

Counting polygon spaces, Boolean functions and majority games

Jean-Claude HAUSMANN

Abstract

We explain why numbers occurring in the classification of polygon spaces coincide with numbers of self-dual equivalence classes of threshold functions, or regular Boolean functions, or of decisive weighted majority games.

1 Introduction

Initiated by K. Walker [16], the classification of polygon spaces with n edges (see [7] and Section 3 hereafter) involves chambers delimited by a hyperplane arrangement in $(\mathbb{R}_{>0})^n$ and so-called virtual genetic codes. The number $c(n)$ of chambers modulo coordinate permutations and the number $v(n)$ of virtual genetic codes were computed by several authors (see [8]) and the currently known figures are as follows

n	3	4	5	6	7	8	9	10	11
$c(n)$	2	3	7	21	135	2,470	175,428	52,980,624	?
$v(n)$	2	3	7	21	135	2,470	319,124	1,214,554,343	$\sim 1.7 \cdot 10^{15}$

(more precisely: $v(11) = 1,706,241, 214, 185, 942$, computed by Minfeng Wang: see [8]). According to the *On-Line Encyclopedia of Integer Sequences (OEIS)*, these numbers occur in other sequences:

1.1. The numbers $c(n)$ of chambers up to permutation coincide with the *Numbers of self-dual equivalence classes of threshold functions of n or fewer variables*, or the *numbers of majority (i.e., decisive and weighted) games with n players*, listed in [14].

1.2. The numbers $v(n)$ of virtual genetic codes coincide with the *numbers of Boolean functions of n variables that are self-dual and regular*, listed for $n \leq 10$ in [15].

The aim of this note is to explain these numerical coincidences by constructing natural bijections between the sets under consideration. In particular, the

above mentioned precise value $v(11)$ may be added in [14]. The principal results are Propositions 4.5, 5.1 and 6.1

The paper is organized as follows. Section 2 presents the transformation group used in various equivalence relations. Section 3 recalls the notations and the classification's result for polygon spaces. In Sections 4 and 5, we introduce threshold functions and majority games and prove the bijections involved in 1.1, while Section 6 concerns the case of 1.2. Finally, we treat in Section 7 the case of non-generic polygon spaces, giving rise to an apparently unknown integer sequence.

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2 The transformation group \mathcal{T}_n

In this section, we define the transformation group \mathcal{T}_n , responsible for several equivalence relations occurring in this paper. Incidentally, a few notation are introduced, which are used throughout the next sections

Fix a positive integer n . If X is a set, the symmetric group Sym_n acts on X^n by permuting the components. This is a right action: an element $x \in X^n$ is formally a map $x : \{1, \dots, n\} \rightarrow X$ ($x_i = x(i)$) and $\sigma \in \text{Sym}_n$ acts by pre-composition, i.e. $x^\sigma = x \circ \sigma$. Note that right actions are most often denoted exponentially in this paper.

Let $\mathcal{A}_n = (\mathbb{Z}_2)^n$, the elementary abelian group of rank n denoted additively. The Sym_n -action on \mathcal{A}_n gives rise to the semi-direct product

$$\mathcal{T}_n = \mathcal{A}_n \rtimes \text{Sym}_n. \quad (2.1)$$

Recall that, as a set, \mathcal{T}_n coincides with $\mathcal{A}_n \times \text{Sym}_n$. We we may use the short notations $\nu = (\nu, \text{id})$ and $\sigma = (0, \sigma)$ (which enables us to consider Sym_n as a subgroup of \mathcal{T}_n). The group \mathcal{T}_n is thus generated by $\nu \in \mathcal{A}_n$ and $\sigma \in \text{Sym}_n$, subject to the relations $\sigma^{-1}\nu\sigma = \nu^\sigma$. Note the formulae $(\nu, \sigma)(\mu, \tau) = (\nu\mu^{\sigma^{-1}}, \sigma\tau)$ and $(\nu, \sigma)^{-1} = (\nu^\sigma, \sigma^{-1})$.

The group \mathcal{T}_n will act on several sets. We finish this section with a few examples.

Example 2.1. *The action of \mathcal{T}_n on \mathbb{R}^n is defined as follows: if $z = (z_1, \dots, z_n) \in \mathbb{R}^n$, the i -th component of $z^{(\nu, \sigma)}$ is*

$$(z^{(\nu, \sigma)})_i = (-1)^{\nu_{\sigma(i)}} z_{\sigma(i)}. \quad (2.2)$$

In particular, $z^\nu = ((-1)^{\nu_1} z_1, \dots, (-1)^{\nu_n} z_n)$ and $z^\sigma = (z_{\sigma(1)}, \dots, z_{\sigma(n)})$. The following lemma will be useful.

Lemma 2.2. *The inclusion $(\mathbb{R}_{\geq 0})^n \hookrightarrow \mathbb{R}^n$ induces a bijection on the orbit sets*

$$(\mathbb{R}_{\geq 0})^n / \text{Sym}_n \xrightarrow{\approx} \mathbb{R}^n / \mathcal{T}_n.$$

Proof. Suppose that $z' = z^{(\nu, \sigma)}$. If $z_i \geq 0$ and $z'_i \geq 0$, Formula (2.2) implies that $\nu_i = 1$ only if $z'_i = 0 = (-1)^{\nu_i} z_{\sigma(i)}$, in which case ν_i may be replaced by 0 without changing z' . Hence, $z' = z^{(\nu, \sigma)} = z^\sigma$, which implies that our map is injective. By Formula (2.2) again, each \mathcal{T}_n -orbit contains an element a with $a_i \geq 0$, so the map is also surjective. \square

