trivalent categories with dim Hom $(1 \rightarrow X^{\otimes 4}) \leq 4$. given in Section 8. A quick argument shows that the braiding guarantees that dim Hom $(1 \rightarrow X^{\otimes 5}) \leq 10$. By the table above, any braided trivalent category with dim Hom $(1 \rightarrow X^{\otimes 4}) \leq 4$ must be OSp(1|2), $SO(3)_q$ or $(G_2)_q$ (the ABA categories are not braided). We also classify all braidings on these categories. Note that these results on braided trivalent categories only use the 1, 0, 1, 1, 4, 10 classifications, and so can be checked by hand in a reasonable amount of time.

1.1. Source code. This article relies on a number of computer calculations. In the interests of verifiability, the source code for all these calculations are bundled with the arXiv source of this article. After downloading the source, you'll find a code/ subdirectory containing a number of Mathematica notebooks. These notebooks are referenced at the necessary points through the text. As described above, equivalent calculations could also be performed by hand except for the computer calculations in Section 6 and parts of Section 7.

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2. TRIVALENT CATEGORIES

In this section, we introduce the notion of a trivalent category, as a pivotal category which is 'generated by a trivalent vertex'. In particular, every morphism in such a category is a linear combination of trivalent graphs (possibly with boundary) embedded in the plane, and indeed any such trivalent graph is allowed as a morphism.

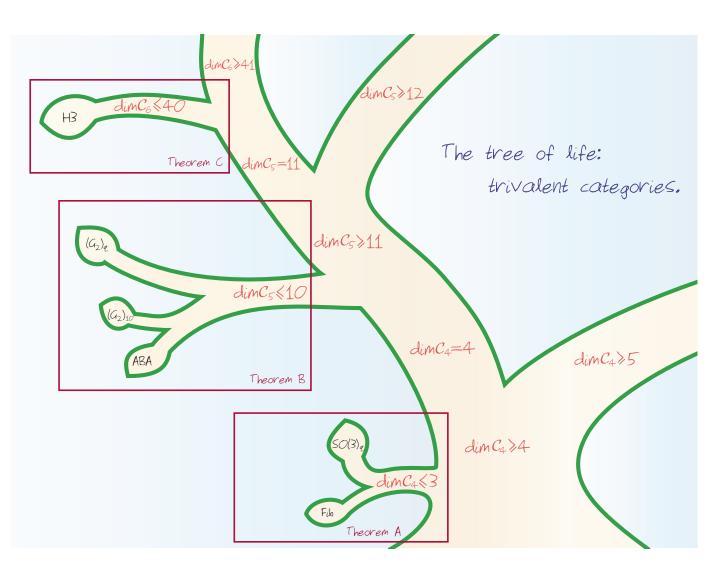


FIGURE 1. The 'tree of life' of trivalent categories, as described in this paper. The rightmost branch, corresponding to trivalent categories with dim $C_4 \geq 5$, certainly has representatives: the (quantum) representation categories of the complex simple Lie algebras (excepting some small cases), with the trivalent vertex the Lie bracket on the adjoint representation. The other two branches, corresponding to dim $C_4 = 4$ and dim $C_5 \geq 12$ or dim $C_4 = 4$, dim $C_5 = 11$, and dim $C_6 \geq 41$, might well be extinct. It is tempting to believe that once some of the first few dim C_n 's are small, it is hard for later ones to be large.

In Appendix A we motivate trivalent categories for a wider audience — particularly graph theorists — by explaining an equivalence between trivalent categories and certain skein theoretic invariants of planar trivalent graphs. While this equivalence is not essential for understanding the paper, reading the appendix may be useful for readers unfamiliar with diagrammatic methods in category theory.

This category theoretic language will be useful, because the examples we will encounter along the way come from fields of mathematics where category theory is very convenient. Nevertheless this point of view is not needed in the proofs of the main theorems. The key to translating category theoretic statements below into graph theoretic statements is to remember that $\operatorname{Hom}_{\mathcal{C}}(1 \to X^{\otimes n})$ is the vector space of formal linear combination of planar trivalent graphs with nboundary points modulo the relevant skein relations for the category \mathcal{C} .

Recall that a (strict) pivotal category is a rigid monoidal category such that $x^{**} = x$ for all x. In this paper, all of our categories will be \mathbb{C} -linear.

Pivotal categories axiomatize the nicest possible theory of duals, and correspondingly have a diagrammatic calculus allowing arbitrary planar isotopies. As usual, we have string diagrams representing morphisms, with oriented strings labelled by objects of the category, and vertices (or 'coupons') labelled by morphisms of the category. The strings may have critical points, which we interpret as the evaluation and coevaluation maps provided by the rigid structure. Arbitrary planar isotopies of a diagram preserve the represented morphism; it is critical that 2π rotations of the vertices are allowed, and this corresponds exactly to strict pivotality.

Given a pivotal category \mathcal{C} and a chosen object X, we use the notation $\mathcal{C}_k = \operatorname{Inv}_{\mathcal{C}}(X^{\otimes k}) = \operatorname{Hom}_{\mathcal{C}}(1 \to X^{\otimes k})$ for the 'invariant spaces' of X. (If you know about planar algebras [Jon99], recall these vector spaces form an unshaded planar algebra.) We say a category \mathcal{C} is <u>evaluable</u> if dim Hom $(1 \to 1) = 1$, and in fact Hom $(1 \to 1)$ may be identified with the ground field by sending the empty diagram to 1. The category \mathcal{C} is <u>nondegenerate</u> if for every morphism $x: a \to b$, there is another morphism $x': b \to a$ so $\operatorname{tr}(xx') \neq 0 \in \operatorname{Hom}(1 \to 1)$.

Definition 2.1. A <u>trivalent</u> category (\mathcal{C}, X, τ) is a nondegenerate evaluable pivotal category over \mathbb{C} with an object X with dim $\mathcal{C}_1 = 0$, dim $\mathcal{C}_2 = 1$, and dim $\mathcal{C}_3 = 1$, with a rotationally invariant morphism $\tau \in \mathcal{C}_3$ called 'the trivalent vertex', such that the category is generated (as a pivotal category) by τ .

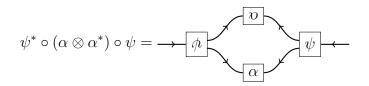
We'll often simply refer to C itself as a trivalent category.

The rotational invariance of τ allows us to drop the "coupon" attached to τ in string diagrams and treat it as an undecorated trivalent vertex.

We want one more simplification to our diagrammatic calculus: in the present situation it turns out that we can always ignore the orientations on strings, because the object X is automatically symmetrically self-dual. Said another way, the 2-valent vertex corresponding to the self-duality $X \cong X^*$ is rotationally symmetric.

Lemma 2.2. The object X is symmetrically self-dual.

Proof. Suppose that $\alpha : X \to X^*$ is any self-duality and that $\psi : X \to X \otimes X$ is an inclusion. Because \mathcal{C} is nondegenerate and dim $\mathcal{C}_3 = 1$, given any two non-zero maps $\beta : X \to X \otimes X$ and $\gamma : X \otimes X \to X^*$, tr $(\alpha^{-1} \circ \gamma \circ \beta) \neq 0$, so $\alpha^{-1} \circ \gamma \circ \beta$ and $\gamma \circ \beta$ are nonzero too. Taking $\gamma = \psi^* \circ (\alpha \otimes \alpha^*)$ and $\beta = \psi$, we see that the map



is nonzero and manifestly rotationally invariant.

Combining the symmetric self-duality of X with the rotational invariance of τ , we can interpret any unoriented planar trivalent graph with n boundary points as an element of C_n .

To any trivalent category we assign several important parameters as follows. Since dim $C_0 = 1$, any diagram with a loop in it is a multiple d of the same diagram missing that loop. The loop value d must be nonzero because it is the pairing of the unique-up-to-scalar element of C_2 with itself, and C is nondegenerate. In addition, we must have a relation

$$(2.1) \qquad \qquad \bigcirc = b \cdot \Big|$$

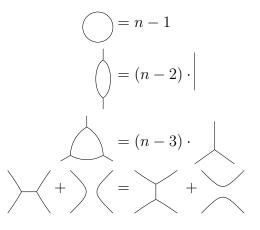
for some parameter b which is again nonzero, since the theta graph must be nonzero. Because one can rescale the trivalent vertex by a constant, without loss of generality we can assume b = 1. Finally, we see that

although in this case the parameter t can be zero. These parameters d and t will be the key parameters in this paper.

We now give a simple example of a trivalent category, the 'chromatic category'. Further examples appear throughout this paper, in particular $SO(3)_q$ (which is essentially the same as the chromatic category) in the proof of Proposition 4.3, $ABA_{(d,t)}$ in the proof of Proposition 5.3, $(G_2)_q$ in Definition 5.22, and the H3 category of [GS12b] which is given a trivalent presentation in Section 6.

The chromatic category has as objects finite subsets of an interval, and a morphism between two such sets is a linear combination of trivalent graphs embedded in a strip between these intervals, subject to the following local

relations:



(Here the object X is just a singleton on an interval, and the morphism τ is just the trivalent vertex.) One can verify that these relations suffice to evaluate any closed trivalent graph: the 'I = H' relation ensures that we can reduce the size of a chosen face without increasing the total number of vertices in the graph, and once there is a small enough face, one of the other relations lets us reduce the total number of vertices. Indeed, in the case that the parameter n is an integer, one can see that this evaluation is exactly the normalized chromatic number of the graph — that is, the number of ways to color the faces of the planar trivalent graph with n colors such that adjacent faces do not share a color, with the 'outer face' of the planar diagram always having a fixed color. The proof of this fact is that in each relation, the total number of colorings is the same on either side of the relation.

Later we will see that this category is actually equivalent to the category we call $SO(3)_q$ below, namely the category of representations of $U_q(\mathfrak{sl}(2))$ consisting of representations whose highest weight is a root, with the equivalence sending the object X to the irreducible 3-dimensional representation and sending the trivalent vertex to some multiple of the quantum determinant. The parameters match up according to the formula $n = q^2 + 2 + q^{-2}$. (This is of course a well known equivalence in quantum topology.)

Similarly, the G_2 spider defined in [Kup96] is an example of a trivalent category, and it is equivalent to the category of representations of $U_q(\mathfrak{g}_2)$ with X the 7-dimensional representation and τ the quantum deformation of the defining invariant antisymmetric trilinear form.

There is a well-known theorem about nondegeneracy and negligibles which we will need.

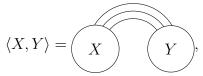
Proposition 2.3. An evaluable pivotal category has a unique maximal ideal, the negligible ideal. (cf. [BMPS12, Proposition 3.5])

Corollary 2.4. Given a collection of linear relations amongst planar trivalent graphs, such that any closed diagram can be reduced to a multiple of the empty diagram by those relations, there is a unique nondegenerate trivalent category satisfying those relations.

Remark 2.5. Any trivalent category is automatically spherical. To see this note that since X is simple we need only check that the dimension of X and the dimension of X^* agree, but since X is self-dual this is obvious.

3. Small graphs

We will write D(n, k) for the collection of trivalent graphs with n boundary points and at most k internal faces having four or more edges. For a fixed trivalent category, we write M(n, k) for the matrix of bilinear inner products, i.e.



of the elements of D(n, k), and $\Delta(n, k)$ is the determinant of M(n, k). Similarly, we will write $D^{\Box}(n, k)$ for the collection of trivalent graphs with n boundary points and at most k internal faces having five or more edges, and analogously define $M^{\Box}(n, k)$ and $\Delta^{\Box}(n, k)$.

Proposition 3.1. For either $\mu = \emptyset$ or $\mu = \Box$, if there is a linear relation amongst diagrams in $D^{\mu}(n,k)$, then $\Delta^{\mu}(n',k') = 0$ for all $n' \ge n$ and $k' \ge k$.

Proof. Take the diagrams appearing in the relation and glue a fixed tree (with n' - n leaves) to a fixed boundary point of each of them. There is then a non-trivial relation amongst the resulting diagrams, and hence $\Delta^{\mu}(n', k') = 0$ also.

Corollary 3.2. If $\Delta^{\mu}(n,k) = 0$ and the diagrams in $D^{\mu}(n,k)$ span C_n , then $\Delta^{\mu}(n',k') = 0$ for all $n' \ge n$, $k' \ge k$.

Remark 3.3. In all our examples, we will have enough conditions on the trivalent category that it will be possible to evaluate each of the $\Delta^{\mu}(n, k)$ that we consider as a rational function in d, b, and t. We will always normalize to set b = 1, but it is worth noting that you can recover the rational function up to an overall power of b, from the b = 1 specialization as follows. Notice that rescaling the trivalent vertex by λ rescales the values of all closed planar trivalent graphs. We can put a grading on closed planar trivalent graphs according to the number of trivalent vertices. Our parameters $d = \bigcirc, b = \bigcirc/d$, and $t = \triangle/bd$ have gradings 0, +2, and +2. It is not difficult to see, by looking at the diagrams involved in the calculation, that $\Delta^{\mu}(n, k)$ is homogenous with respect to this grading. So to recover the rational function from its b = 1 factorization up to an overall power of b, we simply multiply each monomial by a power of b to make it homogenous.

4. DIAGRAMS WITH FOUR BOUNDARY POINTS

Recall that for any trivalent category C we get two numbers (d, t) from the loop and the triangle, and furthermore $d \neq 0$. In this section we prove

Theorem A. [GS12a, Theorem 3.4] A trivalent category C with dim $C_4 \leq 3$ has $P_{SO(3)} = d + t - dt - 2 = 0$ and must be either an $SO(3)_q$ category for $d = q^2 + 1 + q^{-2}$ if $(d, t) \neq (-1, 3/2)$, or OSp(1|2) if (d, t) = (-1, 3/2).

(Throughout this paper we will use P's with various subscripts to denote important polynomials in d and t whose vanishing set corresponds to some existing trivalent category. So, for example, $P_{SO(3)}$ is the polynomial which vanishes when d and t have the values that they have for quantum SO(3). By contrast, we will use $Q_{i,j}$ to denote polynomials whose exact form is not important to the reader. Here i and j are the degrees of the polynomial in dand t respectively. The smaller polynomials are listed in the appendix, and all appear in computer readable form in the polynomials/ directory of the arXiv source of this article.)

This Theorem follows from three propositions.

Proposition 4.1 (Non-existence). For any (d,t) not satisfying $P_{SO(3)} = 0$ there are no trivalent categories with dim span $D(4,0) \leq 3$.

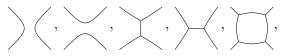
Proposition 4.2 (Uniqueness). For every pair (d,t) satisfying $P_{SO(3)} = 0$ there is at most one trivalent category with dim $C_4 \leq 4$.

Proposition 4.3 (Realization). The $SO(3)_q$ categories are trivalent categories with dim $C_4 \leq 3$, and realize every pair (d,t) satisfying $P_{SO(3)} = 0$, except (-1,3/2). The remaining point (-1,3/2) is realized by OSp(1|2).

Although versions of these propositions were already proved in $[GS12a]^2$, we give a slightly different argument which is easier to automate and thus scale to the needs of the later sections. For the first proposition, we use the dimension assumption to see that $\Delta(4,0)$ and $\Delta(4,1)$ vanish. A short calculation shows that $P_{SO(3)}$ must vanish. (Later, this sort of calculation will be handled by Gröbner bases, but for now it's easy enough to do by hand.) For the second proposition, we fix (d,t) satisfying $P_{SO(3)} = 0$. The proof divides into three phases. We use the degeneracy implied by dim $C_4 \leq 3$ and an easy graph theoretic fact to show that D(n,0) spans C_n for all n. Specializing to n = 4, this shows that the kernel of M(4,0) gives a relation in the category. The fact that this relation suffices to evaluate all closed diagrams shows that there is at most one trivalent category at this value of (d, t). Finally, the third proposition is a straightforward statement about a well-understood family of categories.

Remark 4.4. The proof of Theorem A only needs the weakening of Proposition 4.2 that covers the cases where dim $C_4 \leq 3$. We will need the full strength later in the paper.

Proof of Proposition 4.1 (Non-existence). The diagrams D(4, 1) are



² In [GS12a] the point (d, t) = (-1, 3/2) was not discussed, since in the subfactor context d > 0.

and they have matrix of inner products

$$\begin{pmatrix} \circ \circ & 8 & 9 & \circ \circ & 1 \\ \circ & \circ & 0 & 0 & \circ & 1 \\ \circ & \circ & 0 & \circ & 1 \\ \circ & \circ & 0 & \circ & 1 \\ \circ & \circ & 0 & \circ & 1 \\ \circ & \circ & 0 & \circ & 1 \\ \hline \Box & \Box & \Box & \Box & \Box \end{pmatrix} = \begin{pmatrix} d^2 & d & d & 0 & d \\ d & d^2 & 0 & d & d \\ d & d^2 & 0 & d & d \\ d & 0 & d & td & t^2d \\ 0 & d & td & d & t^2d \\ d & d & t^2d & t^2d & \Box \end{pmatrix} .$$

Recall $\Delta(4, 1)$ is the determinant of this matrix, and $\Delta(4, 0)$ is the determinant of the minor leaving off the last row and column.

