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On Some Quadratic Algebras I $\frac{1}{2}$: Combinatorics of Dunkl and Gaudin Elements, Schubert, Grothendieck, Fuss–Catalan, Universal Tutte and Reduced Polynomials

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Abstract. We study some combinatorial and algebraic properties of certain quadratic algebras related with dynamical classical and classical Yang–Baxter equations.

Key words: braid and Yang–Baxter groups; classical and dynamical Yang–Baxter relations; classical Yang–Baxter, Kohno–Drinfeld and 3-term relations algebras; Dunkl, Gaudin and Jucys–Murphy elements; small quantum cohomology and K-theory of flag varieties; Pieri rules; Schubert, Grothendieck, Schröder, Ehrhart, Chromatic, Tutte and Betti polynomials; reduced polynomials; Chan–Robbins–Yuen polytope; k-dissections of a convex $(n+k+1)$-gon, Lagrange inversion formula and Richardson permutations; multiparameter deformations of Fuss–Catalan and Schröder polynomials; Motzkin, Riordan, Fine, poly-Bernoulli and Stirling numbers; Euler numbers and Brauer algebras; VSASM and CSTCPP; Birman–Ko–Lee monoid; Kronecker elliptic sigma functions

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To the memory of Alain Lascoux 1944–2013, the great Mathematician, from whom I have learned a lot about the Schubert and Grothendieck polynomials.

Contents

Extended abstract

We introduce and study a certain class of quadratic algebras, which are nonhomogeneous in general, together with the distinguish set of mutually commuting elements inside of each, the so-called *Dunkl elements*. We describe relations among the Dunkl elements in the case of a family of quadratic algebras corresponding to a certain splitting of the *universal classical Yang–Baxter relations* into two *three term relations*. This result is a further extension and generalization of analogous results obtained in [45, 117] and [76]. As an application we describe explicitly the set of relations among the Gaudin elements in the group ring of the symmetric group, cf. [108]. We also study relations among the Dunkl elements in the case of (nonhomogeneous) quadratic algebras related with the *universal dynamical classical Yang–Baxter relations*. Some relations of results obtained in papers [45, 72, 75] with those obtained in [54] are pointed out. We also identify a subalgebra generated by the generators corresponding to the *simple roots* in the extended Fomin–Kirillov algebra with the DAHA, see Section 4.3.

The set of generators of algebras in question, naturally corresponds to the set of edges of the *complete graph* K_n (to the set of edges and loops of the complete graph with (simple) loops $\widetilde{K}_n$ in *dynamical* and equivariant cases). More generally, starting from any subgraph Γ of the complete graph with simple loops $\widetilde{K}_n$ we define a (graded) subalgebra $3T_n^{(0)}(\Gamma)$ of the (graded) algebra $3T_n^{(0)}(\widetilde{K}_n)$ [70]. In the case of loop-less graphs $\Gamma \subset K_n$ we state *conjecture*, Conjecture 4.15 in the main text, which relates the *Hilbert polynomial* of the abelian quotient $3T_n^{(0)}(\Gamma)^{ab}$ of the algebra $3T_n^{(0)}(\Gamma)$ and the *chromatic polynomial* of the graph Γ we are started with[1,2]. We check

[1]We *expect* that a similar conjecture is true for any finite (oriented) matroid $\mathcal{M}$. Namely, one (A.K.) can define an analogue of the three term relations algebra $3T^{(0)}(\mathcal{M})$ for any (oriented) matroid $\mathcal{M}$. We *expect* that the abelian quotient $3T^{(0)}(\mathcal{M})^{ab}$ of the algebra $3T^{(0)}(\mathcal{M})$ is isomorphic to the *Orlik–Terao* algebra [114], denoted by $\mathrm{OT}(\mathcal{M})$ (known also as *even* version of the Orlik–Solomon algebra, denoted by $\mathrm{OS}^+(\mathcal{M})$) associated with matroid $\mathcal{M}$ [28]. Moreover, the anticommutative quotient of the *odd* version of the algebra $3T^{(0)}(\mathcal{M})$, as we *expect*, is isomorphic to the Orlik–Solomon algebra $\mathrm{OS}(\mathcal{M})$ associated with matroid $\mathcal{M}$, see, e.g., [11, 49]. In particular,

$$\operatorname{Hilb}\big(3T^{(0)}(\mathcal{M})^{ab}, t\big) = t^{r(\mathcal{M})}\operatorname{Tutte}\big(\mathcal{M}; 1+t^{-1}, 0\big).$$

We *expect* that the Tutte polynomial of a matroid, $\operatorname{Tutte}(\mathcal{M}, x, y)$, is related with the Betti polynomial of a matroid $\mathcal{M}$. Replacing relations $u_{ij}^2 = 0$, $\forall\, i,j$, in the definition of the algebra $3T^{(0)}(\Gamma)$ by relations $u_{ij}^2 = q_{ij}$, $\forall\, i,j$, $(i,j) \in E(\Gamma)$, where $\{q_{ij}\}_{(i,j)\in E(\Gamma)}$, $q_{ij} = q_{ji}$, is a collection of *central* elements, give rise to a *quantization* of the Orlik–Terao algebra $\mathrm{OT}(\Gamma)$. It seems an interesting *task* to clarify combinatorial/geometric significance of *noncommutative* versions of Orlik–Terao algebras (as well as Orlik–Solomon ones) defined as follows: $\mathcal{OT}(\Gamma) := 3T^{(0)}(\Gamma)$, its "quantization" $3T^{(\boldsymbol{q})}(\Gamma)^{ab}$ and K-theoretic analogue $3T^{(\boldsymbol{q})}(\Gamma, \beta)^{ab}$, cf. Definition 3.1, in the theory of hyperplane arrangements. *Note* that a small modification of arguments in [89] as were used for the proof of our Conjecture 4.15, gives rise to a theorem that the algebra $3T_n(\Gamma)^{ab}$ is isomorphic to the Orlik–Terao algebra $\mathrm{OT}(\Gamma)$ studied in [126].

[2]In the case of simple graphs our Conjecture 4.15 has been proved in [89].

our conjecture for the complete graphs K_n and the complete bipartite graphs $K_{n,m}$. Besides, in the case of *complete multipartite graph* $K_{n_1,\ldots,n_r}$, we identify the commutative subalgebra in the algebra $3T_N^{(0)}(K_{n_1,\ldots,n_r})$, $N = n_1 + \cdots + n_r$, generated by the elements

$$\theta_{j,k_j}^{(N)} := e_{k_j}\big(\theta_{N_{j-1}+1}^{(N)}, \ldots, \theta_{N_j}^{(N)}\big),$$
$$1 \le j \le r, \quad 1 \le k_j \le n_j, \quad N_j := n_1 + \cdots + n_j, \quad N_0 = 0,$$

with the cohomology ring $H^*(\mathcal{F}l_{n_1,\ldots,n_r}, \mathbb{Z})$ of the partial flag variety $\mathcal{F}l_{n_1,\ldots,n_r}$. In other words, the set of (additive) Dunkl elements $\big\{\theta_{N_{j-1}+1}^{(N)}, \ldots, \theta_{N_j}^{(N)}\big\}$ plays a role of the *Chern roots* of the tautological vector bundles ξ_j, $j = 1, \ldots, r$, over the partial flag variety $\mathcal{F}l_{n_1,\ldots,n_r}$, see Section 4.1.2 for details. In a similar fashion, the set of *multiplicative* Dunkl elements $\big\{\Theta_{N_{j-1}+1}^{(N)}, \ldots, \Theta_{N_j}^{(N)}\big\}$ plays a role of the *K-theoretic version of Chern roots* of the tautological vector bundle ξ_j over the partial flag variety $\mathcal{F}l_{n_1,\ldots,n_r}$. As a byproduct for a given set of weights $\boldsymbol{\ell} = \{\ell_{ij}\}_{1 \le i < j \le r}$ we compute the *Tutte polynomial* $T(K_{n_1,\ldots,n_k}^{(\boldsymbol{\ell})}, x, y)$ of the $\boldsymbol{\ell}$-weighted complete multipartite graph $K_{n_1,\ldots,n_k}^{(\boldsymbol{\ell})}$, see Section 4, Definition 4.4 and Theorem 4.3. More generally, we introduce *universal Tutte polynomial*

$$T_n(\{q_{ij}\}, x, y) \in \mathbb{Z}[\{q_{ij}\}][x, y]$$

in such a way that for any collection of non-negative integers $\boldsymbol{m} = \{m_{ij}\}_{1 \le i < j \le n}$ and a subgraph $\Gamma \subset K_n^{(\boldsymbol{m})}$ of the weighted complete graph on n labeled vertices with each edge $(i,j) \in K_n^{(\boldsymbol{m})}$ appears with multiplicity m_{ij}, the specialization

$$q_{ij} \longrightarrow 0 \quad \text{if edge} \quad (i,j) \notin \Gamma, \qquad q_{ij} \longrightarrow [m_{ij}]_y := \frac{y^{m_{ij}} - 1}{y - 1} \quad \text{if edge} \quad (i,j) \in \Gamma$$

of the universal Tutte polynomial is equal to the Tutte polynomial of graph Γ multiplied by $(x-1)^{\kappa(\Gamma)}$, see Section 4.1.2, Theorem 4.24, and *comments and examples*, for details.

We also introduce and study a family of (*super*) 6-*term relations* algebras, and suggest a definition of "multiparameter quantum deformation" of the algebra of the curvature of 2-forms of the Hermitian linear bundles over the complete flag variety $\mathcal{F}l_n$. This algebra can be treated as a natural generalization of the (multiparameter) quantum cohomology ring $QH^*(\mathcal{F}l_n)$, see Section 4.2. In a similar fashion as in the case of three term relations algebras, for any subgraph $\Gamma \subset K_n$, one (A.K.) can also define an algebra $6T^{(0)}(\Gamma)$ and projection[3]

$$\mathrm{Ch}\colon\ 6T^{(0)}(\Gamma) \longrightarrow 3T^{(0)}(\Gamma).$$

Note that subalgebra $\mathcal{A}(\Gamma) := \mathbb{Q}[\theta_1, \ldots, \theta_n] \subset 6T^{(0)}(\Gamma)^{ab}$ generated by additive Dunkl elements

$$\theta_i = \sum_{\substack{j \\ (ij) \in E(\Gamma)}} u_{ij}$$

is closely related with problems have been studied in [118, 129], ..., and [137] in the case $\Gamma = K_n$, see Section 4.2.2. We want to draw attention of the reader to the following *problems* related with *arithmetic Schubert*[4] *and Grothendieck calculi*:

(i) Describe (natural) quotient $6T^\dagger(\Gamma)$ of the algebra $6T^{(0)}(\Gamma)$ such that the natural epimorphism $\mathrm{pr}\colon \mathbb{A}(\Gamma) \longrightarrow \mathcal{A}(\Gamma)$ turns out to be *isomorphism*, where we denote by $\mathbb{A}(\Gamma)$ a subalgebra of $6T^\dagger(\Gamma)$ generated over $\mathbb{Q}$ by additive Dunkl elements.

[3] We treat this map as an algebraic version of the homomorphism which sends the curvature of a Hermitian vector bundle over a smooth algebraic variety to its cohomology class, as well as a splitting of classical Yang–Baxter relations (that is six term relations) in a couple of three term relations.

[4] See for example [137] and the literature quoted therein.

(ii) It is not difficult to see [72] that *multiplicative* Dunkl elements $\{\Theta_i\}_{1\le i\le n}$ also mutually commute in the algebra $6T^{(0)}$, cf. Section 3.2. *Problem* we are interested in is to describe commutative subalgebras generated by *multiplicative* Dunkl elements in the algebras $6T^{\dagger}(\Gamma)$ and $6T^{(0)}(\Gamma)^{ab}$. In the latter case one will come to the K-theoretic version of algebras studied in [118], ...

Yet another objective of our paper[5] is to describe several combinatorial properties of some special elements in the associative quasi-classical Yang–Baxter algebras [72], including among others, the so-called *Coxeter element* and the *longest element*. In the *case* of *Coxeter element* we relate the corresponding reduced polynomials introduced in [133, Exercise 6.C5(c)], and independently in [72], cf. [70], with the β-Grothendieck polynomials [42] for some special permutations $\pi_k^{(n)}$. More generally, we identify the β-Grothendieck polynomial $\mathfrak{G}^{(\beta)}_{\pi_k^{(n)}}(X_n)$ with a certain weighted sum running over the set of k-dissections of a convex $(n+k+1)$-gon. In particular we show that the specialization $\mathfrak{G}^{(\beta)}_{\pi_k^{(n)}}(1)$ of the β-Grothendieck polynomial $\mathfrak{G}^{(\beta)}_{\pi_k^{(n)}}(X_n)$ counts the number of k-*dissections* of a convex $(n+k+1)$-gon according to the number of diagonals involved. When the number of diagonals in a k-dissection is the maximal possible (equals to $n(2k-1)-1$), we recover the well-known fact that the number of k-triangulations of a convex $(n+k+1)$-gon is equal to the value of a certain Catalan–Hankel determinant, see, e.g., [129]. In Section 5.4.2 we study multiparameter generalizations of reduced polynomials associated with Coxeter elements.

We also show that for a certain 5-parameters family of vexillary permutations, the specialization $x_i = 1$, $\forall\, i\ge 1$, of the corresponding β-*Schubert* polynomials $\mathfrak{S}^{(\beta)}_w(X_n)$ turns out to be coincide either with the Fuss–Narayana polynomials and their generalizations, or with a (q,β)-deformation of VSASM or that of CSTCPP numbers, see Corollary 5.33B. As examples we show that

(a) the reduced polynomial corresponding to a monomial $x_{12}^n x_{23}^m$ counts the number of (n,m)-*Delannoy paths* according to the number of NE-steps, see Lemma 5.81;

(b) if $\beta=0$, the reduced polynomial corresponding to monomial $(x_{12}x_{23})^n x_{34}^k$, $n\ge k$, counts the number of n up, n down permutations in the symmetric group $\mathbb{S}_{2n+k+1}$, see Proposition 5.82; see also Conjecture 5.83.

We also point out on a conjectural connection between the sets of *maximal* compatible sequences for the permutation $\sigma_{n,2n,2,0}$ and that $\sigma_{n,2n+1,2,0}$ from one side, and the set of VSASM(n) and that of CSTCPP(n) correspondingly, from the other, see Comments 5.48 for details. Finally, in Sections 5.1.1 and 5.4.1 we introduce and study a multiparameter generalization of reduced polynomials considered in [133, Exercise 6.C5(c)], as well as that of the Catalan, Narayana and (small) Schröder numbers.

In the *case* of the *longest element* we relate the corresponding reduced polynomial with the Ehrhart polynomial of the Chan–Robbins–Yuen polytope, see Section 5.3. More generally, we relate the (t,β)-reduced polynomial corresponding to monomial

$$\prod_{j=1}^{n-1} x_{j,j+1}^{a_j} \prod_{j=2}^{n-2}\left(\prod_{k=j+2}^{n} x_{jk}\right), \qquad a_j\in\mathbb{Z}_{\ge 0}, \qquad \forall\, j,$$

[5]This part of our paper had its origin in the study/computation of relations among the additive and multiplicative Dunkl elements in the quadratic algebras we are interested in, as well as the author's attempts to construct a monomial basis in the algebra $3T_n^{(0)}$ and find its Hilbert series for $n\ge 6$. As far as I'm aware these *problems* are still widely open.

with positive t-deformations of the Kostant partition function and that of the Ehrhart polynomial of some flow polytopes, see Section 5.3.

In Section 5.4 we investigate reduced polynomials associated with certain monomials in the algebra $(\widehat{\mathrm{ACYB}})_n^{ab}(\beta)$, known also as Gelfand–Varchenko algebra [67, 72], and study its combinatorial properties. Our main objective in Section 5.4.2 is to study reduced polynomials for Coxeter element treated in a certain multiparameter deformation of the (noncommutative) quadratic algebra $\widehat{\mathrm{ACYB}}_n(\alpha,\beta)$. Namely, to each dissection of a convex $(n+2)$-gon we associate a certain weight and consider the generating function of all dissections of $(n+2)$-gon selected taken with that weight. One can show that the reduced polynomial corresponding to the Coxeter element in the deformed algebra is equal to that generating function. We show that certain specializations of that reduced polynomial coincide, among others, with the Grothendieck polynomials corresponding to the permutation $1\times w_0^{(n-1)}\in\mathbb{S}_n$, the Lagrange inversion formula, as well as give rise to combinatorial (i.e., positive expressions) multiparameters deformations of Catalan and Fuss–Catalan, Motzkin, Riordan and Fine numbers, Schröder numbers and Schröder trees. We *expect* (work in progress) a similar connections between Schubert and Grothendieck polynomials associated with the Richardson permutations $1^k\times w_0^{(n-k)}$, k-dissections of a convex $(n+k+1)$-gon investigated in the present paper, and k-dimensional Lagrange–Good inversion formula studied from combinatorial point of view, e.g., in [22, 50].

1 Introduction

The Dunkl operators have been introduced in the later part of 80's of the last century by Charles Dunkl [35, 36] as a powerful mean to study of harmonic and orthogonal polynomials related with finite Coxeter groups. In the present paper we don't need the definition of Dunkl operators for arbitrary (finite) Coxeter groups, see, e.g., [35], but only for the special case of the symmetric group $\mathbb{S}_n$.

Definition 1.1. Let $P_n=\mathbb{C}[x_1,\dots,x_n]$ be the ring of polynomials in variables $x_1,\dots,x_n$. The type A_{n-1} (additive) rational Dunkl operators $D_1,\dots,D_n$ are the differential-difference operators of the following form

$$D_i=\lambda\frac{\partial}{\partial x_i}+\sum_{j\neq i}\frac{1-s_{ij}}{x_i-x_j}, \tag{1.1}$$

Here s_{ij}, $1\le i<j\le n$, denotes the exchange (or permutation) operator, namely,

$$s_{ij}(f)(x_1,\dots,x_i,\dots,x_j,\dots,x_n)=f(x_1,\dots,x_j,\dots,x_i,\dots,x_n),$$

$\frac{\partial}{\partial x_i}$ stands for the derivative w.r.t. the variable x_i, $\lambda\in\mathbb{C}$ is a parameter.

The key property of the Dunkl operators is the following result.

Theorem 1.2 (C. Dunkl [35])**.** *For any finite Coxeter group (W,S), where $S=\{s_1,\dots,s_l\}$ denotes the set of simple reflections, the Dunkl operators $D_i:=D_{s_i}$ and $D_j:=D_{s_j}$ pairwise commute: $D_iD_j=D_jD_i$, $1\le i,j\le l$.*

Another fundamental property of the Dunkl operators which finds a wide variety of applications in the theory of integrable systems, see, e.g., [56], is the following statement: the operator

$$\sum_{i=1}^{l}(D_i)^2$$

"essentially" coincides with the Hamiltonian of the rational Calogero–Moser model related to the finite Coxeter group (W,S).

Definition 1.3. Truncated (additive) Dunkl operator (or the Dunkl operator at critical level), denoted by $\mathcal{D}_i$, $i = 1, \ldots, l$, is an operator of the form (1.1) with parameter $\lambda = 0$.

For example, the type A_{n-1} rational truncated Dunkl operator has the following form

$$\mathcal{D}_i = \sum_{j \neq i} \frac{1 - s_{ij}}{x_i - x_j}.$$

Clearly the truncated Dunkl operators generate a commutative algebra. The important property of the truncated Dunkl operators is the following result discovered and proved by C. Dunkl [36]; see also [8] for a more recent proof.

Theorem 1.4 (C. Dunkl [36], Yu. Bazlov [8])**.** *For any finite Coxeter group (W, S) the algebra over $\mathbb{Q}$ generated by the truncated Dunkl operators $\mathcal{D}_1, \ldots, \mathcal{D}_l$ is canonically isomorphic to the coinvariant algebra $\mathcal{A}_W$ of the Coxeter group (W, S).*

Recall that for a finite *crystallographic* Coxeter group (W, S) the coinvariant algebra $\mathcal{A}_W$ is isomorphic to the cohomology ring $H^*(G/B, \mathbb{Q})$ of the flag variety G/B, where G stands for the Lie group corresponding to the crystallographic Coxeter group (W, S) we started with.

Example 1.5. In the case when $W = \mathbb{S}_n$ is the symmetric group, Theorem 1.4 states that the algebra over $\mathbb{Q}$ generated by the truncated Dunkl operators $\mathcal{D}_i = \sum\limits_{j \neq i} \frac{1-s_{ij}}{x_i - x_j}$, $i = 1, \ldots, n$, is canonically isomorphic to the cohomology ring of the full flag variety $\mathcal{F}l_n$ of type A_{n-1}

$$\mathbb{Q}[\mathcal{D}_1, \ldots, \mathcal{D}_n] \cong \mathbb{Q}[x_1, \ldots, x_n]/J_n, \tag{1.2}$$

where J_n denotes the ideal generated by the elementary symmetric polynomials $\{e_k(X_n), 1 \leq k \leq n\}$.

Recall that the elementary symmetric polynomials $e_i(X_n)$, $i = 1, \ldots, n$, are defined through the generating function

$$1 + \sum_{i=1}^{n} e_i(X_n) t^i = \prod_{i=1}^{n} (1 + t x_i),$$

where we set $X_n := (x_1, \ldots, x_n)$. It is well-known that in the case $W = \mathbb{S}_n$, the isomorphism (1.2) can be defined over the ring of integers $\mathbb{Z}$.

Theorem 1.4 by C. Dunkl has raised a number of natural questions:

(A) What is the algebra generated by the *truncated*

- trigonometric,
- elliptic,
- super, matrix, ...,
  - (a) additive Dunkl operators?
  - (b) Ruijsenaars–Schneider–Macdonald operators?
  - (c) Gaudin operators?

(B) Describe commutative subalgebra generated by the Jucys–Murphy elements in

- the group ring of the symmetric group;
- the Hecke algebra;
- the Brauer algebra, BMW algebra, ...

(C) Does there exist an analogue of Theorem 1.4 for

- classical and quantum equivariant cohomology and equivariant K-theory rings of the partial flag varieties?
- chomology and K-theory rings of affine flag varieties?
- diagonal coinvariant algebras of finite Coxeter groups?
- complex reflection groups?

The present paper is an extended introduction to a few items from Section 5 of [72].

The main purpose of my paper "On some quadratic algebras, II" is to give some partial answers on the above questions basically in the case of the symmetric group $\mathbb{S}_n$.

The purpose of the *present paper* is to draw attention to an interesting class of nonhomogeneous quadratic algebras closely connected (still mysteriously!) with different branches of Mathematics such as classical and quantum Schubert and Grothendieck calculi, low-dimensional topology, classical, basic and elliptic hypergeometric functions, algebraic combinatorics and graph theory, integrable systems, etc.

What we try to explain in [72] is that upon passing to *a suitable representation* of the quadratic algebra in question, the subjects mentioned above, are a manifestation of certain general properties of that quadratic algebra.

From this point of view, we treat the commutative subalgebra generated (over a universal *Lazard* ring $\mathbb{L}_n$ [88]) by the additive (resp. multiplicative) truncated Dunkl elements in the algebra $3T_n(\beta)$, see Definition 3.1, as *universal cohomology* (resp. *universal K-theory*) ring of the complete flag variety $\mathcal{F}l_n$. The classical or quantum cohomology (resp. the classical or quantum K-theory) rings of the flag variety $\mathcal{F}l_n$ are certain quotients of that *universal ring.*

For example, in [74] we have computed relations among the (truncated) Dunkl elements $\{\theta_i,\, i=1,\dots,n\}$ in the *elliptic representation* of the algebra $3T_n(\beta=0)$. We *expect* that the commutative subalgebra obtained is isomorphic to *elliptic cohomology ring* (not defined yet, but see [48, 52]) of the flag variety $\mathcal{F}l_n$.

Another example from [72]. Consider the algebra $3T_n(\beta=0)$. One can prove [72] the following *identities* in the algebra $3T_n(\beta=0)$:

(A) *summation formula*

$$\sum_{j=1}^{n-1}\left(\prod_{b=j+1}^{n-1}u_{b,b+1}\right)u_{1,n}\left(\prod_{b=1}^{j-1}u_{b,b+1}\right)=\prod_{a=1}^{n-1}u_{a,a+1};$$

(B) *duality transformation formula*, let $m\le n$, then

$$\begin{aligned}\sum_{j=m}^{n-1}&\left(\prod_{b=j+1}^{n-1}u_{b,b+1}\right)\left[\prod_{a=1}^{m-1}u_{a,a+n-1}u_{a,a+n}\right]u_{m,m+n-1}\left(\prod_{b=m}^{j-1}u_{b,b+1}\right)\\&+\sum_{j=2}^{m}\left[\prod_{a=j}^{m-1}u_{a,a+n-1}u_{a,a+n}\right]u_{m,n+m-1}\left(\prod_{b=m}^{n-1}u_{b,b+1}\right)u_{1,n}\\&=\sum_{j=1}^{m}\left[\prod_{a=1}^{m-j}u_{a,a+n}u_{a+1,a+n}\right]\left(\prod_{b=m}^{n-1}u_{b,b+1}\right)\left[\prod_{a=1}^{j-1}u_{a,a+n-1}u_{a,a+n}\right].\end{aligned}$$

One can check that upon passing to the *elliptic representation* of the algebra $3T_n(\beta=0)$, see Section 3.1 or [74], for the definition of *elliptic representation*, the above identities (A) and (B) finally end up correspondingly, to be the *summation formula* and the $N = 1$ case of the *duality transformation formula* for multiple elliptic hypergeometric series (of type A_{n-1}), see, e.g., [63] or Appendix A.6 for the explicit forms of the latter. After passing to the so-called *Fay representation* [72], the identities (A) and (B) become correspondingly to be the summation formula and duality transformation formula for the Riemann theta functions of genus $g > 0$ [72]. These formulas in the case $g \geq 2$ seems to be new.