Example 2.3. *The action of \mathcal{T}_n on Boolean vectors.* We consider another copy of \mathbb{Z}_2^n called \mathcal{B}_n , the set of n -tuples (x_1, \dots, x_n) of Boolean variables. The set \mathcal{B}_1 is thus $\{\text{true}, \text{false}\}$, with its usual numerisation $\text{true} = 1$, $\text{false} = 0$, $\text{xor} = +$, etc. We sometimes use binary strings, e.g. 1010 for $(1, 0, 1, 0)$. The addition law of \mathbb{Z}_2^n produces a right action $\mathcal{B}_n \times \mathcal{A}_n \rightarrow \mathcal{B}_n$ of \mathcal{A}_n on \mathcal{B}_n . Note that the action of 1 on x_i is $(x_i + 1)_{\text{mod} 2} = \bar{x}_i$, the *negation* of x_i ($\bar{0} = 1$ and $\bar{1} = 0$). This is the reason for which an element of \mathcal{A}_n is, in this paper, denoted by $\nu = (\nu_1, \dots, \nu_n)$, the letter ν standing for *negation*. Another useful equality is $\bar{\bar{x}}_i = x_i$ (viewing $\{0, 1\} \subset \mathbb{R}$). This action extends to an action of \mathcal{T}_n on \mathcal{B}_n by the formula

$$(x^{(\nu, \sigma)})_i = x_{\sigma(i)} + \nu_{\sigma(i)}.$$

Example 2.4. *The action of \mathcal{T}_n on $\mathcal{P}(\underline{n})$,* where $\mathcal{P}(\underline{n})$ is the set of subsets of \underline{n} . We use the bijection $\chi: \mathcal{P}(\underline{n}) \rightarrow \mathcal{B}_n$ associating to $J \subset \underline{n}$ its *characteristic* n -tuple $\chi(J)$, whose i -th component is

$$\chi(J)_i = \chi(J)(i) = \text{truth}(i \in J)$$

(i.e. $\chi(J)(i) = 1$ if and only if $i \in J$). Note that $\chi(J) + \chi(K) = \chi(J \Delta K)$, where Δ denotes the symmetric difference. The \mathcal{T}_n -action on $\mathcal{P}(\underline{n})$ is defined so that χ is equivariant, using the \mathcal{T}_n -action of Example 2.3: $\chi(J^{(\nu, \sigma)}) = \chi(J)^{(\nu, \sigma)}$. This amounts to the formulae $J^\nu = J \Delta \chi^{-1}(\nu)$, $J^\sigma = \sigma^{-1}(J)$ and thus

$$J^{(\nu, \sigma)} = \sigma^{-1}(J \Delta \chi^{-1}(\nu)).$$

3 Polygon spaces

In this section, we recall the notations for polygon spaces and their classification (see [7] or [4, § 10.3]). Fix two integers n and d and set $\underline{n} = \{1, 2, \dots, n\}$. For $a = (a_1, \dots, a_n) \in \mathbb{R}^n$, the *polygon space* $\mathcal{N}_d^n(a)$ is defined by

$$\mathcal{N}_d^n(a) = \left\{ z \in (S^{d-1})^n \mid \langle a, z \rangle = 0 \right\} / SO(d), \quad (3.1)$$

where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product in \mathbb{R}^n . Classically, this definition is restricted to $a \in (\mathbb{R}_{>0})^n$, in which case an element of $\mathcal{N}_d^n(a)$ may be visualized as a configuration of n successive segments in \mathbb{R}^d , of length a_1, \dots, a_n , starting and ending at the origin. The vector a is thus called the *length vector*. Following some recent works (see e.g. [2]), we take advantage of Definition (3.1) making sense for $a \in \mathbb{R}^n$. In most of the cases, this extension does not create new polygon spaces up to homeomorphism (see Remark 3.5).

The classification of polygon spaces up to homeomorphism is based on the stratification induced by the *tie hyperplane arrangement* (or just *tie arrangement*) $\mathcal{H}(\mathbb{R}^n)$ in \mathbb{R}^n

$$\mathcal{H}(\mathbb{R}^n) = \{\mathcal{H}_J \mid J \subset \underline{n}\},$$

where the J -tie hyperplane \mathcal{H}_J is defined by

$$\mathcal{H}_J := \left\{ (a_1, \dots, a_n) \in \mathbb{R}^n \mid \sum_{i \in J} a_i = \sum_{i \notin J} a_i \right\}.$$

A tie hyperplane is often called just a *wall*. The stratification associated to $\mathcal{H} = \mathcal{H}(\mathbb{R}^n)$ is defined by the filtration

$$\{0\} = \mathcal{H}^{(0)} \subset \mathcal{H}^{(1)} \subset \dots \subset \mathcal{H}^{(n)} = \mathbb{R}^n,$$

with $\mathcal{H}^{(k)}$ being the subset of those $a \in \mathbb{R}^n$ which belong to at least $n-k$ distinct walls \mathcal{H}_J . A *stratum* of dimension k is a connected component of $\mathcal{H}^{(k)} - \mathcal{H}^{(k-1)}$. Note that a stratum of dimension $k \geq 1$ is an open convex cone in a k -plane of \mathbb{R}^n . Strata of dimension n are called *chambers* and their elements are called *generic*.

Note that the tie arrangement $\mathcal{H}(\mathbb{R}^n)$ is invariant under the action of \mathcal{T}_n . Indeed, using the tools of Example 2.4, one checks that $(\mathcal{H}_J)^{(\nu, \sigma)} = \mathcal{H}_K$ with $K = \sigma(J \triangle \chi^{-1}(\nu))$.

We may restrict the stratification \mathcal{H} to $V = (\mathbb{R}_{>0})^n$, $(\mathbb{R}_{\neq 0})^n$, $(\mathbb{R}_{\geq 0})^n$ or \mathbb{R}^n . Each of these choices for V gives rise to a set $\mathbf{Ch}(V)$ of chambers, contained in the set $\mathbf{Str}(V)$ of corresponding strata.