Fact 4.5. In a trivalent category,

$$\Delta(4,0) = d^4(d+t-dt-2)(d+t+dt).$$

Fact 4.6. In a trivalent category,

 $\Delta(4,1) = -d^4(d+t-dt-2)(2d+2dt-4dt^2+2dt^4+2d^2t^4 - \prod (d+t+dt))$

A trivalent category with dim span $D(4,0) \leq 3$ must have $\Delta(4,0) = \Delta(4,1) = 0$. Since $d \neq 0$, we must have either $P_{SO(3)} = 0$, or (d + t + dt) = 0 and $2d + 2dt - 4dt^2 + 2dt^4 + 2d^2t^4 = 0$. In the latter case, solving gives $(d,t) = \left(\frac{1\pm\sqrt{5}}{2}, \frac{1\mp\sqrt{5}}{2}\right)$ which also satisfies $P_{SO(3)} = 0$.

Remark 4.7. In this paper we work over \mathbb{C} throughout, but it is also interesting to ask these questions in other characteristics. The argument above works with no modifications outside of characteristic 2. In characteristic 2, the argument breaks down since when (d + t + dt) = 0, we have that $\Delta(4, 1)$ is automatically zero. However, in characteristic 2, a closer look shows that $\Delta(4, 0) = d^4 P_{SO(3)}^2$, so the conclusion holds anyway. We will ignore non-zero characteristic in the rest of the paper.

Proof of Proposition 4.2 (Uniqueness). We fix a value of (d, t) satisfying $P_{SO(3)} = 0$.

Lemma 4.8. If there is a relation of the form

(4.1)
$$= \alpha + \beta + \gamma ,$$

then any trivalent graph in \mathcal{C}_n can be reduced to span D(n, 0).

Proof. Applying this relation to the largest face gives a sum of terms with either fewer faces or the same number of faces but with a smaller largest face. By induction, we can write any diagram as a sum of terms with no internal faces. \Box

Lemma 4.9. In a trivalent category with dim span $D(4,0) \leq 3$, the diagrams D(n,0) span C_n for all n.

Proof. If the two diagrams \land and \checkmark are linearly dependent, we obtain a relation as in Equation (4.1) by adding an "H" along the top boundary of both pictures. Otherwise, there must be a relation amongst the four diagrams in D(4,0), with the coefficient of at least one of \land and \checkmark being nonzero.

Rescaling and rotating gives a relation as in Equation (4.1).

Lemma 4.10. In a trivalent category with dim $C_4 \leq 4$, the diagrams D(4,0) span C_4 .

Proof. If dim span $D(4,0) \leq 3$, then the previous Lemma applies. Otherwise, dim span D(4,0) = 4 and the conclusion is immediate.

Note that on $P_{SO(3)} = 0, d \neq 1$.

Lemma 4.11. In a trivalent category where D(4,0) spans C_4 and $P_{SO(3)} = 0$, there is a relation of the form

$$\begin{array}{c} \end{array} + \frac{1}{d-1} \end{array} \begin{pmatrix} -\frac{1}{d-1} \end{pmatrix} = 0.$$

Proof. Since D(4,0) spans, any element of C_4 with inner product 0 with all elements of D(4,0) must vanish. Computing the kernel of M(4,0) as given above, we find this relation.

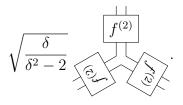
To prove the proposition, we observe that by Lemma 4.10, $C_4 = \text{span } D(4, 0)$, so by Lemma 4.11 there is a relation of the form in Equation (4.1). Finally, Lemma 4.8 shows that this relation suffices to evaluate all closed diagrams as a multiple of the empty diagram, and Corollary 2.4 shows that there is a unique trivalent category at this value of (d, t).

Proof of Proposition 4.3 (Realization). The curve $P_{SO(3)} = 0$ is rational and can be parameterized by $d = \delta^2 - 1$ and $t = \frac{\delta^2 - 3}{\delta^2 - 2}$ where $\delta \neq \pm \sqrt{2}$. It is more usual to change variables to $q + q^{-1} = \delta$ where q is not a primitive 8th root of unity. Under this parameterization q and q^{-1} are sent to the same point.

So long as $\delta \neq 0$, that is $(d, t) \neq (-1, 3/2)$, we have the following realization. Let TL_{δ} be the Temperley-Lieb category, which consists of linear combinations of planar tangles with the circle equal to δ . This is not a trivalent category, since there is no trivalent vertex. However, we can interpret trivalent graphs in TL_{δ} as follows. Take a trivalent graph and replace each strand with a pair of strands attached to the second Jones-Wenzl projection

$$f^{(2)} = \left\langle -\frac{1}{\delta} \right\rangle,$$

and replace each trivalent vertex by



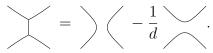
This trivalent category is called $SO(3)_q$ where $\delta = q + q^{-1}$. The trivalent vertex is normalized so that b = 1, and quick calculation shows that $d = \delta^2 - 1 = q^2 + 1 + q^{-2}$ and $t = \frac{\delta^2 - 3}{\delta^2 - 2} = \frac{q^2 - 1 + q^{-2}}{q^2 + q^{-2}}$.

The remaining point (-1, 3/2) is realized in a somewhat different way. The Lie supergroup OSp(1|2) has a standard (1|2)-dimensional representation which we denote X. This representation is simple, and using highest weight theory, $X \otimes X \cong 1 \oplus X \oplus Y$ for some simple object Y distinct from 1 and X. Thus X is self-dual and the map $X \otimes X \to X$ gives a map $X^{\otimes 3} \to 1$. A direct calculation shows that this map factors through the symmetric cube of X, and thus gives a trivalent vertex. We normalize this vertex so that the value of the bigon is 1. This gives a trivalent category which we denote OSp(1|2) which has dim $C_4 = 3$ and d = -1. Thus, it must realize the remaining point (-1, 3/2).

Remark 4.12. We will abuse notation somewhat and use $SO(3)_{\pm i}$ to refer to OSp(1|2). This can be justified by giving an appropriately modified definition of $SO(3)_q$. Specifically the categories of representations of $U_{\pm iq}(\mathfrak{su}(2))$ and $U_q(\mathfrak{osp}(1|2))$ are closely related but non-isomorphic due the appearance of certain signs. However, if you restrict attention to the representations of even highest weight the categories become equivalent because these signs all vanish [Kob93, CW13].

This strange point (-1, 3/2) can also be realized by the nondegenerate quotient of $(G_2)_{\pm i}$. This is a straightforward calculation, but we will delay it until Remark 5.24 since we will discuss $(G_2)_q$ in much more detail in the next section.

Remark 4.13. When d is generic on $P_{SO(3)} = 0$, the dimension of C_n is given by the Motzkin sums 1, 0, 1, 1, 3, 6, 15, 36, ... (A005043 in [OEI]). More specifically, by computing the radical of the inner product, all of these categories have dim $C_4 = 3$ unless d satisfies $d^2 = d + 1$. In this last case (where d is the golden ratio or its Galois conjugate) instead we have dim $C_4 = 2$ and the category satisfies the additional skein relation



These two examples are often called the golden categories or the Fibonacci categories. For these categories the dimensions are given by Fibonacci numbers $1, 0, 1, 1, 2, 3, 5, \ldots$

Remark 4.14. One can make sense of $SO(3)_q$ at an 8th root of unity, by not including the $\sqrt{\frac{\delta}{\delta^2-2}}$ factor in the trivalent vertex. With this normalization, b = 0 and so the category is degenerate. Its nondegenerate quotient has no trivalent vertices.

4.1. Cubic categories.

Definition 4.15. A cubic category is a trivalent category C with dim $C_4 = 4$.

Proposition 4.16. For any cubic category, (a) the diagrams D(4,0) form a basis of C_4 ,

(b)
$$d + t + dt \neq 0$$
 and $P_{SO(3)} = d + t - dt - 2 \neq 0$,
(c) $\square = \frac{2d + 2dt - 4dt^2 + 2dt^4 + 2d^2t^4}{d + t + dt}$, and

(d) the square satisfies the following relation

(4.2)
$$= \frac{dt^2 + t^2 - 1}{dt + d + t} \left(+ \right) + \frac{-t^2 + t + 1}{dt + d + t} \left(+ \right)$$

Proof. By Lemma 4.10 if D(4,0) is dependent then dim $C_4 \leq 3$. Hence, D(4,0) must be independent, and so a basis of C_4 . Hence, $\Delta(4,0) \neq 0$, which implies $d + t + dt \neq 0$ and $d + t - dt - 2 \neq 0$. On the other hand, we must have $\Delta(4,1) = 0$, giving (c). Finally, equation (4.2) is in the radical of the inner product on D(4,1).

We now identify the minimal idempotents in C_4 for any cubic category (subject to a certain quantity being invertible).

Proposition 4.17. Suppose C is a cubic category with parameters d and t. If

$$\xi = \sqrt{d^2t^4 + 2d\left(t^4 - 2t^3 - t^2 + 4t + 2\right) + \left(t^2 - 2t - 1\right)^2}$$

is nonzero, then the four minimal idempotents in C_4 (with respect the multiplication via vertical stacking) are

$$u = \frac{1}{d}$$

$$x =$$

$$y_{\pm} = \frac{-(d+1)t^{2} \pm \xi + 1}{\pm 2\xi} \left(+ \frac{d(t^{2} - 2t - 2) \mp \xi + t^{2} - 2t - 1}{\pm 2d\xi} \right)$$

$$- \frac{d(t+2)t \pm \xi + t^{2} + 1}{\pm 2\xi} \left(+ \frac{dt + d + t}{\pm \xi} \right)$$

with dimensions

$$tr(\iota) = 1$$

$$tr(x) = d$$

$$tr(y_{\pm}) = -\frac{d^3t^2 + d^2(\mp \xi + 2t^2 + 2t - 1) + d(\pm \xi + 2t + 3) \pm \xi - t^2 + 2t + 1}{\pm 2\xi}$$

We note that $tr(y_{\pm})$ never vanishes since $d \neq 0$ and $P_{SO(3)} \neq 0$.

Proof. This is a direct calculation using Equation (4.2), performed in the Mathematica notebook code/idempotents.nb available with the arXiv source of this article. That ι and x are minimal idempotents is clear. We then solve the quadratic equations $y_{\pm}\iota = y_{\pm}x = 0$, $y_{+} + y_{-} = 1 - \iota - x$, and $y_{\pm}^2 = y_{\pm} \neq 0$. \Box

Remark 4.18. When ξ vanishes there is no basis of projections and so the category is not semisimple. This does not happen in any of our examples. (Note that for G_2 at $q = \pm i$ we have that ξ vanishes, but at this special value G_2 is no longer cubic. See Remark 5.5.)

Lemma 4.19. In any cubic category, if n + 2k < 12, all entries of $M^{\Box}(n,k)$ can be written as rational functions in d and t.

Proof. There are fewer than twelve faces in each inner product appearing in $M^{\Box}(n,k)$, because the number of faces is bounded by n + 2k. Any polyhedron with fewer than twelve faces has at least one face which is a square or smaller. The relations for simplifying bigons, triangles and squares (Equations (2.1), (2.2), and (4.2)) then suffice to rewrite this polyhedron as a linear combination of polyhedra with strictly fewer faces; repeating the argument completely evaluates the original polyhedron as a function of d and t.

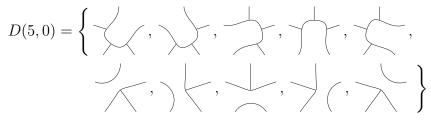
In fact, the denominators in these rational functions are always powers of $Q_{1,1} = dt + d + t$, the denominator appearing in Equation (4.2).

We unapologetically state the values of these determinants as facts, even though for larger n and k they are the result of quite intensive calculations (computing the inner products is already time consuming, and subsequently the determinant is even harder). Of course, a computer is doing these calculations (see code/ComputingInnerProducts.nb).

A careful reader will note that the matrices $M^{\Box}(8,0)$, $M^{\Box}(8,1)$, $M^{\Box}(9,0)$, $M^{\Box}(9,1)$, $M^{\Box}(10,0)$, and $M^{\Box}(11,0)$ are covered by the above lemma, but we do not compute their determinants in what follows. They seem quite difficult without very considerable computational resources. In any case, our analysis of small skein theories meets another hurdle first; we can't even compute the intersections of $\Delta^{\Box}(7,0)$ and $\Delta^{\Box}(7,1)$, or of $\Delta^{\Box}(7,1)$ and $\Delta^{\Box}(7,2)$.

5. DIAGRAMS WITH FIVE BOUNDARY POINTS

In this section we study diagrams with five boundary points. Note that



is a set of 10 diagrams, and

$$D^{\square}(5,1) \setminus D(5,0) = \left\{ \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \\ \end{array} \right\}$$

has just one element.

Theorem B. If C is a cubic category (that is dim $C_4 = 4$) and dim $C_5 \leq 10$, then either

- (1) (d,t) satisfy $P_{ABA} = t^2 t 1 = 0$ and C is one of the ABA categories described below, or
- (2) (d,t) satisfy

$$P_{G_2} = d^2t^5 + 2dt^5 - 4dt^4 - dt^3 + 6dt^2 + 4dt + d + t^5 - 4t^4 + t^3 + 7t^2 - 2 = 0$$

and \mathcal{C} is $(G_2)_q$ with

$$d = q^{10} + q^8 + q^2 + 1 + q^{-2} + q^{-8} + q^{-10},$$

and

$$t = -\frac{q^2 - 1 + q^{-2}}{q^4 + q^{-4}}.$$

Theorem B follows from four propositions.

Proposition 5.1 (Non-existence). For any (d, t) not satisfying $P_{ABA} = 0$ or $P_{G_2} = 0$ there are no trivalent categories with dim $C_4 = 4$ and dim span $D^{\Box}(5, 1) \leq 10$.

Proposition 5.2 (Uniqueness). For every pair (d, t) satisfying $P_{ABA} = 0$ or $P_{G_2} = 0$ there is at most one trivalent category with dim $C_4 = 4$ and dim $C_5 \leq 11$.

Proposition 5.3 (Realization). The trivalent categories $ABA_{(d,t)}$, defined below via a free product construction, satisfy dim $C_4 = 4$ and dim $C_5 \leq 10$, and realize every pair (d, t) satisfying $P_{ABA} = 0$ and $P_{SO(3)} \neq 0$.

Proposition 5.4 (Realization). The $(G_2)_q$ categories are trivalent categories with dim $C_4 = 4$ and dim $C_5 \leq 10$, and realize every pair (d, t) satisfying $P_{G_2} = 0$ and $P_{SO(3)} \neq 0$.

(In fact, in this section we only need the weakening of Proposition 5.2 that covers the cases where dim $C_5 \leq 10$. We will need the full strength in the next section.)

Remark 5.5. By non-degeneracy $d \neq 0$. By Proposition 4.16, any cubic category has $P_{SO(3)} \neq 0$ and $d + t + dt \neq 0$. In this remark we catalog what happens at the remaining special points where one of these polynomials does vanish.

The points on the intersection of the P_{ABA} curve and $P_{SO(3)}(d+t+dt) = 0$ are $(d,t) = (\tau, \bar{\tau})$ and $(\bar{\tau}, \tau)$. These two points do correspond to trivalent, but non-cubic, categories: the golden categories with dim $C_2 = 2$.

The intersection points of the P_{G_2} curve and $P_{SO(3)}(d+t+dt) = 0$ are:

- (1) (d,t) = (-1,3/2), corresponding to q a primitive 4th root of unity.
- (2) (d,t) = (-2,-2), corresponding to q a primitive 3rd or 6th root of unity.
- (3) (d,t) = (2,0), corresponding to q a primitive 12th root of unity.
- (4) The two points $(d, t) = (\tau, \overline{\tau})$ and $(d, t) = (\overline{\tau}, \tau)$, corresponding to q a primitive 30th root of unity.

The first of these cases is the special trivalent non-cubic category $(G_2)_{q=\pm i} = OSp(1|2)$. For the second, the bigon for G_2 at that value is 0, so the non-degenerate quotient is not trivalent. The third case corresponds to the trivalent

non-cubic category $SO(3)_{\zeta_{12}}$. The last case corresponds to the golden categories with dim $C_2 = 2$.

As always we assume $d \neq 0$, corresponding to q is not a primitive 7th, 14th, or 24th root of unity, as otherwise the non-degenerate quotient is trivial.