Worthy to mention that the relation (A) above can be treated as a "non-commutative analogue" of the well-known recurrence relation among the *Catalan numbers*. The study of "descendent relations" in the quadratic algebras in question was originally motivated by the author attempts to construct a *monomial basis* in the algebra $3T_n^{(0)}$, and compute $\operatorname{Hilb}(3T_n^{(0)}, t)$ for $n \geq 6$. These problems are still widely open, but gives rise the author to discovery of several interesting connections with

- classical and quantum Schubert and Grothendieck calculi,
- combinatorics of reduced decomposition of some special elements in the symmetric group,
- combinatorics of generalized *Chan–Robbins–Yuen* polytopes,
- relations among the Dunkl and Gaudin elements,
- computation of Tutte and chromatic polynomials of the weighted complete multipartite graphs, etc.

A few words about the content of the *present paper*. Example 1.5 can be viewed as an illustration of the main problems we are treated in Sections 2 and 3 of the present paper, namely the following ones.

- Let $\{u_{ij},\ 1 \leq i,j \leq n\}$ be a set of generators of a certain algebra over a commutative ring K. The first *problem* we are interested in is to describe "a natural set of relations" among the generators $\{u_{ij}\}_{1\leq i,j\leq n}$ which implies the pair-wise commutativity of *dynamical Dunkl elements*

$$\theta_i = \theta_i^{(n)} =: \sum_{j=1}^{n} u_{ij}, \qquad 1 \leq i \leq n.$$

- Should this be the case then we are interested in to describe the algebra generated by "the integrals of motions", i.e., to describe the quotient of the algebra of polynomials $K[y_1,\dots,y_n]$ by the two-sided ideal $\mathcal{J}_n$ generated by non-zero polynomials $F(y_1,\dots,y_n)$ such that $F(\theta_1,\dots,\theta_n) = 0$ in the algebra over ring K generated by the elements $\{u_{ij}\}_{1\leq i,j\leq n}$.
- We are looking for a set of additional relations which imply that the elementary symmetric polynomials $e_k(Y_n)$, $1 \leq k \leq n$, belong to the set of integrals of motions. In other words, the value of elementary symmetric polynomials $e_k(y_1,\dots,y_n)$, $1 \leq k \leq n$, on the Dunkl elements $\theta_1^{(n)},\dots,\theta_n^{(n)}$ *do not depend* on the variables $\{u_{ij},\ 1 \leq i \neq j \leq n\}$. If so, one can defined *deformation* of elementary symmetric polynomials, and make use of it and the Jacobi–Trudi formula, to define deformed Schur functions, for example. We try to realize this program in Sections 2 and 3.

In Section 2, see Definition 2.3, we introduce the so-called *dynamical classical Yang–Baxter algebra* as "a natural quadratic algebra" in which the Dunkl elements form a pair-wise commuting family. It is the study of the algebra generated by the (truncated) Dunkl elements that is the main objective of our investigation in [72] and the present paper. In Section 2.1 we describe few representations of the dynamical classical Yang–Baxter algebra DCYB_n related with

- quantum cohomology $QH^*(\mathcal{F}l_n)$ of the complete flag variety $\mathcal{F}l_n$, cf. [41];
- quantum equivariant cohomology $QH^*_{T^n\times C^*}(T^*\mathcal{F}l_n)$ of the cotangent bundle $T^*\mathcal{F}l_n$ to the complete flag variety, cf. [54];
- Dunkl–Gaudin and Dunkl–Uglov representations, cf. [108, 138].

In Section 3, see Definition 3.1, we introduce the algebra $3HT_n(\beta)$, which seems to be the most general (noncommutative) deformation of the (even) Orlik–Solomon algebra of type A_{n-1}, such that it's still possible to describe relations among the Dunkl elements, see Theorem 3.8. As an application we describe explicitly a set of relations among the (additive) Gaudin/Dunkl elements, cf. [108]. It should be stressed at this place that we treat the Gaudin elements/operators (either additive or multiplicative) as *images* of the *universal* Dunkl elements/operators (additive or multiplicative) in the *Gaudin representation* of the algebra $3HT_n(0)$. There are several other important representations of that algebra, for example, the Calogero–Moser, Bruhat, Buchstaber–Felder–Veselov (elliptic), Fay trisecant (τ-functions), adjoint, and so on, considered (among others) in [72]. Specific properties of a representation chosen[6] (e.g., *Gaudin representation*) imply some additional relations among the images of the universal Dunkl elements (e.g., *Gaudin elements*) should to be unveiled.

We start Section 3 with definition of algebra $3T_n(\beta)$ and its "Hecke" $3HT_n(\beta)$ and "elliptic" $3MT_n(\beta)$ quotients. In particular we define an elliptic representation of the algebra $3T_n(0)$ [74], and show how the well-known elliptic solutions of the quantum Yang–Baxter equation due to A. Belavin and V. Drinfeld, see, e.g., [9], S. Shibukawa and K. Ueno [130], and G. Felder and V. Pasquier [40], can be plug in to our construction, see Section 3.1. At the end of Section 3.1.1 we point out on a *mysterious* (at least for the author) appearance of the Euler numbers and "traces" of the Brauer algebra in the equivariant Pieri rules hold for the algebra $3TM_n(\beta,\boldsymbol{q},\psi)$ stated in Theorem 3.8.

In Section 3.2 we introduce a *multiplicative* analogue of the Dunkl elements $\{\Theta_j \in 3T_n(\beta), 1\le j\le n\}$ and describe the commutative subalgebra in the algebra $3T_n(\beta)$ generated by multiplicative Dunkl elements [76]. The latter commutative subalgebra turns out to be isomorphic to the quantum equivariant K-theory of the complete flag variety $\mathcal{F}l_n$ [76].

In Section 3.3 we describe relations among the truncated Dunkl–Gaudin elements. In this case the quantum parameters $q_{ij}=p_{ij}^2$, where parameters $\{p_{ij}=(z_i-z_j)^{-1}, 1\le i<j\le n\}$ satisfy the both Arnold and Plücker relations. This observation has made it possible to describe a set of additional *rational relations* among the Dunkl–Gaudin elements, cf. [108].

In Section 3.4 we introduce an equivariant version of multiplicative Dunkl elements, called *shifted Dunkl elements* in our paper, and describe (some) relations among the latter. This result is a generalization of that obtained in Section 3.1 and [76]. However we don't know any geometric interpretation of the commutative subalgebra generated by shifted Dunkl elements.

In Section 4.1 for any subgraph $\Gamma\subset K_n$ of the complete graph K_n we introduce[7] [70, 72], algebras $3T_n(\Gamma)$ and $3T_n^{(0)}(\Gamma)$ which can be seen as analogues of algebras $3T_n$ and $3T_n^{(0)}$ correspondingly[8].

[6]For example, in the cases of either *Calogero–Moser* or *Bruhat* representations one has an additional constraint, namely, $u_{ij}^2=0$ for all $i\ne j$. In the case of *Gaudin* representation one has an additional constraint $u_{ij}^2=p_{ij}^2$, where the (quantum) parameters $\{p_{ij}=\frac{1}{x_i-x_j}, i\ne j\}$, satisfy *simultaneously* the *Arnold* and *Plücker* relations, see Section 2, **II**. Therefore, the (small) quantum cohomology ring of the type A_{n-1} full flag variety $\mathcal{F}l_n$ and the Bethe subalgebra(s) (i.e., the subalgebra generated by Gaudin elements in the algebra $3HT_n(0)$) correspond to *different specializations* of "*quantum parameters*" $\{q_{ij}:=u_{ij}^2\}$ of the *universal cohomology ring* (i.e., the subalgebra/ring in $3HT_n(0)$ generated by (universal) Dunkl elements). For more details and examples, see Section 2.1 and [72].

[7]Independently the algebra $3T_n^{(0)}(\Gamma)$ has been studied in [16], where the reader can find some examples and conjectures.

[8]To avoid confusions, it must be emphasized that the defining relations for algebras $3T_n(\Gamma)$ and $3T_n(\Gamma)^{(0)}$ may have more then three terms.

We want to point out in the Introduction, cf. footnote 1, that an analog of the algebras $3T_n$ and $3T_n^{(\beta)}$, $3HT_n$, etc. treated in the present paper, can be defined for any (oriented or not) *matroid* $\mathcal{M}$. We denote these algebras as $3T(\mathcal{M})$ and $3T^{(\beta)}(\mathcal{M})$. One can show (A.K.) that the *abelianization* of the algebra $3T^{(\beta)}(\mathcal{M})$, denoted by $3T^{(\beta)}(\mathcal{M})^{ab}$, is isomorphic to the *Gelfand–Varchenko* algebra corresponding to a matroid $\mathcal{M}$, whereas the algebra $3T^{(\beta=0)}(\mathcal{M})^{ab}$ is isomorphic to the (even) *Orlik–Solomon* algebra $\mathrm{OS}^+(\mathcal{M})$ of a matroid $\mathcal{M}$.[9] We consider and treat the algebras $3T(\mathcal{M})$, $3HT(\mathcal{M})$, $\ldots$, as *equivariant noncommutative* (or *quantum*) versions of the (even) Orlik–Solomon algebras associated with matroid (including hyperplane, graphic, $\ldots$ arrangements). However a meaning of a quantum deformation of the (even or odd) Orlik–Solomon algebra suggested in the present paper, is missing, even for the braid arrangement of type A_n. Generalizations of the Gelfand–Varchenko algebra has been suggested and studied in [67, 72] and in the present paper under the name quasi-associative Yang–Baxter algebra, see Section 5.

In the present paper we basically study the *abelian quotient* of the algebra $3T_n^{(0)}(\Gamma)$, where graph Γ has no loops and multiple edges, since we expect some applications of our approach to the theory of *chromatic polynomials* of planar graphs, in particular to the complete multipartite graphs $K_{n_1,\ldots,n_r}$ and the grid graphs $G_{m,n}$.[10] Our main results hold for the complete multipartite, cyclic and line graphs. In particular we compute their *chromatic* and *Tutte* polynomials, see Proposition 4.19 and Theorem 4.24. As a byproduct we compute the Tutte polynomial of the $\boldsymbol{\ell}$-weighted complete multipartite graph $K_{n_1,\ldots,n_r}^{(\boldsymbol{\ell})}$ where $\boldsymbol{\ell} = \{\ell_{ij}\}_{1\le i<j\le r}$, is a collection of weights, i.e., a set of non-negative integers.

More generally, for a set of variables $\{\{q_{ij}\}_{1\le i<j\le n}, x, y\}$ we define *universal Tutte polynomial* $T_n(\{q_{ij}\}, x, y) \in \mathbb{Z}[q_{ij}][x, y]$ such that for any collection of non-negative integers $\{m_{ij}\}_{1\le i<j\le n}$ and a subgraph $\Gamma \subset K_n^{(\boldsymbol{m})}$ of the complete graph K_n with each edge (i,j) comes with multiplicity m_{ij}, the specialization

$$q_{ij} \longrightarrow 0 \quad \text{if edge} \quad (i,j) \notin \Gamma, \qquad q_{ij} \longrightarrow [m_{ij}]_y := \frac{y^{m_{ij}} - 1}{y-1} \quad \text{if edge} \quad (i,j) \in \Gamma$$

of the universal Tutte polynomial $T_n(\{q_{ij}\}, x, y)$ is equal to the Tutte polynomial of graph Γ multiplied by the factor $(t-1)^{\kappa(\Gamma)}$:

$$(x-1)^{\kappa(\Gamma)}\mathrm{Tutte}(\Gamma, x, y) := T_n(\{q_{ij}\}, x, y)\Bigg|_{\substack{q_{ij}=0 \text{ if } (i,j)\notin\Gamma \\ q_{ij}=[m_{ij}]_y \text{ if } (i,j)\in\Gamma}}.$$

Here and after $\kappa(\Gamma)$ demotes the number of connected components of a graph Γ. In other words, one can treat the universal Tutte polynomial $T_n(\{q_{ij}\}, x, y)$ as a "reproducing kernel" for the Tutte polynomials of all (loop-less) graphs with the number of vertices not exceeded n.

We also state Conjecture 4.15 that for any loopless graph Γ (possibly with multiple edges) the algebra $3T_{|\Gamma|}^{(0)}(\Gamma)^{ab}$ is isomorphic to the even Orlik–Solomon algebra $\mathrm{OS}^+(\mathcal{A}_\Gamma)$ of the *graphic arrangement* associated with graph Γ in question[11].

At the end we emphasize that the case of the complete graph $\Gamma = K_n$ reproduces the results of the present paper and those of [72], i.e., the case of the full flag variety $\mathcal{F}l_n$. The case of the *complete multipartite graph* $\Gamma = K_{n_1,\ldots,n_r}$ reproduces the analogue of results stated in the present paper for the case of full flag variety $\mathcal{F}l_n$, to the case of the *partial flag* variety $\mathcal{F}_{n_1,\ldots,n_r}$, see [72] for details.

[9]For a definition and basic properties of the Orlik–Solomon algebra corresponding to a matroid, see, e.g., [49, 65].

[10]See http://reference.wolfram.com/language/ref/GridGraph.html for a definition of *grid graph* $G_{m,n}$.

[11]For simple graphs, i.e., without loops and multiple edges, this conjecture has been proved in [89].

In Section 4.1.4 we sketch how to generalize our constructions and some of our results to the case of the Lie algebras of *classical types*[12].

In Section 4.2 we briefly overview our results concerning yet another interesting family of quadratic algebras, namely the *six-term relations algebras* $6T_n$, $6T_n^{(0)}$ and related ones. These algebras also contain a distinguished set of mutually commuting elements called *Dunkl elements* $\{\theta_i,\, i=1,\dots,n\}$ given by $\theta_i=\sum\limits_{j\neq i} r_{ij}$, see Definition 4.48.

In Section 4.2.2 we introduce and study the algebra $6T_n^{\bigstar}$ in greater detail. In particular we introduce a "quantum deformation" of the algebra generated by the curvature of 2-forms of of the Hermitian linear bundles over the flag variety $\mathcal{F}l_n$, cf. [118].

In Section 4.2.3 we state our results concerning the *classical Yang–Baxter algebra* CYB_n and the 6-term relation algebra $6T_n$. In particular we give formulas for the Hilbert series of these algebras. These formulas have been obtained independently in [7] The paper just mentioned, contains a description of a basis in the algebra $6T_n$, and much more.

In Section 4.2.4 we introduce a *super analog* of the algebra $6T_n$, denoted by $6T_{n,m}$, and compute its Hilbert series.

Finally, in Section 4.3 we introduce *extended nil-three term relations* algebra $3\mathfrak{T}_n$ and describe a subalgebra inside of it which is isomorphic to the double affine Hecke algebra of type A_{n-1}, cf. [24].

In Section 5 we describe several combinatorial properties of some special elements in the associative quasi-classical Yang–Baxter algebra[13], denoted by $\widehat{\mathrm{ACYB}}_n$. The main results in that direction were motivated and obtained as a by-product, in the process of the study of the *the structure* of the algebra $3HT_n(\beta)$. More specifically, the main results of Section 5 were obtained in the course of "hunting for descendant relations" in the algebra mentioned, which is an important problem to be solved to construct *a basis* in the nil-quotient algebra $3T_n^{(0)}$. This *problem* is still widely-open.

The results of Section 5.1, see Proposition 5.4, items (1)–(5), are more or less well-known among the specialists in the subject, while those of the item (6) seem to be new. Namely, we show that the polynomial $Q_n(x_{ij}=t_i)$ from [133, Exercise 6.C8(c)], essentially coincides with the β-deformation [42] of the Lascoux–Schützenberger Grothendieck polynomial [86] for some particular permutation. The results of Proposition 5.4(6), point out on a deep connection between reduced forms of monomials in the algebra $\widehat{\mathrm{ACYB}}_n$ and the Schubert and Grothendieck calculi. This observation was the starting point for the study of some combinatorial properties of certain specializations of the Schubert, the β-Grothendieck [43] and the double β-Grothendieck polynomials in Section 5.2. One of the main results of Section 5.2 can be stated as follows.

Theorem 1.6.

(1) *Let $w\in\mathbb{S}_n$ be a permutation, consider the specialization $x_1:=q$, $x_i=1$, $\forall\, i\ge 2$, of the β-Grothendieck polynomial $\mathfrak{G}_w^{(\beta)}(X_n)$. Then*

$$\mathcal{R}_w(q,\beta+1):=\mathfrak{G}_w^{(\beta)}(x_1=q,\, x_i=1,\, \forall\, i\ge 2)\in\mathbb{N}[q,1+\beta].$$

In other words, the polynomial $\mathcal{R}_w(q,\beta)$ has non-negative integer coefficients[14].

For late use we define polynomials

$$\mathfrak{R}_w(q,\beta):=q^{1-w(1)}\mathcal{R}_w(q,\beta).$$

[12] One can define an analogue of the algebra $3T_n^{(0)}$ for the root system of BC_n and $C_n^{\vee}C_n$-types as well, but we are omitted these cases in the present paper.

[13] The algebra $\widehat{\mathrm{ACYB}}_n$ can be treated as "one-half" of the algebra $3T_n(\beta)$. It appears that the basic relations among the Dunkl elements, which do *not* mutually commute anymore, are still *valid*, see Lemma 5.3.

[14] For a more general result see Appendix A.1, Corollary A.7.

(2) *Let $w \in \mathbb{S}_n$ be a permutation, consider the specialization $x_i := q$, $y_i = t$, $\forall\, i \geq 1$, of the double β-Grothendieck polynomial $\mathfrak{G}_w^{(\beta)}(X_n, Y_n)$. Then*

$$\mathfrak{G}_w^{(\beta-1)}(x_i := q,\, y_i := t,\, \forall\, i \geq 1) \in \mathbb{N}[q, t, \beta].$$

(3) *Let w be a permutation, then*

$$\mathfrak{R}_w(1, \beta) = \mathfrak{R}_{1 \times w}(0, \beta).$$

Note that $\mathcal{R}_w(1, \beta) = \mathcal{R}_{w^{-1}}(1, \beta)$, but $\mathcal{R}_w(t, \beta) \neq \mathcal{R}_{w^{-1}}(t, \beta)$, in general.

For the reader convenience we collect some basic definitions and results concerning the β-Grothendieck polynomials in Appendix A.1.

Let us observe that $\mathfrak{R}_w(1,1) = \mathfrak{S}_w(1)$, where $\mathfrak{S}_w(1)$ denotes the specialization $x_i := 1, \forall\, i \geq 1$, of the Schubert polynomial $\mathfrak{S}_w(X_n)$ corresponding to permutation w. Therefore, $\mathfrak{R}_w(1,1)$ is equal to the number of *compatible sequences* [13] (or *pipe dreams*, see, e.g., [129]) corresponding to permutation w.

Problem 1.7. *Let $w \in \mathbb{S}_n$ be a permutation and $l := \ell(w)$ be its length. Denote by $\mathrm{CS}(w) = \{\boldsymbol{a} = (a_1 \leq a_2 \leq \cdots \leq a_l) \in \mathbb{N}^l\}$ the set of compatible sequences [13] corresponding to permutation w.*

- *Define statistics $r(\boldsymbol{a})$ on the set of all compatible sequences $\mathrm{CS}_n := \coprod_{w \in \mathbb{S}_n} \mathrm{CS}(w)$ in a such way that*

$$\sum_{\boldsymbol{a} \in \mathrm{CS}(w)} q^{a_1} \beta^{r(\boldsymbol{a})} = \mathcal{R}_w(q, \beta).$$

- *Find a geometric interpretation, and investigate combinatorial and algebra-geometric properties of polynomials $\mathfrak{S}_w^{(\beta)}(X_n)$, where for a permutation $w \in \mathbb{S}_n$ we denoted by $\mathfrak{S}_w^{(\beta)}(X_n)$ the β-Schubert polynomial defined as follows*

$$\mathfrak{S}_w^{(\beta)}(X_n) = \sum_{\boldsymbol{a} \in \mathrm{CS}(w)} \beta^{r(\boldsymbol{a})} \prod_{i=1}^{l := \ell(w)} x_{a_i}.$$

We *expect* that polynomial $\mathfrak{S}_w^{(\beta)}(1)$ coincides with the Hilbert polynomial of a certain graded commutative ring naturally associated to permutation w.

Remark 1.8. It should be mentioned that, in general, the principal specialization

$$\mathfrak{G}_w^{(\beta-1)}\big(x_i := q^{i-1},\, \forall\, i \geq 1\big)$$

of the $(\beta - 1)$-Grothendieck polynomial may have *negative* coefficients.

Our main objective in Section 5.2 is to study the polynomials $\mathfrak{R}_w(q, \beta)$ for a special class of permutations in the symmetric group $\mathbb{S}_\infty$. Namely, in Section 5.2 we study some combinatorial properties of polynomials $\mathfrak{R}_{\varpi_{\lambda,\phi}}(q, \beta)$ for the five parameters family of *vexillary* permutations $\{\varpi_{\lambda,\phi}\}$ which have the *shape* $\lambda := \lambda_{n,p,b} = (p(n-i+1)+b,\, i = 1, \ldots, n+1)$ and *flag* $\phi := \phi_{k,r} = (k + r(i-1),\ i = 1, \ldots, n+1)$.

This class of permutations is notable for many reasons, including that the specialized value of the Schubert polynomial $\mathfrak{S}_{\varpi_{\lambda,\phi}}(1)$ admits a nice product formula[15], see Theorem 5.29. Moreover,

[15] One can prove a product formula for the principal specialization $\mathfrak{S}_{\varpi_{\lambda,\phi}}(x_i := q^{i-1}, \forall\, i \geq 1)$ of the corresponding Schubert polynomial. We don't need a such formula in the present paper.

we describe also some interesting connections of polynomials $\mathfrak{R}_{\varpi_{\lambda,\phi}}(q,\beta)$ with plane partitions, the Fuss–Catalan numbers[16] and Fuss–Narayana polynomials, k-triangulations and k-dissections of a convex polygon, as well as a connection with two families of ASM. For example, let $\lambda = (b^n)$ and $\phi = (k^n)$ be rectangular shape partitions, then the polynomial $\mathfrak{R}_{\varpi_{\lambda,\phi}}(q,\beta)$ defines a (q,β)-deformation of the number of (ordinary) plane partitions[17] sitting in the box $b \times k \times n$. It seems an interesting *problem* to find an algebra-geometric interpretation of polynomials $\mathfrak{R}_w(q,\beta)$ in the general case.

Question 1.9. *Let a and b be mutually prime positive integers. Does there exist a family of permutations $w_{a,b} \in \mathbb{S}_{ab(a+b)}$ such that the specialization $x_i = 1$, $\forall\, i$ of the Schubert polynomial $\mathfrak{S}_{w_{a,b}}$ is equal to the rational Catalan number $C_{a/b}$? That is*

$$\mathfrak{S}_{w_{a,b}}(1) = \frac{1}{a+b}\binom{a+b}{a}.$$

Many of the computations in Section 5.2 are based on the following determinantal formula for β-Grothendieck polynomials corresponding to grassmannian permutations, cf. [84].