Proposition 3.1. *The inclusions $(\mathbb{R}_{>0})^n \hookrightarrow (\mathbb{R}_{\geq 0})^n \hookrightarrow \mathbb{R}^n$ descend to bijections*

$$\mathbf{Ch}((\mathbb{R}_{>0})^n)/\mathrm{Sym}_n \xrightarrow{\cong} \mathbf{Ch}((\mathbb{R}_{\geq 0})^n)/\mathrm{Sym}_n \xrightarrow{\cong} \mathbf{Ch}(\mathbb{R}^n)/\mathcal{T}_n$$

and

$$\mathbf{Str}((\mathbb{R}_{\geq 0})^n)/\mathrm{Sym}_n \xrightarrow{\cong} \mathbf{Str}(\mathbb{R}^n)/\mathcal{T}_n.$$

Proof. These bijections are direct consequences of Lemma 2.2, except for the first one $\mathbf{Ch}((\mathbb{R}_{>0})^n)/\mathrm{Sym}_n \rightarrow \mathbf{Ch}((\mathbb{R}_{\geq 0})^n)/\mathrm{Sym}_n$. The latter is obviously injective. For the surjectivity, we use that if $\varepsilon > 0$ is small enough and a is generic, we can replace the zero components of a by ε without leaving the chamber of a (see e.g. [5, § 2.1]). \square

Remarks 3.2. (a) The map $\mathbf{Str}((\mathbb{R}_{>0})^n)/\mathrm{Sym}_n \rightarrow \mathbf{Str}((\mathbb{R}_{\geq 0})^n)/\mathrm{Sym}_n$ is clearly injective but it is not surjective. The stratum $\{(0, \dots, 0)\}$ is of course not in the image but also other less degenerate strata, such as the intersection of the two walls $\mathcal{H}_{\{2\}} \cap \mathcal{H}_{\{3\}}$ in $\mathbf{Str}((\mathbb{R}_{\geq 0})^3)$. Indeed, the equations $a_2 = a_1 + a_3$ and $a_3 = a_1 + a_2$ imply that $a_1 = 0$. See also 7.4

(b) In [7] and in the lists of [8], *conventional representatives* for classes in $\mathbf{Ch}((\mathbb{R}_{>0})^n)/\mathrm{Sym}_n$ are used, allowing zero components in length-vectors. These zeros stand there for any small enough positive numbers (e.g. $(0, 1, 1, 1)$ means

$(\varepsilon, 1, 1, 1)$). This actually uses the first bijection of Proposition 3.1 (see also its proof). Thus, in our new setting, these conventional representatives are bona fide representative of $\mathbf{Ch}((\mathbb{R}_{\geq 0})^n)/\text{Sym}_n$. However, using zero length is unsuitable for the symplectic geometry of spatial polygon spaces (see e.g. [6]).

The main theorem for the classification of polygon spaces [7, Theorem 1.1] generalizes, with the same proof, in the following statement.

Theorem 3.3. *Let $a, a' \in \mathbb{R}^n$. If a and a' are two representatives of the same class in $\mathbf{Str}(\mathbb{R}^n)/\mathcal{T}_n$ then $\mathcal{N}_d^n(a)$ and $\mathcal{N}_d^n(a')$ are homeomorphic. \square*

Remark 3.4. For generic a and certain n and d , the converse of Theorem 3.3 is true: if $\mathcal{N}_d^n(a)$ and $\mathcal{N}_d^n(a')$ are homeomorphic, then a and a' represent the same class in $\mathbf{Ch}(\mathbb{R}^n)/\mathcal{T}_n$ (see, e.g. [1, 11, 12]).

Remark 3.5. By Proposition 3.1 and Theorem 3.3, taking generic length vectors in \mathbb{R}^n instead of $(\mathbb{R}_{>0})^n$ does not produce new polygon spaces up to homeomorphism. This is the same for non-generic length vectors, provided that they are taken in $(\mathbb{R}_{\neq 0})^n$ (if $a \in (\mathbb{R}_{\neq 0})^n$, then $b = a^\nu \in (\mathbb{R}_{>0})^n$ for some $\nu \in \mathcal{A}_n$ and the map $x \mapsto x^\nu$ gives a homeomorphism from $\mathcal{N}_d^n(b)$ to $\mathcal{N}_d^n(a)$). But the non-generic strata of $\mathcal{H}(\mathbb{R}_{\geq 0})^n$ (see Remark 3.2.(a)) produce, in general, new polygon spaces. For example, for $(0, 1, 1) \in \mathcal{H}_{\{2\}} \cap \mathcal{H}_{\{3\}} \in \mathbf{Str}((\mathbb{R}_{\geq 0})^3)$, one has

$$\begin{aligned} \mathcal{N}_3^3(0, 1, 1) &= \{(x, y, z) \in (S^2)^3 \mid y + z = 0\}/SO(3) \\ &\approx (S^2 \times S^2)/SO(3) \\ &\approx pt \times S^2/SO(2) \approx [-1, 1], \end{aligned}$$

which is not homeomorphic to a polygon space $\mathcal{N}_3^3(a)$ for $a \in (\mathbb{R}_{>0})^3$. Indeed, $\mathbf{Str}((\mathbb{R}_{>0})^3)/\text{Sym}_3$ contains 3 strata, giving $\mathcal{N}_3^3(0, 0, 1) = \emptyset$, $\mathcal{N}_3^3(1, 1, 2) = pt$ and $\mathcal{N}_3^3(1, 1, 1) = pt$.