Remark 5.6. If \mathcal{C} and \mathcal{D} are two pivotal categories, then their tensor product $\mathcal{C} \boxtimes \mathcal{D}$ has morphisms consisting of a red planar diagram and a blue planar diagram superimposed on each other, where the blue parts live in \mathcal{C} and the red parts live in \mathcal{D} and the red diagrams are allowed to cross the blue diagrams in a symmetric way. Clearly the dimensions of the box spaces multiply, as

$$\operatorname{Hom}_{\mathcal{C}\boxtimes\mathcal{D}}(1\to (X\boxtimes Y)^{\otimes n})\cong \operatorname{Hom}_{\mathcal{C}}(1\to X^{\otimes n})\otimes \operatorname{Hom}_{\mathcal{D}}(1\to Y^{\otimes n}),$$

and the invariants of closed diagrams also just multiply. Let \mathcal{G}_{τ} denote the Golden category with loop value τ (and triangle value $\bar{\tau}$). The tensor products $\mathcal{G}_{\tau} \boxtimes \mathcal{G}_{\tau}, \ \mathcal{G}_{\tau} \boxtimes \mathcal{G}_{\bar{\tau}}, \ \text{and} \ \mathcal{G}_{\bar{\tau}} \boxtimes \mathcal{G}_{\bar{\tau}} \ \text{give pivotal categories with a distinguished}$ trivalent vertex with *n*-boundary point spaces of dimension $(1, 0, 1, 1, 4, 9, \ldots)$. It is natural to wonder how they fit into our classification. The category $\mathcal{G}_{\tau} \boxtimes \mathcal{G}_{\bar{\tau}}$ has (d,t) = (-1,-1) which lies on the G_2 curve and corresponds to q a primitive 20th root. The categories $\mathcal{G}_{\tau} \boxtimes \mathcal{G}_{\tau}$ and $\mathcal{G}_{\bar{\tau}} \boxtimes \mathcal{G}_{\bar{\tau}}$ have (d, t) being $(\tau^2, \bar{\tau}^2)$ and $(\bar{\tau}^2, \tau^2)$. These points lie on the SO(3) curve corresponding to q being a primitive 20th root of unity (half of which give $d = \tau^2$ and half of which give $d = \bar{\tau}^2$). In particular, these categories are not generated by the trivalent vertex. Instead, they correspond to the even part of the D_6 subfactor and its Galois conjugate. These categories contain a trivalent subcategory $SO(3)_{\zeta_{20}}$ and a single extra generating 4-box satisfying a version of the relations from [MPS10]. The relationship between D_6 and the tensor products of golden categories is explained by level-rank duality between $SO(3)_4$ and $SO(4)_3$ together with the coincidence of Lie algebras between $\mathfrak{so}(4)$ and $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$, as explained in [MPS11, Theorem 4.1].

We follow the same outline as in the previous section.

Proof of Proposition 5.1 (Non-existence).

Fact 5.7. In any trivalent category,

$$\Delta(5,0) = d^{10} P_{ABA}^2 P_{SO(3)}^4 Q_{1,2}.$$

Fact 5.8. In any trivalent category,

$$\Delta^{\Box}(5,1) = d^{10} P^2_{ABA} P^4_{SO(3)} \left(-5dt \left(dt^5 + 2t^5 - 2t^4 - 2t^3 + 2t^2 + t \right) + Q_{1,2} \right),$$

and in a cubic category, this specializes to

$$\Delta^{\Box}(5,1) = d^{11} P^3_{ABA} P^5_{SO(3)} P_{G_2} Q^{-2}_{1,1}.$$

A cubic category with dim span $D^{\square}(5,1) \leq 10$ must have $\Delta^{\square}(5,1) = 0$. Since $d \neq 0$, we must have one of $P_{SO(3)} = 0$, $P_{ABA} = 0$, or $P_{G_2} = 0$. Since $P_{SO(3)} \neq 0$, the proposition is proved.

Proof of Proposition 5.2 (Uniqueness). We fix a value of (d, t) satisfying P_{G_2} or P_{ABA} .

It is well-known that any planar trivalent graph has a pentagon or smaller face. We want an analogue of this result for open planar trivalent graphs, that is, planar trivalent graphs having boundary. (This is similar to the analysis in [Kup96].) In order to state this result we introduce some language.

Definition 5.9. An <u>open</u> planar trivalent graph is a planar trivalent graph in the disc which meets the boundary in $n \ge 1$ specified points. An open planar trivalent graph has two kinds of faces, <u>internal faces</u> and <u>boundary faces</u>, depending on whether they touch the boundary of the disc.

We say a planar trivalent graph is <u>connected</u> if the vertices and edges (whether internal edges or boundary edges) form a connected topological space.

We say that an open planar trivalent graph is <u>boundary connected</u> if every component meets the boundary. Note that a face need not be topologically a disc, unless the graph is boundary connected.

A <u>boundary region</u> of a connected planar trivalent graph is a small neighborhood of some contiguous proper subset of the boundary faces. A boundary region is called a <u>growth region</u> if the number of edges meeting the boundary is greater than the number of edges not meeting the boundary. Figure 2 illustrates these definitions.

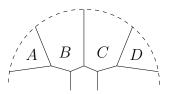


FIGURE 2. Some boundary faces in a portion of an open planar trivalent graph. The boundary region which is a small neighborhood of faces A, B, C, and D is a growth region – it has five edges meeting the boundary and two which don't. The boundary region which is a small neighborhood of B and C is not a growth region — it has three edges meeting the boundary and four edges which don't.

We now restrict our attention to boundary connected open trivalent graphs. We assign a <u>charge</u> to each face as follows. Let n be the number of edges meeting that face, and let m be the number of disjoint boundary intervals meeting that face. Now assign the charge 6 - n - 2m. In particular, an internal n-gon face is assigned 6 - n, so the only positively charged internal faces are pentagons or smaller. A boundary face that only meets the boundary once and touches n edges is given charge 4 - n. A standard Euler characteristic argument shows that the total charge of the graph is 6.

Lemma 5.10. In a connected open planar trivalent graph, any boundary region for which the sum of the charges on its constituent boundary faces is at least 2 is a growth region. *Proof.* Once we assume a graph is connected, each boundary face meets the boundary in a single interval. Now, if a boundary region has I incoming edges and O outgoing edges, then the total charge of the boundary faces is O - I + 1.

We say that an internal face is small if it has 5 or fewer sides.

Lemma 5.11. Any connected open planar trivalent graph has an internal small face or a growth region.

Proof. Since the total charge is 6, either the boundary charge is at least 6, or there is a positively charged internal face, which must be a small face. If the boundary charge is 6 or more, then there are at least two boundary faces because a single boundary face has charge 4 - n (where n is the number of edges it touches). Thus, we can divide the boundary into two proper sub-regions. One of these has charge three or greater, so by Lemma 5.10 this is a growth region.

It follows that we can easily inductively enumerate all boundary connected open graphs with n boundary points and no internal small faces by first enumerating the connected graphs by attaching growth regions to open graphs with strictly fewer boundary points and with no internal small faces, and then writing down the all the planar unions of such graphs.

Corollary 5.12. If $n \leq 5$, then any boundary connected open planar trivalent graph with n boundary points and no small faces is in D(n, 0).

Of course, when n = 6 there's an open graph with a single hexagonal face which does not lie in D(6, 0).

Lemma 5.13. Suppose C is a category generated by a trivalent vertex, with relations reducing n-gons for each $n \leq 4$. Suppose further there is some relation between the diagrams in $D^{\Box}(5,1)$. Then there is a relation reducing the 5-gon (as linear combination of diagrams in D(5,0)).

Proof. If the relation between the diagrams in $D^{\Box}(5,1)$ already includes the pentagon, we are done. Otherwise there must be a relation only amongst the diagrams in D(5,0). If this relation involves any of the diagrams of the form \swarrow we can add an H to the boundary of this relation and obtain a relation writing a pentagon as a linear combination of diagrams without internal faces. Otherwise, if there's a relation only involving the diagrams of the form \checkmark , we find this diagram as a sub-diagram of a pentagon consisting of a vertex and its opposite edge. We then apply the relation inside the pentagon, and obtain another relation writing a pentagon as a linear combination of diagrams without internal faces. (One can readily verify no pentagons appear in other terms.)

Corollary 5.14. Given fixed relations reducing n-gons for each $n \leq 4$, and some relation between the diagrams in $D^{\Box}(5,1)$, there is at most one trivalent category satisfying these relations.

Proof. As any closed trivalent graphs contains an *n*-gon with $n \leq 5$, and by the previous Lemma we can reduce this diagram, we see that the available relations suffice to evaluate all closed diagrams. By Corollary 2.4 we are done.

Lemma 5.15. Suppose C is a cubic category, with a relation between the diagrams in $D^{\Box}(5,1)$. The C_5 is spanned by D(5,0).

Proof. By Lemma 5.13, we can reduce any diagram in C_5 to a linear combination of diagrams without small faces. By Corollary 5.12, these are in D(5,0). \Box

Lemma 5.16. In a cubic category where $D^{\square}(5,1)$ spans C_5 and $P_{ABA} = 0$ there are relations:

(5.1)
$$(5.1) + \zeta^{2} + \zeta^{2} + \zeta^{3} + \zeta^{4} = 0$$

(5.2)
$$(5.2) + \zeta^{-1} + \zeta^{-2} + \zeta^{-3} + \zeta^{-4} = 0$$

with $\zeta^{5} = 1$ and $t + \zeta^{2} + \zeta^{3} = 0$ (so if $t = \frac{1 + \sqrt{5}}{2}$, $\zeta = \exp(\pm 2\pi i/5)$, and

with $\zeta^5 = 1$ and $t + \zeta^2 + \zeta^3 = 0$ (so if $t = \frac{1+\sqrt{5}}{2}$, $\zeta = \exp(\pm 2\pi i/5)$, and if $t = \frac{1-\sqrt{5}}{2}$, $\zeta = \exp(\pm 4\pi i/5)$).

Proof. By non-degeneracy and $D^{\Box}(5,1)$ spanning, anything in the kernel of M(5,1) must be a relation. When $P_{ABA} = t^2 - t - 1 = 0$, we obtain the relations above. (See the calculation in code/ABA.nb.)

Now we turn our attention to the case where $P_{G_2} = 0$. The following Lemma is well-known [Kup94].

Lemma 5.17. The curve $P_{G_2} = 0$ is rational and can be parameterized by $d = x^5 + x^4 - 5x^3 - 4x^2 + 6x + 3,$

and

$$t = -\frac{x-1}{x^2 - 2}$$

where $x \neq \pm \sqrt{2}$.

It is more usual to change variables so that $x = q^2 + q^{-2}$ in order to relate this to quantum groups with the usual variables. Note that this change of variables is typically 4-to-1 with $\pm q^{\pm 1}$ all corresponding to the same pair (d, t). In these variables we have

$$d = q^{10} + q^8 + q^2 + 1 + q^{-2} + q^{-8} + q^{-10},$$

and

$$t = -\frac{q^2 - 1 + q^{-2}}{q^4 + q^{-4}},$$

where q is not a primitive 16th root of unity. Recall that we also have that q is not a primitive 3rd or 6th root of unity since $d + t + dt \neq 0$.

Lemma 5.18. In a cubic category where $D^{\square}(5,1)$ spans C_5 and $P_{G_2} = 0$ there is a relation:

$$= \alpha \left(\checkmark + rotations \right) + \beta \left(\checkmark + rotations \right),$$

where

$$\alpha = -\frac{1}{(q^2 + 1 + q^{-2})(q^4 + q^{-4})}$$

$$\beta = -\frac{1}{(q^2 + 1 + q^{-2})^2(q^4 + q^{-4})^2}$$

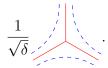
Proof. When $P_{G_2} = 0$ these relations are in the radical of the inner product on $M^{\Box}(5,1)$. (See the calculation in code/G2.nb.)

We now prove the proposition. Suppose $D^{\square}(5,1)$ is dependent. Then Lemma 5.15 shows that D(5,0) spans, and hence trivially $D^{\square}(5,1)$ spans also. On the other hand, if $D^{\square}(5,1)$ is independent, it also must span, since we are assuming dim $\mathcal{C}_5 \leq 11$. Then Lemma 5.16 and Lemma 5.18 ensure that there are relations amongst the diagrams in $D^{\square}(5,1)$. By Corollary 5.14 there is a unique cubic category at the given values of d and t.

Proof of Proposition 5.3 (Realization). Note that for any (d, t) with $P_{ABA} = t^2 - t - 1 = 0$ we can rewrite $(d, t) = (\delta^2 \tau, \bar{\tau})$ where bar is the Galois conjugate $(1 + \sqrt{5})/2 \leftrightarrow (1 - \sqrt{5})/2$, by taking τ to be the Galois conjugate of t and $\delta = \sqrt{d\tau^{-1}}$. This change of variables is generally 2-to-1, and there is a symmetry $\delta \leftrightarrow -\delta$.

Let TL_{δ} be the Temperley-Lieb category of planar tangles with loop value δ , and let \mathcal{G}_{τ} be the golden category with loop value τ (so that the triangle value will be $\bar{\tau}$). If \mathcal{C} and \mathcal{D} are two pivotal categories, then their free product $\mathcal{C} * \mathcal{D}$ consists of planar diagrams with connected components labelled blue and red, where the blue parts live in \mathcal{C} and the red parts live in \mathcal{D} (cf. [IMP13] and [Lan02, Section 8]). We've shown all blue components in what follows with dashed lines. Consider the free product $TL_{\delta} * \mathcal{G}_{\tau}$. This is a category of planar diagrams with connected components labelled blue and red, where the blue strands have no vertices and have loop value δ , while red strands allow trivalent vertices and have loop value τ .

Inside the free product $TL_{\delta} * \mathcal{G}_{\tau}$ we can find a trivalent category, which we call $ABA_{(\delta^2\tau,\bar{\tau})}$ as follows. Given a trivalent graph interpret each strand as a red strand then a blue strand then a red strand (hence the name ABA), and think of each trivalent vertex as given by

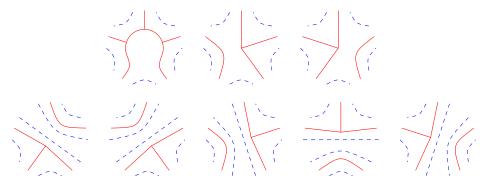


Here the normalization factor ensures that the bigon factor b is 1. We call this category $ABA'_{(d,t)}$ and then define $ABA_{(d,t)}$ to be the quotient by the negligibles.

Remark 5.19. As far as we know, it may be that $ABA'_{(d,t)}$ has no negligibles and so $ABA'_{(d,t)} = ABA_{(d,t)}$. Indeed, when dt^{-1} is a positive number bigger than 4 the pairing is positive definite because it is the restriction of an obviously positive definite pairing on the tensor product. Hence when d is generic, $ABA'_{(d,t)}$ has no negligibles.

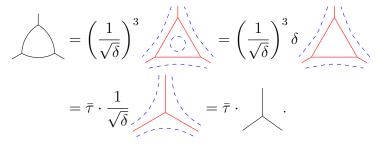
We can now prove the proposition. We first find a spanning set for the *n*-boundary point space for *ABA*. First note that the blue (A-labelled) lines of the *ABA* diagram give a noncrossing partition of the red boundary points, where two red boundary points are in the same partition if you can get between them without crossing a blue line. Second, recall that the dimension of the space of *m*-boundary point red diagrams is F_{m-1} (the m – 1st Fibonacci number). There is a standard explicit basis given by fixing a trivalent tree connecting all the boundary vertices and then picking a subset of internal edges to delete, with the condition that each vertex have either two or three edges coming out of it.

Putting the above two steps together, for each noncrossing partition (specifying the location of the blue lines), we can find a spanning set for diagrams compatible with that partition consisting of $\prod_{p \in \pi} F_{|p|-1}$ diagrams (where π is a partition, p ranges over parts of the partition, and |p| is the size of the part). This process gives spanning sets for the *n*-boundary point spaces for $0 \le n \le 5$ with sizes 1, 0, 1, 1, 4, 8. We illustrate this below by giving the spanning set for n = 5.



By computing inner products we see that these spanning sets actually form bases unless $P_{SO(3)} = 0$ (in which case the nondegenerate quotient is just a golden category). Hence the ABA categories are cubic categories with dim $C_5 \leq 11$.

Finally, it is straightforward to check that the loop value is $\delta^2 \tau = d$. We see that the triangle value is $\bar{\tau}$:



This completes the proof of Proposition 5.3.

Remark 5.20. By Remark 5.19, when *d* is generic there are no further relations, and so the dimension of the *n*-boundary point space is given by $\sum_{\pi} \prod_{p \in \pi} F_{|p|-1}$. This sequence begins 1, 0, 1, 1, 4, 8, 25, 64, ... and its ordinary generating function satisfies the relation $G(x) = \frac{1-xG(x)}{1-xG(x)-x^2G(x)^2}$. Therefore it is given by OEIS A046736 [OEI], which counts the number of ways to place non-intersecting diagonals on a n + 2-gon so as to create no triangles. To find this generating function identity, we use a general recipe due to Speicher [Spe94] for any such weighted counting of non-crossing partitions. (This approach thus applies to the pivotal tensor category generated by ABA in the free product of Temperley-Lieb with an arbitrary trivalent category.) Let a_i be an arbitrary sequence with $a_0 = 1$, let $b_n = \sum_{\pi} \prod_{p \in \pi} a_{|p|}$, and let A(x) and B(x) be their ordinary generating functions. Then rewriting [Sta99, Exercise 5.35] yields B(x) = A(xB(x)). In our particular example, we use that the generating function for the shifted Fibonacci sequence is $\frac{1-x}{1-x-x^2}$.

Remark 5.21. It is not difficult to work out the simple objects in the ABA categories. They are of the form $A^{(n_1)}BA^{(n_2)}BA^{(n_3)}B\dots BA^{(n_k)}$ where $A^{(n)}$ denotes the *n*th Jones–Wenzl made of blue strands (so for generic δ you allow the n_i to be any positive number, while for $\delta = \zeta + \zeta^{-1}$ there is a corresponding bound on n_i). The fusion rules are given by concatenation and applying the usual SU(2) fusion rules for blue Jones-Wenzl's and $B^2 = B + 1$ for red strands. So, for example, $(ABA)(ABA) = ABA^{(2)}BA + ABA + A^{(2)} + 1$.