Theorem 1.10 (see Comments 5.37(b))**.** *If $w = \sigma_\lambda$ is the grassmannian permutation with shape $\lambda = (\lambda_, \ldots, \lambda_n)$ and a unique descent at position n, then*[18]

$$\text{(A)} \quad \mathfrak{G}^{(\beta)}_{\sigma_\lambda}(X_n) = \mathrm{DET}\left| h^{(\beta)}_{\lambda_j+i,j}(X_n)\right|_{1\le i,j\le n} = \frac{\mathrm{DET}\left| x_i^{\lambda_j+n-j}(1+\beta x_i)^{j-1}\right|_{1\le i,j\le n}}{\prod\limits_{1\le i<j\le n}(x_i-x_j)},$$

where $X_n = (x_i, x_1, \ldots, x_n)$, and for any set of variables X,

$$h^{(\beta)}_{n,k}(X) = \sum_{a=0}^{k-1}\binom{k-1}{a} h_{n-k+a}(X)\beta^a,$$

and $h_k(X)$ denotes the complete symmetric polynomial of degree k in the variables from the set X.

$$\text{(B)} \quad \mathfrak{G}_{\sigma_\lambda}(X,Y) = \frac{\mathrm{DET}\left| \prod\limits_{a=1}^{\lambda_j+n-j}(x_i+y_a+\beta x_i y_a)(1+\beta x_i)^{j-1}\right|_{1\le i,j\le n}}{\prod\limits_{1\le i<j\le n}(x_i-x_j)}.$$

[16] We define the (generalized) Fuss–Catalan numbers to be $\mathrm{FC}^{(p)}_n(b) := \frac{1+b}{1+b+(n-1)p}\binom{np+b}{n}$. Connection of the Fuss–Catalan numbers with the *p-ballot numbers* $\mathrm{Bal}_p(m,n) := \frac{n-mp+1}{n+m+1}\binom{n+m+1}{m}$ and the *Rothe numbers* $R_n(a,b) := \frac{a}{a+bn}\binom{a+bn}{n}$ can be described as follows

$$\mathrm{FC}^{(p)}_n(b) = R_n(b+1,p) = \mathrm{Bal}_{p-1}(n,(n-1)p+b).$$

[17] Let λ be a partition. An ordinary plane partition (plane partition for short)bounded by d and shape λ is a filling of the shape λ by the numbers from the set $\{0,1,\ldots,d\}$ in such a way that the numbers along columns and rows are weakly *decreasing*. A *reverse* plane partition bounded by d and shape λ is a filling of the shape λ by the numbers from the set $\{0,1,\ldots,d\}$ in such a way that the numbers along columns and rows are weakly *increasing*.

[18] The equality

$$\mathfrak{G}^{(\beta)}_{\sigma_\lambda}(X_n) = \frac{\mathrm{DET}\left| x_i^{\lambda_j+n-j}(1+\beta x_i)^{j-1}\right|_{1\le i,j\le n}}{\prod\limits_{1\le i<j\le n}(x_i-x_j)},$$

has been proved independently in [107].

In Sections 5.2.2 and 5.4.2 we study connections of Grothendieck polynomial associated with the Richardson permutation $w_k^{(n)} = 1^k \times w_0^{(n-k)}$, k-dissections of a convex $(n+k+1)$-gon, generalized reduced polynomial corresponding to a certain monomial in the algebra $\widehat{\mathrm{ACYB}}_n$ and the Lagrange inversion formula. In the case of generalized Richardson permutation $w_{n,p}^{(k)}$ corresponding to the k-shifted dominant permutations $w^{(p,n)}$ associated with the Young diagram $\lambda_{p,n} := p(n-1, n-2, \dots, 1)$, namely, $w_{n,p}^{(k)} = 1^k \times w^{(p,n)}$, we treat only the case $k = 1$, see also [39]. In the case $k \geq 2$ one comes to a *task* to count and find a lattice path type interpretation for the number of k-**p***gulations* of a convex n-gon that is the number of partitioning of a convex n-gon on parts which are all equal to a convex $(p+2)$-gon, by a (maximal) family of diagonals such that each diagonal has at most k internal intersections with the members of a family of diagonals selected.

In Section 5.3 we give a partial answer on Question 6.C8(d) by R. Stanley [133]. In particular, we relate the reduced polynomial corresponding to monomial

$$\left(x_{12}^{a_2} \cdots x_{n-1,n}{}^{a_n}\right) \prod_{j=2}^{n-2} \prod_{k=j+2}^{n} x_{jk}, \qquad a_j \in \mathbb{Z}_{\geq 0}, \qquad \forall\, j,$$

with the Ehrhart polynomial of the generalized Chan–Robbins–Yuen polytope, if $a_2 = \cdots = a_n = m+1$, cf. [101], with a t-deformation of the Kostant partition function of type A_{n-1} and the Ehrhart polynomials of some flow polytopes, cf. [103].

In Section 5.4 we investigate certain specializations of the reduced polynomials corresponding to monomials of the form

$$x_{12}^{m_1} \cdots x_{n-1,n}^{m_n}, \qquad m_j \in \mathbb{Z}_{\geq 0}, \qquad \forall\, j.$$

First of all we observe that the corresponding specialized reduced polynomial appears to be a *piece-wise polynomial function* of parameters $\boldsymbol{m} = (m_1, \dots, m_n) \in (\mathbb{R}_{\geq 0})^n$, denoted by $P_{\boldsymbol{m}}$. It is an interesting *problem* to compute the *Laplace transform* of that piece-wise polynomial function. In the present paper we compute the value of the function $P_{\boldsymbol{m}}$ in the dominant chamber $\mathcal{C}_n = (m_1 \geq m_2 \geq \cdots \geq m_n \geq 0)$, and give a combinatorial interpretation of the values of that function in points (n, m) and (n, m, k), $n \geq m \geq k$.

For the reader convenience, in Appendices A.1–A.6 we collect some useful auxiliary information about the subjects we are treated in the present paper.

Almost all results in Section 5 state that some two specific sets have the same number of elements. Our proofs of these results are pure algebraic. It is an interesting *problem* to find *bijective proofs* of results from Section 5 which generalize and extend remarkable bijective proofs presented in [103, 129, 135, 142] to the *cases* of

- the β-Grothendieck polynomials,
- the (small) Schröder numbers,
- k-dissections of a convex $(n+k+1)$-gon,
- special values of reduced polynomials.

We are planning to treat and present these bijections in separate publication(s).

We *expect* that the reduced polynomials corresponding to the higher-order powers of the Coxeter elements also admit an interesting combinatorial interpretation(s). Some preliminary results in this direction are discussed in Comments 5.67.

At the end of introduction I want to add a few remarks.

(a) After a suitable modification of the algebra $3HT_n$, see [75], and the case $\beta \neq 0$ in [72], one can compute the set of relations among the (additive) Dunkl elements (defined in Section 2, equation (2.1)). In the case $\beta = 0$ and $q_{ij} = q_i \delta_{j-i,1}$, $1 \le i < j \le n$, where $\delta_{a,b}$ is the Kronecker delta symbol, the commutative algebra generated by additive Dunkl elements (2.3) appears to be "almost" isomorphic to the equivariant quantum cohomology ring of the flag variety $\mathcal{F}l_n$, see [75] for details. Using the *multiplicative* version of Dunkl elements, see Section 3.2, one can extend the results from [75] to the case of equivariant quantum K-theory of the flag variety $\mathcal{F}l_n$, see [72].

(b) As it was pointed out previously, one can define an analogue of the algebra $3T_n^{(0)}$ for any (oriented) matroid $\mathcal{M}_n$, and state a conjecture which connects the Hilbert polynomial of the algebra $3T_n^{(0)}((\mathcal{M}_n)^{ab}, t)$ and the chromatic polynomial of matroid $\mathcal{M}_n$. We *expect* that algebra $3T_n^{(\beta=1)}(\mathcal{M}_n)^{ab}$ is isomorphic to the *Gelfand–Varchenko* algebra associated with matroid $\mathcal{M}$. It is an interesting *problem* to find a combinatorial meaning of the algebra $3T_n^{(\beta)}(\mathcal{M}_n)$ for $\beta = 0$ and $\beta \neq 0$.

(c) Let R be a (graded) ring (to be specified later) and $\mathfrak{F}_{n^2}$ be the free associative algebra over R with the set of generators $\{u_{ij},\ 1 \le i, j \le n\}$. In the subsequent text we will distinguish the set of generators $\{u_{ii}\}_{1 \le i \le n}$ from that $\{u_{ij}\}_{1 \le i \neq j \le n}$, and set

$$x_i := u_{ii}, \qquad i = 1, \dots, n.$$

A guiding idea to choose definitions and perform constructions in the present paper is to impose a set of relations $\mathcal{R}_n$ among the generators $\{x_i\}_{1 \le i \le n}$ and that $\{u_{ij}\}_{1 \le i \neq j \le n}$ which ensure the mutual commutativity of the following elements

$$\theta_i^{(n)} := \theta_i = x_i + \sum_{j \neq i}^{n} u_{ij}, \qquad i = 1, \dots, n,$$

in the algebra $\mathcal{F}_{n^2}/\mathcal{R}_n$, as well as to have a good chance to describe/compute

• "Integral of motions", that is finding a big enough set of algebraically independent polynomials (quite possibly that polynomials are trigonometric or elliptic ones) $I_\alpha^{(n)}(y_1, \dots, y_n) \in R[Y_n]$ such that

$$I_\alpha^{(n)}\big(\theta_1^{(n)}, \dots, \theta_n^{(n)}\big) \in R[X_n], \qquad \forall\, \alpha,$$

in other words, the latter specialization of any integral of motion has to be independent of the all generators $\{u_{ij}\}_{1 \le i \neq j \le n}$.

• Give a presentation of the algebra $\mathcal{I}_n$ generated by the integral of motions that is to find a set of defining relations among the elements $\theta_1, \dots, \theta_n$, and describe a R-basis $\big\{m_\alpha^{(n)}\big\}$ in the algebra $\mathcal{I}_n$.

• Generalized Littlewood–Richardson and Murnaghan–Nakayama problems. Given an integral of motion $I_\beta^{(m)}(Y_m)$ and an integer $n \ge m$, find an explicit positive (if possible) expression in the quotient algebra $\mathcal{F}_{n^2}/\mathcal{R}_n$ of the element

$$I_\beta^{(m)}\big(\theta_1^{(n)}, \dots, \theta_m^{(n)}\big).$$

For example in the case of the 3-term relations algebra $3T_n^{(0)}$ (as well as its equivariant, quantum, etc. versions) the generalized Littlewood–Richardson problem is to find a positive expression in the algebra $3T_n^{(0)}$ for the element $\mathfrak{S}_w\big(\theta_1^{(n)}, \dots, \theta_m^{(n)}\big)$, where $\mathfrak{S}_w(Y_n)$ stands for the Schubert polynomial corresponding to a permutation $w \in \mathbb{S}_n$.

Generalized Murnaghan–Nakayama problem consists in finding a combinatorial expression in the algebra $3T_n^{(0)}$ for the element $\sum_{i=1}^{m} (\theta_i^{(n)})^k$.

Partial results concerning these problems have been obtained as far as we aware in [45, 70, 72, 73, 104, 117].

• "Partition functions". Assume that the (graded) algebra $\mathcal{I}_n$ generated over R by the elements $\theta_1, \dots, \theta_n$ has finite dimension/rank, and the (non zero) maximal degree component $\mathcal{I}_{\max}^{(n)}$ of that algebra has dimension/rank one and generated by an element ω. For any element $g \in \mathcal{F}_{n^2}$ let us denote by $\operatorname{Res}_\omega(g)$ an element in R such that

$$\overline{g} = \operatorname{Res}_\omega(g)\omega,$$

where we denote by $\overline{g}$ the image of element g in the component $\mathcal{I}_{\max}^{(n)}$.

We define *partition function* associated with the algebra $\mathcal{I}_n$ as follows

$$\mathcal{Z}(\mathcal{I}_n) = \operatorname{Res}_w\left(\exp\left(\sum_\alpha q_\alpha m_\alpha^{(n)}\right)\right),$$

where $\{q_\alpha\}$ is a set of parameters which is consistent in one-to-one correspondence with a basis $\{m_\alpha^{(n)}\}$ chosen.

We are interesting in to find a closed formula for the partition function $\mathcal{Z}(\mathcal{I}_n)$ as well as that for a *small* partition function

$$\mathcal{Z}^{(0)}(\mathcal{I}_n) := \operatorname{Res}_\omega\left(\exp\left(\sum_{1\le i,j\le n}\lambda_{ij}u_{ij}\right)\right),$$

where $\{\lambda_{ij}\}_{1\le i,j\le n}$ stands for a set of parameters. One can show [68] that the partition function $\mathcal{Z}(\mathcal{I}_n)$ associated with algebra $3T_n^{\boldsymbol{q}}$ satisfies the famous Witten–Dijkraaf–Verlinde–Verlinde equations.

As a preliminary steps to perform our guiding idea we

(i) investigate properties of the abelianization of the algebra $\mathcal{F}_{n^2}/\mathcal{R}_n$. Some unexpected connections with the theory of hyperplane arrangements and graph theory are discovered;

(ii) investigate a variety of descendent relations coming from the defining relations. Some polynomials with interesting combinatorial properties are naturally appear.

To keep the size of the present paper reasonable, several new results are presented as exercises.

We conclude Introduction by a short historical remark. As far as we aware, the commutative version of 3-term relations which provided the framework for a definition of the FK algebra $\mathcal{E}_n$ [45] and a plethora of its generalizations, have been frequently used implicitly in the theory of elliptic functions and related topics, starting at least from the middle of the 19th century, see, e.g., [141] for references, and up to now, and for sure will be used for ever. The key point is that the Kronecker sigma function

$$\sigma_z(w) := \frac{\sigma(z-w)\theta'(0)}{\sigma(z)\sigma(-w)},$$

where $\sigma(z)$ denotes the Weierstrass sigma function, satisfies the quadratic three terms *addition formula* or *functional equation* discovered, as far as we aware, by K. Weierstrass. In fact this functional equation is really equivalent[19] to the famous Jacobi–Riemann three term relation of degree four between the Riemann theta functions $\theta(x)$. In the rational degeneration of theta functions, the three term relation between Kronecker sigma functions turns to the famous three term Jacobi identity which can be treated as an associative analogue of the Jacobi identity in the theory of Lie algebras.

[19] We refer the reader to a nice written paper by Tom H. Koornwinder [79] for more historical information.

To our best knowledge, in an abstract form that is as a set of defining relations in a certain algebra, an anticommutative version of three term relations had been appeared in a remarkable paper by V.I. Arnold [3]. Nowadays these relations are known as *Arnold relations*. These relations and its various generalizations play fundamental role in the theory of arrangements, see, e.g., [113], in topology, combinatorics and many other branches of Mathematics.

In commutative set up abstract form of 3-term relations has been invented by O. Mathieu [96]. In the context of the braided Hopf algebras (of type A) 3-term relations like algebras (as some examples of the Nichols algebras) have appeared in papers by A. Milinski and H.-J. Schneider (2000), N. Andruskiewitsch (2002), S. Madjid (2004), I. Heckenberger (2005) and many others[20].

It is well-known that the Nichols algebra associated with the symmetric group $\mathbb{S}_n$ and trivial conjugacy class is a quotient of the algebra FK_n. It is still an open *problem* to prove (or disprove) that these two algebras are isomorphic.

2 Dunkl elements

Having in mind to fulfill conditions suggested by our guiding line mentioned in Introduction as far as it could be done till now, we are led to introduce the following algebras[21].

Definition 2.1 (additive Dunkl elements)**.** The (additive) Dunkl elements θ_i, $i = 1, \dots, n$, in the algebra $\mathcal{F}_n$ are defined to be

$$\theta_i = x_i + \sum_{\substack{j=1 \\ j \neq i}}^{n} u_{ij}. \tag{2.1}$$

We are interested in to find "natural relations" among the generators $\{u_{ij}\}_{1 \leq i,j \leq n}$ such that the Dunkl elements (2.1) are pair-wise *commute*. One of the natural conditions which is the commonly accepted in the theory of integrable systems, is

- locality conditions:

$$\begin{aligned} &(a) \quad [x_i, x_j] = 0 \qquad \text{if} \ \ i \neq j, \\ &(b) \quad u_{ij}u_{kl} = u_{kl}u_{ij} \qquad \text{if} \ \ i \neq j, \quad k \neq l \ \ \text{and} \ \ \{i,j\} \cap \{k,l\} = \varnothing. \end{aligned} \tag{2.2}$$

Lemma 2.2. *Assume that elements $\{u_{ij}\}$ satisfy the locality condition* (2.1)*. If $i \neq j$, then*

$$[\theta_i, \theta_j] = \Bigg[x_i + \sum_{k \neq i,j} u_{ik}, u_{ij} + u_{ji}\Bigg] + \Bigg[u_{ij}, \sum_{k=1}^{n} x_k\Bigg] + \sum_{k \neq i,j} w_{ijk},$$

where

$$w_{ijk} = [u_{ij}, u_{ik} + u_{jk}] + [u_{ik}, u_{jk}] + [x_i, u_{jk}] + [u_{ik}, x_j] + [x_k, u_{ij}]. \tag{2.3}$$

Therefore in order to ensure that the Dunkl elements form a pair-wise *commuting* family, it's natural to assume that the following conditions hold

[20] We refer the reader to the site https://en.wikipedia.org/wiki/Nichols_algebra for basic definitions and results concerning Nichols' algebras and references on vast literature treated different aspects of the theory of Nichols' algebras and braided Hopf algebras.

[21] Surprisingly enough, in many cases to find relations among the elements $\theta_1, \dots, \theta_n$ there is no need to require that the elements $\{\theta_i\}_{1 \leq i \leq n}$ are pairwise commute.

- unitarity:
$$[u_{ij}+u_{ji},u_{kl}]=0=[u_{ij}+u_{ji},x_k] \qquad \text{for all distinct} \ \ i,\ j,\ k,\ l, \tag{2.4}$$
i.e., the elements $u_{ij}+u_{ji}$ are *central*.

- "conservation laws":
$$\left[\sum_{k=1}^{n} x_k, u_{ij}\right]=0 \qquad \text{for all} \ \ i,\ j, \tag{2.5}$$
i.e., the element $E:=\sum\limits_{k=1}^{n} x_k$ is *central*,

- unitary dynamical classical Yang–Baxter relations:
$$[u_{ij},u_{ik}+u_{jk}]+[u_{ik},u_{jk}]+[x_i,u_{jk}]+[u_{ik},x_j]+[x_k,u_{ij}]=0, \tag{2.6}$$
if i, j, k are pair-wise distinct.

Definition 2.3 (dynamical six term relations algebra $6DT_n$)**.** We denote by $6DT_n$ (and frequently will use also notation DCYB_n) the quotient of the algebra $\mathcal{F}_n$ by the two-sided ideal generated by relations (2.2)–(2.6).

Clearly, the Dunkl elements (2.1) generate a commutative subalgebra inside of the algebra $6DT_n$, and the sum $\sum\limits_{i=1}^{n}\theta_i=\sum\limits_{i=1}^{n}x_i$ belongs to the center of the algebra $6DT_n$.

Remark 2.4. Occasionally we will call the Dunkl elements of the form (2.1) by *dynamical Dunkl elements* to distinguish the latter from *truncated Dunkl elements*, corresponding to the case $x_i=0$, $\forall\, i$.

2.1 Some representations of the algebra $6DT_n$

2.1.1 Dynamical Dunkl elements and equivariant quantum cohomology

(I) (cf. [41]). Given a set $q_1,\dots,q_{n-1}$ of mutually commuting parameters, define
$$q_{ij}=\prod_{a=i}^{j-1}q_a \qquad \text{if} \quad i<j,$$
and set $q_{ij}=q_{ji}$ in the case $i>j$. Clearly, that if $i<j<k$, then $q_{ij}q_{jk}=q_{ik}$.

Let $z_1,\dots,z_n$ be a set of (mutually commuting) variables. Denote by $P_n:=\mathbb{Z}[z_1,\dots,z_n]$ the corresponding ring of polynomials. We consider the variable z_i, $i=1,\dots,n$, also as the operator acting on the ring of polynomials P_n by multiplication on the variable z_i.

Let $s_{ij}\in\mathbb{S}_n$ be the transposition that swaps the letters i and j and fixes the all other letters $k\neq i,j$. We consider the transposition s_{ij} also as the operator which acts on the ring P_n by interchanging z_i and z_j, and fixes all other variables. We denote by
$$\partial_{ij}=\frac{1-s_{ij}}{z_i-z_j}, \qquad \partial_i:=\partial_{i,i+1},$$
the divided difference operators corresponding to the transposition s_{ij} and the simple transposition $s_i:=s_{i,i+1}$ correspondingly. Finally we define operator (cf. [41])
$$\partial_{(ij)}:=\partial_i\cdots\partial_{j-1}\partial_j\partial_{j-1}\cdots\partial_i \qquad \text{if} \ \ i<j.$$
The operators $\partial_{(ij)}$, $1\le i<j\le n$, satisfy (among other things) the following set of relations (cf. [41])

- $[z_j, \partial_{(ik)}] = 0$ if $j \notin [i,k]$, $\left[\partial_{(ij)}, \sum\limits_{a=i}^{j} z_a\right] = 0$,
- $[\partial_{(ij)}, \partial_{(kl)}] = \delta_{jk}[z_j, \partial_{(il)}] + \delta_{il}[\partial_{(kj)}, z_i]$ if $i < j$, $k < l$.

Therefore, if we set $u_{ij} = q_{ij}\partial_{(ij)}$ if $i < j$, and $u_{ij} = -u_{ji}$ if $i > j$, then for a triple $i < j < k$ we will have

$$\begin{aligned}
&[u_{ij}, u_{ik} + u_{jk}] + [u_{ik}, u_{jk}] + [z_i, u_{jk}] + [u_{ik}, z_j] + [z_k, u_{jk}] \\
&\qquad = q_{ij}q_{jk}[\partial_{(ij)}, \partial_{(jk)}] + q_{ik}[\partial_{(ik)}, z_j] = 0.
\end{aligned}$$

Thus the elements $\{z_i, i = 1, \dots, n\}$ and $\{u_{ij}, 1 \le i < j \le n\}$ define a representation of the algebra DCYB_n, and therefore the Dunkl elements

$$\theta_i := z_i + \sum_{j \ne i} u_{ij} = z_i - \sum_{j<i} q_{ji}\partial_{(ji)} + \sum_{j>i} q_{ij}\partial_{(ij)}$$

form a pairwise commuting family of operators acting on the ring of polynomials

$$\mathbb{Z}[q_1, \dots, q_{n-1}][z_1, \dots, z_n],$$

cf. [41]. This representation has been used in [41] to construct the small quantum cohomology ring of the complete flag variety of type A_{n-1}.

(II) Consider degenerate affine Hecke algebra $\mathfrak{H}_n$ generated by the central element h, the elements of the symmetric group $\mathbb{S}_n$, and the mutually commuting elements $y_1, \dots, y_n$, subject to relations

$$s_i y_i - y_{i+1} s_i = h, \quad 1 \le i < n, \qquad s_i y_j = y_j s_i, \quad j \ne i, i+1,$$

where s_i stand for the simple transposition that swaps only indices i and $i+1$. For $i < j$, let $s_{ij} = s_i \cdots s_{j-1} s_j s_{j-1} \cdots s_i$ denotes the permutation that swaps only indices i and j. It is an easy exercise to show that

- $[y_j, s_{ik}] = h[s_{ij}, s_{jk}]$ if $i < j < k$,
- $y_i s_{ik} - s_{ik} y_k = h + h s_{ik} \sum\limits_{i<j<k} s_{jk}$ if $i < k$.

Finally, consider a set of mutually commuting parameters $\{p_{ij}, 1 \le i \ne j \le n, p_{ij} + p_{ji} = 0\}$, subject to the constraints

$$p_{ij}p_{jk} = p_{ik}p_{ij} + p_{jk}p_{ik} + hp_{ik}, \qquad i < j < k.$$

Comments 2.5. If parameters $\{p_{ij}\}$ are *invertible*, and satisfy relations

$$p_{ij}p_{jk} = p_{ik}p_{ij} + p_{jk}p_{ik} + \beta p_{ik}, \qquad i < j < k,$$

then one can rewrite the above displayed relations in the following form

$$1 + \frac{\beta}{p_{ik}} = \left(1 + \frac{\beta}{p_{ij}}\right)\left(1 + \frac{\beta}{p_{jk}}\right), \qquad 1 \le i < j < k \le n.$$

Therefore there exist parameters $\{q_1, \dots, q_n\}$ such that $1 + \beta/p_{ij} = q_i/q_j$, $1 \le i < j \le n$. In other words, $p_{ij} = \frac{\beta q_j}{q_j - q_j}$, $1 \le i < j \le n$. However in general, there are many other types of solutions, for example, solutions related to the Heaviside function[22] $H(x)$, namely, $p_{ij} = H(x_i - x_j)$, $x_i \in \mathbb{R}$, $\forall\, i$, and its discrete analogue, see Example **(III)** below. In the both cases $\beta = -1$; see also Comments 2.12 for other examples.

[22] See https://en.wikipedia.org/wiki/Heaviside_step_function.

To continue presentation of Example **(II)**, define elements $u_{ij} = p_{ij} s_{ij}$, $1 \leq i \neq j \leq n$.

Lemma 2.6 (dynamical classical Yang–Baxter relations)**.**

$$[u_{ij}, u_{ik} + u_{jk}] + [u_{ik}, u_{jk}] + [u_{ik}, y_j] = 0, \qquad 1 < i < j < k \leq n. \tag{2.7}$$

Indeed,

$$u_{ij} u_{jk} = u_{ik} u_{ij} + u_{jk} u_{ik} + h p_{ik} s_{ij} s_{jk}, \qquad u_{jk} u_{ij} = u_{ij} u_{ik} + u_{ik} u_{jk} + h p_{ik} s_{jk} s_{ij},$$

and moreover, $[y_j, u_{ik}] = h p_{ik} [s_{ij}, s_{jk}]$.