We now restrict ourselves to the generic case. If $a \in \mathbb{R}^n$ is generic then $\sum_{i \in J} a_i \neq \sum_{i \notin J} a_i$ for all $J \subset \underline{n}$. When $\sum_{i \in J} a_i < \sum_{i \notin J} a_i$, the set J is called *a-short* (or just *short*) and its complement is *a-long* (or just *long*). Short subsets form a subset $\text{Sh}(a)$ of $\mathcal{P}(\underline{n})$. Define $\text{Sh}_n(a) = \{J \in \{1, \dots, n-1\} \mid J \cup \{n\} \in \text{Sh}(a)\}$. As J is short if and only if $\bar{J} = \underline{n} - J$ is long, the set $\text{Sh}_n(a)$ determines $\text{Sh}(a)$. Indeed, either $n \in J$ or $n \in \bar{J}$, and thus $\text{Sh}_n(a)$ tells us whether $J \in \text{Sh}(a)$ or $\bar{J} \in \text{Sh}(a)$.

The chamber of a is obviously determined by $\text{Sh}(a)$ and thus by $\text{Sh}_n(a)$. This permits us to characterize $\alpha \in \mathbf{Ch}((\mathbb{R}_{\geq 0})^n)/\mathcal{T}_n$ by a subset $\text{gc}(\alpha)$ of $\mathcal{P}(\underline{n})$, called the *genetic code* of α . For this, we consider, using Proposition 3.1, the only representative a of α such that $0 \geq a_1 \geq \dots \geq a_n$. Define a partial order “ \hookrightarrow ” on $\mathcal{P}(\underline{n})$ by saying that $A \hookrightarrow B$ if and only if there exists a non-decreasing map $\varphi : A \rightarrow B$ such that $\varphi(x) \geq x$. Note that, if B is short, so is A if $A \hookrightarrow B$.

The *genetic code* $\text{gc}(\alpha)$ of α is the set of elements A_1, \dots, A_k of $\text{Sh}_n(a)$ which are maximal with respect to the order “ \hookrightarrow ”. The chamber of a (and thus α) is determined by $\text{gc}(\alpha)$. We also use the notation $\text{gc}(a)$ for $\text{gc}(\alpha)$.

For instance $\text{gc}((0, 0, 1)) = \{\emptyset\}$, $\text{gc}((0, 1, 1, 1)) = \{\{4, 1\}\}$, $\text{gc}((1, 1, 2, 3, 3, 5)) = \{\{6, 2, 1\}, \{6, 3\}\}$, etc (see [7]).

An algorithm is designed in [7] to list all the possible genetic code. A first step is to observe that all $A, B \in \text{gc}(\alpha)$ satisfy

- (a) $A \not\leftrightarrow B$ if $A \neq B$ and
- (b) $\bar{A} \not\leftrightarrow A$.

Indeed, Condition (a) holds true by maximality of A and, if $\bar{A} \leftrightarrow A$, then A would be both short and long which is impossible. A finite set $\{A_1, \dots, A_k\} \subset \mathcal{P}_m(\underline{n})$, satisfying Conditions (a) and (b) is called a *virtual genetic code* (of type n).

4 Self-dual threshold functions

Fix a positive integer n . We use the set \mathcal{B}_n of Boolean vectors with its \mathcal{T}_n -action introduced in Example 2.3. A Boolean function on n variables is a map $\mathcal{B}_n \rightarrow \mathcal{B}_1 = \mathbb{Z}_2$. The group \mathcal{T}_n acts on the right on the set of Boolean functions by $f^{(\nu, \sigma)}(x) = f(x^{(\nu, \sigma)^{-1}})$, which gives the formula

$$f^{(\nu, \sigma)}(x_1, \dots, x_n) = (x_{\sigma^{-1}(1)} + \nu_1, \dots, x_{\sigma^{-1}(n)} + \nu_n). \quad (4.1)$$

This \mathcal{T}_n -action on the set of Boolean functions produces the equivalence relation used in 1.2.

A Boolean function f is called *self-dual* if $f(\bar{x}) = \overline{f(x)}$.

Lemma 4.1. *Self-duality is preserved by the action of \mathcal{T}_n .*

Proof. Let $f: \mathcal{B}_n \rightarrow \mathcal{B}_1$ be a Boolean function which is self-dual and let $(\nu, \sigma) \in \mathcal{T}_n$. Note that $\bar{x} = x^{(\mathbf{1}, \text{id})}$ where $\mathbf{1} = (1, \dots, 1)$. Then,

$$\begin{aligned} f^{(\nu, \sigma)}(\bar{x}) &= f^{(\nu, \sigma)}(x^{(\mathbf{1}, \text{id})}) \\ &= f(x^{(\mathbf{1}, \text{id})(\nu, \sigma)^{-1}}) \\ &= f(x^{(\nu, \sigma)^{-1}(\mathbf{1}, \text{id})}) \quad \text{since } (\mathbf{1}, \text{id}) \text{ is in the center of } \mathcal{T}_n \\ &= \overline{f(x^{(\nu, \sigma)^{-1}})} \\ &= \overline{f(x^{(\nu, \sigma)^{-1}})} \quad \text{since } f \text{ is self-dual} \\ &= \overline{f^{(\nu, \sigma)}(x)}, \end{aligned}$$

which proves that $f^{(\nu, \sigma)}$ is self-dual. □

Let $w = (w_1, \dots, w_n) \in \mathbb{R}^n$ and $t \in \mathbb{R}$. We consider the Boolean function

$$f_{(w, t)}(x) = \text{truth}(\langle x, w \rangle \geq t) = \begin{cases} 1 & \text{if } \langle x, w \rangle \geq t \\ 0 & \text{otherwise,} \end{cases}$$

where \langle, \rangle denotes the standard scalar product in \mathbb{R}^n . The function $f_{(w,t)}$ is called the *threshold function* with *weights* w_1, w_2, \dots, w_n and *threshold* t (see [3] and Remark 4.6 below). Reference [3] emphasize the importance of threshold functions in neuron-like systems. The few subsequent lemmas gather several properties of threshold functions.