Proof of Proposition 5.4 (Realization). We recall Kuperberg's skein theoretic description of the quantum G_2 spider categories [Kup96, Kup94] (warning, there is a sign error in [Kup96]). We change conventions in two ways: Kuperberg's q is our q^2 (which agrees with the usual quantum group conventions) and we normalize the trivalent vertex so that the bigon equals the strand (which is possible so long as q is not a primitive 3rd or 6th or 16th root of unity).

Definition 5.22. If q is not a primitive 3rd, 6th, or 16th root of unity, let $(G_2)'_q$ be the pivotal category generated by a trivalent vertex, modulo the following skein relations, and let $(G_2)_q$ be the nondegenerate quotient of $(G_2)'_q$ by its negligible ideal.

$$\bigcirc = \Phi_7 \Phi_{14} \Phi_{24} = q^{10} + q^8 + q^2 + 1 + q^{-2} + q^{-8} + q^{-10}$$

$$\bigcirc = 0$$

$$\diamondsuit = 0$$

$$= 0$$

$$= -\frac{\Phi_{12}}{\Phi_{16}} = -\frac{q^2 - 1 + q^{-2}}{q^4 + q^{-4}}$$

$$= \frac{\Phi_8}{\Phi_3 \Phi_6 \Phi_{16}} \left(+ \right) + \frac{1}{\Phi_3 \Phi_6 \Phi_{16}^2} \left(+ \right) + \frac{1}{\Phi_3 \Phi_6 \Phi_{16}^2} \left(+ \right) + \frac{1}{\Phi_3 \Phi_6 \Phi_{16}^2} \left(+ \right) + \frac{1}{\Phi_3^2 \Phi_6^2 \Phi_{16}^2} + \frac{1}{\Phi_3^2 \Phi_6^2 \Phi_{16}^2} \right),$$

where Φ_k is the *k*th symmetrized cyclotomic polynomial. That is, $\Phi_k = \prod_{\zeta} \left(q^{\frac{1}{2}} - \zeta q^{-\frac{1}{2}}\right)$ where the product is taken over all primitive *k*th roots of unity. Explicitly,

$$\begin{split} \Phi_3 &= q + 1 + q^{-1} \\ \Phi_6 &= q - 1 + q^{-1} \\ \Phi_7 &= q^3 + q^2 + q + 1 + q^{-1} + q^{-2} + q^{-3} \\ \Phi_8 &= q^2 + q^{-2} \\ \Phi_{12} &= q^2 - 1 + q^{-2} \\ \Phi_{14} &= q^3 - q^2 + q - 1 + q^{-1} - q^{-2} + q^{-3} \\ \Phi_{16} &= q^4 + q^{-4} \\ \Phi_{24} &= q^4 - 1 + q^{-4}. \end{split}$$

Now, suppose that in addition, q is not a primitive 7th, 14th, or 24th root of unity. We want to show that $(G_2)_q$ is a trivalent category. We've already seen in Corollary 5.12 that for $n \leq 5$ the *n*-boundary point space of $(G_2)_q$ is spanned by diagrams in D(n,0) and hence has dimensions bounded by 1, 0, 1, 1, 4, 10. However, a priori these relations might collapse everything.

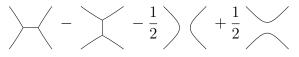
In [SW07], Sikora and Westbury introduce the notion of confluence, and claim that the above relations are confluent (we have verified this calculation). By definition, confluence means that if we start with a graph and then use one of the above relations to simplify one face or another face, then we can apply more simplifications on faces until both expressions become equal. By the Diamond Lemma [New42] this shows that any two reductions give the same answer. This means that the inner product of diagrams is well-defined, and then taking inner products lets us prove that the obvious spanning set is a basis for $n \leq 4$, except when we are also on the SO(3) curve (see the following remark). In particular, $(G_2)_q$ is a cubic category. Finally we observe that the formulas for d and t agree with the ones in the parameterization of $P_{G_2} = 0$.

Lemma 5.23. If q is not a 3rd, 6th, 16th, 7th, 14th, or 24th root of unity, then the dimension of the n-boundary point spaces of $(G_2)_q$ are 1, 0, 1, 1, 4, 10 unless q is a primitive 20th root of unity in which case they 1, 0, 1, 1, 4, 9.

Proof. Unless (d, t) lies on the $Q_{1,2}$ curve, the 10 diagrams in D(5, 0) are linearly independent. The only points on the intersection of the $Q_{1,2}$ and G_2 curves are

 $(-2, -2), (-1, -1), (2, 0), (\tau, \overline{\tau}), \text{ and } (\overline{\tau}, \tau)$. These correspond, respectively, to a primitive 3rd or 6th root of unity, a primitive 20th root of unity, a primitive 12th root of unity, and a primitive 30th root of unity. The only case not excluded by our assumptions is q a primitive 20th root of unity. Calculating the determinant of a 9-by-9 minor of M(5,0) shows that the 4-boundary point space is 9 dimensional at this value.

Remark 5.24. When $q = \pm i$ is a primitive 4th root of unity $(G_2)_q$ can still be defined by the above relations and confluence argument. Since we have an explicit spanning set for the 4-boundary point space, a direct calculation shows that the relation



does lie in the radical of the inner product, so in the nondegenerate quotient $(G_2)_{\pm i}$ the dimension of the 4-box space drops from 4 down to 3. This gives an alternate proof that the point (d, t) = (-1, 3/2) on $P_{SO(3)}$ can be realized.

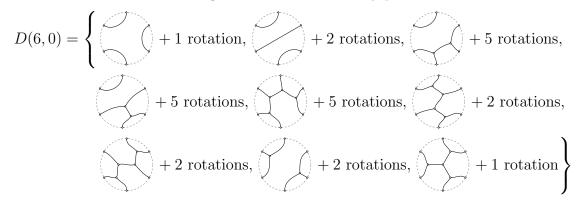
Remark 5.25. When q is one of the bad values (a primitive 3rd, 6th, or 16th root of unity) the above definition can be modified by normalizing the trivalent vertex differently to give a well-defined category. However, in this category the value of the bigon will be 0 and so the trivalent vertex will be zero.

Remark 5.26. The category $(G_2)_q$ gets its name from its relationship to the quantum group $U_q(\mathfrak{g}_2)$. If q is generic, then the category of maps between tensor powers of the standard 7-dimensional representation of $U_q(\mathfrak{g}_2)$ are given by the above diagrams. In fact, by the results of this section it is clear that the subcategory of $\operatorname{Rep}(U_q(\mathfrak{g}_2))$ generated by the trivalent vertex must be $(G_2)_q$ and then Kuperberg showed that the dimensions match up so the subcategory is the whole category.

When q is a root of unity, the correct algebraic category is the category of tilting modules. There is a map $(G_2)'_q \to \operatorname{Rep}^{\operatorname{tilting}}(U_q(\mathfrak{g}_2))$, but it is not clear whether this is surjective, or if it descends to a map from the nondegenerate quotient $(G_2)_q$ to the non-degenerate quotient of the category of tilting modules.

6. DIAGRAMS WITH SIX BOUNDARY POINTS

We now move on to the diagrams with six boundary points. We have



$$D^{\square}(6,1) \setminus D(6,0) = \left\{ \begin{array}{c} & & \\ & & \\ & & \\ & & \\ D^{\square}(6,2) \setminus D^{\square}(6,1) = \left\{ \begin{array}{c} & & \\$$

so $\#D(6,0) = 34, \#D^{\square}(6,1) = 41$, and $\#D^{\square}(6,2) = 44$.

In this section we prove

Theorem C. If C is a trivalent category with dim $C_4 = 4$, dim $C_5 = 11$, and dim $C_6 \leq 40$ then $d^2 - 3d - 1 = 0$, $t = -\frac{2}{3}d + \frac{5}{3}$, and C is the H3 fusion category constructed by Grossman and Snyder [GS12b] (which is Morita equivalent to the even parts of the Haagerup subfactor [AH99]) or its Galois conjugate.

This theorem follows from four propositions.

Proposition 6.1 (Non-existence). There are no trivalent categories with $\dim C_4 = 4$, $\dim C_5 = 11$, and D(6, 0) linearly dependent.

Proposition 6.2 (Non-existence). For any (d, t) not satisfying $d^2 - 3d - 1 = 0$ and $t = -\frac{2}{3}d + \frac{5}{3}$ there are no trivalent categories with dim $C_4 = 4$, dim $C_5 = 11$, and dim span $D^{\Box}(6, 2) \leq 40$.

Proposition 6.3 (Uniqueness). For each pair (d, t) satisfying $d^2 - 3d - 1 = 0$ and $t = -\frac{2}{3}d + \frac{5}{3}$, there is at most one trivalent category with dim $C_4 = 4$, with dim $C_5 = 11$, and with dim $C_6 \leq 40$.

Proposition 6.4 (Realization). The H3 fusion category and its Galois conjugate are trivalent and have dim $C_4 = 4$, dim $C_5 = 11$, and dim $C_6 = 37 \le 40$.

These two categories exhaust the possibilities allowed by the first three propositions.

Proof of Proposition 6.1 (Non-existence). We have the following values of determinants.

Fact 6.5. In any cubic category,

$$\Delta(6,0) = -d^{34}Q_{1,1}^{-8}Q_{0,1}^2Q_{0,2}^9Q_{2,4,a}Q_{3,5}^2Q_{6,9}P_{SO(3)}^{19}$$

Fact 6.6. In any cubic category,

$$\Delta^{\Box}(6,1) = d^{41}Q_{1,1}^{-29}Q_{0,2}^{16}Q_{2,3}Q_{3,4}^2Q_{7,11}P_{SO(3)}^{27}P_{G_2}^6.$$

Fact 6.7. In any cubic category,

$$\Delta^{\Box}(6,2) = d^{44}Q_{1,1}^{-44}Q_{0,2}^{19}Q_{2,3}Q_{4,5}^2Q_{8,12}P_{SO(3)}^{33}P_{G_2}^9$$

Fact 6.8. In any cubic category,

$$\Delta(7,0) = -d^{112}Q_{1,1}^{-70}Q_{0,2}^{48}Q_{11,19}Q_{36,60}^2P_{SO(3)}^{76}Q_{2,4,b}$$

If D(6,0) is dependent, then $\Delta(6,0)$ must vanish, and indeed $\Delta^{\Box}(6,1)$, $\Delta^{\Box}(6,2)$ and $\Delta(7,0)$ must vanish also. This only happens at finitely many points, all of which are on the G_2 or SO(3) curves. (This calculation is curious; finding intersections of $\Delta(6,0)$ with the other varieties appears to be rather hard. However the Gröbner basis calculation showing $\Delta^{\Box}(6,1)$ and $\Delta^{\Box}(6,2)$ intersect at finitely many points besides $Q_{0,2}Q_{2,3}P_{SO(3)}P_{G_2} = 0$ is quite manageable, and after that we can easily find the complete intersection.) This calculation can be found in the file code/GroebnerBasisCalculations.nb available with the arXiv source of this article. Now, using the full strength of Proposition 5.2, we see that any trivalent category with $C_5 \leq 11$ at one of these points must actually be an ABA or $(G_2)_q$ category, contradicting our assumption that dim $C_5 = 11$.

Proof of Proposition 6.2 (Non-existence). Beyond the determinant calculations in the previous section, with high probability we have the following two determinants.

Conjecture 6.9. In any cubic category,

$$\Delta^{\Box}(7,1) = -d^{155}Q_{1,1}^{-242}Q_{0,2}^{91}Q_{2,3}^{7}Q_{21,33}Q_{51,69}^{2}P_{SO(3)}^{133}P_{G_2}^{35}$$

and

$$\Delta^{\Box}(7,2) = -d^{183}Q_{1,1}^{-403}Q_{0,2}^{119}Q_{4,5}^{14}Q_{22,36}Q_{54,78}^2P_{SO(3)}^{189}P_{G_2}^{63}.$$

Lemma 6.10. With a probability of $1 - 10^{-500}$, each of these conjectures is correct.

Proof. We begin by explaining what we mean. The Schwartz-Zippel lemma [Sch80, Zip79, DL78] (pointed out to us by Dylan Thurston) gives a method of probabilistically checking polynomial identities (we first clear denominators if necessary): if $P \in k[x_1, \ldots, x_n]$ has total degree bounded by D, and the x_i are drawn uniformly and independently from a finite subset $S \subset k$ of size N, then either P is identically zero or $P(x_1, \ldots, x_n) \neq 0$ with probability at least $1 - \frac{D}{N}$.

We can easily bound the total degree of $\Delta^{\square}(7,1)$ at 1611, and of $\Delta^{\square}(7,2)$ at 2664. Evaluating the determinant of $M^{\square}(7,1)$ at a pair of values (d,t) drawn uniformly from positive integers at most 10^{40} takes on the order of 3 minutes (on a 12-core Xeon E5), while the determinant of $M^{\square}(7,2)$ takes 6 minutes. This gives us a probability of error of at most one part in 10^{12} , for $\Delta^{\square}(7,1)$, or 10^{6} , for $\Delta^{\square}(7,2)$, per minute of running time; we stopped after reaching 10^{500} . These checks are implemented in code/SchwartzZippel.nb.

Remark 6.11. To guess these polynomials in the first place, we adopted the following ad-hoc strategy. Suppose we have some large matrix M(d,t)with entries in $\mathbb{Q}(d,t)$, and want to evaluate the determinant. Arithmetic in $\mathbb{Q}(d,t)$ is difficult, so we avoid this by first specializing one variable, d, to various different primes. We work with a set of primes \mathcal{P} , which is chosen to be big enough that results we obtain below are successfully verified by the Schwartz-Zippel lemma argument given above!

At each prime $p \in \mathcal{P}$ we can compute det M(p, t) as a rational function in $\mathbb{Q}(t)$ relatively quickly. We now want to recover M(d,t). In our examples, these are not irreducible, and it turns out to be most efficient to first factorize each det M(p, t) into products of powers (possibly negative) of polynomials in $\mathbb{Z}[t]$. For large enough primes p, the factorization is uniform, in the sense that degrees and multiplicities of the irreducible factors of the different det M(p, t)are in bijection, and so we obtain det $M(p,t) = \prod_{i \in \mathcal{I}} \mathcal{K}_{p,i}(t)^{n_i}$, for some fixed index set \mathcal{I} and exponents n_i for each $i \in \mathcal{I}$. Write $\mathcal{K}_{p,i}(t) = \sum_r \mathcal{L}_{p,i,r} t^r$ for some integers $\mathcal{L}_{p,i,r}$.

We now want to recover an irreducible polynomial $K_i(d,t) = \sum_r L_{i,r}(d)t^r =$ $\sum_{r,s} L_{i,r,s} d^s t^r$ so $\mathcal{K}_{p,i}(t) = K_i(p,t)$. This requires that $L_{i,r}(p) = \mathcal{L}_{p,i,r}$ for each p. In particular, this says that the constant term $L_{i,r,0}$ of $L_{i,r}$ satisfies

$$L_{i,r,0} \equiv L_{i,r}(0) \equiv \mathcal{L}_{p,i,r} \pmod{p}$$

By the Chinese remainder theorem, we then know $L_{i,r,0} \pmod{\prod_{p \in \mathcal{P}} p}$, and we guess that it is actually equal to this residue. We then continue making guesses recursively, using the identities

$$L_{i,r,s} \equiv \left(\mathcal{L}_{p,i,r} - \sum_{t=0}^{s-1} L_{i,r,t} p^t \right) p^{-s} \pmod{p}$$

and the Chinese remainder theorem. This method is implemented in the notebook code/GuessDeterminants.nb.

We will give two separate proofs of Proposition 6.2. The first is easy to follow but depends on Conjecture 6.9, while the second is more difficult but unconditional.

Lemma 6.12. Fix some (d, t) not satisfying $d^2 - 3d - 1 = 0$ and $t = -\frac{2}{3}d + \frac{5}{3}$. If Conjecture 6.9 holds at this (d,t) then there are no trivalent categories with $\dim \mathcal{C}_4 = 4$ and $\dim \mathcal{C}_5 = 11$, and $\dim \operatorname{span} D^{\square}(6, 2) \leq 40$.

Proof. Since dim span $D^{\square}(6,2) \leq 40$ and there are 41 diagrams in $D^{\square}(6,1)$, we must have a relation amongst $D^{\square}(6,1)$. Thus (d,t) must give a solution to $\Delta^{\square}(6,1) = \Delta^{\square}(6,2) = \Delta^{\square}(7,1) = \Delta^{\square}(7,2) = 0$, and the possibilities are (see code/GroebnerBasisCalculations.nb):

(a) (d, t) is on the P_{ABA} or P_{G_2} curve, (b) $d^2 - 3d - 1 = 0$ and $t = -\frac{2}{3}d + \frac{5}{3}$,

(c) d is a root of the degree 33 polynomial $S_a(d)$, and $t = T_a(d)$, or

(d) d is a root of the degree 63 polynomial $S_b(d)$, and $t = T_b(d)$.

The polynomials S_a, T_a, S_b , and T_b (the last of which is stupendously large) are available in the file code/BadPoints.nb.