Therefore, the elements

$$\theta_i = y_i - h \sum_{j<i} u_{ij} + h \sum_{i<j} u_{ij}, \qquad i = 1, \dots, n,$$

form a mutually commuting set of elements in the algebra $\mathbb{Z}[\{p_{ij}\}] \otimes_{\mathbb{Z}} \mathfrak{H}_n$.

Theorem 2.7. *Define matrix* $M_n = (m_{i,j})_{1 \leq i,j \leq n}$ *as follows*

$$m_{i,j}(u; z_1, \dots, z_n) = \begin{cases} u - z_i & \text{if } i = j, \\ -h - p_{ij} & \text{if } i < j, \\ p_{ij} & \text{if } i > j. \end{cases}$$

Then

$$\mathrm{DET} \left| M_n(u; \theta_1, \dots, \theta_n) \right| = \prod_{j=1}^{n} (u - y_j).$$

Moreover, let us set $q_{ij} := h^2 (p_{ij} + p_{ij}^2) = h^2 q_i q_j (q_i - q_j)^{-2}$, $i < j$, *then*

$$e_k(\theta_1, \dots, \theta_n) = e_k^{(\boldsymbol{q})}(y_1, \dots, y_n), \qquad 1 \leq k \leq n,$$

where $e_k(x_1, \dots, x_n)$ *and* $e_k^{(\boldsymbol{q})}(x_1, \dots, x_n)$ *denote correspondingly the classical and multiparameter quantum* [45] *elementary polynomials*[23]*.*

Let's stress that the elements y_i and θ_j *do not* commute in the algebra $\mathfrak{H}_n$, but the symmetric functions of $y_1, \dots, y_n$, i.e., the center of the algebra $\mathfrak{H}_n$, do.

A few remarks in order. First of all, $u_{ij}^2 = p_{ij}^2$ are central elements. Secondly, in the case $h = 0$ and $y_i = 0$, $\forall\, i$, the equality

$$\mathrm{DET} \left| M_n(u; x_1, \dots, x_n) \right| = u^n$$

describes the set of *polynomial* relations among the Dunkl–Gaudin elements (with the following choice of parameters $p_{ij} = (q_i - q_j)^{-1}$ are taken). And our final remark is that according to [54, Section 8], the quotient ring

$$\mathcal{H}_n^{\boldsymbol{q}} := \mathbb{Q}[y_1, \dots, y_n]^{\mathbb{S}_n} \otimes \mathbb{Q}[\theta_1, \dots, \theta_n] \otimes \mathbb{Q}[h] \Big/ \left\langle M_n(u; \theta_1, \dots, \theta_n) = \prod_{j=1}^{n} (u - y_j) \right\rangle$$

[23] For the reader convenience we remind [45] a definition of the quantum elementary polynomial $e_k^{\boldsymbol{q}}(x_1, \dots, x_n)$. Let $\boldsymbol{q} := \{q_{ij}\}_{1 \leq i < j \leq n}$ be a collection of "quantum parameters", then

$$e_k^{\boldsymbol{q}}(x_1, \dots, x_n) = \sum_{\ell} \sum_{\substack{1 \leq i_1 < \cdots < i_\ell \leq n \\ j_1 > i_1, \dots, j_\ell > i_\ell}} e_{k-2\ell}(X_{\overline{I \cup J}}) \prod_{a=1}^{\ell} q_{i_a, j_a},$$

where $I = (i_1, \dots, i_\ell)$, $J = (j_1, \dots, j_\ell)$ should be distinct elements of the set $\{1, \dots, n\}$, and $X_{\overline{I \cup J}}$ denotes set of variables x_a for which the subscript a is neither one of i_m nor one of the j_m.

is isomorphic to the quantum equivariant cohomology ring of the cotangent bundle $T^*\mathcal{F}l_n$ of the complete flag variety of type A_{n-1}, namely,

$$\mathcal{H}_n^{\boldsymbol{q}} \cong QH^*_{T^n\times\mathbb{C}^*}(T^*\mathcal{F}l_n)$$

with the following choice of quantum parameters: $Q_i := hq_{i+1}/q_i$, $i = 1, \dots, n-1$.

On the other hand, in [75] we computed the so-called *multiparameter deformation* of the equivariant cohomology ring of the complete flag variety of type A_{n-1}.

A deformation defined in [75] depends on parameters $\{q_{ij},\ 1 \le i < j \le n\}$ without any constraints are imposed. For the special choice of parameters

$$q_{ij} := h^2 \frac{q_i\ q_j}{(q_i - q_j)^2}$$

the multiparameter deformation of the equivariant cohomology ring of the type A_{n-1} complete flag variety $\mathcal{F}l_n$ constructed in [75], is isomorphic to the ring $\mathcal{H}_n^{\boldsymbol{q}}$.

Comments 2.8. Let us fix a set of independent parameters $\{q_1, \dots, q_n\}$ and define new parameters

$$\left\{ q_{ij} := hp_{ij}(p_{ij} + h) = h^2 \frac{q_iq_j}{(q_i - q_j)^2} \right\}, \quad 1 \le i < j \le n, \qquad \text{where} \quad p_{ij} = \frac{q_j}{q_i - q_j}, \quad i < j.$$

We set $\deg(q_{ij}) = 2$, $\deg(p_{ij}) = 1$, $\deg(h) = 1$.

The new parameters $\{q_{ij}\}_{1\le i<j\le n}$, do not free anymore, but satisfy rather complicated algebraic relations. We display some of these relations soon, having in mind a question: is there some intrinsic meaning of the algebraic variety defined by the set of defining relations among the "quantum parameters" $\{q_{ij}\}$?

Let us denote by $\mathcal{A}_{n,h}$ the quotient ring of the ring of polynomials $\mathbb{Q}[h][x_{ij},\ 1 \le i < j \le n]$ modulo the ideal generating by polynomials $f(x_{ij})$ such that the specialization $x_{ij} = q_{ij}$ of a polynomial $f(x_{ij})$, namely $f(q_{ij})$, is equal to zero. The algebra $\mathcal{A}_{n,h}$ has a natural filtration, and we denote by $\mathcal{A}_n = \operatorname{gr} \mathcal{A}_{n,h}$ the corresponding associated graded algebra.

To describe (a part of) relations among the parameters $\{q_{ij}\}$ let us observe that parameters $\{p_{ij}\}$ and $\{q_{ij}\}$ are related by the following identity

$$q_{ij}q_{jk} - q_{ik}(q_{ij} + q_{jk}) + h^2 q_{ik} = 2p_{ij}p_{ik}p_{jk}(p_{ik} + h) \qquad \text{if} \quad i < j < k.$$

Using this identity we can find the following relations among parameters in question

$$\begin{aligned} &q_{ij}^2q_{jk}^2 + q_{ij}^2q_{ik}^2 + h^4q_{ik}^2q_{jk}^2 - 2q_{ij}q_{ik}q_{jk}(q_{ij} + q_{jk} + q_{ik}) \\ &\qquad - 2h^2q_{ik}(q_{ij}q_{jk} + q_{ij}q_{ik} + q_{jk}q_{ik}) = 8hq_{ij}q_{ik}q_{jk}p_{ik}, \end{aligned} \tag{2.8}$$

if $1 \le i < j < k \le n$.

Finally, we come to a relation of degree 8 among the "quantum parameters" $\{q_{ij}\}$

$$\big(\text{l.h.s. of (2.8)}\big)^2 = 64h^2q_{ij}^2q_{ik}^3q_{jk}^2, \qquad 1 \le i < j < k \le n.$$

There are also higher degree relations among the parameters $\{q_{ij}\}$ some of whose in degree 16 follow from the deformed Plücker relation between parameters $\{p_{ij}\}$:

$$\frac{1}{p_{ik}p_{jl}} = \frac{1}{p_{ij}p_{kl}} + \frac{1}{p_{il}p_{jk}} + \frac{h}{p_{ij}p_{jk}p_{kl}}, \qquad i < j < k < l.$$

However, we don't know how to describe the algebra $\mathcal{A}_{n,h}$ generated by quantum parameters $\{q_{ij}\}_{1\le i<j\le n}$ even for $n=4$.

The algebra $\mathcal{A}_n = \mathrm{gr}(\mathcal{A}_{n,h})$ is isomorphic to the quotient algebra of $\mathbb{Q}[x_{ij},\ 1\le i<j\le n]$ modulo the ideal generated by the set of relations between "quantum parameters"

$$\left\{\overline{q}_{ij} := \left(\frac{1}{z_i - z_j}\right)^2\right\}_{1\le i<j\le n},$$

which correspond to the Dunkl–Gaudin elements $\{\theta_i\}_{1\le i\le n}$, see Section 3.2 below for details. In this case the parameters $\{\overline{q}_{ij}\}$ satisfy the following relations

$$\overline{q}_{ij}^2\overline{q}_{jk}^2 + \overline{q}_{ij}^2\overline{q}_{ik}^2 + \overline{q}_{jk}^2\overline{q}_{ik}^2 = 2\overline{q}_{ij}\overline{q}_{ik}\overline{q}_{jk}(\overline{q}_{ij} + \overline{q}_{jk} + \overline{q}_{jk})$$

which correspond to the relations (2.8) in the special case $h=0$. One can find a set of relations in degrees 6, 7 and 8, namely for a given pair-wise distinct integers $1\le i,j,k,l\le n$, one has

- one relation in degree 6

$$\begin{aligned}&\overline{q}_{ij}^2\overline{q}_{ik}^2\overline{q}_{il}^2 + \overline{q}_{ij}^2\overline{q}_{jk}^2\overline{q}_{jl}^2 + \overline{q}_{ik}^2\overline{q}_{jk}^2\overline{q}_{kl}^2 + \overline{q}_{il}^2\overline{q}_{jl}^2\overline{q}_{kl}^2\\ &\quad - 2\overline{q}_{ij}\overline{q}_{ik}\overline{q}_{il}\overline{q}_{jk}\overline{q}_{jl}\overline{q}_{kl}\left(\frac{\overline{q}_{ij}}{\overline{q}_{kl}} + \frac{\overline{q}_{kl}}{\overline{q}_{ij}} + \frac{\overline{q}_{ik}}{\overline{q}_{jl}} + \frac{\overline{q}_{jl}}{\overline{q}_{ik}} + \frac{\overline{q}_{il}}{\overline{q}_{jk}} + \frac{\overline{q}_{jk}}{\overline{q}_{il}}\right)\\ &\quad + 8\overline{q}_{ij}\overline{q}_{ik}\overline{q}_{il}\overline{q}_{jk}\overline{q}_{jl}\overline{q}_{kl} = 0;\end{aligned}$$

- three relations in degree 7

$$\begin{aligned}&\overline{q}_{ik}\big(\overline{q}_{ij}\overline{q}_{il}\overline{q}_{kl} - \overline{q}_{ij}\overline{q}_{il}\overline{q}_{jk} + \overline{q}_{ij}\overline{q}_{jk}\overline{q}_{kl} - \overline{q}_{il}\overline{q}_{jk}\overline{q}_{kl}\big)^2\\ &\quad = 8\overline{q}_{ij}^2\overline{q}_{ik}^2\overline{q}_{jk}\overline{q}_{kl}\big(\overline{q}_{jk} + \overline{q}_{jl} + \overline{q}_{kl}\big) - 4\overline{q}_{ij}^2\overline{q}_{il}^2\overline{q}_{jl}\big(\overline{q}_{jk}^2 + \overline{q}_{kl}^2\big),\end{aligned}$$

- one relation in degree 8

$$\overline{q}_{ij}^2\overline{q}_{il}^2\overline{q}_{jk}^2\overline{q}_{kl}^2 + \overline{q}_{ij}^2\overline{q}_{ik}^2\overline{q}_{jl}^2\overline{q}_{kl}^2 + \overline{q}_{ik}^2\overline{q}_{il}^2\overline{q}_{jk}^2\overline{q}_{jl}^2 = 2\overline{q}_{ij}\overline{q}_{ik}\overline{q}_{il}\overline{q}_{jk}\overline{q}_{jl}\overline{q}_{kl}\big(\overline{q}_{ij}\overline{q}_{kl} + \overline{q}_{ik}\overline{q}_{jl} + \overline{q}_{il}\overline{q}_{jk}\big),$$

However we don't know does the list of relations displayed above, contains the all independent relations among the elements $\{\overline{q}_{ij}\}_{1\le i<j\le n}$ in degrees 6, 7 and 8, even for $n=4$. In degrees ≥ 9 and $n\ge 5$ some independent relations should appear.

Notice that the parameters $\left\{p_{ij} = \frac{hq_j}{q_i-q_j},\ i<j\right\}$ satisfy the so-called *Gelfand–Varchenko* relations, see, e.g., [67]

$$p_{ij}p_{jk} = p_{ik}p_{ij} + p_{jk}p_{ik} + hp_{ik}, \qquad i<j<k,$$

whereas parameters $\left\{\overline{p}_{ij} = \frac{1}{q_i-q_j},\ i<j\right\}$ satisfy the so-called *Arnold* relations

$$\overline{p}_{ij}\overline{p}_{jk} = \overline{p}_{ik}\overline{p}_{ij} + \overline{p}_{jk}\overline{p}_{ik}, \qquad i<j<k.$$

Project 2.9. *Find Hilbert series* $\mathrm{Hilb}(\mathcal{A}_n,t)$ *for* $n\ge 4$.[24]

[24] This is a particular case of more general problem we are interested in. Namely, let $\{f_\alpha \in \mathbb{R}[x_1,\dots,x_n]\}_{1\le\alpha\le N}$ be a collection of linear forms, and $k\ge 2$ be an integer. Denote by $I(\{f_\alpha\})$ the ideal in the ring of polynomials $\mathbb{R}[z_1,\dots,z_N]$ generated by polynomials $\Phi(z_1,\dots,z_N)$ such that

$$\Phi\big(f_1^{-k},\dots,f_N^{-k}\big) = 0.$$

Compute the Hilbert series (polynomial?) of the quotient algebra $\mathbb{R}[z_1,\dots,z_N]/I(\{f_\alpha\})$.

For example, $\mathrm{Hilb}(\mathcal{A}_3,t)=\frac{(1+t)(1+t^2)}{(1-t)^2}$.

Finally, if we set $q_i:=\exp(hz_i)$ and take the limit $\lim\limits_{h\to 0}\frac{h^2q_iq_j}{(q_i-q_j)^2}$, as a result we obtain the Dunkl–Gaudin parameter $\overline{q}_{ij}=\frac{1}{(z_i-z_j)^2}$.

(III) Consider the following representation of the degenerate affine Hecke algebra $\mathfrak{H}_n$ on the ring of polynomials $P_n=\mathbb{Q}[x_1,\dots,x_n]$:

- the symmetric group $\mathbb{S}_n$ acts on P_n by means of operators

$$\overline{s}_i=1+(x_{i+1}-x_i-h)\partial_i,\qquad i=1,\dots,n-1,$$

- y_i acts on the ring P_n by multiplication on the variable x_i: $y_i(f(x))=x_if(x)$, $f\in P_n$. Clearly,

$$y_i\overline{s_i}-y_{i+1}\overline{s_i}=h\qquad\text{and}\qquad y_i(\overline{s}_i-1)=(\overline{s}_i-1)y_{i+1}+x_{i+1}-x_i-h.$$

In the subsequent discussion we will identify the operator of multiplication by the variable x_i, namely the operator y_i, with x_i.

This time define $u_{ij}=p_{ij}(\overline{s}_i-1)$, if $i<j$ and set $u_{ij}=-u_{ji}$ if $i>j$, where parameters $\{p_{ij}\}$ satisfy the same conditions as in the previous example.

Lemma 2.10. *The elements $\{u_{ij},\ 1\le i<j\le n\}$, satisfy the dynamical classical Yang–Baxter relations displayed in Lemma 2.6, equation (2.7).*

Therefore, the Dunkl elements

$$\overline{\theta}_i:=x_i+\sum_{\substack{j\\ j\neq i}}u_{ij},\qquad i=1,\dots,n,$$

form a commutative set of elements.

Theorem 2.11 ([54])**.** *Define matrix $\overline{M}_n=(\overline{m}_{ij})_{1\le i,j\le n}$ as follows*

$$\overline{m}_{i,j}(u;z_1,\dots,z_n)=\begin{cases}u-z_i+\sum\limits_{j\neq i}hp_{ij} & \text{if } i=j,\\ -h-p_{ij} & \text{if } i<j,\\ p_{ij} & \text{if } i>j.\end{cases}$$

Then

$$\mathrm{DET}\left|\overline{M}_n(u;\overline{\theta}_1,\dots,\overline{\theta}_n)\right|=\prod_{j=1}^{n}(u-x_j).$$

Comments 2.12. Let us list a few more representations of the *dynamical* classical Yang–Baxter relations.

- Trigonometric Calogero–Moser representation. Let $i<j$, define

$$u_{ij}=\frac{x_j}{x_i-x_j}(s_{ij}-\epsilon),\qquad \epsilon=0\text{ or }1,$$
$$s_{ij}(x_i)=x_j,\qquad s_{ij}(x_j)=x_i,\qquad s_{ij}(x_k)=x_k,\qquad \forall\,k\neq i,j.$$

- Mixed representation:

$$u_{ij} = \left(\frac{\lambda_j}{\lambda_i - \lambda_j} - \frac{x_j}{x_i - x_j} \right) (s_{ij} - \epsilon), \qquad \epsilon = 0 \text{ or } 1, \qquad s_{ij}(\lambda_k) = \lambda_k, \qquad \forall\, k.$$

We set $u_{ij} = -u_{ji}$, if $i > j$. In all cases we define Dunkl elements to be $\theta_i = \sum_{j \neq i} u_{ij}$.

Note that operators

$$r_{ij} = \left(\frac{\lambda_i + \lambda_j}{\lambda_i - \lambda_j} - \frac{x_i + x_j}{x_i - x_j} \right) s_{ij}$$

satisfy the three term relations: $r_{ij} r_{jk} = r_{ik} r_{ij} + r_{jk} r_{ik}$, and $r_{jk} r_{ij} = r_{ij} r_{jk} + r_{ik} r_{jk}$, and thus satisfy the *classical* Yang–Baxter relations.

2.1.2 Step functions and the Dunkl–Uglov representations of the degenerate affine Hecke algebras [138]

Consider step functions $\eta^{\pm} \colon \mathbb{R} \longrightarrow \{0,1\}$

$$\text{(Heaviside function)} \qquad \eta^{+}(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0, \end{cases} \qquad \eta^{-}(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

For any two real numbers x_i and x_j set $\eta_{ij}^{\pm} = \eta^{\pm}(x_i - x_j)$.

Lemma 2.13. *The functions η_{ij} satisfy the following relations*

$$\eta_{ij}^{\pm} + \eta_{ji}^{\pm} = 1 + \delta_{x_i,x_j}, \qquad (\eta_{ij}^{\pm})^2 = \eta_{ij}^{\pm},$$
$$\eta_{ij}^{\pm} \eta_{jk}^{\pm} = \eta_{ik}^{\pm} \eta_{ij}^{\pm} + \eta_{jk}^{\pm} \eta_{ik}^{\pm} - \eta_{ik}^{\pm},$$

where $\delta_{x,y}$ denotes the Kronecker delta function.

To introduce the Dunkl–Uglov operators [138] we need a few more definitions and notation. To start with, denote by $\Delta_i^{\pm}$ the finite difference operators: $\Delta_i^{\pm}(f)(x_1, \ldots, x_n) = f(\ldots, x_i \pm 1, \ldots)$. Let as before, $\{s_{ij},\, 1 \leq i \neq j \leq n,\, s_{ij} = s_{ji}\}$, denotes the set of transpositions in the symmetric group $\mathbb{S}_n$. Recall that $s_{ij}(x_i) = x_j$, $s_{ij}(x_k) = x_k$, $\forall\, k \neq i, j$. Finally define Dunkl–Uglov operators $d_i^{\pm} \colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ to be

$$d_i^{\pm} = \Delta_i^{\pm} + \sum_{j<i} \delta_{x_i,x_j} - \sum_{j<i} \eta_{ji}^{\pm} s_{ij} + \sum_{j>i} \eta_{ij}^{\pm} s_{ij}.$$

To simplify notation, set $u_{ij}^{\pm} := \eta_{ij}^{\pm} s_{ij}$ if $i < j$, and $\widetilde{\Delta}_i^{\pm} = \Delta_i^{\pm} + \sum_{j<i} \delta_{x_i,x_j}$.

Lemma 2.14. *The operators $\{u_{ij}^{\pm},\, 1 \leq i < j \leq n\}$ satisfy the following relations*

$$\left[u_{ij}^{\pm}, u_{ik}^{\pm} + u_{jk}^{\pm}\right] + \left[u_{ik}^{\pm}, u_{jk}^{\pm}\right] + \left[u_{ik}^{\pm}, \sum_{j<i} \delta_{x_i,x_j}\right] = 0 \qquad \textit{if } \; i < j < k.$$

From now on we *assume that* $x_i \in \mathbb{Z}$, $\forall\, i$, that is, we will work with the restriction of the all operators defined at beginning of Example 2.28(c), to the subset $\mathbb{Z}^n \subset \mathbb{R}^n$. It is easy to see that under the assumptions $x_i \in \mathbb{Z}$, $\forall\, i$, we will have

$$\Delta_j^{\pm} \eta_{ij}^{\pm} = (\eta_{ij}^{\pm} \mp \delta_{x_i,x_j}) \Delta_i^{\pm}. \tag{2.9}$$

Moreover, using relations (2.12), (2.13) one can prove that

Lemma 2.15.

- $[u_{ij}^{\pm}, \widetilde{\Delta}_i^{\pm} + \widetilde{\Delta}_j^{\pm}] = 0$,
- $[u_{ik}^{\pm}, \widetilde{\Delta}_j^{\pm}] = \big[u_{ik}^{\pm}, \sum\limits_{j<i} \delta_{x_i,x_j}\big]$, $i < j < k$.

Corollary 2.16.

- *The operators* $\{u_{ij}^{\pm}, 1 \le i < j < k \le n\}$, *and* $\widetilde{\Delta}_i^{\pm}$, $i = 1, \dots, n$ *satisfy the dynamical classical Yang–Baxter relations*

$$\big[u_{ij}^{\pm}, u_{ik}^{\pm} + u_{jk}^{\pm}\big] + \big[u_{ik}^{\pm}, u_{jk}^{\pm}\big] + \big[u_{ik}^{\pm}, \widetilde{\Delta}_j\big] = 0 \qquad \text{if} \;\; i < j < k.$$

- *The operators* $\{s_i := s_{i,i+1}, 1 \le i < n$, *and* $\widetilde{\Delta}_j^{\pm}$, $1 \le j \le n\}$ *give rise to two representations of the degenerate affine Hecke algebra* $\mathfrak{H}_n$. *In particular, the Dunkl–Uglov operators are mutually commute:* $[d_i^{\pm}, d_j^{\pm}] = 0$ [138].

2.1.3 Extended Kohno–Drinfeld algebra and Yangian Dunkl–Gaudin elements

Definition 2.17. Extended Kohno–Drinfeld algebra is an associative algebra over $\mathbb{Q}[\beta]$ generated by the elements $\{z_1, \dots, z_n\}$ and $\{y_{ij}\}_{1 \le i \ne j \le n}$ subject to the set of relations

(i) The elements $\{y_{ij}\{_{1 \le i \ne j \le n}$ satisfy the Kohno–Drinfeld relations

- $y_{ij} = y_{ji}$, $[y_{ij}, y_{kl}] = 0$ if i, j, k, l are distinct,
- $[y_{ij}, y_{ik} + y_{jk}] = 0 = [y_{ij} + y_{ik}, y_{jk}]$ if $i < j < k$.

(ii) The elements $z_1, \dots, z_n$ generate the free associative algebra $\mathcal{F}_n$.

(iii) Crossing relations:

- $[z_i, y_{jk}] = 0$ if $i \ne j, k$, $[z_i, z_j] = \beta[y_{ij}, z_i]$ if $i \ne j$.

To define the (Yangian) Dunkl–Gaudin elements, cf. [54], let us consider a set of elements $\{p_{ij}\}_{1 \le i \ne j \le n}$ subject to relations

- $p_{ij} + p_{ji} = \beta$, $[p_{ij}, y_{kl}] = 0 = [p_{ij}, z_k]$ for all i, j, k,
- $p_{ij} p_{jk} = p_{ik}(p_{jk} - p_{ji})$ if $i < j < k$.