Lemma 4.2. *Threshold functions are preserved by the action of \mathcal{T}_n . More precisely, let $(w, t) \in \mathbb{R}^n \times \mathbb{R}$ and let $(\nu, \sigma) \in \mathcal{T}_n$. Then $f_{(w,t)}^{(\nu,\sigma)} = f_{(w',t')}$ where*

$$w' = w^{(\nu,\sigma)} \quad \text{and} \quad t' = t - \langle \nu, w^\sigma \rangle.$$

Proof. As we are dealing with an action of \mathcal{T}_n , it is enough to prove the lemma for $\sigma = (0, \sigma)$ and for $\nu = (\nu, \text{id})$. In the first case, one has

$$f_{(w,t)}^\sigma(x) = \text{truth}(\langle x^{\sigma^{-1}}, w \rangle \geq t) = \text{truth}(\langle x, w^\sigma \rangle \geq t) = f_{(w^\sigma, t)}.$$

In the other case, one first prove, using the truth tables of x_i and ν_i , that

$$(x_i)^{\nu_i} w_i = x_i (-1)^{\nu_i} w_i + \nu_i w_i.$$

This implies that $\langle x^\nu, w \rangle = \langle x, w^\nu \rangle + \langle \nu, w \rangle$. Therefore,

$$\begin{aligned} f_{(w,t)}^\nu(x) &= f_{(w,t)}(x^\nu) \\ &= \text{truth}(\langle x^\nu, w \rangle \geq t) \\ &= \text{truth}(\langle x, w^\nu \rangle \geq t - \langle \nu, w \rangle) \\ &= f_{(w',t)}(x). \end{aligned}$$

□

For $w \in \mathbb{R}^n$, define $\lfloor w \rfloor = \frac{1}{2} \sum_{i=1}^n w_i$. The relationship between threshold and self-dual functions is the following.

Lemma 4.3. *A threshold function $f_{(w,t)}$ is self-dual if and only if the following two conditions hold.*

- (a) $f_{(w,t)} = f_{(w, \lfloor w \rfloor)}$ and
- (b) $\langle x, w \rangle \neq \lfloor w \rfloor$ for any $x \in \mathcal{B}_n$.

Proof. Conditions (a) and (b) are clearly sufficient for $f_{(w,t)}$ being self-dual. Conversely, if $f = f_{(w,t)}$ is self-dual, then

$$\langle x, w \rangle \geq t \Leftrightarrow f(x) = 1 \Leftrightarrow f(\bar{x}) = 0 \Leftrightarrow \langle \bar{x}, w \rangle < t.$$

This, together with the same argument exchanging x and \bar{x} , proves that

$$\langle x, w \rangle < \langle \bar{x}, w \rangle \quad \text{or} \quad \langle x, w \rangle > \langle \bar{x}, w \rangle$$

for all $x \in \mathcal{B}_n$. As $\langle x, w \rangle + \langle \bar{x}, w \rangle = 2\lfloor w \rfloor$, this proves (a) and (b). □

Lemma 4.4. *Let $w \in \mathbb{R}^n$ and $(\nu, \sigma) \in \mathcal{T}_n$. Then $f_{(w, \lfloor w \rfloor)}^{(\nu, \sigma)} = f_{(w', \lfloor w' \rfloor)}$ (where w' is given by Lemma 4.2).*

Proof. As we are dealing with an action of \mathcal{T}_n , it is enough to prove the lemma for $(0, \sigma)$ and (ν, id) . The case $(0, \sigma)$ is straightforward by Lemma 4.2 and the equality $\lfloor w^{\sigma^{-1}} \rfloor = \lfloor w \rfloor$. For (ν, id) , Lemma 4.2 again tells us that $f_{(w, \lfloor w \rfloor)}^{(\nu, \text{id})} = f_{(w', t')}$ with $w' = w^\nu$ and

$$t' = \lfloor w \rfloor - \langle \nu, w \rangle = \frac{1}{2} \left(\sum w^i - 2 \langle \nu, w \rangle \right) = \lfloor w^\nu \rfloor. \quad \square$$

We are ready to prove the main result of this section. To a generic $a \in \mathbb{R}^n$, we associate the threshold function $f_{(a, \lfloor a \rfloor)}$, which is self dual by Lemma 4.3. If a and a' belong to the same chamber of $\mathcal{H}(\mathbb{R}^n)$, then

$$f_{(a, \lfloor a \rfloor)}^{-1}(\{0\}) = \text{Sh}(a) = \text{Sh}(a') = f_{(a', \lfloor a' \rfloor)}^{-1}(\{0\}),$$

which proves that $f_{(a, \lfloor a \rfloor)} = f_{(a', \lfloor a' \rfloor)}$. We thus get a map

$$\tilde{\Xi} : \mathbf{Ch}(\mathbb{R}^n) \rightarrow \mathbf{SDT}(n),$$

where $\mathbf{SDT}(n)$ denotes the set of self-dual threshold functions on \mathcal{B}_n .