In the first case, the full strength of Proposition 5.2 shows that $\dim C_5 < 11$. It remains to eliminate the last two cases. For both those values of (d, t) the rank of $M^{\Box}(6,2)$ is 43 which is incompatible with dim span $D^{\Box}(6,2) < 40$. (This calculation is done twice in the last section of code/BadPoints.nb. We check slowly and directly that the rank is exactly 43 by doing arithmetic in the number field, but we also quickly see that the rank is at least 43 by calculating the rank of the matrix modulo a prime in the number field, thereby reducing

the question to calculating the rank over $\mathbb{Z}/11\mathbb{Z}$ and $\mathbb{Z}/41\mathbb{Z}$ respectively. The latter approach was suggested to us by David Roe.)

Now we turn to the unconditional proof of Proposition 6.2, which follows immediately from the following lemma.

Lemma 6.13. If C is a trivalent category with dim $C_4 = 4$ and dim $C_5 = 11$, and dim span $D^{\Box}(6,2) \leq 40$, then Conjecture 6.9 holds.

Proof. In order for there to be such a trivalent category, we must have that (d,t) is a solution to $\Delta^{\square}(6,1) = \Delta^{\square}(6,2) = 0$. We consider each factor of $\Delta^{\square}(6,1)$ separately. The P_{G_2} and P_{ABA} factors contradict dim $\mathcal{C}_5 = 11$ by Proposition 5.2.

On the elliptic curve $Q_{2,3}$ any rational function can be written in the form $\alpha(t)d + \beta(t)$ for some rational functions α and β , and algebra can be done efficiently on functions of this form just as it is done with ordinary rational functions. We can then compute the value of the determinants $\Delta^{\Box}(7,1)$ and $\Delta^{\Box}(7,2)$ exactly and verify Conjecture 6.9 for these points. (This calculation is performed in code/DeterminantsOnEllipticCurve.nb.)

The other two factors $Q_{3,4}$ and $Q_{7,11}$ of $\Delta^{\square}(6,1)$ intersect $\Delta^{\square}(6,2) = 0$ in finitely many points (see code/GroebnerBasisCalculations.nb for this calculation). For each of these finitely many points we can compute the determinants $\Delta^{\square}(7,1)$ and $\Delta^{\square}(7,2)$ exactly (cf. code/DeterminantsOnExtraPoints.nb). Thus Lemma 6.12 applies, and we note that the points described in item (c) of Lemma 6.12 all lie on $Q_{2,3}$.

Proof of Proposition 6.3 (Uniqueness). In the proof of Proposition 5.2 (Uniqueness) we used an analogue for open graphs of the well-known theorem that any closed planar trivalent graph has pentagonal or smaller face. That theorem plays a key role in the proof of the (easy) 5-color theorem, and the study and eventual proof of the 4-color theorem lead to an enormous number of similar results proved using the <u>discharging method</u>. The typical illustration of the discharging method is the following well-known lemma.

Definition 6.14. A <u>very small</u> face is a square, triangle, or bigon. A <u>pentapent</u> is a pair of adjacent pentagons. A <u>hexapent</u> is a pentagon and an adjacent hexagon.

Lemma 6.15. Any closed trivalent graph contains either a very small face, a pentapent, or a hexapent.

Proof. Suppose that the graph has no very small faces. We assign a charge of 6 - n to every *n*-gon face. By measuring Euler characteristic, the total charge is 12, which is certainly positive. We now "discharge" the pentagons, distributing their charge equally among the neighboring faces. Since this does not change the total charge of the graph there must be a face with positive charge. Such a face must either be a pentagon next to a pentagon, a hexagon next to a pentagon, or a 7-gon next to at least 6 pentagons. In the final case, at least two of those six pentagons are adjacent so there's a pentapent.

Just as before, in order to apply this technique to our setting, we need an analogue of this lemma for open graphs.

Definition 6.16. A lonely pentagon of an open planar trivalent graph is a pentagon which touches at most two internal faces. A lonely pentagon is either a corner pentagon or a bridge pentagon. A <u>corner pentagon</u> is a pentagon which touches at most two internal faces which are adjacent to each other, and a <u>bridge pentagon</u> is a pentagon which touches exactly two internal faces which are not adjacent to each other.

Lemma 6.17 (Planar subgraphs). Every connected open planar trivalent graph has either a very small face, a pentapent, a hexapent, a growth region, or a corner pentagon.

The proof of this Lemma below will use the discharging method following the same outline as in the closed case. We assign the usual charges of 6 - n to each interior *n*-gon face and 4 - n to each boundary face touching *n* edges. By measuring Euler characteristic, the total charge is 6. We then 'discharge' the internal pentagons, distributing their charge equally among their neighboring internal faces. Since this does not change the total charge of the graph, we go looking for faces with positive charge and find that positive charge indicates that we have one of the features listed in Lemma 6.17. This last step turns out to be somewhat delicate, so we first prove a slightly weaker lemma:

Lemma 6.18. Every connected open planar trivalent graph has either a very small face, a pentapent, a hexapent, a growth region, or a lonely pentagon.

Proof. Suppose a connected open planar trivalent graph T has no growth regions. By discharging we will show that T has either a very small face, a pentapent, a hexapent, or a lonely pentagon.

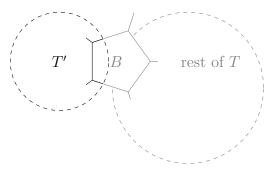
Assign a charge as described above. Since T is connected, a quick argument shows that its Euler characteristic is 6. If the boundary has charge 6 or more, it has at least two boundary faces because a single boundary face has charge 4 - n (where n is the number of edges it touches). Thus, we can divide the boundary into two proper sub-regions. One of these will have charge 3 or more, and hence be a growth region by Lemma 5.10.

If the boundary has charge less than 6, then there must be a positive charge among the internal faces. We now have the internal pentagons discharge according to the following rule: each distributes its charge of 1 evenly among the adjacent internal faces. Suppose there are no lonely pentagons. Then each pentagon distributes its charge among at least 3 faces, and so faces receive at most $\frac{1}{3}$ charge from each neighboring pentagon. The total charge has not changed, so there must be some face with positive charge.

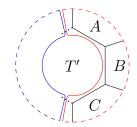
We now consider the ways in which an *n*-gon may end up with positive charge. If $n \ge 7$, either it neighbors a pair of adjacent pentagons and we are done, or it neighbors at most $\lfloor \frac{n}{2} \rfloor$ adjacent pentagons. In the latter case, the total charge is at most $6 - n + \frac{1}{3} \lfloor \frac{n}{2} \rfloor \le 0$. If n = 5 or 6, the positively-charged face must have received some charge from an adjacent pentagon, showing the existence of a pentapent or hexapent. Finally, we may have had a very small face all along.

Proof of Lemma 6.17. Lemma 6.18 is almost what we need, except it proves we must have a lonely pentagon, not necessarily a corner pentagon.

Suppose that a connected open planar trivalent graph T has bridge pentagons. Each bridge pentagon, when removed, disconnects the graph into two parts which each have fewer bridge pentagons. By descent, there must be a subgraph which is bridge-pentagon-free and connected to the rest of the graph by a single bridge pentagon. Call this subgraph T' and the bridge pentagon connecting it to the rest of the graph B.



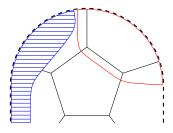
Observe T' is still connected, by virtue of containing an edge of B. Now consider the boundary of T'. If it has total charge of 6 or greater, then after B has been attached it still has charge of 2 or greater. To see this, consider the boundary face of T' which B attaches to, and its two boundary neighbors. A small neighborhood of these three faces has four outgoing edges, and at least one incoming edge (otherwise B would be a corner pentagon), hence its charge is at most 4. Thus, the boundary complement of this region (the region enclosed in blue lines below) has charge at least 2 and so is a growth region by Lemma 5.10.



Alternately, if the boundary of T' had total charge of 5 or less, then there is a net positive charge and at least one small face in the interior. As above, we discharge any pentagons and conclude we must have a very small face, pentapent, hexapent, or corner pentagon in T' (recall above we ensured that T' had no bridge pentagons). The very small face, pentapent, or hexapent also appears in T. A corner pentagon of T' is either also a corner pentagon of T, or is adjacent to B, hence part of a pentapent.

Lemma 6.19. Let T be a connected open planar trivalent graph with no very small faces, pentapents, hexapents or corner pentagons. If we remove a maximal growth region from T, the remaining graph (call it T') also has no very small faces, pentapents, hexapents or corner pentagons.

Proof. Since any face of T' is a face of T, T' has no very small faces, pentapents or hexapents. If we created a corner pentagon by removing a growth region from T, the growth region that we removed was not maximal. To see why, consider a corner pentagon in T which was not in T'. It must look like the following diagram:



Here the dashed curve is the boundary of T, the shaded blue region is the growth region, and the graph (perhaps also the growth region) continues below the bottom of the picture. Then the union of the blue region with the region enclosed by red is a larger growth region.

Lemma 6.20. We can enumerate connected planar trivalent graph with no very small faces, pentapents, or hexapents by starting with the empty diagram, and

- sequentially adding growth regions;
- in the final step, simultaneously adding some "H"s to create corner pentagons.

We can enumerate all planar trivalent graphs (not necessarily connected) with no very small faces, pentapents, or hexapents by taking planar disjoint unions of such connected planar trivalent graphs.

Proof. Consider a graph T which has no very small faces, pentapents or hexapents. From each corner pentagon, remove an "H" neighborhood of one of its sides which touches an external face. Since T has no pentapents, the regions we remove will not overlap.

Let T' be the graph with an "H" removed from each corner pentagon. T' has no corner pentagons, so by repeatedly applying Lemma 6.19, we get a sequence of growth regions building T' up from the empty diagram.

This lemma shows in particular that there are a finite number of such planar trivalent graphs with any given number of trivalent vertices, or with any given number of boundary points and internal faces.

Corollary 6.21. Every planar trivalent graph with no very small faces and at most 6 boundary points is in $D^{\Box}(n, 1)$ or has an internal pentapent or an internal hexapent.

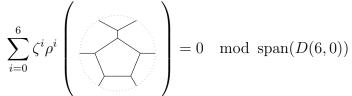
This corollary is proved by having a computer write down all the graphs in $D^{\Box}(n \leq 6, 1)$ with no very small faces, pentapents, or hexapents, according to the algorithm of Lemma 6.20.

Corollary 6.22. In a cubic category with reduction relations for pentapents and hexapents, $D^{\Box}(n, 1)$ spans C_n for $n \leq 6$.

Inside $D^{\Box}(6,1)$ there are 6 'pentafork' diagrams. We next analyze relations amongst these diagrams, up to lower order terms (i.e. terms with strictly fewer vertices, which in this case is exactly the diagrams in D(6,0)).

Lemma 6.23. We let ρ denote the operator which rotates an open graph by one click counterclockwise. A trivalent category with relations reducing n-gons for $n \leq 4$ and a relation amongst the 6 pentaforks modulo $\operatorname{span}(D(6,0))$ must also have a relation reducing a pentapent to something in $\operatorname{span}(D^{\square}(6,1))$ and a relation reducing a hexapent into $\operatorname{span}(D^{\square}(7,1))$

Proof. If there is a relation amongst the 6 pentaforks, then there is one of the form

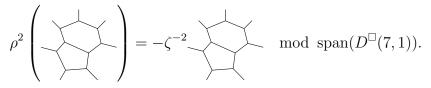


for some (not necessarily primitive) 6-th root of unity ζ . Gluing an 'H' diagram onto the upper right boundary face of each diagram, we obtain

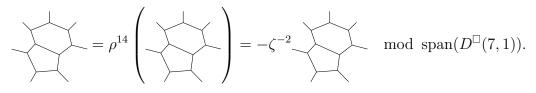
Applying this relation three times we see that

This is only possible if the pentapent is zero modulo $\operatorname{span}(D^{\square}(6,1))$.

Now we turn our attention to the hexapent. Gluing lower two points of the tree \checkmark to the upper right boundary face of the pentafork relation gives two hexapents, three pentapents, and a pentagon connected via an edge to a square. Applying the square reduction relation and the new pentapent relation, we get the following relation among the hexapents modulo lower order terms:



Applying this relation seven times, we see that



But $\zeta^{-2} \neq -1$ because ζ is a sixth root of unity and so cannot be a primitive fourth root of unity. Hence, we see that the hexapent is zero modulo $\operatorname{span}(D^{\square}(7,1))$.

Lemma 6.24. If dim $C_4 = 4$, dim $C_5 = 11$, and dim span $D^{\square}(6, 1) \leq 39$, then there is a relation amongst the 6 pentaforks modulo span(D(6, 0)).

Proof. By Proposition 6.1, the 34 diagrams in D(6, 0) must be linearly independent. Hence, modulo span(D(6, 0)), there must either be two relations amongst the pentaforks, or a relation writing the hexagon as a linear combination of pentaforks along with a relation amongst the pentaforks.

Lemma 6.25. If dim $C_4 = 4$, dim $C_5 = 11$, and dim $C_6 \leq 40$, then $D^{\Box}(6, 1)$ spans.

Proof. Suppose dim span $D^{\Box}(6,1) \leq 39$. By the previous relation, there's a relation among the pentaforks modulo lower terms. Thus, Lemma 6.23 shows there are reduction relations for pentapents and hexapents, and finally Corollary 6.22 shows that $D^{\Box}(6,1)$ spans.

Alternatively, if dim span $D^{\Box}(6,1) = 40$, then clearly $D^{\Box}(6,1)$ spans because dim $C_6 \leq 40$.

We're now ready to give the proof of Proposition 6.3. Its statement assumes the hypotheses of Lemma 6.25, and so we can now assume that $D^{\Box}(6, 1)$ spans. In particular, any element of the kernel of $M^{\Box}(6, 1)$, evaluated at the given values of d and t, must be a relation in the category. By explicit calculation, we see the kernel of $M^{\Box}(6, 1)$ is four dimensional. By Proposition 6.1 we know that D(6, 0) is linearly independent, so there must be 4 relations among the pentaforks and the hexagon modulo D(6, 0). In particular, there must be at least 3 relations among the pentaforks modulo D(6, 0). Then Lemma 6.23 implies that there are relations reducing pentapents and hexapents, Lemma 6.15 implies that we have enough relations to evaluate all closed diagrams, and Corollary 2.4 says that there is a unique cubic category at the given values of d and t.

Remark 6.26. Although we don't need to know the form of the pentafork, pentapent, and hexapent relations to prove uniqueness, we can obtain them explicitly as follows. By looking at the radical of the inner product on $D^{\Box}(6, 1)$, we obtain three relations amongst the pentaforks (modulo lower order terms), and one relation giving a hexagon in terms of the pentaforks and lower order terms. The pentafork relations have rotational eigenvalues $-1, \omega, \omega^2$, for ω a primitive cube root of unity. One can then use the argument from Lemma 6.23 to obtain explicit pentapent and hexapent reductions. See code/H3_relations.nb.

Remark 6.27. The pentafork relations guarantee pentapent and hexapent relations. We do not know whether one can use non-degeneracy to go the other way and recover the pentafork relations from the pentapent and hexapent relations.

Proof of Proposition 6.4 (Realization). First we recall the definition of the H3 category. Then we must show it is trivalent, has $d = \frac{3+\sqrt{13}}{2}$ and $t = -\frac{2}{3}d + \frac{5}{3}$, and has dimension sequence dim $C_4 = 4$, dim $C_5 = 11$, and dim $C_6 = 37 \leq 40$. The construction of the Haagerup fusion category and hence H3 involves a choice of square root of 13. Thus taking Galois conjugation $\sqrt{13} \mapsto -\sqrt{13}$ gives another category which realizes the other value of (d, t).

Recall that H2 is the Haagerup category with three invertible objects $(1, h, and h^2)$ and three non-invertible objects $(Y, hY and h^2Y)$ with fusion rules $hY = Yh^2$ and $Y^2 = 1 + Y + hY + h^2Y$. Since the associator on the 1, h, h^2 subcategory is trivial, there's a canonical algebra $\mathbb{C}[\mathbb{Z}/\mathbb{Z}3]$ in H2. As defined in [GS12b], H3 is the tensor category of bimodule objects in H2 over this algebra. Recall that H3 has the same Grothendieck ring as category H2, with $g^3 = 1$, $gX = Xg^2$ and $X^2 = 1 + X + gX + g^2X$. We will need two additional facts about H3. First, as shown in [GS12b], there's no algebra structure on 1 + X in H3. (In fact, this is the property that was used to show H3 is not the same as H2, because the Haagerup subfactor gives an algebra structure on 1 + Y in H2. The second fact is that H3 is isomorphic to its complex conjugate. This follows from H2 being isomorphic to its complex conjugate and the structure constants for the algebra $\mathbb{C}[\mathbb{Z}/\mathbb{Z}3]$ being real.

From the fusion rules for H3 we see that dim $Inv(X^{\otimes 3}) = 1$, and thus that there is a map $f: X \otimes X \to X$. We need to see that this is a trivalent vertex, and that it generates the category. We know that f must be a rotational eigenvector, but we do not yet know the eigenvalue. Note that since conjugation by g permutes X, gX, and g^2X we must have the same rotational eigenvalue for each of the three maps $X \otimes X \to X, gX \otimes gX \to gX$, and $g^2X \otimes g^2X \to g^2X$. Since H3 is isomorphic to its complex conjugate, we see that these eigenvalues must all be 1.