Let us set $u_{ij} = p_{ij} y_{ij}$, $i \ne j$, and define the (Yangian) Dunkl–Gaudin elements as follows

$$\theta_i = z_i + \sum_{j \ne i} u_{ij}, \qquad i = 1, \dots, n.$$

Proposition 2.18 (cf. [54, Lemma 3.5])**.** *The elements* $\theta_1, \dots, \theta_n$ *form a mutually commuting family.*

Indeed, let $i < j$, then

$$\begin{aligned}[\theta_i, \theta_j] &= [z_i, z_j] + \beta[z_i, y_{ij}] + p_{ij}[y_{ij}, z_i + z_j] \\ &\quad + \sum_{k \ne i,j} \big(p_{ik} p_{jk}\big[y_{ij} + y_{ik}, y_{jk}\big] + p_{ik} p_{ji}\big[y_{ij}, y_{ik} + y_{jk}\big]\big) = 0.\end{aligned}$$

A representation of the extended Kohno–Drinfeld algebra has been constructed in [54], namely one can take

$$y_{ij} := T_{ij}^{(1)} T_{ji}^{(1)} - T_{jj}^{(1)} = y_{ji}, \qquad z_i := \beta T_{ii}^{(2)} - \frac{\beta}{2} T_{ii}^{(1)}\big(T_{ii}^{(1)} - 1\big),$$

$$p_{ij} := \frac{\beta q_j}{q_i - q_j}, \qquad i \neq j,$$

where $q_1, \ldots, q_n$ stands for a set of mutually commuting *quantum* parameters, and $\{T_{ij}^{(s)}\}_{\substack{1 \le i,j \le n \\ s \in \mathbb{Z}_{\ge 0}}}$ denotes the set of generators of the Yangian $Y(\mathfrak{gl}_n)$, see, e.g., [106].

A proof that the elements $\{z_i\}_{1 \le i \le n}$ and $\{y_{ij}\}_{1 \le i \neq j \le n}$ satisfy the extended Kohno–Drinfeld algebra relations is based on the following relations, see, e.g., [54, Section 3],

$$\big[T_{ij}^{(1)}, T_{kl}^{(s)}\big] = \delta_{il} T_{kj}^{(s)} - \delta_{jk} T_{il}^{(s)}, \qquad i,j,k,l = 1, \ldots, n, \qquad s \in \mathbb{Z}_{\ge 0}.$$

2.2 "Compatible" Dunkl elements, Manin matrices and algebras related with weighted complete graphs rK_n

Let us consider a collection of generators $\{u_{ij}^{(\alpha)},\ 1 \le i,j \le n,\ \alpha = 1, \ldots, r\}$, subject to the following relations

- either the unitarity (the case of sign "+") or the symmetry relations (the case of sign "−")[25]

$$u_{ij}^{(\alpha)} \pm u_{ji}^{(\alpha)} = 0, \qquad \forall\, \alpha, i, j, \tag{2.10}$$

- *local* 3-*term relations*:

$$u_{ij}^{(\alpha)} u_{jk}^{(\alpha)} + u_{jk}^{(\alpha)} u_{ki}^{(\alpha)} + u_{ki}^{(\alpha)} u_{ij}^{(\alpha)} = 0, \qquad i,j,k \ \text{ are distinct}, \quad 1 \le \alpha \le r. \tag{2.11}$$

We define *global* 3-term relations algebra $3T_{n,r}^{(\pm)}$ as "compatible product" of the local 3-term relations algebras. Namely, we require that the elements

$$U_{ij}^{(\boldsymbol{\lambda})} := \sum_{\alpha=1}^{r} \lambda_\alpha u_{ij}^{(\alpha)}, \qquad 1 \le i,j \le n,$$

satisfy the global 3-term relations

$$U_{ij}^{(\boldsymbol{\lambda})} U_{jk}^{(\boldsymbol{\lambda})} + U_{jk}^{(\boldsymbol{\lambda})} U_{ki}^{(\boldsymbol{\lambda})} + U_{ki}^{(\boldsymbol{\lambda})} U_{ij}^{(\boldsymbol{\lambda})} = 0$$

for all values of parameters $\{\lambda_i \in \mathbb{R},\ 1 \le \alpha \le r\}$.

It is easy to check that our request is equivalent to a validity of the following sets of relations among the generators $\{u_{ij}^{(\alpha)}\}$

(a) *local* 3-*term relations*: $u_{ij}^{(\alpha)} u_{jk}^{(\alpha)} + u_{jk}^{(\alpha)} u_{ki}^{(\alpha)} + u_{ki}^{\alpha)} u_{ij}^{(\alpha)} = 0$,

(b) 6-*term crossing relations*:

$$u_{ij}^{(\alpha)} u_{jk}^{(\beta)} + u_{ij}^{(\beta)} u_{jk}^{(\alpha)} + u_{k,i}^{(\alpha)} u_{ij}^{(\beta)} u_{ki}^{(\alpha)} + u_{jk}^{(\alpha)} u_{ki}^{(\beta)} + u_{jk}^{(\beta)} u_{ki}^{(\alpha)} = 0,$$

i, j, k are distinct, $\alpha \neq \beta$.

[25]More generally one can impose the q-symmetry conditions

$$u_{ij} + q u_{ji} = 0, \qquad 1 \le i < j \le n$$

and ask about relations among the local Dunkl elements to ensure the commutativity of the global ones. As one might expect, the matrix $Q := \big(\theta_j^{(\alpha)}\big)_{\substack{1 \le a \le r \\ 1 \le j \le n}}$ composed from the local Dunkl elements should be a q-Manin matrix. See, e.g., [25], or https://en.wikipedia.org/wiki/Manin.matrix for a definition and basic properties of the latter.

Now let us consider *local* Dunkl elements

$$\theta_i^{(\alpha)} := \sum_{j\neq i} u_{ij}^{(\alpha)}, \qquad j=1,\dots,n, \quad \alpha=1,\dots,r.$$

It follows from the local 3-term relations (2.11) that for a fixed $\alpha\in[1,r]$ the local Dunkl elements $\big\{\theta_i^{(\alpha)}\big\}_{\substack{1\le i\le n\\ 1\le\alpha\le r}}$ either mutually commute (the sign "+"), or pairwise anticommute (the sign "−"). Similarly, the global 3-term relations imply that the global Dunkl elements

$$\theta_i^{(\lambda)} := \lambda_1\theta_i^{(1)}+\dots+\lambda_r\theta_i^{(r)} = \sum_{j\neq i} U_{ij}^{(\lambda)}, \qquad i=1,\dots,n,$$

also either mutually commute (the case "+") or pairwise anticommute (the case "−").

Now we are looking for a set of relations among the local Dunkl elements which is a consequence of the commutativity (anticommutativity) of the global Dunkl elements. It is quite clear that if $i<j$, then

$$\big[\theta_i^{(a)},\theta_j^{(b)}\big]_{\pm} = \sum_{a=1}^{r}\lambda_a^2\big[\theta_i^{(a)},\theta_j^{(a)}\big]_{\pm} + \sum_{1\le a<b\le r}\lambda_a\lambda_b\big(\big[\theta_i^{(a)},\theta_j^{(b)}\big]_{\pm}+\big[\theta_i^{(b)},\theta_j^{(a)}\big]_{\pm}\big),$$

and the commutativity (or anticommutativity) of the global Dunkl elements for all $(\lambda_1,\dots,\lambda_r)\in\mathbb{R}^r$ is equivalent to the following set of relations

- $[\theta_i^{(a)},\theta_j^{(a)}]_{\pm}=0$,
- $[\theta_i^{(a)},\theta_j^{(b)}]_{\pm}+[\theta_i^{(b)},\theta_j^{(a)}]_{\pm}=0$, $a<b$ and $i<j$, where by definition we set $[a,b]_{\pm}:=ab\mp ba$.

In other words, the matrix $\Theta_n := \big(\theta_i^{(a)}\big)_{\substack{1\le a\le r\\ 1\le i\le n}}$ should be either a *Manin matrix* (the case "+"), or its super analogue (the case "−"). Clearly enough that a similar construction can be applied to the algebras studied in Section 2, **I**–**III**, and thus it produces some interesting examples of the Manin matrices. It is an interesting *problem* to describe the algebra generated by the local Dunkl elements $\big\{\theta_i^{(a)}\big\}_{\substack{1\le a\le r\\ 1\le i\le n}}$ and a commutative subalgebra generated by the global Dunkl elements inside the former. It is also an interesting *question* whether or not the coefficients $C_1,\dots,C_n$ of the column characteristic polynomial $\operatorname{Det}^{\mathrm{col}}|\Theta_n-tI_n| = \sum_{k=0}^{n} C_k t^{n-k}$ of the Manin matrix Θ_n generate a commutative subalgebra? For a definition of the column determinant of a matrix, see, e.g., [25].

However a close look at this problem and the question posed needs an additional treatment and has been omitted from the content of the present paper.

Here we are looking for a "natural conditions" to be imposed on the set of generators $\{u_{ij}^{\alpha}\}_{\substack{1\le\alpha\le r\\ 1\le i,j\le n}}$ in order to ensure that the local Dunkl elements satisfy the commutativity (or anticommutativity) relations:

$$\big[\theta_i^{(\alpha)},\theta_j^{(\beta)}\big]_{\pm}=0, \qquad \text{for all } 1\le i<j\le n, \qquad 1\le\alpha,\beta\le r.$$

The "natural conditions" we have in mind are

- *locality relations*:

$$\big[u_{ij}^{(\alpha)},u_{kl}^{(\beta)}\big]_{\pm}=0, \tag{2.12}$$

- *twisted classical Yang–Baxter relations*:

$$\left[u_{ij}^{(\alpha)}, u_{jk}^{(\beta)}\right]_{\pm} + \left[u_{ik}^{(\alpha)}, u_{ji}^{(\beta)}\right]_{\pm} + \left[u_{ik}^{(\alpha)}, u_{jk}^{(\beta)}\right]_{\pm} = 0, \tag{2.13}$$

if i, j, k, l are distinct and $1 \le \alpha, \beta \le r$.

Finally we define a multiple analogue of the three term relations algebra, denoted by $3T^{\pm}(rK_n)$, to be the quotient of the global 3-term relations algebra $3T^{\pm}_{n,r}$ modulo the two-sided ideal generated by the left hand sides of relations (2.12), (2.13) and that of the following relations

- $\left(u_{ij}^{(\alpha)}\right)^2 = 0$, $\left[u_{ij}^{(\alpha)}, u_{ij}^{(\beta)}\right]_{\pm} = 0$, for all $i \ne j$, $\alpha \ne \beta$.

The outputs of this construction are

- commutative (or anticommutative) quadratic algebra $3T^{(\pm)}(rK_n)$ generated by the elements $\left\{u_{ij}^{(\alpha)}\right\}_{\substack{1\le i<j\le n\\ \alpha=1,\dots,r}}$,
- a family of nr either mutually commuting (the case "+"), or pair-wise anticommuting (the case "−") local Dunkl elements $\left\{\theta_i^{(\alpha)}\right\}_{\substack{i=1,\dots,n\\ \alpha=1,\dots,r}}$.

We *expect* that the subalgebra generated by local Dunkl elements in the algebra $3T^{+}(rK_n)$ is closely related (isomorphic for $r = 2$) with the coinvariant algebra of the diagonal action of the symmetric group $\mathbb{S}_n$ on the ring of polynomials $\mathbb{Q}\left[X_n^{(1)}, \dots, X_n^{(r)}\right]$, where $X_n^{(j)}$ stands for the set of variables $\left\{x_1^{(j)}, \dots, x_n^{(j)}\right\}$. The algebra $3T^{-}(2K_n)^{\mathrm{anti}}$ has been studied in [72] and [12]. In the present paper we state only our old conjecture.

Conjecture 2.19 (A.N. Kirillov, 2000)**.**

$$\mathrm{Hilb}\left(3T^{-}(3K_n)^{\mathrm{anti}}, t\right) = (1+t)^n(1+nt)^{n-2},$$

where for any algebra A we denote by A^{anti} the quotient of algebra A by the two-sided ideal generated by the set of anticommutators $\{ab + ba \mid (a,b) \in A \times A\}$.

According to observation of M. Haiman [55], the number $2^n(n+1)^{n-2}$ is thought of as being equal to the dimension of the space of triple coinvariants of the symmetric group $\mathbb{S}_n$.

2.3 Miscellany

2.3.1 Non-unitary dynamical classical Yang–Baxter algebra DCYB_n

Let $\widetilde{\mathcal{A}_n}$ be the quotient of the algebra $\mathfrak{F}_n$ by the two-sided ideal generated by the relations (2.2), (2.5) and (2.6). Consider elements

$$\theta_i = x_i + \sum_{a\ne i} u_{ia} \qquad \text{and} \qquad \bar\theta_j = -x_j + \sum_{b\ne j} u_{bj}, \qquad 1 \le i < j \le n.$$

Clearly, if $i < j$, then

$$[\theta_i, \bar\theta_j] + [x_i, x_j] = \left[\sum_{k=1}^{n} x_k, u_{ij}\right] + \sum_{k\ne i,j} w_{ikj},$$

where the elements w_{ijk}, $i < j$, have been defined in Lemma 2.2, equation (2.3).

Therefore the elements θ_i and $\bar\theta_j$ *commute* in the algebra $\widetilde{A}_n$.

In the case when $x_i = 0$ for all $i = 1, \ldots, n$, the relations

$$w_{ijk} := [u_{ij}, u_{ik} + u_{jk}] + [u_{ik}, u_{jk}] = 0 \qquad \text{if } i,\ j,\ k \text{ are all distinct,}$$

are well-known as the *non-unitary classical Yang–Baxter relations*. Note that for a given triple of pair-wise distinct (i, j, k) one has in fact 6 relations. These six relations imply that $[\theta_i, \bar{\theta}_j] = 0$. However, in general,

$$[\theta_i, \theta_j] = \left[\sum_{k \neq i,j} u_{ik}, u_{ij} + u_{ji}\right] \neq 0.$$

Dynamical classical Yang–Baxter algebra DCYB_n**.** In order to ensure the commutativity relations among the Dunkl elements (2.1), i.e., $[\theta_i, \theta_j] = 0$ for all i, j, let us remark that if $i \neq j$, then

$$[\theta_, \theta_j] = [x_i + u_{ij}, x_j + u_{ji}] + [x_i + x_j, u_{ij}] + \left[u_{ij}, \sum_{k=1}^{n} x_k\right] + \sum_{\substack{k=1 \\ k \neq i,j}}^{n} [u_{ij} + u_{ik}, u_{jk}] + [u_{ik}, u_{ji}] + [x_i, u_{jk}] + [u_{ik}, x_j] + [x_k, u_{ij}].$$

Definition 2.20. Define *dynamical non-unitary classical Yang–Baxter algebra* DNUCYB_n to be the quotient of the free associative algebra $\mathbb{Q}\langle\{x_i,\ 1 \le i \le n\},\ \{u_{ij}\}_{1 \le i \neq j \le n}\rangle$ by the two-sided ideal generated by the following set of relations

- zero curvature conditions:

$$[x_i + u_{ij}, x_j + u_{ji}] = 0, \qquad 1 \le i \neq j \le n, \tag{2.14}$$

- conservation laws conditions:

$$\left[u_{ij}, \sum_{k=1}^{n} x_k\right] = 0 \qquad \text{for all} \quad i \neq j, k.$$

- crossing relations:

$$[x_i + x_j, u_{ij}] = 0, \qquad i \neq j.$$

- twisted dynamical classical Yang–Baxter relations:

$$[u_{ij} + u_{ik}, u_{jk}] + [u_{ik}, u_{ji}] + [x_i, u_{jk}] + [u_{ik}, x_j] + [x_k, u_{ij}] = 0,$$

  i, j, k are distinct.

It is easy to see that the twisted classical Yang–Baxter relations

$$[u_{ij} + u_{ik}, u_{jk}] + [u_{ik}, u_{ji}] = 0, \qquad i, j, k \quad \text{are distinct,} \tag{2.15}$$

for a fixed triple of distinct indices i, j, k contain in fact 3 different relations whereas the non-unitary classical Yang–Baxter relations

$$[u_{ij} + u_{ik}, u_{jk}] + [u_{ij}, u_{ik}], \qquad i, j, k \quad \text{are distinct,}$$

contain 6 different relations for a fixed triple of distinct indices i, j, k.

Definition 2.21.

- Define *dynamical classical Yang–Baxter algebra* DCYB_n to be the quotient of the algebra DNUCYB_n by the two-sided ideal generated by the elements
$$\sum_{k \neq i,j} [u_{ik}, u_{ij} + u_{ji}] \qquad \text{for all} \ \ i \neq j.$$
- Define *classical Yang–Baxter algebra* CYB_n to be the quotient of the dynamical classical Yang–Baxter algebra DCYB_n by the set of relations
$$x_i = 0 \qquad \text{for} \ \ i = 1, \dots, n.$$

Example 2.22. Define

$$p_{ij}(z_1, \dots, z_n) = \begin{cases} \dfrac{z_i}{z_i - z_j} & \text{if } 1 \le i < j \le n, \\ -\dfrac{z_j}{z_j - z_i} & \text{if } n \ge i > j \ge 1. \end{cases}$$

Clearly, $p_{ij} + p_{ji} = 1$. Now define operators $u_{ij} = p_{ij} s_{ij}$, and the truncated Dunkl operators to be $\theta_i = \sum_{j \neq i} u_{ij}$, $i = 1, \dots, n$. All these operators act on the field of rational functions $\mathbb{Q}(z_1, \dots, z_n)$; the operator $s_{ij} = s_{ji}$ acts as the exchange operator, namely, $s_{ij}(z_i) = z_j$, $s_{ij}(z_k) = z_k$, $\forall\, k \neq i, j$, $s_{ij}(z_j) = z_i$.

Note that this time one has

$$p_{12} p_{23} = p_{13} p_{12} + p_{23} p_{13} - p_{13}.$$

It is easy to see that the operators $\{u_{ij},\ 1 \le i \neq j \le n\}$ satisfy relations (3.1), and therefore, satisfy the twisted classical Yang–Baxter relations (2.13). As a corollary we obtain that the truncated Dunkl operators $\{\theta_i,\ i = 1, \dots, n\}$ are pair-wise commute. Now consider the Dunkl operator $D_i = \partial_{z_i} + h\theta_i$, $i = 1, \dots, n$, where h is a parameter. Clearly that $[\partial_{z_i} + \partial_{z_j}, u_{ij}] = 0$, and therefore $[D_i, D_j] = 0$, $\forall\, i, j$. It easy to see that

$$s_{i,i+1} D_i - D_{i+1} s_{i,i+1} = h, \qquad [D_i, s_{j,j+1}] = 0 \qquad \text{if} \ \ j \neq i, i+1.$$

In such a manner we come to the well-known representation of the degenerate affine Hecke algebra $\mathfrak{H}_n$.

2.3.2 Dunkl and Knizhnik–Zamolodchikov elements

Assume that $\forall\, i$, $x_i = 0$, and generators $\{u_{ij},\ 1 \le i < j \le n\}$ satisfy the locality conditions (2.2) and the classical Yang–Baxter relations

$$[u_{ij}, u_{ik} + u_{jk}] + [u_{ik}, u_{jk}] = 0 \qquad \text{if} \ \ 1 \le i < j < k \le n.$$

Let $y, z, t_1, \dots, t_n$ be parameters, consider the rational function

$$F_{\mathrm{CYB}}(z; \boldsymbol{t}) := F_{\mathrm{CYB}}(z; t_1, \dots, t_n) = \sum_{1 \le i < j \le n} \frac{(t_i - t_j) u_{ij}}{(z - t_i)(z - t_j)}.$$

Then

$$[F_{\mathrm{CYB}}(z; \boldsymbol{t}), F_{\mathrm{CYB}}(y; \boldsymbol{t})] = 0 \qquad \text{and} \qquad \operatorname{Res}_{z = t_i} F_{\mathrm{CYB}}(z; \boldsymbol{t}) = \theta_i.$$

Now assume that a set of generators $\{c_{ij},\ 1 \le i \ne j \le n\}$ satisfy the locality and symmetry (i.e., $c_{ij} = c_{ji}$) conditions, and the Kohno–Drinfeld relations:

$$[c_{ij}, c_{kl}] = 0 \qquad \text{if} \ \ \{i,j\} \cap \{k,l\} = \varnothing,$$
$$[c_{ij}, c_{jk} + c_{ik}] = 0 = [c_{ij} + c_{ik}, c_{jk}], \qquad i < j < k.$$

Let $y, z, t_1, \dots, t_n$ be parameters, consider the rational function

$$F_{\mathrm{KD}}(z; \boldsymbol{t}) := F_{\mathrm{KD}}(z; t_1, \dots, t_n) = \sum_{1 \le i \ne j \le n} \frac{c_{ij}}{(z - t_i)(t_i - t_j)} = \sum_{1 \le i < j \le n} \frac{c_{ij}}{(z - t_i)(z - t_j)}.$$

Then

$$[F_{\mathrm{KD}}(z; \boldsymbol{t}), F_{\mathrm{KD}}(y; \boldsymbol{t})] = 0 \qquad \text{and} \qquad \mathrm{Res}_{z=t_i} F_{\mathrm{KD}}(z; \boldsymbol{t}) = \mathrm{KZ}_i,$$

where

$$\mathrm{KZ}_i = \sum_{\substack{j=1 \\ j \ne i}}^{n} \frac{c_{ij}}{t_i - t_j}$$

denotes the truncated *Knizhnik–Zamolodchikov* element.

2.3.3 Dunkl and Gaudin operators

(a) Rational Dunkl operators. Consider the quotient of the algebra DCYB_n, see Definition 2.3, by the two-sided ideal generated by elements

$$\{[x_i + x_j, u_{ij}]\} \qquad \text{and} \qquad \{[x_k, u_{ij}],\ k \ne i, j\}.$$

Clearly the Dunkl elements (2.1) mutually commute. Now let us consider the so-called *Calogero–Moser* representation of the algebra DCYB_n on the ring of polynomials $R_n := \mathbb{R}[z_1, \dots, z_n]$ given by

$$x_i(p(z)) = \lambda \frac{\partial p(z)}{\partial z_i}, \qquad u_{ij}(p(z)) = \frac{1}{z_i - z_j}(1 - s_{ij})p(z), \qquad p(z) \in R_n.$$

The symmetric group $\mathbb{S}_n$ acts on the ring R_n by means of transpositions $s_{ij} \in \mathbb{S}_n$: $s_{ij}(z_i) = z_j$, $s_{ij}(z_j) = z_i$, $s_{ij}(z_k) = z_k$ if $k \ne i, j$.

In the Calogero–Moser representation the Dunkl elements θ_i becomes the rational Dunkl operators [35], see Definition 1.1. Moreover, one has $[x_k, u_{ij}] = 0$ if$k \ne i, j$, and

$$x_i u_{ij} = u_{ij} x_j + \frac{1}{z_i - z_j}(x_i - x_j - u_{ij}), \qquad x_j u_{ij} = u_{ij} x_i - \frac{1}{z_i - z_j}(x_i - x_j - u_{ij}).$$

(b) Gaudin operators. The Dunkl–Gaudin representation of the algebra DCYB_n is defined on the field of rational functions $K_n := \mathbb{R}(q_1, \dots, q_n)$ and given by

$$x_i(f(q)) := \lambda \frac{\partial f(q)}{\partial q_i}, \qquad u_{ij} = \frac{s_{ij}}{q_i - q_j}, \qquad f(q) \in K_n,$$

but this time we *assume* that $w(q_i) = q_i$, $\forall\, i \in [1, n]$ and for all $w \in \mathbb{S}_n$. In the Dunkl–Gaudin representation the Dunkl elements becomes the rational Gaudin operators, see, e.g., [108]. Moreover, one has $[x_k, u_{ij}] = 0$, if $k \ne i, j$, and

$$x_i u_{ij} = u_{ij} x_j - \frac{u_{ij}}{q_i - q_j}, \qquad x_j u_{ij} = u_{ij} x_i + \frac{u_{ij}}{q_i - q_j}.$$

Comments 2.23. It is easy to check that if $f \in \mathbb{R}[z_1,\ldots,z_n]$, and $x_i := \frac{\partial}{\partial z_i}$, then the following commutation relations are true

$$x_i f = f x_i + \frac{\partial}{\partial_{z_i}}(f), \qquad u_{ij} f = s_{ij}(f) u_{ij} + \partial_{z_i,z_j}(f).$$

Using these relations it easy to check that in the both cases ($\boldsymbol{a}$) and (**b**) the elementary symmetric polynomials $e_k(x_1,\ldots,x_n)$ commute with the all generators $\{u_{ij}\}_{1\le i,j\le n}$, and therefore commute with the all Dunkl elements $\{\theta_i\}_{1\le i\le n}$. Let us *stress* that $[\theta_i, x_k] \ne 0$ for all $1 \le i,k \le n$.

Project 2.24. *Describe a commutative algebra generated by the Dunkl elements* $\{\theta_i\}_{1\le i\le n}$ *and the elementary symmetric polynomials* $\{e_k(x_1,\ldots,x_n)\}_{1\le k\le n}$.

2.3.4 Representation of the algebra $3T_n$ on the free algebra $\mathbb{Z}\langle t_1,\ldots,t_n\rangle$

Let $\mathcal{F}_n = \mathbb{Z}\langle t_1,\ldots,t_n\rangle$ be free associative algebra over the ring of integers $\mathbb{Z}$, equipped with the action of the symmetric group $\mathbb{S}_n$: $s_{ij}(t_i) = t_j$, $s_{ij}(t_k) = t_k$, $\forall\, k \ne i,j$.