Proposition 4.5. *The above map $\tilde{\Xi}$ descends to a bijection*

$$\Xi : \mathbf{Ch}(\mathbb{R}^n)/\mathcal{T}_n \xrightarrow{\cong} \mathbf{SDT}(n)/\mathcal{T}_n.$$

Proof. By Lemma 4.4, the map $\tilde{\Xi}$ is \mathcal{T}_n -equivariant, so the orbit map Ξ is well defined. It is surjective by Lemma 4.3. For the injectivity, let α and β be two chambers, represented by length vectors a and b . If $\Xi(\alpha) = \Xi(\beta)$, then $f_{(b, \lfloor b \rfloor)} = f_{(a, \lfloor a \rfloor)}^{(\nu, \sigma)}$ for some $(\nu, \sigma) \in \mathcal{T}_n$. By Lemma 4.4, one has $f_{(b, \lfloor b \rfloor)} = f_{(a^{(\nu, \sigma)}, \lfloor a^{(\nu, \sigma)} \rfloor)}$. Therefore

$$\text{Sh}(a^{(\nu, \sigma)}) = f_{(a^{(\nu, \sigma)}, \lfloor a^{(\nu, \sigma)} \rfloor)}^{-1}(\{0\}) = f_{(b, \lfloor b \rfloor)}^{-1}(\{0\}) = \text{Sh}(b),$$

which implies that $\beta = \alpha^{(\nu, \sigma)}$. □

Remarks 4.6. (a) There are several variants in the literature for the definition of a threshold function, for instance by requiring that w_i and/or t be integers (see e.g. [10, p. 75]). As chambers contain integral representative, Proposition 4.5 holds true as well for these versions.

(b) Variables corresponding to zero weights are idle for $f_{(a, \lfloor a \rfloor)}$, so the latter depends on fewer than n variables. This is the reason of the words “ n or fewer variables” in 1.1.

(c) Composing the bijection Ξ of Proposition 4.5 with the bijection $\mathbf{Ch}((\mathbb{R}_{>0})^n)/\text{Sym}_n \xrightarrow{\cong} \mathbf{Ch}(\mathbb{R}^n)/\mathcal{T}_n$ of Proposition 3.1, produces a bijection

$$\mathbf{Ch}((\mathbb{R}_{>0})^n)/\text{Sym}_n \xrightarrow{\cong} \mathbf{SDT}(n)/\mathcal{T}_n. \quad (4.2)$$

This explains why the numbers of the first line of the table in the introduction are equal to those of [14].

By Proposition 3.1, the bijection of (4.2) factors through the bijection $\mathbf{Ch}((\mathbb{R}_{>0})^n)/\mathrm{Sym}_n \xrightarrow{\cong} \mathbf{SDT}(n)/\mathcal{T}_n$. A direct consequence is the following lemma, which will be useful later.

Lemma 4.7. *Let (w, t) and (w', t') be two elements of $\mathbb{R}^n \times \mathbb{R}$. Suppose that w and w' are generic and that $f_{(w,t)}$ and $f_{(w',t')}$ are in the same \mathcal{T}_n -orbit. If $w_i \geq 0$ and $w'_i \geq 0$ for all $i \in \underline{n}$, then w and w' are in the same Sym_n -orbit.*

5 Decisive weighted majority games

In this section, we describe the equivalence between $\mathbf{Ch}((\mathbb{R}_{\geq 0})^n)/\mathrm{Sym}_n$ and the strategic equivalence classes of decisive weighted majority games with n players, as mentioned in 1.1. Our references for game theory are [13, Chapter 10] and [9].

A *game* on a set of n players (indexed by \underline{n}) is a set of subsets of \underline{n} called the *winning sets*, such that any set containing a winning set is also winning. The game is *decisive* (or *simple*) if for any $S \subset \underline{n}$, either S or its complement is winning but not both. Two games are *strategically equivalent* if there is a bijection between their players identifying the families of winning sets. A game with n players may be extended to $m > n$ players by adding $m - n$ “voteless” players or *dummies*: a subset S of \underline{m} is thus winning if and only if $S \cap \underline{n}$ is winning.

We see that a game \mathcal{G} defines and is determined by a Boolean function $f_{\mathcal{G}}: \mathcal{B}_n \rightarrow \mathcal{B}_1$, given by $f(x) = 1$ if and only if x is the characteristic n -tuple of a set $S \subset \underline{n}$ which is a winning set. As any superset of a winning set wins, the function $f_{\mathcal{G}}$ is *monotone*, i.e. its value does not change from 1 to 0 when any of its variables changes from 0 to 1 [10, p. 55]. That \mathcal{G} is decisive translates into $f_{\mathcal{G}}$ being self-dual. Two games \mathcal{G} and \mathcal{G}' are strategically equivalent if and only if $f_{\mathcal{G}}$ and $f_{\mathcal{G}'}$ are in the same Sym_n -orbit.

A game \mathcal{G} is a *weighted majority game* if there exists $w \in \mathbb{R}^n$ such that $f_{\mathcal{G}} = f_{(w, \lfloor w \rfloor)}$. We write $\mathcal{G} = \mathcal{G}(w)$. If $w_i \leq 0$, then the player i is a dummy. Indeed, since every superset of a winning set wins, $w_i \leq 0$ implies that $|w_i|$ is so small that it makes no difference. Therefore, one can replace the negative weights by 0 without changing the strategic equivalence class of the game. We can thus suppose that no weight is negative. By Lemma 4.7, two decisive majority games $\mathcal{G}(w)$ and $\mathcal{G}(w')$ are then strategically equivalent if and only if w and w' are in the same Sym_n -orbit.

Note that if $a \in (\mathbb{R}_{\geq 0})^n$ is a length vector, the winning set of \mathcal{G}_a are the a -long subsets of \underline{n} . Also, a is generic if and only if \mathcal{G}_a is decisive. The above considerations, together with Propositions 3.1 and 4.5, gives the following result.

Proposition 5.1. *The map $a \mapsto \mathcal{G}(a)$ induces a bijection from $\mathbf{Ch}((\mathbb{R}_{\geq 0}^n))/\mathrm{Sym}_n$ to the set of strategic equivalence classes of decisive weighted majority games.*

6 Self-dual regular Boolean functions

A partial order on \mathcal{B}_n is defined by saying that $x \preceq y$ if $x_1 + \cdots + x_k \leq y_1 + \cdots + y_k$ for $1 \leq k \leq n$ [10, p. 92]. Note that $x \preceq y$ if and only if $\bar{y} \preceq \bar{x}$: indeed, $x_1 + \cdots + x_k \leq y_1 + \cdots + y_k$ if and only if $k - x_1 - \cdots - x_k \geq k - y_1 - \cdots - y_k$.