Next, we consider the pivotal subcategory $H3' \subset H3$ generated by the trivalent vertex f. Since 1 + X does not have the structure of an algebra object in H3, we see that H3 cannot take a functor from the SO(3) category at this value of (d, t). Hence, dim $H3'_4$ cannot be smaller than 4.

If dim $\operatorname{Inv}_{H3'}(X^{\otimes 5}) < 11$, by Theorem B, H3' is either a $(G_2)_q$ category or an ABA category. In both cases we compute $\operatorname{tr}(y_+) = \dim gX = \frac{3+\sqrt{13}}{2} = \dim g^2 X = \operatorname{tr}(y_-)$ using the formulas from Theorem 4.17 and get a contradiction. If H3' were an ABA category then $d = \frac{3+\sqrt{13}}{2}$ and $t^2 - t - 1 = 0$. This splits into two cases based on the choice of t: either $\operatorname{tr}(y_+) \approx -6.344$ while $\operatorname{tr}(y_-) \approx 12.9496$ if $t = \frac{1+\sqrt{5}}{2}$ (which is a contradiction) or $\operatorname{tr}(y_+) \approx 1.04123$ while $\operatorname{tr}(y_-) \approx 5.56432$ if $t = \frac{1-\sqrt{5}}{2}$ which is again a contradiction.

Similarly, we can rule out H3' being a $(G_2)_q$ category, although the calculations are messier. We first find the possible values of q so

$$\frac{3+\sqrt{13}}{2} = q^{10} + q^8 + q^2 + 1 + q^{-2} + q^{-8} + q^{-10}$$

and for each, verify ξ (with $t = -\frac{q^2-1+q^{-2}}{q^4+q^{-4}}$) is invertible and $\operatorname{tr}(y_+) \neq \operatorname{tr}(y_-)$. (This calculation is contained in code/idempotents.nb.) It is clear from the fusion rules that $d = \frac{3+\sqrt{13}}{2}$. Since dim $\text{Inv}_{H3'}(X^{\otimes 6}) \leq \dim \text{Inv}_{H3}(X^{\otimes 6}) = 37$, then by Proposition 6.2 we must have $t = -\frac{2}{3}d + \frac{5}{3}$.

Our final task is to show that H3' = H3. Since the principal graph for H3 is depth 3, H3 is generated by its morphisms in $Inv_{H3}(X^{\otimes k})$ for $k \leq 6$. Hence it is enough to show that dim $Inv_{H3'}(X^{\otimes 6}) = 37$. Consider the diagrams in $D^{\Box}(6, 1)$, leaving out (any) 4 of the pentaforks, and compute the determinant of the corresponding 37-by-37 matrix. This is the nonzero number

$$-\frac{\frac{12874212105079943047176987387755967947399861}{278128389443693511257285776231761}}{-\frac{3570663990466532246521487414951846015270252}{278128389443693511257285776231761}\sqrt{13}}$$

when $d = \frac{3+\sqrt{13}}{2}$ and $t = -\frac{2}{3}d + \frac{5}{3}$. (See code/H3_relations.nb for this calculation.) Hence these 37 diagrams must in fact be linearly independent in $H3'_6$. Thus dim $\operatorname{Inv}_{H3'}(X^{\otimes 6}) \geq 37$, and in fact $H3'_6 = H3_6$, with $D^{\Box}(6,1)$ spanning. Thus H3' = H3, so H3 is a trivalent category satisfying the conditions of the theorem.

7. Non-trivial rotational eigenvalues

Suppose that \mathcal{C} is a pivotal category with the sequence dim \mathcal{C}_n beginning 1, 0, 1, 1 which is generated by the map $1 \to X \otimes X \otimes X$. This map must be an eigenvector for rotation, whose eigenvalue must be a cube root of unity. Thus far we have considered the case where the rotational eigenvalue is 1, and in this section we consider the case where the rotational eigenvalue for counterclockwise rotation is a primitive cube root of unity ω . We call such a category a twisted trivalent category. Similarly to before, a twisted trivalent category gives an invariant of planar trivalent graphs which are decorated with a choice of direction at each vertex (which we denote by placing a dot in one of the three regions adjacent to the vertex) subject to the following skein relation

$$\bullet = \rho \left(\begin{array}{c} \\ \\ \bullet \end{array} \right) = \omega$$

As before we normalize the bigon (with dots inward) to be 1.

In a twisted trivalent category, the triangle with all dots pointing inward is some multiple of the trivalent vertex, but it's manifestly rotationally invariant and hence must be zero. This significantly simplifies the analysis, because all the determinants considered above become polynomials in just one variable, the loop value d. It is then easy to detect intersections between the corresponding varieties, by factoring into irreducible polynomials.

We now quickly run through the analogues of all the above results in the twisted case. Let $D_{\omega}^{\Box}(n,k)$ be defined as before as diagrams with n boundary points and no more than k faces, none of which are squares or smaller. Note that there is an ambiguity here; for each diagram you must fix the location of the dots. We let $M_{\omega}^{\Box}(n,k)$ be the matrix of inner products and $\Delta_{\omega}^{\Box}(n,k)$ be the determinant of this matrix. Note that $M_{\omega}^{\Box}(n,k)$ is well-defined only up to rescaling rows by a power of ω , and thus $\Delta_{\omega}^{\Box}(n,k)$ is only well-defined up to an

overall rescaling by a power of ω . Below we always fix this normalization in a way that makes the determinant a polynomial with real coefficients (though it is not clear that this is possible in general).

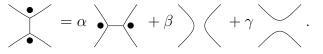
Many of the small determinants used below can be calculated by hand. The larger ones, however, rely on a newer software implementation of our methods, which unfortunately is not ready for release. (The older implementation, used and described above, was not designed to keep track of rotations of vertices.) Our plans for further investigations of small skein theories will make use of the newer implementation, so we defer a description and release until a later paper.

Proposition 7.1. For any $d \neq 2$ there are no twisted trivalent categories with dim span $D_{\omega}(4,0) \leq 3$.

Proof. If $D_{\omega}(4,0) \leq 3$ then the determinant $\Delta_{\omega}(4,0) = d^5(d-2)$ vanishes. Since $d \neq 0$, we see that this forces d = 2.

Proposition 7.2. When d = 2 there is at most one trivalent category with $\dim C_4 \leq 4$.

Proof. If $D_{\omega}(4,0)$ is linearly dependent then we get a relation of the form



As before, this relation shows that $D_{\omega}(n,0)$ spans \mathcal{C}_n . On the other hand if $D_{\omega}(4,0)$ is linearly independent then it is a basis of \mathcal{C}_4 . Thus computing the kernel of the inner product we see that

$$\left(\begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array} - \left(\begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array}\right) = \left(\begin{array}{c} \circ \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{array}\right).$$

In either case, the relation shows that $D_{\omega}^{\Box}(n,0)$ spans \mathcal{C}_n and hence the relation is enough to evaluate all closed diagrams. Thus there is at most one such category.

Remark 7.3. Note that this uniqueness statement applies for a particular choice of primitive cube root of unity for the rotational eigenvalue; typically, as below, examples will appear in pairs corresponding to both choices.

For realization we use [EGO04] which gives Chmutova's classification of fusion categories of global dimension 6. We recall the following notation from [Ost03, §3]. Suppose that H is a subgroup of G, that $\xi \in Z^3(G, \mathbb{C}^{\times})$ is a 3-cocycle and that $\psi \in C^2(H, \mathbb{C}^{\times})$ is a 2-cochain whose coboundary is the restriction of ξ . Then we have a fusion category $\operatorname{Vec}(G, \xi)$ of twisted G-graded vector spaces, and the twisted group ring $\mathbb{C}_{\psi}[H]$ is an algebra object. Let $\mathcal{C}(G, H, \xi, \psi)$ denote the category of bimodules over the twisted group ring. Recall that $H^3(S_3, \mathbb{C}^{\times}) = \mathbb{Z}/6\mathbb{Z}$, $H^3(S_2, \mathbb{C}^{\times}) = \mathbb{Z}/2\mathbb{Z}$, and $H^2(S_2, \mathbb{C}^{\times}) = 0$. If ξ is an element of order 3 in $H^3(S_3, \mathbb{C}^{\times})$, then its restriction to $H^3(S_2, \mathbb{C}^{\times})$ is trivial. So there's a 2-cochain ψ on S_2 (which is unique up to homology) such that $d\psi = \xi$ on S_2 . **Proposition 7.4.** If ξ is a 3-cocycle of order 3 in $H^3(S_3, \mathbb{C}^{\times})$ and ψ a 2-cochain on S_2 such that $d\psi = \xi$ on S_2 , then $\mathcal{C}(S_3, S_2, \xi, \psi)$ gives a trivalent category with d = 2 and dim $\mathcal{C}_4 = 3$.

Proof. These two categories (one for each choice of ξ) each have three objects 1, X, g with $g^2 = 1$, gX = Xg, and $X^2 = 1 + 2X + g$. (In other words, they are near group categories [Sie03, EG14, Izu15] for the group $\mathbb{Z}/2\mathbb{Z}$.) So a direct calculation shows that the dimensions of the Hom spaces are given by $1, 0, 1, 1, 3, \ldots$ Since these are distinct from $\operatorname{Rep}(S_3) = SO(3)_{\zeta_{12}}$, our previous classification shows that the rotational eigenvalue cannot be 1 so they must both be twisted trivalent categories.

Combining these results we get the following classification.

Theorem D. A twisted trivalent category C with dim $C_4 \leq 3$ must be one of the two $C(S_3, S_2, \xi, \psi)$ categories.

(Here we haven't specified which values of ξ correspond to which rotational eigenvalues, although it must be a bijective correspondence; it would be interesting to work this out.)

Now, a twisted cubic category C (that is, one with dim $C_4 = 4$) must (by the analogue of Theorem 4.16) have $d \neq 2$, $D_{\omega}(4,0)$ a basis of C_4 and satisfy the relation

$$\underbrace{) \bullet \bullet}_{\bullet \bullet} = -\frac{1}{d} \left(\underbrace{) \bullet \bullet}_{\bullet} + \underbrace{) \bullet \bullet}_{\bullet} \right) + \frac{1}{d} \left(\underbrace{) \leftarrow}_{\bullet} + \underbrace{) \bullet}_{\bullet} \right)$$

Using only this relation (along with the known values d for loops, 1 for bigons, and 0 for triangles), we can readily compute all the following determinants.

$$\begin{aligned} \Delta_{\omega}(5,0) &= d^{10}(d-2)^5 \\ \Delta_{\omega}^{\Box}(5,1) &= d^9(d-2)^6 \\ \Delta_{\omega}(6,0) &= -d^{26}(d-2)^{23}(d-1)\left(d^2-d-1\right)\left(d^3-2d^2-3d+1\right)\left(d^4-4d^3+3d^2-d-1\right) \\ \Delta_{\omega}^{\Box}(6,1) &= -d^{12}(d-2)^{31}(d-1)^2(d+1)^2\left(d^2-3d-1\right)^4\left(d^4-2d^3-3d^2-d+2\right) \\ \Delta_{\omega}^{\Box}(7,1) &= -d^{-86}(d-2)^{141}(d+1)^{16}\left(d^2-3d-1\right)^{35}Q_{\omega,9}Q_{\omega,60} \end{aligned}$$

(The polynomials $Q_{\omega,i}$ appear in the appendix.)

Proposition 7.5. In any twisted cubic category $D_{\omega}^{\square}(5,1)$ is linearly independent.

Proof. Since d is not 0 or 2, we have $\Delta_{\omega}^{\Box}(5,1) \neq 0$.

It follows that there are no twisted cubic categories with dim $C_5 \leq 10$. It also follows that if dim $C_5 = 11$, then $D_{\omega}^{\Box}(5,1)$ is a basis for C_5 .

Proposition 7.6. If C is a twisted cubic category with dim $C_5 = 11$ and dim $C_6 \leq 40$, then d = -1 or $d = \frac{3\pm\sqrt{13}}{2}$.

Proof. First, dim $C_6 \leq 40$ implies that $D_{\omega}^{\Box}(6,1)$ is linearly dependent and hence $\Delta_{\omega}^{\Box}(6,1)$ and $\Delta_{\omega}^{\Box}(7,1)$ both vanish. But their only shared factors are d+1 and $d^2 - 3d - 1$.

Proposition 7.7. There is at most one twisted cubic category with dim $C_5 = 11$ and dim $C_6 \leq 40$ for each of the three points d = -1 or $d = \frac{3\pm\sqrt{13}}{2}$.

Proof. Again we must have a pentafork relation. The method of Lemma 6.23 can still be used to guarantee pentapent and hexapent reductions and thus uniqueness, as follows. The first argument there shows that we have a relation for reducing the pentapent unless $1 = (-\zeta^{-2}\omega)^3 = -1$. The second argument there shows that we have a relation for reducing the hexapent unless $1 = (-\zeta^{-2}\omega)^3 = -\zeta^{-2}\omega$ which can only occur when $\zeta^2 = -\omega$. Here ζ^2 is a third root of unity, while $-\omega$ is a primitive sixth root of unity, so this can not happen.

The two cases with $d = \frac{3+\sqrt{13}}{2}$ are realized by the twisted Haagerup fusion categories conjectured in [EG11]. Ostrik observed that these can be constructed as follows. Start with the construction of two $\mathbb{Z}/9\mathbb{Z}$ Izumi near group categories in [EG14]. Following Izumi and Evans-Gannon, the center of one of these categories contains a copy of Rep($\mathbb{Z}/3\mathbb{Z}$) as a symmetric tensor subcategory. One can then de-equivariantize [DGNO10] by this to get a new category which has objects $1, g, g^2, X, gX, g^2X$ but where the three invertible elements have nontrivial associator. We will denote the categories obtained this way H_{ω} (one for each primitive cube root of unity). The two with $d = \frac{3-\sqrt{13}}{2}$ are realized by the Galois conjugates of the H_{ω} .

Remark 7.8. Note that the untwisted H2 (one of the even parts of the Haagerup subfactor) and H3 can be constructed in a similar way. Start with the unique $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ Izumi near-group category. The center of this category has two different copies of $\operatorname{Rep}(\mathbb{Z}/3\mathbb{Z})$ (one is the diagonal and the other the anti-diagonal). De-equivariantizing by each of these gives H2 and H3 (although it's not clear which is which).

Proposition 7.9. The two categories H_{ω} are twisted cubic categories with $\dim \mathcal{C}_5 = 11$, $\dim \mathcal{C}_6 = 37$, and $d = \frac{3+\sqrt{13}}{2}$.

Proof. This argument closely follows the argument from 6.4. Again let H' be the subcategory generated by the trivalent vertex. The same argument as before shows that H' must be a trivalent or twisted trivalent category. By the fusion rules we know that $d = \frac{3+\sqrt{13}}{2}$. We need to see that it is twisted and show that $H' = H_{\omega}$ which would show that H_{ω} is trivalent and, from the fusion rules, that the dimensions of the invariant spaces begin 1, 0, 1, 1, 4, 11, 37. If H' were untwisted, then it would have to be on our list of untwisted trivalent categories. First, H' cannot be $SO(3)_q$ because 1 + X is not an algebra (if it were then g would be in the normalizer of that algebra contradicting the nontrivial associator). Second, H' cannot be an ABA or $(G_2)_q$ category for the same dimensional considerations that showed H3' couldn't lie in those familes. Third, H' can't be H3 because of the nontriviality of the associator. Hence it is a twisted trivalent category. Finally, in order to show that $H_{\omega} = H'$ it is enough to show that dim $\operatorname{Inv}_{H'}(X^{\otimes 6}) = 37$ which follows calculating that the 41-by-41 matrix $M_{\omega}^{\Box}(6,1)$ has rank at least 37 at this value of d. This rank calculation can be easily done modulo a prime sitting above 3 in $\mathbb{Z}[d]$. \Box

Question 7.10. Does there exist a twisted trivalent category \mathcal{Q}_{ω} with d = -1, dim $\mathcal{C}_5 = 11$ and dim $\mathcal{C}_6 \leq 40$?

Such a category is unique if it exists. It would have dim $C_6 = 39$ and would satisfy the following two relations³ (where ζ is the primitive sixth root of unity which is a square root of ω):

$$- \underbrace{\sum_{i=0}^{5} \rho^{i} \left(\underbrace{- \underbrace{0}}_{\bullet \bullet} \right)}_{i=0} + \sum_{i=0}^{5} \rho^{i} \left(\underbrace{- \underbrace{0}}_{\bullet \bullet} \right) + \sum_{i=0}^{1} \rho^{i} \left(\underbrace{- \underbrace{0}}_{\bullet \bullet} \right) + \sum_{i=0}^{2} \rho^{i}$$

and

$$0 = \sum_{i=0}^{5} \zeta^{i} \rho^{i} \left(\underbrace{- \underbrace{\bullet}}_{i=0} \right) + \sum_{i=0}^{5} \zeta^{i} \rho^{i} \left(\underbrace{- \underbrace{\bullet}}_{i=0} \right) + \sum_{i=0}^{5} \zeta^{i} \rho^{i} \left(\underbrace{- \underbrace{\bullet}}_{i=0} \right) + \sum_{i=0}^{5} \zeta^{i} \rho^{i} \left(\underbrace{- \underbrace{\bullet}}_{i=0} \right).$$

Such a \mathcal{Q}_{ω} cannot come from any operator algebraic construction since d = -1 < 0. As we will see in the next section, \mathcal{Q}_{ω} is not braided and so \mathcal{Q}_{ω} is not a Drinfel'd-Jimbo quantum group. Furthermore, the dimensions of the objects at depth 2 are not real (they are conjugate primitive sixth roots of unity), so \mathcal{Q}_{ω} must have infinitely many simple objects. We do not think it comes from any well understood construction, but it has also passed every test we have attempted to use to rule it out. We also note that the above relations are particularly nice as they involve only 24 terms each.