Define the action of $u_{ij} \in 3T_n$ on the set of generators of the algebra $\mathcal{F}_n$ as follows

$$u_{ij}(t_k) = \delta_{i,k} t_i t_j - \delta_{j,k} t_j t_i.$$

The action of generator u_{ij} on the whole algebra $\mathcal{F}_n$ is defined by linearity and the twisted Leibniz rule:

$$u_{ij}(1) = 0, \qquad u_{ij}(a+b) = u_{ij}(a) + u_{ij}(b), \qquad u_{ij}(ab) = u_{ij}(a) b + s_{ij}(a) u_{ij}(b).$$

It is easy to see from (2.14) that

$$s_{ij} u_{jk} = u_{ik} s_{ij}, \qquad s_{ij} u_{kl} = u_{kl} s_{ij} \quad \text{ if } \ \{i,j\} \cap \{k,l\} = \varnothing, \qquad u_{ij} + u_{ji} = 0.$$

Now let us consider operator

$$u_{ijk} := u_{ij} u_{jk} - u_{jk} u_{ik} - u_{ik} u_{ij}, \qquad 1 \le i < j < k \le n.$$

Lemma 2.25.

$$u_{ijk}(ab) = u_{ijk}(a) b + s_{ij} s_{jk}(a) u_{ijk}(b), \qquad a,b \in \mathcal{F}_n.$$

Lemma 2.26.

$$u_{ijk}(a) = 0 \qquad \forall\, a \in \mathcal{F}_n.$$

Indeed,

$$\begin{aligned}
u_{ijk}(t_i) &= -u_{jk}(u_{ij}(t_i)) - u_{ik}(u_{ij}(t_i)) = -t_i u_{jk}(t_k) - u_{ik}(t_i) t_j = t_i(t_k t_j) - (t_i t_k) t_j = 0,\\
u_{ijk}(t_k) &= u_{ij}(u_{jk}(t_k)) - u_{jk}(u_{ik}(t_k)) = -u_{ij}(t_k t_j) + u_{jk}(t_k t_i) = t_k(u_{ij}(t_j) + u_{jk}(t_k) t_i = 0,\\
u_{ijk}(t_j) &= u_{ij}(u_{jk}(t_j)) - u_{ik}(u_{ij}(t_j)) = -u_{ij}(t_j) t_k - t_j u_{ik}(t_i) = (t_j t_i) t_k - t_j(t_i t_k) = 0.
\end{aligned}$$

Therefore Lemma 2.26 follows from Lemma 2.25.

Let $\mathcal{F}_n^{\bullet}$ be the quotient of the free algebra $\mathcal{F}_n$ by the two-sided ideal generated by elements $t_i^2 t_j - t_j t_i^2$, $1 \le i \ne j \le n$. Since $u_{i,j}^2(t_i) = t_i t_j^2 - t_j^2 t_i$, one can define a representation of the algebra $3T_n^{(0)}$ on that $\mathcal{F}_n^{\bullet}$. One can also define a representation of the algebra $3T_n^{(0)}$ on that $\mathcal{F}_n^{(0)}$, where $\mathcal{F}_n^{(0)}$ denotes the quotient of the algebra $\mathcal{F}_n$ by the two-sided ideal generated by elements $\{t_i^2,\, 1 \le i \le n\}$. Note that $(u_{i,k} u_{j,k} u_{i,j})(t_k) = [t_i t_j t_i, t_k] \ne 0$ in the algebra $\mathcal{F}_n^{(0)}$, but the elements $u_{i,j}u_{i,k}u_{j,k}u_{i,j}$, $1 \le i < j < k \le n$, which belong to the kernel of the Calogero–Moser representation [72], act trivially both on the algebras $\mathcal{F}_n^{(0)}$ and that $\mathcal{F}_n^{\bullet}$.

Note finally that the algebra $\mathcal{F}_n^{(0)}$ is *Koszul* and has Hilbert series $\operatorname{Hilb}\big(\mathcal{F}_n^{(0)}, t\big) = \frac{1+t}{1-(n-1)t}$, whereas the algebra $\mathcal{F}_n^{\bullet}$ is *not* Koszul for $n \ge 3$, and

$$\operatorname{Hilb}(\mathcal{F}_n^{\bullet}, t) = \frac{1}{(1-t)(1-(n-1)t)(1-t^2)^{n-1}}.$$

In Appendix A.5 we apply the representation introduced in this section to the study of relations in the subalgebra $Z_n^{(0)}$ of the algebra $3T_n^{(0)}$ generated by the elements $u_{1,n}, \dots, u_{n-1,n}$. To distinguish the generators $\{u_{ij}\}$ of the algebra $3T_n^{(0)}$ from the introduced in this section operators u_{ij} acting on it, in Appendix A.5 we will use for the latter notation $\nabla_{ij} := u_{ij}$.

2.3.5 Kernel of Bruhat representation

Bruhat representations, classical and quantum, of algebras $3T_n^{(0)}$ and $3QT_n$ can be seen as a connecting link between commutative subalgebras generating by either additive or multiplicative Dunkl elements in these algebras, and classical and quantum Schubert and Grothendieck calculi.

(**Ia**) **Bruhat representation of algebra** $3T_n^{(0)}$, cf. [45]. Define action of $u_{i,j} \in 3T_n^{(0)}$ on the group ring of the symmetric group $\mathbb{Z}[\mathbb{S}_n]$ as follows: let $w \in \mathbb{S}_n$, then

$$u_{i,j} w = \begin{cases} w s_{ij} & \text{if } \ l(w s_{ij}) = l(w) + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let us remind that $s_{ij} \in \mathbb{S}_n$ denotes the transposition that interchanges i and j and fixes each $k \ne i, j$; for each permutation $u \in \mathbb{S}_n$, $l(u)$ denotes its length.

(**Ib**) **Quantum Bruhat representation of algebra** $3QT_n$, cf. [45]. Let us remind that algebra $3QT_n$ is the quotient of the 3-term relations algebra $3T_n$ by the two-sided ideal generated by the elements

$$\{u_{ij}^2, |j-i| \ge 2\} \bigcup \{u_{i,i+1}^2 = q_i, \ i = 1, \dots, n-1\}.$$

Define the $\mathbb{Z}[q]-$linear action of $u_{i,j} \in 3QT_n$, $i < j$, on the extended group ring of the symmetric group $\mathbb{Z}[q][\mathbb{S}_n]$ as follows: let $w \in \mathbb{S}_n$, and $q_{ij} = q_i q_{i+1} \cdots q_{j-1}$, $i < j$, then

$$u_{i,j} w = \begin{cases} w s_{ij} & \text{if } \ l(w s_{ij}) = l(w) + 1, \\ q_{ij} w s_{ij} & \text{if } \ l(w s_{ij}) = l(w) - l(s_{ij}), \\ 0 & \text{otherwise.} \end{cases}$$

Let us remind, see, e.g., [92], that in general one has

$$l(w s_{ij}) = \begin{cases} l(w) - 2e_{ij} - 1 & \text{if } \ w(i) > w(j), \\ l(w) + 2\ e_{ij} + 1 & \text{if } \ w(i) < w(j). \end{cases}$$

Here $e_{ij}(w)$ denotes the number of k such that $i < k < j$ and $w(k)$ lies between $w(i)$ and $w(j)$. In particular, $l(w s_{ij}) = l(w) + 1$ iff $e_{ij}(w) = 0$ and $w(i) < w(j)$; $l(w s_{ij}) = l(w) - l(s_{ij}) = l(w) - 2(j-i) + 1$ iff $w(i) > w(j)$ and $e_{ij} = j - i - 1$ is the maximal possible.

(**II**) **Kernel of the Bruhat representation.** It is not difficult to see that the following elements of degree three and four belong to the kernel of the Bruhat representation:

$$(\mathbf{IIa}) \quad u_{i,j} u_{i,k} u_{i,j} \qquad \text{and} \qquad u_{i,k} u_{j,k} u_{i,k} \qquad \text{if } \ 1 \le i < j < k \le n;$$

(IIb) $u_{i,k}u_{i,l}u_{j,l}$ and $u_{j,l}u_{i,l}u_{i,k}$;

(IIc) $u_{il}u_{ik}u_{jl}u_{il}$, $u_{il}u_{ij}u_{kl}u_{il}$, $u_{ik}u_{il}u_{jk}u_{ik}$,
$u_{ij}u_{ik}u_{il}u_{ij}$, $u_{ik}u_{il}u_{ij}u_{ik}$ if $1 \le i < j < k < l \le n$.

This observation motivates the following definition.

Definition 2.27. *The reduced 3-term relation algebra* $3T_n^{\text{red}}$ is defined to be the quotient of the algebra $3T_n^{(0)}$ by the two-sided ideal generated by the elements displayed in **IIa–IIc** above.

Example 2.28.

$$\operatorname{Hilb}\big(3T_3^{\text{red}}, t\big) = (1,3,4,1), \qquad \dim\big(3T_3^{\text{red}}\big) = 9,$$
$$\operatorname{Hilb}\big(3T_4^{\text{red}}, t\big) = (1,6,19,32,19,6,1), \qquad \dim\big(3T_4^{\text{red}}\big) = 84,$$
$$\operatorname{Hilb}\big(3T_5^{\text{red}}, t\big) = (1,10,55,190,383,370,227,102,34,8,1), \qquad \dim\big(3T_5^{\text{red}}\big) = 1374.$$

We *expect* that $\dim(3T_n^{red})_{\binom{n}{2}-1} = 2(n-1)$ if $n \ge 3$.

Theorem 2.29.

1. *The algebra* $3T_n^{\text{red}}$ *is finite-dimensional, and its Hilbert polynomial has degree* $\binom{n}{2}$.
2. *The maximal degree* $\binom{n}{2}$ *component of the algebra* $3T_n^{\text{red}}$ *has dimension one and generated by any element which is equal to the product (in any order) of all generators of the algebra* $3T_n^{\text{red}}$.
3. *The subalgebra in* $3T_n^{\text{red}}$ *generated by the elements* $\{u_{i,i+1},\ i = 1,\dots,n-1\}$ *is canonically isomorphic to the nil-Coxeter algebra* NC_n. *In particular, its Hilbert polynomial is equal to* $[n]_t! := \prod_{j=1}^{n} \frac{(1-t^j)}{1-t}$, *and the element* $\prod_{j=1}^{n-1}\prod_{a=j}^{1} u_{a,a+1}$ *of degree* $\binom{n}{2}$ *generates the maximal degree component of the algebra* $3T_n^{\text{red}}$.
4. *The subalgebra over* $\mathbb{Z}$ *generated by the Dunkl elements* $\{\theta_1,\dots,\theta_n\}$ *in the algebra* $3T_n^{\text{red}}$ *is canonically isomorphic to the cohomology ring* $H^*(\mathcal{F}l_n,\mathbb{Z})$ *of the type A flag variety* $\mathcal{F}l_n$.

A definition of the nil-Coxeter algebra NC_n one can find in Section 4.1.1. It is known, see [8] or Section 4.1.1, that the subalgebra generated by the elements $\{u_{i,i+1},\ i = 1,\dots,n-1\}$ in the whole algebra $3T_n^{(0)}$ is canonically isomorphic to the nil-Coxeter algebra NC_n as well.

We **expect** that the kernel of the Bruhat representation of the algebra $3T_n^{(0)}$ is generated by all *monomials* of the form $u_{i_1,j_1}\cdots u_{i_k,j_k}$ such that the sequence of transpositions $t_{i_1,j_1},\dots,t_{i_k,j_k}$ does not correspond to *a path* in the Bruhat graph of the symmetric group $\mathbb{S}_n$. For example if $1 \le i < j < k < l \le n$, the elements $u_{i,k}u_{i,l}u_{j,l}$ and $u_{j,l}u_{i,l}u_{i,k}$ do belong to the kernel of the Bruhat representation.

Problem 2.30.

1. *The image of the Bruhat representation of the algebra* $3T_n^{(0)}$ *defines a subalgebra*

   $$\operatorname{Im}\big(3T_n^{(0)}\big) \subset \operatorname{End}_{\mathbb{Q}}(\mathbb{Q}[\mathbb{S}_n]).$$

   Does this image isomorphic to the algebra $3T_n^{\text{red}}$*? Compute Hilbert polynomials of algebras* $\operatorname{Im}\big(3T_n^{(0)}\big)$ *and* $3T_n^{\text{red}}$.
2. *Describe the image(s) of the affine nil-Coxeter algebra* $\widetilde{\mathrm{NC}}_n$, *see Section 4.1.1, in the algebras* $3T_n^{\text{red}}$ *and* $\operatorname{End}_{\mathbb{Q}}(\mathbb{Q}[\mathbb{S}_n])$.

2.3.6 The Fulton universal ring [47], multiparameter quantum cohomology of flag varieties [45] and the full Kostant–Toda lattice [29, 80]

Let $X_n = (x_1, \dots, x_n)$ be be a set of variables, and

$$\boldsymbol{g} := \boldsymbol{g}^{(n)} = \{g_a[b] \,|\, a \geq 1,\ b \geq 1,\ a+b \leq n\}$$

be a set of parameters; we put $\deg(x_i) = 1$ and $\deg(g_a[b]) = b+1$, and set $g_k[0] := x_k$, $k = 1, \dots, n$. For a subset $S \subset [1,n]$ we denote by X_S the set of variables $\{x_i \,|\, i \in S\}$.

Let t be an auxiliary variable, denote by $M = (m_{ij})_{1\leq i,j\leq n}$ the matrix of size n by n with the following elements:

$$m_{i,j} = \begin{cases} x_i + t & \text{if } \ i = j, \\ g_i[j-i] & \text{if } \ j > i, \\ -1 & \text{if } \ i - j = 1, \\ 0 & \text{if } \ i - j > 1. \end{cases}$$

Let $P_n(X_n, t) = \det |M|$.

Definition 2.31. The Fulton universal ring $\mathcal{R}_{n-1}$ is defined to be the quotient[26]

$$\mathcal{R}_{n-1} = \mathbb{Z}\big[\boldsymbol{g}^{(n)}\big][x_1, \dots, x_n]/\langle P_n(X_n, t) - t^n\rangle.$$

Lemma 2.32. *Let* $P_n(X_n, t) = \sum\limits_{k=0}^{n} c_k(n) t^{n-k}$, $c_0(n) = 1$. *Then*

$$c_k(n) := c_k\big(n; X_n, \boldsymbol{g}^{(n)}\big) = \sum_{\substack{1\leq i_1<i_2<\cdots<i_s<n \\ j_1\geq 1,\dots,j_s\geq 1 \\ m:=\sum(j_a+1)\leq n}} \prod_{a=1}^{s} g_{i_a}[j_a] e_{k-m}\big(X_{[1,n]\setminus \bigcup_{a=1}^{s}[i_a, i_a+j_a]}\big), \tag{2.16}$$

where in the summation we assume additionally that the sets $[i_a, i_a+j_a] := \{i_a, i_a+1, \dots, i_a+j_a\}$, $a = 1, \dots, s$, *are pair-wise disjoint.*

It is clear that $\mathcal{R}_{n-1} = \mathbb{Z}[\boldsymbol{g}^{(n)}][x_1, \dots, x_n]/\langle c_n(1), \dots, c_n(n)\rangle$. One can easily see that the coefficients $c_k(n)$ and $g_m[k]$ satisfy the following recurrence relations [47]:

$$c_k(n) = c_k(n-1) + \sum_{a=0}^{k-1} g_{n-a}[a] c_{k-a-1}(n-a-1), \qquad c_0(n) = 1,$$

$$g_m[k] = c_{k+1}(m+k) - c_{k+1}(m+k-1) - \sum_{a=0}^{k-1} g_{m+k-a}[a] c_{k-a}(m+k-a),$$

$$g_m[0] := x_m.$$

On the other hand, let $\{q_{ij}\}_{1\leq i<j\leq n}$ be a set of (quantum) parameters, and $e_k^{(\boldsymbol{q})}(X_n)$ be the multiparameter quantum elementary polynomial introduced in [45]. We are interested in to describe a set of relations between the parameters $\{g_i[j]\}_{\substack{i\geq 1, j\geq 1 \\ i+j\leq n}}$ and the quantum parameters $\{q_{ij}\}_{1\leq i<j\leq n}$ which implies that

$$c_k(n) = e_k^{(\boldsymbol{q})}(X_n) \qquad \text{for} \quad k = 1, \dots, n.$$

[26] If $P(t, X_n) = \sum\limits_{k\geq 1} f_k(X_n) t^k$, $f_k(X_n) \in \mathbb{Q}[Xn]$ is a polynomial, we denote by $\langle P(t, X_n)\rangle$ the ideal in the polynomial ring $\mathbb{Q}[X_n]$ generated by the coefficients $\{f_1, f_2, \dots\}$.

To start with, let us recall the recurrence relations among the quantum elementary polynomials, cf. [117]. To do so, consider the generating function

$$E_n\big(X_n;\{q_{ij}\}_{1\le i<j\le n}\big)=\sum_{k=0}^{n} e_k^{(\boldsymbol{q})}(X_n)t^{n-k}.$$

Lemma 2.33 ([41, 117])**.** *One has*

$$E_n\big(X_n;\{q_{ij}\}_{1\le i<j\le n}\big)=(t+x_n)E_{n-1}\big(X_{n-1};\{q_{ij}\}_{1\le i<j\le n-1}\big)$$
$$+\sum_{j=1}^{n-1} q_{jn}E_{n-2}\big(X_{[1,n-1]\setminus\{j\}};\{q_{a,b}\}_{\substack{1\le a<b\le n-1\\ a\ne j,b\ne j}}\big).$$

Proposition 2.34. *Parameters $\{g_a[b]\}$ can be expressed polynomially in terms of quantum parameters $\{q_{ij}\}$ and variables $x_1,\dots,x_n$, in a such way that*

$$c_k(n)=e_k^{(\boldsymbol{q})}(X_n),\qquad \forall\, k,n.$$

Moreover,

- $g_a[b]=\sum\limits_{k=1}^{a} q_{k,a+b}\prod\limits_{j=a+1}^{a+b-1}(x_j-x_k)+$ *lower degree polynomials in* $x_1,\dots,x_n$*,*
- *the quantum parameters $\{q_{ij}\}$ can be presented as rational functions in terms of variables $x_1,\dots,x_n$ and polynomially in terms of parameters $\{g_a[b]\}$ such that the equality $c_k(n)=e_k^{(\boldsymbol{q})}(X_n)$ holds for all k, n.*

In other words, the transformation

$$\{q_{ij}\}_{1\le i<j\le n}\longleftrightarrow\{g_a[b]\}_{\substack{a+b\le n\\ a\ge 1,\, b\ge 1}}$$

defines a "birational transformation" between the algebra $\mathbb{Z}[\boldsymbol{g}^{(n)}][X_n]/\langle P_n(X_n,t)-t^n\rangle$ and multiparameter quantum deformation of the algebra $H^*(\mathcal{F}l_n,\mathbb{Z})$.

Example 2.35. Clearly,

$$g_{n-1}[1]=\sum_{j=1}^{n-1}q_{j,n},\quad n\ge 2\qquad\text{and}\qquad g_{n-2}[2]=\sum_{j=1}^{n-2}q_{jn}(x_{n-1}-x_j),\quad n\ge 3.$$

Moreover

$$g_1[3]=q_{14}\big((x_2-x_1)(x_3-x_1)+q_{23}-q_{12}\big)+q_{24}\big(q_{13}-q_{12}\big),$$
$$g_2[3]=q_{15}\big((x_3-x_1)(x_4-x_1)+q_{24}+q_{34}-q_{12}-q_{13}\big)$$
$$+q_{25}\big((x_3-x_2)(x_4-x_2)+q_{14}+q_{34}-q_{12}-q_{23}\big)+q_{35}\big(q_{14}+q_{24}-q_{13}-q_{23}\big).$$

Comments 2.36. The full Kostant–Toda lattice (FKTL for short) has been introduced in the end of $70's$ of the last century by B. Kostant and since that time has been extensively studied both in Mathematical and Physical literature. We refer the reader to the original paper by B. Kostant [29, 80] for the definition of the FKTL and its basic properties. In the present paper we just want to point out on a connection of the Fulton universal ring and hence the multiparameter deformation of the cohomology ring of complete flag varieties, and polynomial integral of motion of the FKTL. Namely,

Polynomials $c_k(n;X_n,\boldsymbol{g}^{(n)})$ defined by (2.16) coincide with the polynomial integrals of motion of the FKTL.

It seems an interesting task to clarify a meaning of the FKTL rational integrals of motion in the context of the universal Schubert calculus [47] and the algebra $3HT_n(0)$, as well as any meaning of universal Schubert or Grothendieck polynomials in the context of the Toda or full Kostant–Toda lattices.

3 Algebra $3HT_n$

Consider the twisted classical Yang–Baxter relation

$$[u_{ij}+u_{ik},u_{jk}]+[u_{ik},u_{ji}]=0,$$

where i, j, k are distinct. Having in mind applications of the Dunkl elements to combinatorics and algebraic geometry, we split the above relation into two relations

$$u_{ij}u_{jk}=u_{jk}u_{ik}-u_{ik}u_{ji} \qquad \text{and} \qquad u_{jk}u_{ij}=u_{ik}u_{jk}-u_{ji}u_{ik} \tag{3.1}$$

and impose the following *unitarity* constraints

$$u_{ij}+u_{ji}=\beta,$$

where β is a central element. Summarizing, we come to the following definition.

Definition 3.1. Define algebra $3T_n(\beta)$ to be the quotient of the free associative algebra

$$\mathbb{Z}[\beta]\langle u_{ij},\, 1\le i<j\le n\rangle$$

by the set of relations

- *locality*: $u_{ij}u_{kl}=u_{kl}u_{ij}$ if $\{i,j\}\cap\{k,l\}=\varnothing$,
- 3-*term relations*: $u_{ij}u_{jk}=u_{ik}u_{ij}+u_{jk}u_{ik}-\beta u_{ik}$, and $u_{jk}u_{ij}=u_{ij}u_{ik}+u_{ik}u_{jk}-\beta u_{ik}$ if $1\le i<j<k\le n$.

It is clear that the elements $\{u_{ij},\, u_{jk},\, u_{ik},\, 1\le i<j<k\le n\}$ satisfy the classical Yang–Baxter relations, and therefore, the elements $\big\{\theta_i:=\sum\limits_{j\ne i}u_{ij},\, 1=1,\dots,n\big\}$ form a mutually commuting set of elements in the algebra $3T_n(\beta)$.

Definition 3.2. We will call $\theta_1,\dots,\theta_n$ by the (universal) additive Dunkl elements.

For each pair of indices $i<j$, we define element $q_{ij}:=u_{ij}^2-\beta u_{ij}\in 3T_n(\beta)$.

Lemma 3.3.

1. *The elements* $\{q_{ij},\, 1\le i<j\le n\}$ *satisfy the Kohno–Drinfeld relations* (*known also as the horizontal four term relations*)

$$q_{ij}q_{kl}=q_{kl}q_{ij} \qquad \textit{if} \;\; \{i,j\}\cap\{k,l\}=\varnothing,$$
$$[q_{ij},q_{ik}+q_{jk}]=0, \qquad [q_{ij}+q_{ik},q_{jk}]=0 \qquad \textit{if} \;\; i<j<k.$$

2. *For a triple* $(i<j<k)$ *define* $u_{ijk}:=u_{ij}-u_{ik}+u_{jk}$. *Then*

$$u_{ijk}^2=\beta u_{ijk}+q_{ij}+q_{ik}+q_{jk}.$$

3. *Deviation from the Yang–Baxter and Coxeter relations:*

$$u_{ij}u_{ik}u_{jk}-u_{jk}u_{ik}u_{ij}=[u_{ik},q_{ij}]=[q_{jk},u_{ik}],$$
$$u_{ij}u_{jk}u_{ij}-u_{jk}u_{ij}u_{jk}=q_{ij}u_{ik}-u_{ik}q_{jk}.$$

Comments 3.4. It is easy to see that the horizontal 4-term relations listed in Lemma 3.3(1), are consequences of the locality conditions among the generators $\{q_{ij}\}$, together with the commutativity conditions among the Jucys–Murphy elements

$$d_i := \sum_{j=i+1}^{n} q_{ij}, \qquad i = 2, \dots, n,$$

namely, $[d_i, d_j] = 0$. In [72] we describe some properties of a commutative subalgebra generated by the Jucys–Murphy elements in the (nil[27]) Kohno–Drinfeld algebra. It is well-known that the Jucys–Murphy elements generate a maximal commutative subalgebra in the group ring of the symmetric group $\mathbb{S}_n$. It is an open *problem*

$$\boxed{\text{describe defining relations among the Jucys–Murphy elements in the group ring } \mathbb{Z}[\mathbb{S}_n].}$$

Finally we introduce the "Hecke quotient" of the algebra $3T_n(\beta)$, denoted by $3HT_n(\beta)$.