A Boolean $f: \mathcal{B}_n \rightarrow \mathcal{B}_1$ is *regular* if $f(x) \preceq f(y)$ whenever $x \preceq y$ [10, p. 93]. For example, a threshold function $f_{(w,t)}$ is regular if $w_1 \geq w_2 \geq \cdots \geq w_n \geq 0$. Let $\mathbf{SDR}(n)$ be the set of self dual regular Boolean functions on \mathcal{B}_n .

Proposition 6.1. *There is a bijection between $\mathbf{SDR}(n)$ and the set of virtual genetic codes (see Section 3).*

Before proving Proposition 6.1, we note the following lemma, in which $\mathcal{B}_n^1 = \{x \in \mathcal{B}_n \mid x_1 = 1\}$.

Lemma 6.2. *A self-dual Boolean $f: \mathcal{B}_n \rightarrow \mathcal{B}_1$ is determined by its restriction to \mathcal{B}_n^1 .*

Proof. We use that $x \in \mathcal{B}_n^1$ if and only if $\bar{x} \notin \mathcal{B}_n^1$. As f is self-dual, one has

$$f(x) = \begin{cases} f(x) & \text{if } x \in \mathcal{B}_n^1 \\ \overline{f(\bar{x})} & \text{otherwise.} \end{cases} \quad (6.1)$$

□

Proof of Proposition 6.1. To $f \in \mathbf{SDR}(n)$, we associate its *code* $\gamma(f)$ which is a subset of \mathcal{B}_n^1 . By definition, $\gamma(f)$ is the set of \preceq -maximal elements $x \in \mathcal{B}_n^1$ for which $f(x) = 0$. For instance $\gamma(f_{((2,1,1,1),5/2)}) = \{1000\}$ and $\gamma(f_{((2,2,2,1),7/2)}) = \{1001\}$. If $\gamma(f) = \{b_1, \dots, b_k\}$, then

- (i) $b_i \not\preceq b_j$ for all $i \neq j$ and
- (ii) $\bar{b}_i \not\preceq b_j$ for all i, j .

Let Γ_n be the set of subsets of \mathcal{B}_n^1 satisfying (i) and (ii). We first establish that $\gamma: \mathbf{SDR}(n) \rightarrow \Gamma_n$ is a bijection. Indeed, $\gamma(f)$ clearly determines the restriction of f to \mathcal{B}_n^1 , and then determines f by Lemma 6.2; this proves that γ is injective. For the surjectivity, let $R \in \Gamma_n$. The formula

$$f(x) = \begin{cases} 0 & \text{if } \exists r \in R \text{ with } x \preceq r \\ 1 & \text{otherwise} \end{cases}$$

defines a function on \mathcal{B}_n^1 which can be extended to $f: \mathcal{B}_n \rightarrow \mathcal{B}_1$ by (6.1). Such a definition guarantees that f is self-dual. For the regularity, let $x \preceq y$ be two elements in \mathcal{B}_n . The condition $f(x) \preceq f(y)$ is automatic if $f(y) = 1$. We can thus assume that $f(y) = 0$, so we must prove that $f(x) = 0$. There are four cases.

Case 1: $x_1 = 1 = y_1$. As $f(y) = 0$, there exists $r \in R$ with $y \preceq r$. As $x \preceq y$, then $x \preceq r$ and thus $f(x) = 0$.

Case 2: $x_1 < y_1$. If $f(x) = 1$, then $\bar{x} \preceq r$ for some $r \in R$. As $f(y) = 0$, there exists $s \in R$ with $y \preceq s$. Therefore, $\bar{r} \preceq x \preceq y \preceq s$ which contradicts (ii).

Case 3: $x_1 > y_1$. This is impossible since $x \preceq y$.

Case 4: $x_1 = 0 = y_1$. If $f(x) = 1$, then $f(\bar{x}) = 0$. As $f(\bar{y}) = 1$ and $\bar{y} \preceq \bar{x}$, the pair (\bar{y}, \bar{x}) would contradict Case 1, already established.

It remains to establish a bijection from Γ_n and the set of virtual genetic codes. To $x \in \mathcal{B}_n^1$ one associates $x^\sharp \subset \underline{n}$ by the rule

$$x^\sharp = \{i \in \underline{n} \mid x_{n+1-i} = 1\}.$$

For instance, $(1000)^\sharp = \{4\}$ while $(1010)^\sharp = \{2, 4\}$. Obviously, $x \mapsto x^\sharp$ is a bijection between \mathcal{B}_n^1 and the subsets of \underline{n} containing n . Conditions (i) and (ii) above are intertwined with Conditions (a) and (b) of [7, p. 37]. The latter define a virtual genetic code. Hence, the correspondence $x \mapsto x^\sharp$ maps Γ_n bijectively to the set of virtual genetic codes. \square

Remark 6.3. For $n \geq 9$, not every self-dual regular Boolean function is equivalent to a threshold function. As an example, the function with $\gamma(f) = \{100101010\}$, corresponding to the genetic code $\{9, 6, 4, 2\}$ (see [7, Lemma 4.5]).

7 Non-generic strata

In this section, we give an analogue of the bijection Ξ of Proposition 4.5, extended to possibly non-generic strata, taking advantage of 3-valued (3V) Boolean functions. A *3V-Boolean function* is a map $f: \mathcal{B}_n \rightarrow \{-1, 0, 1\}$. It is *self-dual* if $f(\bar{x}) = -f(x)$. As in Section 4, a right \mathcal{T}_n -action on the set of 3V-Boolean functions using (4.1). As in Lemma 4.1, one proves that self-duality is preserved by this \mathcal{T}_n -action.