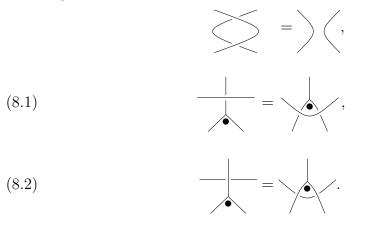
In conclusion, we have the following classification of twisted trivalent categories.

dimension bounds	new examples
$1,0,1,1,3,\ldots$	$\mathcal{C}(S_3, S_2, \xi, \psi)$
$1,0,1,1,4,10,\ldots$	nothing
$1,0,1,1,4,11,37,\ldots$	H_{ω}
$1,0,1,1,4,11,39,\ldots$	\mathcal{Q}_{ω} , if it exists
$1,0,1,1,4,11,39,\ldots$ $1,0,1,1,4,11,40,\ldots$	nothing more

³Although the computer alerted us to the existence of these relations, we actually computed them by hand, since it is difficult to read off from our computer program where the dots belong. This by-hand calculation following [Kup94] took two people-days.

8. Braided Trivalent Categories

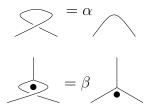
Definition 8.1. We call a trivalent or twisted trivalent category braided if there is an element in the 4-boundary point space, which we write using a crossing, which satisfies the following relations (in the untwisted case ignore the dots):



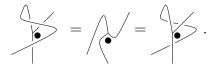
Note that the latter two relations above imply the Reidemeister 3 relations also hold via the Kauffman trick (since the category is generated by the trivalent vertex and so the crossing can be written in terms of trivalent vertices). As usual in quantum topology, the Reidemeister 1 relation need not hold.

Lemma 8.2. There are no braided twisted trivalent categories.

Proof. By dimensional considerations we have the following relations



for some numbers α and β . If we use the other crossing in these pictures we get the same relations with α^{-1} and β^{-1} . We can now compute the action of rotation on the twisted trivalent vertex in two different ways:



Thus, $\alpha^{-1}\beta^2 = \omega = \alpha\beta^{-2}$, and so $\omega = \omega^{-1}$ which is a contradiction.

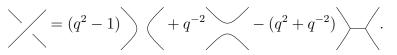
Remark 8.3. The same argument shows that if X is a simple object in a braided tensor category and $f: 1 \to X^{\otimes n}$ is an eigenvector for both rotation and braiding, then the rotational eigenvalue is ± 1 .

Lemma 8.4. If C is a braided trivalent category with dim $C_4 \leq 4$, then dim $C_5 \leq 10$.

Proof. Since dim $C_4 \leq 4$ we get that D(4,0) forms a spanning set for C_4 . Look at Equations (8.1) and (8.2) and expand every crossing as a sum of diagrams in D(4,0). These each give a relation between diagrams in $D^{\Box}(5,1)$. We claim that at least one of these relations is nontrivial. Since the crossings are rotations of each other, at least one of the crossings has a nontrivial coefficient of either \times or \times . In the former case the expanded relation has a nontrivial coefficient of \pm , and in the latter case the expanded relation has a nontrivial coefficient of the pentagon. Thus we have a nontrivial relation among $D^{\Box}(5,1)$, so by Lemma 5.15, we get that D(5,0) spans C_5 . In particular, dim $C_5 \leq 10$.

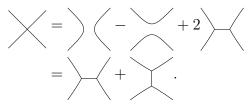
This is enough to give a complete classification of braided trivalent categories, but before stating this classification we list the examples that occur.

Example 8.5. The standard braiding on $SO(3)_q$ is



Note that although $SO(3)_{\pm q^{\pm 1}}$ are the same fusion category, there are two distinct braided tensor categories corresponding to $\pm q$ and $\pm q^{-1}$. Finally, note that when $q = \pm i$ this formula gives the standard symmetric braiding on OSp(1|2).

Example 8.6. Let X be the standard 2-dimensional representation of S_3 and $X \otimes X \to X$ be a non-zero map (which is unique up to scalar), then this generates a trivalent category. The standard symmetric braiding is



It is easy to see from our classification that this category agrees with $SO(3)_q$ for q a primitive 12th root of unity as a trivalent category. However, the standard symmetric braiding on representations of S_3 does not agree with the standard braiding for $SO(3)_q$. We will denote this braided trivalent category by S_3 , to distinguish it from $SO(3)_{\zeta_{12}}$.

Example 8.7. The standard braiding for $(G_2)_q$ for $q \neq \pm i$ is

$$= \frac{1}{q+q^{-1}} \left(q^3 \right) \left(+ q^{-3} \right)$$
$$- \frac{q^6 + q^4 + q^2 + q^{-2} + q^{-4} + q^{-6}}{q+q^{-1}} \left(q \right) \left(+ q^{-1} \right).$$

Note that although $(G_2)_{\pm q^{\pm 1}}$ give the same fusion category, there are two distinct braided tensor categories corresponding to $\pm q$ and $\pm q^{-1}$.

Corollary 8.8. The only braided trivalent categories with dim $C_4 \leq 4$ are $SO(3)_q$ (for any q including $SO(3)_{\pm i} = OSp(1|2)$), S_3 , and $(G_2)_q$ for $q \neq \pm i$.

Proof. By our classification, we need only classify all braidings for the $SO(3)_q$, ABA, and $(G_2)_q$ categories. In the G_2 case, when q is not a fourth or twentieth root of unity, this was done by Kuperberg in [Kup94]. Following Kuperberg, you write down a general element of the 4-boundary point space and check whether it satisfies the braiding relations. Since this is somewhat tedious and since the hardest case was already done by Kuperberg, we will skip much of the details here.

For the $SO(3)_q$ categories (which agree for $\pm q^{\pm 1}$), generically there are exactly two braidings on corresponding to one corresponding to the standard braiding of $SO(3)_{\pm q}$ and the other to $SO(3)_{\pm q^{-1}}$. When $q = \pm 1$ or $q = \pm i$, these two braidings agree and yield the symmetric braidings on SO(3) and OSp(1|2). When q is a primitive 12th root of unity, there are three braidings, two corresponding to the $SO(3)_q$ braidings and one corresponding to the symmetric braiding which corresponds to the standard braiding on S_3 .

A direct calculation shows that the ABA categories do not have a braiding. When δ , the loop value for A, is not the golden ratio or its conjugate, there is a simple more conceptual approach. Namely, the fusion rules are noncommutative since $A^{(2)}(ABA) = A^{(3)}BA + ABA$ while $(ABA)A^{(2)} = ABA^{(3)} + ABA$. (When δ is the golden ratio, the fusion rules are commutative so one must use the brutal approach.)

Finally, for $(G_2)_q$, Kuperberg proved that when the 5-boundary point space is 10-dimensional, there are exactly two braidings for such a category, one corresponding to the standard braiding of $(G_2)_{\pm q}$ and the other to $(G_2)_{\pm q^{-1}}$. The only remaining cases are q is a primitive 4th or 20th root of unity. The former case was already dealt with above, and it turns out in the latter case there are still only the two standard braidings.

We expect this theorem to give Wenzl-style recognition results [KW93, TW05] for $(G_2)_q$ and Deligne's S_t , showing that they are the only braided tensor categories with their Grothendieck rings.

Corollary 8.9. The only symmetric trivalent categories with dim $C_4 \leq 4$ are SO(3), S_3 , (G_2) , and OSp(1|2).

Note that by Deligne's theorem [Del02], any symmetric abelian category of exponential growth must be the category of representations of a supergroup, so this corollary is not surprising. The exponential growth condition is satisfied in our setting, but this corollary does not follow directly from Deligne's theorem because *a priori* there could be symmetric trivalent categories which do not come from abelian categories.

9. Prospects

There are several obstacles to pushing the above techniques further in the study of trivalent categories. First, even though in principal just knowing d and t should be enough to calculate the determinants $D^{\Box}(n,k)$ for n+2k < 12, as we

saw in Conjecture 6.9, in practice arithmetic of two variable rational functions is sufficiently difficult that we cannot compute all of these determinants exactly. If one is willing to accept probabilistic proofs then we can compute more of these determinants. Second, if we want to go beyond n + 2k = 12 then we need to introduce the dodecahedron as a third variable, which will make things quite a bit more complicated. Finally, we are already pushing up against the limits of practical Gröbner basis calculations: for example we already cannot directly intersect $\Delta(7,0)$ and $\Delta^{\Box}(7,1)$. For all these reasons, it is unlikely that we will be able to push the classification of trivalent categories much further without new ideas.

Instead we plan to continue investigating these ideas in other settings than the trivalent setting. There are numerous good candidates for investigation, including the following.

- Braided trivalent categories with dimensions bounded by 1, 0, 1, 1, 5. This includes the conjectured exceptional series of Deligne and Vogel [Del96, Vog11].
- Skein theoretic invariants of planar graphs together with a 2-coloring of the vertices and a 2-coloring of the faces. These correspond to quadrilaterals of subfactors and we hope to strengthen the classification results of Grossman, Izumi, and Jones [GJ07, GI08].
- Categories generated by a 4-valent vertex with a checkerboard shading. These were studied by Bisch, Jones, and Liu in [BJ00, BJ03, BJL14], and we hope to push their techniques beyond what can be done by hand.
- Categories generated by a 2n-valent vertex with a checkerboard shading. For n = 2 this is the previous example, and for n = 3 they were studied by Dylan Thurston [Thu04a].
- Skein theoretic invariants of virtual knots. This includes the representation theory of the Higman–Sims sporadic finite simple group [Jae92, Kup97].

APPENDIX A. SKEIN THEORETIC INVARIANTS AND PIVOTAL CATEGORIES

The goal of this section is to provide background so that this paper is accessible to knot theorists, graph theorists, and other readers unfamiliar with tensor categories or planar algebras.

Suppose we want to study certain numerical invariants f of planar trivalent graphs. Assume that f of the empty diagram is 1, that $f(\bigcirc)$ and $f(\bigcirc)$ are nonzero, and that f satisfies the following multiplicative conditions:

$$(0) f\left(\begin{array}{c} X & Y \\ \end{array}\right) = f(X) \cdot f(Y)$$

$$(1) f\left(\begin{array}{c} X - Y \\ \end{array}\right) = 0$$

$$(2) f\left(\begin{array}{c} X & Y \\ \end{array}\right) = f\left(\begin{array}{c} X \\ \end{array}\right) \cdot f\left(\begin{array}{c} Y \\ \end{array}\right) / f\left(\begin{array}{c} Y \\ \end{array}\right)$$

$$(3) f\left(\begin{array}{c} X - Y \\ \end{array}\right) = f\left(\begin{array}{c} X \\ \end{array}\right) \cdot f\left(\begin{array}{c} Y \\ \end{array}\right) / f\left(\begin{array}{c} Y \\ \end{array}\right) / f\left(\begin{array}{c} Y \\ \end{array}\right)$$

Thus the invariant of any k-disconnected graph for $k \leq 3$ is determined by the invariants of the pieces.

Example A.1. An almost trivial example of a multiplicative invariant of graphs is $a^{\#V}$, for some number a, where #V denotes the number of trivalent vertices in the graph.

Example A.2. An important example of a multiplicative invariant of graphs is the number of *n*-colorings of the faces of the graph, divided by n. (The division by n is a normalization factor ensuring that the empty graph is assigned 1 instead of n.) This example can be generalized by considering non-integer specializations of the chromatic polynomial.

Question. What examples are there of such multiplicative invariants of trivalent planar graphs?

While the question appears to be an elementary question about planar trivalent graphs, we discover that the examples are actually related to quite distant subjects in mathematics. In particular, we are able to identify each of the small examples we encounter with some suprising or exotic object coming from representation theory or the theory of subfactors!

In order to understand the main results of the paper in the language of graph invariants, we first want to extend this invariant of closed trivalent graphs to an invariant of planar graphs with boundary. That is, we extract a sequence of vector spaces, the 'open graphs, modulo negligibles'. We now describe how these vector spaces have the structure of a pivotal tensor category (or planar algebra).

Let $\hat{\mathcal{C}}_n$ denote the (infinite dimensional) vector space with basis the planar trivalent graphs drawn in the disc, with *n* fixed boundary points, up to isotopy rel boundary. This vector space has a natural bilinear pairing, given by gluing two open graphs together (starting at a preferred boundary point), to obtain a closed planar graph, which we then evaluate to a number using our multiplicative invariant f. The kernel of this bilinear pairing is called 'the negligible elements'. Let \mathcal{C}_n^f denote the quotient vector space of $\hat{\mathcal{C}}_n$ by negligible elements.

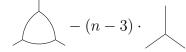
One may assemble these vector spaces into a single algebraic structure, variously axiomatized as an (unshaded) planar algebra [Jon99], a spider [Kup96] or a pivotal tensor category [BW99]. We'll only describe the last in any detail. The category, which we'll call C^f , has as objects the natural numbers. We'll first describe a bigger category of trivalent graphs, which we call \hat{C} and which does not depend at all on our multiplicative invariant. In \hat{C} , the morphisms from n to m are simply the formal linear combinations of planar graphs drawn in a rectangle with n points along the bottom edge and m points along the top edge, i.e. the vector space \hat{C}_{n+m} . We can compose morphisms in the obvious way, by stacking rectangles. This category is a tensor category, with the tensor product given by drawing diagrams side by side. Finally it is a pivotal category, with the evaluation and coevaluation maps given by caps and cups.

Inside $\hat{\mathcal{C}}$, the negligible (with respect to f) elements form a planar ideal — if some (linear combination of) graphs pair with arbitrary other graphs to give

zero, then glueing more graph to the boundary preserves this property. We thus define the category \mathcal{C}^f to be the quotient of $\hat{\mathcal{C}}$ by the negligible ideal. This "ideal" property says that we can treat the negligible elements as skein relations: they can be applied locally in any part of a graph. Furthermore, typically this ideal is finitely generated by a few particular skein relations.

Thus, in \mathcal{C}^f , the objects are still the natural numbers and the morphisms from n to m are just \mathcal{C}^f_{n+m} . The category \mathcal{C}^f is still a pivotal tensor category, and now it is <u>evaluable</u> (i.e. dim $\mathcal{C}^f_0 = 1$, and in fact \mathcal{C}^f_0 may be identified with the ground field by sending the empty diagram to 1) and <u>non-degenerate</u> (i.e. for every morphism $x : a \to b$, there is another morphism $x' : b \to a$ so $\langle x, x' \rangle \neq 0 \in \mathcal{C}^f_0$). Writing X for the generating object in \mathcal{C}^f (i.e. 1 in the natural numbers!), we see that X is a symmetrically self-dual object, with duality pairings and copairings given by the cap and cup diagrams. Moreover, the trivalent vertex is a rotationally symmetric map $1 \to X \otimes X \otimes X$.

Example A.3. If the invariant is the normalized number of *n*-colorings described in Example A.2, then a linear combinations of graphs is negligible if and only if for any coloring of the boundary faces the given linear combination of the numbers of ways of extending that coloring to the interior is zero. For example, the following element of C_3 is negligible:



In particular, this gives a skein relation in C^f which says that you can remove a triangle and multiply by (n-3). There are also other negligible elements; in fact after renormalizing the trivalent vertex, C^f becomes equivalent to the pivotal category $SO(3)_q$ coming from quantum groups where q is a number satisfying $(q + q^{-1})^2 = n$ (see Section 4 for a description of $SO(3)_q$).

Proposition A.4. The construction of C^f from f gives a bijective correspondence between trivalent categories and multiplicative invariants of planar graphs.

Proof. First we prove that the category C^f constructed from a multiplicative invariant f is trivalent. Consider C_0^f . The empty diagram is not negligible, so we need only show that any closed diagram is a multiple of the empty diagram. If α is a closed diagram and β is the empty diagram, then $\alpha - f(\alpha)\beta$ is negligible, so in C_0^f we have that $\alpha = f(\alpha)\beta$. Now we look at C_1^f . By multiplicativity we have that any diagram with one boundary point is negligible, so dim $C_1^f = 0$. The remaining cases are similar.

Given a trivalent category C, we need to construct a multiplicative invariant of planar graphs. The usual diagrammatic calculus for pivotal categories shows that any trivalent category gives an invariant of closed graphs just by interpreting the graphs as elements of C_0 and sending the empty diagram to 1.

We want to check that this invariant is multiplicative, in which case it is clear that it provides an inverse to $f \mapsto \mathcal{C}^f$. We first check that the loop and the theta are nonzero. The single strand in \mathcal{C}_2 must be nonzero, because if it

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were zero then all nonempty diagrams would be zero. Since dim $C_2 = 1$, we see that any diagram in C_2 is a multiple of the single strand, hence nondegeneracy says that the inner product of the strand with itself is nonzero, hence the loop value is nonzero. Similarly, by considering C_3 we see that the theta value is nonzero. Next we want to prove the multiplicative properties. Each of these are similar, so we only prove (2). We have that X is some multiple of the single strand, so we see that $X = (X)/() \cdot |$ (by pairing with the strand). Substituting this into the LHS of (2) gives the RHS.