Definition 3.5. Define algebra $3HT_n(\beta)$ to be the quotient of the algebra $3T_n(\beta)$ by the set of relations

$$q_{ij} q_{kl} = q_{kl} q_{ij} \qquad \text{for all} \ \ i,\ j,\ k,\ l.$$

In other words we assume that the all elements $\{q_{ij},\ 1 \le i < j \le n\}$ are *central* in the algebra $3T_n(\beta)$. From Lemma 3.3 follows immediately that in the algebra $3HT_n(\beta)$ the elements $\{u_{ij}\}$ satisfy the multiplicative (or quantum) Yang–Baxter relations

$$u_{ij} u_{ik} u_{jk} = u_{jk} u_{ik} u_{ij} \qquad \text{if} \ \ i < j < k. \tag{3.2}$$

To underline the dependence of the algebra $3HT_n(\beta)$ on the central elements $\boldsymbol{q} := \{q_{ij}\}$, we will use for the former the notation $3T_n^{(\boldsymbol{q})}(\beta)$ as well.

Exercises 3.6 (some relations in the algebra $3T_n^{(\boldsymbol{q})}(\beta)$)**.**

1. Noncommutative analogue of recurrence relation among the Catalan numbers [70, 72], cf. Section 5.1. Let k, n be positive integers, $k < n$ and $i_1, \dots, i_k$, $1 \le i_k < n$, be a collection of pairwise distinct integers. *Prove* the following identity in the algebra $3T_n^{(\boldsymbol{q})}(\beta)$[28]

$$\prod_{a=1}^{k} u_{i_a, i_{a+1}} + \sum_{r=2}^{k+1} \left(\prod_{a=r}^{n} \beta(u_{i_a,i_{a+1}}) u_{i_{a_{k+1}}, i_{a_1}} \left(\prod_{a=1}^{r-2} u_{i_a, i_{a+1}} \right) \right)$$
$$= \beta \prod_{a=1}^{k} u_{i_a, i_{a+1}} - \beta \big(u_{i_1, i_{k+1}} R_{I \setminus \{i_{k+1}\}} - R_{I \setminus \{i_{k+1}\}} u_{i_k, i_{k+1}} \big),$$

where R_I denotes the r.h.s. of the above identity. For example,

$$12\,23 + 23\,31 + 31\,12 = \beta(12 - 13 + 23),$$
$$12\,23\,34 + 23\,34\,41 + 34\,41\,12 + 41\,12\,23$$
$$\quad = \beta(12\,23 - 14(12 - 13 + 23) + (12 - 13 + 23)34),$$

[27] That is the quotient of the Kohno–Drinfeld algebra generated by the elements $\{q_{ij}\}$ by the two-sided ideal generated by the elements $\{q_{ij}^2\}_{1 \le i,j \le n}$.

[28] *Hint*: denote the r.h.s. of of the identity stated in item (1) by R_I. One possible proof is based on induction and examination of the element $R_{I \cup \{i_{k+2}\}} := u_{i_{a_1}, i_{a+2}} R_I - R_I u_{i_{a+1}, i_{a_{i+2}}}$.

where we use short notation $ij := u_{ij}$. See Introduction, *summation formula*, **A**, for an interpretation of the above formula in the case $\beta = 0$, $q_{ij} = 0$, $\forall\, i,j$. *Note* that the above formula does not depend on deformation (or quantum) parameters $\{q_{ij}\}$, in particular it also true for the algebras $3T(\Gamma)$ associated with a simple graph Γ, and gives rise to quantum as well as K-theoretic deformations of the Orlik–Terao algebra of a simple graph, cf. [89].

2. Cyclic relations, cf. [45]. Let $i_1, i_2, \dots, i_k$, $1 \le i_a \le n$ be a collection of pairwise distinct integers. *Show* that

$$\sum_{r=1}^{k-1} \left(\prod_{a=r+1}^{k} u_{i_1,i_a} \right) \left(\prod_{a=2}^{r} u_{i_1,i_a} \right) u_{i_{r+1},i_1} = - \left(\sum_{a=2}^{k} q_{i_1,i_a} \left(\prod_{b=a+1}^{k} u_{i_a,i_b} \right) \left(\prod_{b=2}^{a-1} u_{i_a,i_b} \right) \right).$$

For example, $12\,13\,14\,21 + 13\,14\,12\,31 + 14\,12\,13\,41 = -q_{12}\,23\,24 - q_{13}\,34\,32 - q_{14}\,42\,43$.

Note that the r.h.s. does not depend on parameter β.

3.1 Modified three term relations algebra $3MT_n(\beta,\psi)$

Let β, $\{q_{ij} = q_{ji},\, \psi_{ij} = \psi_{ji},\, 1 \le i,j \le n\}$, be a set of mutually commuting elements.

Definition 3.7. Modified 3-term relation algebra $3MT_n(\beta, \boldsymbol{q}, \psi)$ is an associative algebra over the ring of polynomials $\mathbb{Z}[\beta, q_{ij}, \psi_{ij}]$ with the set of generators $\{u_{ij},\, 1 \le i,j \le n\}$ subject to the set of relations

- $u_{ij} + u_{ji} = \beta$, $u_{ij}u_{kl} = u_{kl}u_{ij}$ if $\{i,j\} \cap \{k,l\} = \varnothing$,
- *three term relations*:

$$u_{ij}u_{jk} + u_{ki}u_{ij} + u_{jk}u_{ki} = \beta(u_{ij} + u_{ik} + u_{jk}) \qquad \text{if } \ i,\ j,\ k \ \text{ are distinct,}$$

- $u_{ij}^2 = \beta u_{uj} + q_{ij} + \psi_{ij}$ if $i \neq j$,
- $u_{ij}\psi_{kl} = \psi_{kl}u_{ij}$ if $\{i,j\} \cap \{k,l\} = \varnothing$,
- *exchange relations*: $u_{ij}\psi_{jk} = \psi_{ik}u_{ij}$ if i, j, k are distinct,
- elements β, $\{q_{ij},\, 1 \le i,j \le n\}$ are *central*.

It is easy to see that in the algebra $3MT_n(\beta, \boldsymbol{q}, \boldsymbol{\psi})$ the generators $\{u_{ij}\}$ satisfy the modified *Coxeter* and modified *quantum Yang–Baxter relations*, namely

- *modified Coxeter relations*: $u_{ij}u_{jk}u_{ij} - u_{jk}u_{ij}u_{jk} = (q_{ij} - q_{jk})u_{ik}$,
- *modified quantum Yang–Baxter relations*:

$$u_{ij}u_{ik}u_{jk} - u_{jk}u_{ik}u_{ij} = (\psi_{jk} - \psi_{ij})u_{ik}$$

  if i, j, k are distinct.

Clearly the additive Dunkl elements $\big\{\theta_i := \sum_{j\neq i} u_{ij},\, i = 1,\dots,n\big\}$ generate a commutative subalgebra in $3MT_n(\beta,\psi)$.

It is still possible to describe relations among the additive Dunkl elements [72], cf. [74]. However we don't know any geometric interpretation of the commutative algebra obtained. It is not unlikely that this commutative subalgebra is a common generalization of the small quantum cohomology and elliptic cohomology (remains to be defined!) of complete flag varieties.

The algebra $3MT_n(\beta = 0, \boldsymbol{q} = \boldsymbol{0}, \psi)$ has an elliptic representation [72, 74]. Namely,

$$u_{ij} := \sigma_{\lambda_i - \lambda_j}(z_i - z_j)s_{ij}, \qquad q_{ij} = \wp(\lambda_i - \lambda_j), \qquad \psi_{ij} = -\wp(z_i - z_j),$$

where $\{\lambda_i,\ i=1,\dots,n\}$ is a set of parameters (e.g., complex numbers), and $\{z_1,\dots,z_n\}$ is a set of variables; s_{ij}, $i<j$, denotes the transposition that swaps i on j and fixes all other variables;

$$\sigma_\lambda(z) := \frac{\theta(z-\lambda)\theta'(0)}{\theta(z)\theta(\lambda)}$$

denotes the *Kronecker sigma function*; $\wp(z)$ denotes the *Weierstrass P-function*.

"Multiplicative" version of the elliptic representation. Let q be parameter. In this place we will use the same symbol $\theta(x)$ to denote the "multiplicative" version of the Riemann theta function

$$\theta(x) := \theta(x;q) = (x;q)_\infty(q/x;q)_\infty,$$

where by definition $(x;q)_\infty = (x)_\infty = \prod_{k\ge 0}(1-x\ q^k)$. Let us state some well-known properties of the Riemann theta function:

- $\theta(qx;q) = \theta(1/x;q) = -x^{-1}\theta(x;q)$,
- functional equation:

$$x/y\theta\big(ux^{\pm 1}\big)\theta\big(yv^{\pm 1}\big) + \theta\big(uv^{\pm 1}\big)\theta\big(xy^{\pm 1}\big) = \theta\big(uy^{\pm 1}\big)\theta\big(xv^{\pm 1}\big),$$

  where by definition $\theta(xy^{\pm 1}) := \theta(xy)\theta(xy^{-1})$.
- Jacobi triple product identity:

$$(q;q)_\infty\theta(x;q) = \sum_{n\in\mathbb{Z}}(-x)^n q^{\binom{n}{2}}.$$

One can easily check that after the change of variables

$$x := \left(\frac{z^2}{\lambda w}\right)^{1/2}, \qquad y := \left(\frac{w}{\lambda}\right)^{1/2}, \qquad u := \left(\frac{w}{\lambda\mu^2}\right)^{1/2}, \qquad v := (w\lambda)^{1/2},$$

the functional equation for the Riemann theta function $\theta(x)$ takes the following form

$$\sigma_\lambda(z)\ \sigma\mu(w) = \sigma_{\lambda\mu}(z)\sigma_\mu(w/z) + \sigma_{\lambda\mu}(w)\sigma_\lambda(z/w),$$

where

$$\sigma_\lambda(z) := \frac{\theta(z/\lambda)}{\theta(z)\theta\big(\lambda^{-1}\big)}$$

denotes the (multiplicative) *Kronecker sigma function*. Therefore, the operators

$$u_{ij}(f) := \sigma_{\lambda_i/\lambda_j}(z_i/z_j)s_{ij}(f),$$

where s_{ij} denotes the exchange operator which swaps the variables z_i and z_j, namely $s_{ij}(z_i)=z_j$, $s_{ij}(z_j)=z_i$, $s_{ij}(z_k)=z_k$, $\forall\, k\ne i,j$, and s_{ij} acts trivially on dynamical parameters λ_i, namely, $s_{ij}(\lambda_k)=\lambda_k$, $\forall\, k$, *give rise* to a representation of the algebra $3MT_n(\beta=0,\boldsymbol{q}=\boldsymbol{0},\psi)$.

The 3-term relations among the elements $\{u_{ij}\}$ are consequence (in fact equivalent) to the famous *Jacobi–Riemann* 3-term relation of degree 4 among the theta function $\theta(z)$, see, e.g., [141, p. 451, Example 5]. In several cases, see Introduction, relations (**A**) and (**B**), identities among the Riemann theta functions can be rewritten in terms of the elliptic Kronecker sigma functions and turn out to be a consequence of certain relations in the algebra $3MT_n(\beta = 0, \boldsymbol{q} = \boldsymbol{0}, \psi)$ for some integer n, and vice versa[29].

The algebra $3HT_n(\beta)$ is the quotient of algebra $3MT_n(\beta, \boldsymbol{q}, \psi)$ by the two-sided ideal generated by the elements $\{\psi_{ij}\}$. Therefore the elements $\{u_{ij}\}$ of the algebra $3HT_n(\beta)$ satisfy the quantum Yang–Baxter relations $u_{ij}u_{ik}u_{jk} = u_{jk}u_{ik}u_{ij}$, $i < j < k$, and as a consequence, the multiplicative Dunkl elements

$$\Theta_i = \prod_{a=i-1}^{1} (1 + hu_{a,i})^{-1} \prod_{a=i+1}^{n} (1 + hu_{i,a}), \qquad i = 1, \dots, n, \qquad u_{0,i} = u_{i,n+1} = 0$$

generate a commutative subalgebra in the algebra $3HT_n(\beta)$, see Section 3.1. We emphasize that the Dunkl elements Θ_j, $j = 1, \dots, n$, do not pairwise commute in the algebra $3MT_n(\beta, \boldsymbol{q}, \boldsymbol{\psi})$, if $\psi_{ij} \neq 0$ for some $i \neq j$. One way to construct a multiplicative analog of additive Dunkl elements $\theta_i := \sum_{j \neq i} u_{ij}$ is to add a new set of mutually commuting generators denoted by $\{\rho_{ij}, \rho_{ij} + \rho_{ji} = 0, 1 \leq i \neq j \leq n\}$ subject to the crossing relations

- ρ_{ij} commutes with β, q_{kl} and $\psi_{k,l}$ for all i, j, k, l,
- $\rho_{ij}u_{kl} = u_{kl}\rho_{ij}$ if $\{i, j\} \cap \{k, l\} = \varnothing$, $\rho_{ij}u_{jk} = u_{jk}\rho_{ik}$ if i, j, k are distinct,
- $\rho_{ij}^2 - \beta\rho_{ij} + \psi_{ij} = \rho_{jk}^2 - \beta\rho_{jk} + \psi_{jk}$ for all triples $1 \leq i < j < k \leq n$.

Under these assumptions one can check that elements

$$R_{ij} := \rho_{ij} + u_{ij}, \qquad 1 \leq i < j \leq n$$

satisfy the quantum Yang–Baxter relations

$$R_{ij}R_{ik}R_{jk} = R_{jk}R_{ik}R_{ij}, \qquad i < j < k.$$

In the case of *elliptic representation* defined above, one can take

$$\rho_{ij} := \sigma_\mu(z_i - z_j),$$

where $\mu \in \mathbb{C}^*$ is a parameter. This solution to the quantum Yang–Baxter equation has been discovered in [130]. It can be seen as an operator form of the famous (finite-dimensional) solution to QYBE due to A. Belavin and V. Drinfeld [9]. One can go to one step more and add to the algebra in question a new set of generators corresponding to the shift operators $T_{i,q}\colon z_i \longrightarrow qz_i$, cf. [40]. In this case one can define *multiplicative Dunkl elements* which are closely related with the elliptic Ruijsenaars–Schneider–Macdonald operators.

3.1.1 Equivariant modified three term relations algebra

Let $\boldsymbol{h} = (h_2, \dots, h_n)$ be a set of parameters. We define *equivariant modified* 3*-term relations algebra* $3EMT_n(\beta, h, \boldsymbol{q}, \psi)$ to be the extension of the algebra $3TM_n(\beta, \boldsymbol{q}, \psi)$ by the set of mutually commuting generators $\{y_1, \dots, y_n\}$ subject to the crossing relations

- $y_iu_{jk} = u_{jk}y_i$ if $i \neq j, k$, $y_iu_{ij} = u_{ij}y_j + h_j$, $y_ju_{ij} = u_{ij}y_i - h_j$, $i < j$,
- $[y_k, q_{ij}] = 0 = [y_k, \psi_{ij}]$ for all i, j, k.

[29] It is commonly believed that any identity between the Riemann theta functions is a consequence of the Jacobi–Riemann three term relations among the former. However we do not expect that the all hypergeometric type identities among the Riemann theta functions can be obtained from certain relations in the algebra $3MT_n(\beta = 0, \boldsymbol{q} = \boldsymbol{0}, \psi)$ after applying the *elliptic representation* of the latter.

It is clear that the additive Dunkl elements $\theta_i = y_i + \sum_{j \neq i} u_{ij}$, $i = 1, \dots, n$, are pair-wise commute. For simplicity's sake, we shall restrict our consideration to the case $\beta = 0$.

Theorem 3.8 (generalized Pieri's rule, cf. [72, 74, 117])**.** *Let $1 \leq m \leq n$, then*

$$e_k^{(\boldsymbol{h},\boldsymbol{q})}\big(\theta_1^{(n)}, \dots, \theta_m^{(n)}\big) := \sum_{\substack{A \subset [1,m],\, |A|=2r \\ B \subset [1,m] \setminus A, |B|=2s}} H_r(A) M_B(\{q_{ij}\}) e_{k-2r-2s}(\Theta_{[1,m] \setminus (A \cup B)})$$
$$= \sum_{A \subset [1,m]} Y_A \sum_{\substack{B \subset [1,m] \setminus A \\ |B|=2s}} (-1)^s M_B(\{\psi_{ij}\}) \sum_{\substack{I \subset [1,n] \setminus A,\, I \cap B = \varnothing \\ |A|+|B|+|I|=k}} \prod_{\substack{(i_\alpha, j_\alpha) \in I \times I \\ 1 \leq i_\alpha \leq m < j_\alpha \leq n,\, \forall \alpha \\ i_1, \dots, i_I \text{ are distinct}}} u_{i_\alpha, j_\alpha},$$

where for any subset $C \subset [1,n]$ we put $Y_C := \prod_{c \in C} y_c$, and $e_\ell(\Theta_C) = e_m(\{\theta_c\}_{c \in C})$ stands for the degree ℓ elementary symmetric polynomial of the elements $\{\theta_c\}_{c \in C}$, $e_k(\{\theta_c\}_{c \in C}) = \delta_{0,k}$ if $k \leq 0$; if $B \subset [1,n]$, $|B| = 2s$, we set

$$M_B(\{\psi_{ij}) = \prod_{\substack{L \subset B,\, |L|=s \\ (i_1, \dots, i_s) \subset L \\ (j_1, \dots, j_s) \subset B \setminus L,\, i_\alpha < j_\alpha, j_\alpha \in B \setminus L,\, i_\alpha < n\, \forall\, \alpha}} \psi_{i_\alpha, j_\alpha};$$

in a similar manner one can define $M_B(\{q_{ij}\})$; finally we set

$$H_r(A) = h_{a_{2r}} \left(\sum_{(a_1, \dots, a_{r-1}) \subset A \setminus \{a_{2r}\}} \prod_{j=1}^{r-1} \max(a_j - 2j + 1, 0)\ h_{a_j} \right).$$

It is not difficult to *show* that

$$H_r(A)\big|_{h_a = 1,\, a \in A} = (2r-1)!!,$$

as well as the number of different monomials which appear in $H_r([1, 2r])$ is equal to the Catalan number Cat_r. For example,

$$H_3([1,6]) = h_6(h_{24} + 2h_{25} + 2h_{34} + 4h_{35} + 6h_{45}),$$
$$H_4([1,8]) = h_8\big(h_{246} + 2h_{247} + 2h_{256} + 4h_{257} + 6h_{267} + 2h_{346} + 4h_{347} + 4h_{356}$$
$$+ 8h_{357} + 12h_{367} + 6h_{456} + 12h_{457} + 18h_{467} + 24h_{567}\big).$$

Exercise 3.9. Write

$$H_r([1, 2r]) = h_{a_{2r}} \left(\sum_{\substack{A := (a_1, \dots, a_{r-1}) \subset [1, 2r-1] \\ a_j \geq 2j}} c_A^{(r)} h_A \right),$$

where $c_A^{(r)} := \prod_{j=1}^{r-1} \max(a_j - 2j + 1, 0)$ and $h_A := \prod_{a \in A} h_a$. *Show* that

$$\sum_{\substack{A := (a_1, \dots, a_{r-1}) \subset [1, 2r-1] \\ a_j \geq 2j}} \big(c_A^{(r)}\big)^2 = E_r, \tag{3.3}$$

where E_r denotes the r-th Euler number, see, e.g., [131, A000364].

Find representation theoretic interpretation of numbers $\{c_A^{(r)}\}$ and the identity (3.3).

Clearly,

$$\sum_{\substack{A:=(a_1,\ldots,a_{r-1})\subset[1,2r-1]\\ a_j\geq 2j}} c_A^{(r)} = (2r-1)!!.$$

Question 3.10. *Does there exist a semisimple algebra $\mathfrak{A}(r)$, $\dim(\mathfrak{A}(r)) = E_r$ such that the all irreducible representations $\pi_A^{(r)}$ of the algebra $\mathfrak{A}(r)$ are in one-to-one correspondence with the set $\mathcal{P}(r) := \{A = (a_1,\ldots,a_{r-1}) \subset [1,2r-1],\ a_j \geq 2j,\ \forall\, j\}$ and $\dim(\pi_A) = c_A^{(r)}$, $\forall\, A \in \mathcal{P}(r)$?*

It is worth noting that the Dunkl element θ_i, $1 \leq i \leq n$, doesn't commute either with y_j, $j \neq i$ or any ψ_{kl}. On the other hand one can check easily that $[e_k(y_1,\ldots,y_n),\theta_i] = 0$, $\forall\, k,i$.

3.2 Multiplicative Dunkl elements

Since the elements u_{ij}, u_{ik} and u_{jk}, $i<j<k$, satisfy the classical and *quantum* Yang–Baxter relations (3.1) and (3.2), one can define a multiplicative analogue denoted by Θ_i, $1 \leq i \leq n$, of the Dunkl elements θ_i. Namely, to start with, we define elements

$$h_{ij} := h_{ij}(t) = 1 + t u_{ij}, \qquad i \neq j.$$

We consider $h_{ij}(t)$ as an element of the algebra $\widetilde{3HT_n} := 3HT_n(\beta) \otimes \mathbb{Z}[[q_{ij}^{\pm 1}, t, x, y, \ldots]]$, where we assume that the all parameters $\{q_{ij}, t, x, y, \ldots\}$ are *central* in the algebra $\widetilde{3HT_n}$.

Lemma 3.11.

(1*a*) $h_{ij}(x)h_{ij}(y) = h_{ij}(x+y+\beta xy) + q_{ij}xy$,

(1*b*) $h_{ij}(x)h_{ji}(y) = h_{ij}(x-y) + \beta y - q_{ij}xy$ *if* $i<j$.

It follows from (1*b*) *that $h_{ij}(t)h_{ji}(t) = 1 + \beta t - t^2 q_{ij}$ if $i<j$, and therefore the elements $\{h_{ij}\}$ are invertible in the algebra $\widetilde{3HT_n}$.*

(2) $h_{ij}(x)h_{jk}(y) = h_{jk}(y)h_{ik}(x) + h_{ik}(y)h_{ij}(x) - h_{ik}(x+y+\beta xy)$,

(3) *multiplicative Yang–Baxter relations:*

$$h_{ij}h_{ik}h_{jk} = h_{jk}h_{ik}h_{ij} \qquad \textit{if} \;\; i<j<k,$$

(4) *define multiplicative Dunkl elements* (*in the algebra* $\widetilde{3HT_n}$) *as follows*

$$\Theta_j := \Theta_j(t) = \left(\prod_{a=j-1}^{1} h_{aj}^{-1}\right)\left(\prod_{a=n}^{j+1} h_{ja}\right), \qquad 1 \leq j \leq n.$$

Then the multiplicative Dunkl elements pair-wise commute.

Clearly

$$\prod_{j=1}^{n} \Theta_j = 1, \qquad \Theta_j = 1 + t\theta_j + t^2(\cdots) \qquad \text{and} \qquad \Theta_I \prod_{\substack{i\notin I,\, j\in I\\ i<j}} \big(1 + t\beta - t^2 q_{ij}\big) \in 3HT_n(\beta).$$

Here for a subset $I \subset [1,n]$ we use notation $\Theta_I = \prod_{a\in I} \Theta_a$. Note, that the element Θ_I is a product of (exactly!) $k(n-k)$ terms of a form $h_{i_\alpha j_\alpha}$, where $k := |I|$.

Our main result of this section is a description of relations among the multiplicative Dunkl elements.

Theorem 3.12 (A.N. Kirillov and T. Maeno [76]). *In the algebra $3HT_n(\beta)$ the following relations hold true*

$$\sum_{\substack{I\subset[1,n]\\|I|=k}} \Theta_I \prod_{\substack{i\notin I,\, j\in J\\ i<j}} \left(1+t\beta - t^2 q_{ij}\right) = \begin{bmatrix} n \\ k \end{bmatrix}_{1+t\beta}.$$

Here $\begin{bmatrix} n \\ k \end{bmatrix}_q$ denotes the q-Gaussian polynomial.

Corollary 3.13. *Assume that $q_{ij}\neq 0$ for all $1\le i<j\le n$. Then the all elements $\{u_{ij}\}$ are invertible and $u_{ij}^{-1}=q_{ij}^{-1}(u_{ij}-\beta)$. Now define elements $\Phi_i\in\widetilde{3HT_n}$ as follows*

$$\Phi_i = \left\{\prod_{a=i-1}^{1} u_{ai}^{-1}\right\}\left\{\prod_{a=n}^{i+1} u_{ia}\right\}, \qquad i=1,\dots,n.$$

Then we have

(1) *relationship among Θ_j and Φ_j:*

$$t^{n-2j+1}\Theta_j\left(t^{-1}\right)|_{t=0} = (-1)^j\Phi_j,$$

(2) *the elements $\{\Phi_i,\, 1\le i\le n,\}$ generate a commutative subalgebra in the algebra $\widetilde{3HT_n}$,*

(3) *for each $k=1,\dots,n$, the following relation in the algebra $3HT_n$ among the elements $\{\Phi_i\}$ holds*

$$\sum_{\substack{I\subset[1,n]\\|I|=k}} \prod_{\substack{i\notin I,\, j\in I\\ i<j}} (-q_{ij})\Phi_I = \beta^{k(n-k)},$$

where $\Phi_I := \prod_{a\in I}\Phi_a$.