Let $w = (w_1, \dots, w_n) \in \mathbb{R}^n$ and $t \in \mathbb{R}$. The 3V-Boolean function

$$f_{(w,t)}^{3V}(x) = \begin{cases} 1 & \text{if } \langle x, w \rangle > t \\ 0 & \text{if } \langle x, w \rangle = t \\ -1 & \text{if } \langle x, w \rangle < t \end{cases}$$

is called the *3V-threshold function* with *weights* w_1, w_2, \dots, w_n and *threshold* t . With essentially the same proofs, Lemma 4.1–4.7 remain valid without change for 3V-threshold functions, except for Lemma 4.3 which requires the hypothesis $0 \notin \text{Image}(f_{(w,t)}^{3V})$. If this is not the case, one easily proves the following lemma.

Lemma 7.1. *Let $(w, t) \in \mathbb{R}^n \times \mathbb{R}$ such that $0 \in \text{Image}(f_{(w,t)}^{3V})$. Then, $f_{(w,t)}^{3V}$ is self-dual if and only if $t = \lfloor w \rfloor$. \square*

Let $\mathbf{SDT}^{3V}(n)$ be the set of self-dual 3V-threshold functions on \mathcal{B}_n . As in Section 4, we define a map $\tilde{\Xi}^{3V}: \mathbf{Str}(\mathbb{R}^n) \rightarrow \mathbf{SDT}^{3V}(n)$ by associating to $S \in \mathbf{Str}(\mathbb{R}^n)$ the 3V-threshold function $f_{(a, \lfloor a \rfloor)}^{3V}$ for $a \in S$. We check that $\tilde{\Xi}^{3V}$ is well defined and \mathcal{T}_n -equivariant, thus inducing a map $\tilde{\Xi}^{3V}: \mathbf{Str}(\mathbb{R}^n)/\mathcal{T}_n \rightarrow \mathbf{SDT}^{3V}(n)/\mathcal{T}_n$. The same proof as for Proposition 4.5 gives following proposition.

Proposition 7.2. *The map $\Xi^{3V} : \mathbf{Str}(\mathbb{R}^n)/\mathcal{T}_n \rightarrow \mathbf{SDT}^{3V}(n)/\mathcal{T}_n$ is a bijection.*
 \square

Remark 7.3. The relationship between the maps Ξ and Ξ^{3V} of Propositions 4.5 and 7.2 is as follows. There is an obvious injection $j : \mathbf{SDT}(n)/\mathcal{T}_n \rightarrow \mathbf{SDT}^{3V}(n)/\mathcal{T}_n$ induced by $f \mapsto \epsilon \circ f$ where $\epsilon(u) = (-1)^u$. Its image is the set of 3V-Boolean functions f such that $0 \notin \text{Image}(f)$. One has a commutative diagram

$$\begin{array}{ccc} \mathbf{Ch}(\mathbb{R}^n)/\mathcal{T}_n & \xrightarrow{\quad} & \mathbf{Str}(\mathbb{R}^n)/\mathcal{T}_n \\ \Xi \downarrow \approx & & \approx \downarrow \Xi^{3V} \\ \mathbf{SDT}(n)/\mathcal{T}_n & \xrightarrow{j} & \mathbf{SDT}^{3V}(n)/\mathcal{T}_n \end{array} .$$

7.4. Computing the number of strata. Consider the following numbers

- $c(n) = \sharp(\mathbf{Ch}(\mathbb{R}^n)/\mathcal{T}_n)$
- $k(n) = \sharp(\mathbf{Str}((\mathbb{R}_{\neq 0})^n)/\mathcal{T}_n)$
- $tk(n) = \sharp(\mathbf{Str}(\mathbb{R}^n)/\mathcal{T}_n)$.

For example, $c(1) = 0$ and $k(1) = tk(1) = 1$ (the stratum of (0)). For $n = 2$, one has $c(2) = 1$ (the chamber of $(0, 1)$), $k(2) = 2$ (the previous chamber and the stratum of $(1, 1)$), while $tk(2) = 3$ because the stratum of (0) in $\mathbf{Str}(\mathbb{R}^1)$ gives rise to that of $(0, 0)$ in $\mathbf{Str}(\mathbb{R}^2)$. In general, the injection $\mathbb{R}^{n-1} \approx \{0\} \times \mathbb{R}^{n-1} \hookrightarrow \mathbb{R}^n$ induces an injection $[\mathbf{Str}(\mathbb{R}^{n-1}) - \mathbf{Ch}(\mathbb{R}^{n-1})]/\mathcal{T}_{n-1} \hookrightarrow \mathbf{Str}(\mathbb{R}^n)/\mathcal{T}_n$. This proves the recursion formula

$$tk(n) = k(n) + tk(n-1) - c(n-1). \quad (7.1)$$

The number $k(n)$ was computed in [7, § 5] for $n \leq 8$. Thanks to Proposition 3.1, the values of $c(n)$ may be taken from the table in the introduction. Using Formula (7.1), we thus get the following table.

n	1	2	3	4	5	6	7	8
$c(n)$	0	1	2	3	7	21	135	2,470
$k(n)$	1	2	3	7	21	117	1506	62254
$tk(n)$	1	3	5	10	28	138	1623	63742

The sequences $k(n)$ and $tk(n)$ do not seem to occur in the *On-Line Encyclopedia of Integer Sequences*.

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Jean-Claude HAUSMANN
 Mathématiques – Université
 B.P. 64, CH–1211 Geneva 4, Switzerland
jean-claude.hausmann@unige.ch