APPENDIX B. POLYNOMIALS APPEARING IN DETERMINANTS

This appendix contains some of the irreducible factors of determinants appearing in this paper. The other irreducible factors, which are very large, are contained in text files packaged with the **arXiv** source of this paper, and described here. Each polynomial is named as $Q_{i,j}$, where *i* is the largest exponent of *d* and *j* is the largest exponent of *t*. Where two polynomials have the same pair of largest exponents, we name them with an additional character in the subscript, as in $Q_{2,4,a}$ and $Q_{2,4,b}$.

$$\begin{split} P_{SO(3)} &= d(t-1) - t + 2 \\ P_{ABA} &= t^2 - t - 1 \\ P_{G_2} &= d^2 t^5 + d \left(2t^5 - 4t^4 - t^3 + 6t^2 + 4t + 1 \right) + t^5 - 4t^4 + t^3 + 7t^2 - 2 \\ Q_{0,1} &= t + 1 \\ Q_{1,1} &= d(t+1) + t \\ Q_{1,2} &= d \left(2t^2 + 2t + 1 \right) + 3t^2 - 2 \\ Q_{2,3} &= d^2 \left(t^3 + t^2 - 2t - 1 \right) + d \left(2t^3 - 2t^2 + t \right) + t^3 - 3t^2 + t + 4 \\ Q_{3,4} &= d^3 \left(t^4 + 3t^3 - t^2 - 3t - 1 \right) + d^2 \left(2t^4 + t^2 + 2t + 1 \right) + \\ &+ d \left(t^4 - 3t^3 + 3t^2 + 6t + 1 \right) - t^2 + 2t + 2 \\ Q_{3,5} &= d^3 \left(3t^5 + 4t^4 - 2t^3 - 6t^2 - 4t - 1 \right) + d^2 \left(8t^5 + 2t^4 - 11t^3 - 5t^2 + 5t + 3 \right) + \\ &+ d \left(7t^5 - 6t^4 - 6t^3 + 7t^2 + 3t - 1 \right) + 2t^5 - 4t^4 + t^3 + 5t^2 - 2t - 2 \\ Q_{2,4,a} &= d^2 \left(t^4 - t^3 - 4t^2 - 3t - 1 \right) + d \left(2t^4 - 6t^3 - 7t^2 + t + 3 \right) + t^4 - 5t^3 + t^2 + 2t - 2 \\ Q_{2,4,b} &= d^2 \left(t^4 + 2t^3 - t^2 - 2t - 1 \right) + d \left(2t^4 - 2t^3 - 2t^2 + 3t + 4 \right) + t^4 - 4t^3 + 5t^2 + 2t - 4 \\ Q_{4,5} &= d^4t^5 + d^3 \left(3t^5 - 3t^4 - 3t^3 + 7t^2 + 5t + 1 \right) + d^2 \left(3t^5 - 5t^4 - 5t^3 + 10t^2 + 12t + 2 \right) + \\ &+ d \left(t^5 - t^4 - 5t^3 + 3t^2 + 9t + 5 \right) + t^4 - 3t^3 + 4t + 1 \end{split}$$

$$\begin{split} Q_{6,9} &= d^6 \left(4t^8 + t^7 - 15t^6 - 20t^5 - 6t^4 + 8t^3 + 10t^2 + 5t + 1 \right) + \\ &+ d^5 \left(2t^9 + 12t^8 - 19t^7 - 54t^6 - 17t^5 + 21t^4 - 11t^3 - 43t^2 - 30t - 7 \right) + \\ &+ d^4 \left(6t^9 - 6t^8 - 31t^7 + 11t^6 - 119t^4 - 130t^3 - 21t^2 + 35t + 14 \right) + \\ &+ d^3 \left(2t^9 - 32t^8 + 72t^7 + 59t^6 - 227t^5 - 258t^4 + 59t^3 + 164t^2 + 43t - 3 \right) + \\ &+ d^2 \left(-10t^9 + 10t^8 + 123t^7 - 136t^6 - 305t^5 + 103t^4 + 225t^3 + 23t^2 - 38t - 13 \right) + \\ &+ d^2 \left(-12t^9 + 56t^8 - 9t^7 - 149t^6 - 16t^5 + 175t^4 + 46t^3 - 89t^2 - 17t + 16 \right) + \\ &- 4t^9 + 28t^8 - 49t^7 - 4t^6 + 69t^5 - 54t^4 - 9t^3 + 54t^2 - 14t - 20 \\ Q_{\omega,9} &= d^9 - 7d^8 + 15d^7 - 2d^6 - 14d^5 - 16d^4 + 41d^3 - 23d^2 + d + 5 \\ Q_{\omega,60} &= d^{60} - 42d^{59} + 825d^{58} - 10050d^{57} + 84827d^{56} - 524435d^{55} + 2444075d^{54} \\ &- 8680920d^{53} + 23364055d^{52} - 46267136d^{51} + 62172868d^{50} \\ &- 43026307d^{49} - 10724689d^{48} + 19327948d^{47} + 113757871d^{46} \\ &- 289556454d^{45} + 161677043d^{44} + 403173198d^{43} - 822414523d^{42} \\ &+ 340360209d^{41} + 658154819d^{40} - 734499791d^{39} - 499750302d^{38} \\ &+ 1417408819d^{37} - 680996389d^{36} - 701113119d^{35} + 1161482902d^{34} \\ &- 934417344d^{33} + 751648667d^{32} - 23523738d^{31} - 1359642298d^{30} \\ &+ 1528218917d^{29} + 342409869d^{28} - 1836361788d^{27} + 946900947d^{26} \\ &+ 763927401d^{25} - 1172104767d^{24} + 652553812d^{23} - 193252562d^{22} \\ &- 352541742d^{21} + 857069723d^{20} - 561108191d^{19} - 289399926d^{18} \\ &+ 602082003d^{17} - 186224613d^{16} - 206339296d^{15} + 185432097d^{14} \\ &- 10906225d^{13} - 54265030d^{12} + 26840191d^{11} + 547786d^{10} \\ &- 5118901d^9 + 1967134d^8 - 218389d^7 - 37050d^6 + 47054d^5 \\ &- 35063d^4 + 10325d^3 - 903d^2 - 49d + 7 \end{split}$$

The other factors, $Q_{7,11}$, $Q_{8,12}$, $Q_{11,19}$, $Q_{21,33}$, $Q_{22,36}$, $Q_{51,69}$, $Q_{54,78}$, and $Q_{36,60}$ are available in LATEX and Mathematica formats in the polynomials/ subdirectory of the arXiv source as files Q_i,j.tex and Q_i,j.m, and also in the Mathematica notebook code/GroebnerBasisCalculations.nb

References

- [BJ00] Dietmar Bisch and Vaughan F. R. Jones. Singly generated planar algebras of small dimension. <u>Duke Math. J.</u>, 101(1):41–75, 2000. MR1733737.
- [BJ03] Dietmar Bisch and Vaughan Jones. Singly generated planar algebras of small dimension. II. <u>Adv. Math.</u>, 175(2):297–318, 2003. MR1972635 DOI:10.1016/S0001-8708(02)00060-9.
- [BJL14] Dietmar Bisch, Vaughan F.R. Jones, and Zhengwei Liu. Singly generated planar algebras of small dimension, part III, 2014. arXiv:1410.2876.

- [BMPS12] Stephen Bigelow, Scott Morrison, Emily Peters, and Noah Snyder. Constructing the extended Haagerup planar algebra. <u>Acta Math.</u>, 209(1):29–82, 2012. arXiv:0909.4099 MR2979509 DOI:10.1007/s11511-012-0081-7.
- [BW99] John W. Barrett and Bruce W. Westbury. Spherical categories. <u>Adv. Math.</u>, 143(2):357-375, 1999. arXiv:hep-th/9310164 MR1686423 DOI:10.1006/aima.1998.1800.
- [CW13] Sean Clark and Weiqiang Wang. Canonical basis for quantum osp(1|2). Lett. Math. Phys., 103(2):207-231, 2013. MR3010460 DOI:10.1007/s11005-012-0592-3.
- [Del96] Pierre Deligne. La série exceptionnelle de groupes de Lie. <u>C. R. Acad. Sci. Paris</u> Sér. I Math., 322(4):321–326, 1996. MR1378507.
- [Del02] P. Deligne. Catégories tensorielles. Mosc. Math. J., 2(2):227-248, 2002. Dedicated to Yuri I. Manin on the occasion of his 65th birthday. MR1944506 http://www. ams.org/distribution/mmj/vol2-2-2002/deligne.pdf.
- [DGNO10] Vladimir Drinfeld, Shlomo Gelaki, Dmitri Nikshych, and Victor Ostrik. On braided fusion categories. I. <u>Selecta Math. (N.S.)</u>, 16(1):1–119, 2010. arXiv:0906. 0620 MR2609644 DOI:10.1007/s00029-010-0017-z.
- [DL78] Richard Demillo and Richard Lipton. A probabilistic remark on algebraic program testing. <u>Information Processing Letters</u>, 7(4):193-195, 1978. available at http: //rjlipton.files.wordpress.com/2009/11/demillolipton.pdf.
- [EG11] David E. Evans and Terry Gannon. The exoticness and realisability of twisted Haagerup-Izumi modular data. <u>Comm. Math. Phys.</u>, 307(2):463-512, 2011. arXiv:1006.1326 MR2837122 DOI:10.1007/s00220-011-1329-3.
- [EG14] David E. Evans and Terry Gannon. Near-group fusion categories and their doubles. <u>Adv. Math.</u>, 255:586-640, 2014. arXiv:1208.1500 MR3167494 DOI:10.1016/j.aim.2013.12.014.
- [EGO04] Pavel Etingof, Shlomo Gelaki, and Viktor Ostrik. Classification of fusion categories of dimension pq. Int. Math. Res. Not., 2004(57):3041-3056, 2004. arXiv:math.QA/ 0304194 MR2098028 DOI:10.1155/S1073792804131206.
- [GI08] Pinhas Grossman and Masaki Izumi. Classification of noncommuting quadrilaterals of factors. Internat. J. Math., 19(5):557-643, 2008. arXiv:0704.1121 MR2418197 DOI:10.1142/S0129167X08004807.
- [GJ07] Pinhas Grossman and Vaughan F. R. Jones. Intermediate subfactors with no extra structure. J. Amer. Math. Soc., 20(1):219-265 (electronic), 2007. arXiv:0412423, MR2257402 DOI:10.1090/S0894-0347-06-00531-5.
- [GS12a] Pinhas Grossman and Noah Snyder. The Brauer-Picard group of the Asaeda-Haagerup fusion categories, 2012. arXiv:1202.4396, to appear in <u>Trans. Am.</u> Math. Soc.
- [GS12b] Pinhas Grossman and Noah Snyder. Quantum subgroups of the Haagerup fusion categories. <u>Comm. Math. Phys.</u>, 311(3):617-643, 2012. arXiv:1102.2631 MR2909758 DOI:10.1007/s00220-012-1427-x.
- [IMP13] Masaki Izumi, Scott Morrison, and David Penneys. Quotients of $A_2 * T_2$, 2013. DOI:10.4153/CJM-2015-017-4, extended version available as "Fusion categories between $\mathcal{C} \boxtimes \mathcal{D}$ and $\mathcal{C} * \mathcal{D}$ " at arXiv:1308.5723.
- [Izu15] Masaki Izumi. A Cuntz algebra approach to the classification of near-group categories, 2015. arXiv:1512.04288.
- [Jae92] François Jaeger. Strongly regular graphs and spin models for the Kauffman polynomial. Geom. Dedicata, 44(1):23–52, 1992. MR1188082.
- [Jon85] Vaughan F. R. Jones. A polynomial invariant for knots via von Neumann algebras. <u>Bull. Amer. Math. Soc. (N.S.)</u>, 12(1):103–111, 1985. MR766964 DOI:10.1090/S0273-0979-1985-15304-2.
- [Jon99] Vaughan F. R. Jones. Planar algebras, I, 1999. arXiv:math.QA/9909027.
- [Kob93] Ken-ichiro Kobayashi. Representation theory of $osp(1, 2)_q$. Z. Phys. C, 59(1):155–158, 1993. MR1230843 DOI:10.1007/BF01555850.

- [Kup94] Greg Kuperberg. The quantum G₂ link invariant. <u>Internat. J. Math.</u>, 5(1):61–85, 1994. arXiv:math.QA/9201302 MR1265145 DOI:10.1142/S0129167X94000048.
- [Kup96] Greg Kuperberg. Spiders for rank 2 Lie algebras. Comm. Math. Phys., 180(1):109– 151, 1996. arXiv:q-alg/9712003 MR1403861 euclid.cmp/1104287237.
- [Kup97] Greg Kuperberg. Jaeger's Higman-Sims state model and the B₂ spider. J. Algebra, 195(2):487-500, 1997. arXiv:9601221 MR1469634 DOI:10.1006/jabr.1997.7045.
- [KW93] David Kazhdan and Hans Wenzl. Reconstructing monoidal categories. In <u>I. M.</u> <u>Gel'fand Seminar</u>, volume 16 of <u>Adv. Soviet Math.</u>, pages 111–136. Amer. Math. Soc., Providence, RI, 1993. MR1237835 (preview at google books).
- [Lan02] Zeph A. Landau. Exchange relation planar algebras. In <u>Proceedings of the</u> <u>Conference on Geometric and Combinatorial Group Theory</u>, Part II (Haifa, 2000), volume 95, pages 183–214, 2002.
- [MPS10] Scott Morrison, Emily Peters, and Noah Snyder. Skein theory for the D_{2n} planar algebras. J. Pure Appl. Algebra, 214(2):117-139, 2010. arXiv:math/0808.0764 MR2559686 DOI:10.1016/j.jpaa.2009.04.010.
- [MPS11] Scott Morrison, Emily Peters, and Noah Snyder. Knot polynomial identities and quantum group coincidences. <u>Quantum Topol.</u>, 2(2):101–156, 2011. arXiv:1003. 0022 MR2783128 DOI:10.4171/QT/16.
- [MS12] Scott Morrison and Noah Snyder. Subfactors of index less than 5, Part 1: The principal graph odometer. <u>Comm. Math. Phys.</u>, 312(1):1–35, 2012. arXiv:1007. 1730 MR2914056 DOI:10.1007/s00220-012-1426-y.
- [New42] M. H. A. Newman. On theories with a combinatorial definition of "equivalence". Ann. of Math. (2), 43:223–243, 1942. MR0007372.
- [OEI] The On-Line Encyclopedia of Integer Sequences. http://oeis.org.
- [Ost03] Viktor Ostrik. Module categories over the Drinfeld double of a finite group. Int. Math. Res. Not., 2003(27):1507-1520, 2003. arXiv:math/0202130 MR1976233 D0I:10.1155/S1073792803205079.
- [Sch80] J. T. Schwartz. Fast probabilistic algorithms for verification of polynomial identities. J. Assoc. Comput. Mach., 27(4):701-717, 1980. MR594695 DOI:10.1145/322217.322225.
- [Sem09] Secret Blogging Seminar. How to almost prove the 4-color theorem, 2009. http://sbseminar.wordpress.com/2009/10/07/ how-to-almost-prove-the-4-color-theorem/ (accessed 2014-12-28).
- [Sie03] Jacob Siehler. Near-group categories. <u>Algebr. Geom. Topol.</u>, 3:719-775, 2003. arXiv:math/0209073 MR1997336 DOI:10.2140/agt.2003.3.719.
- [Spe94] Roland Speicher. Multiplicative functions on the lattice of noncrossing partitions and free convolution. <u>Math. Ann.</u>, 298(4):611–628, 1994. MR1268597 DOI:10.1007/BF01459754.
- [Sta99] Richard P. Stanley. Enumerative combinatorics. Vol. 2, volume 62 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1999. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin MR1676282 DOI:10.1017/CB09780511609589.
- [SW07] Adam S. Sikora and Bruce W. Westbury. Confluence theory for graphs. <u>Algebr. Geom. Topol.</u>, 7:439-478, 2007. arXiv:math.QA/0609832 MR2308953 DOI:10.2140/agt.2007.7.439.
- [Thu04a] D. P. Thurston. From Dominoes to Hexagons, May 2004. arXiv:math/0405482.
- [Thu04b] Dylan P. Thurston. The F_4 and E_6 families have a finite number of points, 2004. available at www.math.columbia.edu/~dpt/writing/F4E6.ps.
- [TW05] Imre Tuba and Hans Wenzl. On braided tensor categories of type BCD. J. Reine Angew. Math., 581:31-69, 2005. arXiv:math.QA/0301142 MR2132671 D0I:10.1515/crll.2005.2005.581.31.
- [Vog11] Pierre Vogel. Algebraic structures on modules of diagrams. J. Pure Appl. Algebra, 215(6):1292–1339, 2011. MR2769234 DOI:10.1016/j.jpaa.2010.08.013.

[Zip79] Richard Zippel. Probabilistic algorithms for sparse polynomials. In <u>Symbolic</u> and algebraic computation (EUROSAM '79, Internat. Sympos., Marseille, 1979), volume 72 of <u>Lecture Notes in Comput. Sci.</u>, pages 216–226. Springer, Berlin-New York, 1979. MR575692.