In fact the element Φ_i admits the following "reduced expression" (i.e., one with the minimal number of terms involved) which is useful for proofs and applications

$$\Phi_i = \left\{\overrightarrow{\prod_{j\in I}}\left\{\overrightarrow{\prod_{\substack{i\in I_+^c\\ i<j}}} u_{ij}^{-1}\right\}\right\}\left\{\overrightarrow{\prod_{j\in I_+^c}}\left\{\overrightarrow{\prod_{\substack{i\in I\\ i<j}}} u_{ij}\right\}\right\}. \tag{3.4}$$

Let us explain notations. For any (totally) ordered set $I=(i_1<i_2<\cdots<i_k)$ we denote by I_+ the set I with the opposite order, i.e., $I_+=(i_k>i_{k-1}>\cdots>i_1)$; if $I\subset[1,n]$, then we set $I^c:=[1,n]\backslash I$. For any (totally) ordered set I we denote by $\overrightarrow{\prod_{i\in I}}$ the ordered product according to the order of the set I.

Note that the total number of terms in the r.h.s. of (3.4) is equal to $i(n-i)$.

Finally, from the "reduced expression" (3.4) for the element Φ_i one can see that

$$\prod_{\substack{i\notin I, j\in I\\ i<j}} (-q_{ij})\Phi_I = \left\{\overrightarrow{\prod_{j\in I}}\left\{\overrightarrow{\prod_{\substack{i\in I_+^c\\ i<j}}} (\beta - u_{ij})\right\}\right\}\left\{\overrightarrow{\prod_{j\in I_+^c}}\left\{\overrightarrow{\prod_{\substack{i\in I\\ i<j}}} u_{ij}\right\}\right\} := \widetilde{\Phi_I}\in 3HT_n.$$

Therefore the identity

$$\sum_{\substack{I\subset[1,n]\\|I|=k}} \widetilde{\Phi_I} = \beta^{k(n-k)}$$

is true in the algebra $3HT_n$ for any set of parameters $\{q_{ij}\}$.

Comments 3.14. In fact from our proof of Theorem 3.8 we can deduce more general statement, namely, consider integers m and k such that $1 \le k \le m \le n$. Then

$$\sum_{\substack{I\subset[1,m]\\|I|=k}} \Theta_I \prod_{\substack{i\in[1,m]\setminus I,\, j\in J\\ i<j}} \left(1+t\beta-t^2q_{ij}\right) = \begin{bmatrix} m\\ k\end{bmatrix}_{1+t\beta} + \sum_{\substack{A\subset[1,n],\,B\subset[1,n]\\|A|=|B|=r}} u_{A,B}, \tag{3.5}$$

where, by definition, for two sets $A=(i_1,\dots,i_r)$ and $B=(j_1,\dots,j_r)$ the symbol $u_{A,B}$ is equal to the (ordered) product $\prod\limits_{a=1}^{r} u_{i_a,j_a}$. Moreover, the elements of the sets A and B have to satisfy the following conditions:

- for each $a=1,\dots,r$ one has $1 \le i_a \le m < j_a \le n$, and $k \le r \le k(n-k)$.

Even more, if $r=k$, then sets A and B have to satisfy the following additional conditions:

- $B=(j_1 \le j_2 \le \cdots \le j_k)$, and the elements of the set A are pair-wise distinct.

In the case $\beta=0$ and $r=k$, i.e., in the case of additive (truncated) Dunkl elements, the above statement, also known as the quantum Pieri formula, has been stated as conjecture in [45], and has been proved later in [117].

Corollary 3.15 ([76])**.** *In the case when $\beta=0$ and $q_{ij}=q_i\delta_{j-i,1}$, the algebra over $\mathbb{Z}[q_1,\dots,q_{n-1}]$ generated by the multiplicative Dunkl elements $\{\Theta_i$ and Θ_i^{-1}, $1\le i\le n\}$ is canonically isomorphic to the quantum K-theory of the complete flag variety $\mathcal{F}l_n$ of type A_{n-1}.*

It is still an open *problem* to describe explicitly the set of monomials $\{u_{A,B}\}$ which appear in the r.h.s. of (3.5) when $r>k$.

3.3 Truncated Gaudin operators

Let $\{p_{ij},\ 1\le i\ne j\le n\}$ be a set of mutually commuting parameters. We assume that parameters $\{p_{ij}\}_{1\le i<j\le n}$ are invertible and satisfy the Arnold relations

$$\frac{1}{p_{ik}} = \frac{1}{p_{ij}} + \frac{1}{p_{jk}}, \qquad i<j,k.$$

For example one can take $p_{ij}=(z_i-z_j)^{-1}$, where $z=(z_1,\dots,z_n)\in(\mathbb{C}\backslash 0)^n$.

Definition 3.16. Truncated (rational) Gaudin operator corresponding to the set of parameters $\{p_{ij}\}$ is defined to be

$$G_i = \sum_{j\ne i} p_{ij}^{-1} s_{ij}, \qquad 1\le i\le n,$$

where s_{ij} denotes the exchange operator which switches variables x_i and x_j, and fixes parameters $\{p_{ij}\}$.

We consider the Gaudin operator G_i as an element of the group ring $\mathbb{Z}[\{p_{ij}^{\pm 1}\}][\mathbb{S}_n]$, call this element $G_i\in\mathbb{Z}[\{p_{ij}^{\pm 1}\}][\mathbb{S}_n]$, $i=1,\dots,n$, by *Gaudin element* and denoted it by $\theta_i^{(n)}$.

It is easy to see that the elements $u_{ij}:=p_{ij}^{-1}s_{ij}$, $1\le i\ne j\le n$, define a representation of the algebra $3HT_n(\beta)$ with parameters $\beta=0$ and $q_{ij}=u_{ij}^2=p_{ij}^2$.

Therefore one can consider the (truncated) Gaudin elements as a special case of the (truncated) Dunkl elements. Now one can rewrite the relations among the Dunkl elements, as well as the quantum Pieri formula [45, 117], in terms of the Gaudin elements.

The key observation which allows to rewrite the quantum Pieri formula as a certain relation among the Gaudin elements, is the following one: parameters $\{p_{ij}^{-1}\}$ satisfy the Plücker relations

$$\frac{1}{p_{ik}p_{jl}} = \frac{1}{p_{ij}p_{kl}} + \frac{1}{p_{il}p_{jk}} \qquad \text{if} \ \ i<j<k<l.$$

To describe relations among the Gaudin elements $\theta_i^{(n)}$, $i=1,\dots,n$, we need a bit of notation. Let $\{p_{ij}\}$ be a set of invertible parameters as before, $i_a<j_a$, $a=1,\dots,r$. Define polynomials in the variables $\boldsymbol{h}=(h_1,\dots,h_n)$

$$G_{m,k,r}^{(n)}(\boldsymbol{h},\{p_{ij}\}) = \sum_{\substack{I\subset[1,n-1]\\ |I|=r}} \frac{1}{\prod\limits_{i\in I} p_{in}} \sum_{\substack{J\subset[1,n]\\ |I|+m=|J|+k}} \binom{n-|I\cup J|}{n-m-|I|}\tilde{h}_J, \tag{3.6}$$

where

$$\tilde{h}_J = \sum_{\substack{K\subset J,\, L\subset J\\ |K|=|L|,\, K\bigcap L=\varnothing}} \prod_{j\in J\setminus(K\cup L)} h_j \prod_{k_a\in K,\, l_a\in L} p_{k_a,l_a}^2,$$

and summation runs over subsets $K=\{k_1<k_2<\cdots<k_r\}$ and $L=\{l_1<l_2<\cdots<l_r\}\subset J\}$, such that $k_a<l_a$, $a=1,\dots,r$.

Theorem 3.17 (relations among the *Gaudin* elements [72], cf. [108])**.**

(1) *Under the assumption that elements $\{p_{ij},\ 1\le i<j\le n\}$ are invertible, mutually commute and satisfy the Arnold relations, one has*

$$\begin{aligned} &G_{m,k,r}^{(n)}\big(\theta_1^{(n)},\dots,\theta_n^{(n)},\{p_{ij}\}\big)=0 \qquad \textit{if} \ \ m>k,\\ &G_{0,0,r}^{(n)}\big(\theta_1^{(n)},\dots,\theta_n^{(n)},\{p_{ij}\}\big)=e_r(d_2,\dots,d_n), \end{aligned} \tag{3.7}$$

where $d_2,\dots,d_n$ denote the Jucys–Murphy elements in the group ring $\mathbb{Z}[\mathbb{S}_n]$ of the symmetric group $\mathbb{S}_n$, see Comments 3.4 for a definition of the Jucys–Murphy elements.

(2) *Let $J=\{j_1<j_2<\cdots<j_r\}\subset[1,n]$, define matrix $M_J:=(m_{a,b})_{1\le a,b\le r}$, where*

$$m_{a,b}:=m_{a,b}(\boldsymbol{h};\{p_{ij}\})=\begin{cases} h_{j_a} & \textit{if} \ \ a=b,\\ p_{j_a,j_b} & \textit{if} \ \ a<b,\\ -p_{j_b,j_a} & \textit{if} \ \ a>b.\end{cases}$$

Then

$$\tilde{h}_J=\mathrm{DET}\,|M_J|.$$

Examples 3.18.

(1) Let us display the polynomials $G_{m,k,r}^{(n)}(\boldsymbol{h},\{p_{ij}\})$ a few cases

$$G_{m,0,r}^{(n)}(\boldsymbol{h},\{p_{ij}\}) = \sum_{\substack{I\subset[1,n-1]\\ |I|=r}} \prod_{i\in I} p_{in}^{-1}\Bigg(\sum_{\substack{J\subset[1,n]\\ |J|=m+r,\, I\subset J}} \tilde{h}_J\Bigg),$$

$$G_{m,k,0}^{(n)}(\boldsymbol{h},\{p_{ij}\}) = \binom{n-m+k}{k} e_{m-k}^{\boldsymbol{q}}(h_1,\dots,h_n),$$

$$G_{m,1,r}^{(n)}(\boldsymbol{h},\{p_{ij}\}) = \sum_{\substack{I\subset[1,n-1]\\ |I|=r}} \prod_{i\in I} p_{in}^{-1}\Bigg(\sum_{\substack{J\subset[1,n]\\ I\subset J,\, |J|=m+r}} (n-m-r+1)\tilde{h}_J + \sum_{\substack{J\subset[1,n]\\ |J|=m+r-1,\, |I\cup J|=m+r}} \tilde{h}_J\Bigg).$$

(2) Let us list the relations (3.7) among the Gaudin elements in the case $n=3$. First of all, the Gaudin elements satisfy the "standard" relations among the Dunkl elements $\theta_1+\theta_2+\theta_3=0$, $\theta_1\theta_2+\theta_1\theta_3+\theta_2\theta_3+q_{12}+q_{13}+q_{23}=0$, $\theta_1\theta_2\theta_3+q_{12}\theta_3+q_{13}\theta_2+q_{23}\theta_1=0$. Moreover, we have additional relations which are specific for the Gaudin elements

$$G_{2,0,1}^{(3)}=\frac{1}{p_{13}}(\theta_1\theta_2+\theta_1\theta_3+q_{12}+q_{13})+\frac{1}{p_{23}}(\theta_1\theta_2+\theta_2\theta_3+q_{12}+q_{23})=0,$$

the elements $p_{23}\theta_1+p_{13}\theta_2$ and $\theta_1\theta_2$ are central.

It is well-known that the elementary symmetric polynomials $e_r(d_2,\dots,d_n):=C_r$, $r=1,\dots,n-1$, generate the center of the group ring $\mathbb{Z}[p_{ij}^{\pm 1}][\mathbb{S}_n]$, whereas the Gaudin elements $\{\theta_i^{(n)},\ i=1,\dots,n\}$, generate a maximal commutative subalgebra $\mathcal{B}(p_{ij})$, the so-called *Bethe subalgebra*, in $\mathbb{Z}[p_{ij}^{\pm 1}][\mathbb{S}_n]$. It is well-known, see, e.g., [108], that $\mathcal{B}(p_{ij})=\bigoplus_{\lambda\vdash n}\mathcal{B}_\lambda(p_{ij})$, where $\mathcal{B}_\lambda(p_{ij})$ is the λ-isotypic component of $\mathcal{B}(p_{ij})$. On each λ-isotypic component the value of the central element C_k is the explicitly known constant $c_k(\lambda)$.

It follows from [108] that the relations (3.7) together with relations

$$G_{0,0,r}\big(\theta_1^{(n)},\dots,\theta_n^{(n)},\{p_{ij}\}\big)=c_r(\lambda)$$

are the defining relations for the algebra $\mathcal{B}_\lambda(p_{ij})$.

Let us remark that in the definition of the Gaudin elements we can use *any* set of mutually commuting, invertible elements $\{p_{ij}\}$ which satisfies the Arnold conditions. For example, we can take

$$p_{ij}:=\frac{q^{j-2}(1-q)}{1-q^{j-i}},\qquad 1\le i<j\le n.$$

It is not difficult to see that in this case

$$\lim_{q\to 0}\frac{\theta_J^{(n)}}{p_{1j}}=-d_j=-\sum_{a=1}^{j-1}s_{aj},$$

where as before, d_j denotes the Jucys–Murphy element in the group ring $\mathbb{Z}[\mathbb{S}_n]$ of the symmetric group $\mathbb{S}_n$. Basically from relations (3.7) one can deduce the relations among the Jucys–Murphy elements $d_2,\dots,d_n$ after plugging in (3.6) the values $p_{ij}:=\frac{q^{j-2}(1-q)}{1-q^{j-i}}$ and passing to the limit $q\to 0$. However the real computations are rather involved.

Finally we note that the *multiplicative* Dunkl/Gaudin elements $\{\Theta_i,\ 1,\dots,n\}$ also generate a maximal commutative subalgebra in the group ring $\mathbb{Z}[p_{ij}^{\pm 1}][\mathbb{S}_n]$. Some relations among the elements $\{\Theta_l\}$ follow from Theorem 3.12, but we don't know an analogue of relations (3.7) for the multiplicative Gaudin elements, but see [108].

Exercises 3.19. Let $A=(a_{i,j})$ be a $2m\times 2m$ *skew-symmetric* matrix. The *Pfaffian* and *Hafnian* of A are defined correspondingly by the equations

$$\mathrm{Pf}(A)=\frac{1}{2^m m!}\sum_{\sigma\in S_{2m}}\mathrm{sgn}(\sigma)\prod_{i=1}^m a_{\sigma(2i-1),\sigma(2i)},$$

$$\mathrm{Hf}(A)=\frac{1}{2^m m!}\sum_{\sigma\in S_{2m}}\prod_{i=1}^m a_{\sigma(2i-1),\sigma(2i)},$$

where $\mathbb{S}_{2m}$ is the symmetric group and $\mathrm{sgn}(\sigma)$ is the signature of a permutation $\sigma\in\mathbb{S}_{2m}$.[30]

[30] See, e.g., https://en.wikipedia.org/wiki/Pfaffian.

Now let n be a positive integer, and $\{p_{ij},\ 1 \le i \ne j \le n,\ p_{ij} + p_{ji} = 0\}$ be a set of skew-symmetric, invertible and mutually commuting elements. We set $p_{ii} = 0$ for all i, and $\boldsymbol{q} := \{p_{ij}^2\}_{1 \le i < j \le n}$.

Now let us *assume* that the elements $\{p_{ij}\}_{1\le i<j\le n}$ satisfy the Plüker relations for the elements $\{p_{ij}^{-1}\}_{1 \le i < j \le n}$, namely,

$$\frac{1}{p_{ik}p_{jl}} = \frac{1}{p_{ij}p_{kl}} + \frac{1}{p_{il}p_{jk}} \qquad \text{for all} \quad 1 \le i < j < k < l \le n.$$

(a) Let n be an *even* positive integer. Let us define $A_n(p_{ij}) := (p_{ij})_{1\le i,j\le n}$ to be the $n \times n$ skew-symmetric matrix corresponding to the family $\{p_{ij}\}_{1\le i<j\le n}$. Show that

$$\mathrm{DET}\,|A_n(p_{ij})| = \mathrm{Hf}\left(A_n(p_{ij}^2)\right).$$

(b) Let n be a positive integer, and $z_1, \dots, z_n$ be a set of mutually commuting variables, define polynomials $H_i(z_1, \dots, z_n \,|\, \{p_{ij}\})$, $i = 1, \dots, n$ from the equation

$$\mathrm{DET}\,|\,\mathrm{diag}(t + z_1, \dots, t + z_n) + A_n(p_{ij})| = t^n + \sum_{i=1}^{n} H_i(z_1, \dots, z_n \,|\, \{p_{ij}\}) t^{n-i},$$

where $\mathrm{diag}(t + z_1, \dots, t + z_n)$ means the diagonal matrix.

Show that for $k = 1, \dots, n$ the polynomial $H_k(z_1, \dots, z_n \,|\, \{p_{ij}\})$ is equal to the multiparameter quantum elementary polynomial $e_k^{(\boldsymbol{q})}(z_1, \dots, z_n)$, see, e.g., [45], or Theorem 2.7.

For example, take $n = 4$, then

$$\begin{aligned}\mathrm{DET}\,|A(p_{ij})| &= (p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23})^2 = p_{12}^2p_{34}^2 + p_{13}^2p_{24}^2 + p_{14}^2p_{23}^2 \\ &\quad - 2p_{12}p_{13}p_{23}p_{14}p_{24}p_{34}\left(\frac{1}{p_{12}p_{34}} - \frac{1}{p_{13}p_{24}} + \frac{1}{p_{14}p_{23}}\right) \\ &= p_{12}^2p_{34}^2 + p_{13}^2p_{24}^2 + p_{14}^2p_{23}^2 = \mathrm{Hf}\left(A_4\left(\{p_{ij}^2\}\right)\right).\end{aligned}$$

The last equality follows from the Plücker relations for parameters $\{p_{ij}^{-1}\}$.

On the other hand, if one assumes that a set of skew symmetric parameters $\{r_{ij}\}_{1\le i<j\le n}$, $r_{ij} + r_{ji} = 0$, satisfies the "standard" Plücker relations, namely

$$r_{ik}r_{jl} = r_{ij}r_{kl} + r_{il}r_{jk}, \qquad i < j < k < l,$$

then $\mathrm{DET}\,|A_n(r_{ij})| = 0$.

3.4 Shifted Dunkl elements $\mathfrak{d}_i$ and $\mathfrak{D}_i$

As it was stated in Corollary 3.15, the *truncated* additive and multiplicative Dunkl elements in the algebra $3HT_n(0)$ generate over the ring of polynomials $\mathbb{Z}[q_1, \dots, q_{n-1}]$ correspondingly the *quantum cohomology* and *quantum K-theory* rings of the full flag variety $\mathcal{F}l_n$. In order to describe the corresponding *equivariant* theories, we will introduce the *shifted* additive and multiplicative Dunkl elements. To start with we need at first to introduce an extension of the algebra $3HT_n(\beta)$.

Let $\{z_1, \dots, z_n\}$ be a set of mutually commuting elements and $\{\beta, \boldsymbol{h} = (h_2, \dots, h_n), t, q_{ij} = q_{ji},\ 1 \le i, j \le n\}$ be a set of parameters. We set $h_n := 0$.

Definition 3.20 (cf. Definition 3.1)**.** Define algebra $\overline{3TH_n(\beta, \boldsymbol{h})}$ to be the semi-direct product of the algebra $3TH_n(\beta)$ and the ring of polynomials $\mathbb{Z}[\boldsymbol{h}, t][z_1, \dots, z_n]$ with respect to the crossing relations

(1) $z_i u_{kl} = u_{kl} z_i$ if $i \notin \{k, l\}$,

(2) $z_i u_{ij} = u_{ij} z_j + \beta z_i + h_j$, $z_j u_{ij} = u_{ij} z_i - \beta z_i - h_j$ if $1 \le i < j < k \le n$.

Now we set as before $h_{ij} := h_{ij}(t) = 1 + t u_{ij}$.

Definition 3.21.

- Define shifted additive Dunkl elements to be

$$\mathfrak{d}_i = z_i + \sum_{i<j} u_{ij} - \sum_{i>j} u_{ji}.$$

- Define shifted multiplicative Dunkl elements to be

$$\mathfrak{D}_i = \left(\prod_{a=i-1}^{1} h_{ai}^{-1} \right) (1 + z_i) \left(\prod_{a=n}^{i+1} h_{ia} \right).$$

Lemma 3.22.

$$[\mathfrak{d}_i, \mathfrak{d}_j] = 0, \qquad [\mathfrak{D}_i, \mathfrak{D}_j] = 0 \qquad \textit{for all} \quad i,\ j.$$

Now we stated an analogue of Theorem 3.8 for shifted multiplicative Dunkl elements. As a preliminary step, for any subset $I \subset [1, n]$ let us set $\mathfrak{D}_I = \prod_{a \in I} \mathfrak{D}_a$. It is clear that

$$\mathfrak{D}_I \prod_{\substack{i \notin I,\, j \in I \\ i<j}} \left(1 + t\beta - t^2 q_{ij}\right) \in \overline{3HT_n(\beta, \boldsymbol{h})}.$$

Theorem 3.23. *In the algebra $\overline{3HT_n(\beta, \boldsymbol{h})}$ the following relations hold true*

$$\sum_{\substack{I \subset [1,n] \\ |I|=k}} \mathfrak{D}_I \prod_{\substack{i \notin I, j \in J \\ i<j}} \left(1 + t\beta - t^2 q_{ij}\right)$$

$$= \sum_{\substack{I \subset [1,n] \\ I=\{1 \le i_1 < \dots < i_k \le n\}}} \prod_{a=1}^{k} (1+t\beta)^{n-k-i_a+a} \left(z_{i_a} (1+t\beta)^{i_a - a} + 1 + h_{i_a} \frac{(1+t\beta)^{i_a-a} - 1}{\beta} \right).$$

In particular, if $\beta = 0$, we will have

Corollary 3.24. *In the algebra $\overline{3HT_n(0, \boldsymbol{h})}$ the following relations hold*

$$\sum_{\substack{I \subset [1,n] \\ |I|=k}} \mathfrak{D}_I \prod_{\substack{i \notin I, j \in J \\ i<j}} \left(1 - t^2 q_{ij}\right) = \sum_{\substack{I \subset [1,n] \\ I=\{1 \le i_1, \dots, i_k \le n\}}} \prod_{a=1}^{k} \left(z_{i_a} + 1 + t h_{i_a} (i_a - a) \right). \tag{3.8}$$

Conjecture 3.25. *If $h_1 = \dots = h_{n-1} = 1$, $t = 1$ and $q_{ij} = \delta_{i,j+1}$, then the subalgebra generated by multiplicative Dunkl elements $\mathfrak{D}_i$, $i = 1, \dots, n$, in the algebra $\overline{3HT_n(0, \boldsymbol{h} = \mathbf{1})}$ (and $t = 1$), is isomorphic to the equivariant quantum K-theory of the complete flag variety $\mathcal{F}l_n$.*

Our proof is based on induction on k and the following relations in the algebra $\overline{3HT_n(\beta, \boldsymbol{h})}$

$$h_{ji} \cdot (1 + x_j) = h_j + (1+\beta) x_j - x_i + (1 + x_i) h_{ji}, \qquad h_{ji} h_{jk} = h_{jk} h_{ki} + h_{ik} h_{ji} - 1 - \beta,$$

if $i < j < k$, and we set $h_{ij} := h_{ij}(1)$. These relations allow to reduce the left hand side of the relations listed in Theorem 3.23 to the case when $z_i = 0$, $h_i = 0$, $\forall\, i$. Under these assumptions one needs to proof the following relations in the algebra $3HT_n(\beta)$, see Theorem 3.12,

$$\sum_{\substack{I\subset[1,n]\\|I|=k}} \mathfrak{D}_I \prod_{\substack{i\notin I,\, j\in J\\ i<j}} (1+t\beta-t^2q_{ij}) = \begin{bmatrix} n\\ k\end{bmatrix}_{1+t\beta}.$$

In the case $\beta = 0$ the identity (3.8) has been proved in [76].

One of the main steps in our proof of Theorem 3.8 is the following explicit formula for the elements $\mathfrak{D}_I$.

Lemma 3.26. *One has*

$$\widetilde{\mathfrak{D}_I} := \mathfrak{D}_I \prod_{\substack{i\notin I,\, j\in I\\ i<j}} (1+t\beta-t^2q_{ij}) = \overset{\nearrow}{\prod_{b\in I}} \Bigg(\overset{\searrow}{\prod_{\substack{a\notin I\\ a<b}}} h_{ba} \Bigg) \overset{\nearrow}{\prod_{a\in I}} \Bigg((1+z_a) \overset{\searrow}{\prod_{\substack{b\notin I\\ a<b}}} h_{ab} \Bigg).$$

Note that if $a < b$, then $h_{ba} = 1 + \beta t - u_{ab}$. Here we have used the symbol

$$\overset{\nearrow}{\prod_{b\in I}} \Bigg(\overset{\searrow}{\prod_{\substack{a\notin I\\ a<b}}} h_{ba} \Bigg)$$

to denote the following product. At first, for a given element $b \in I$ let us define the set $I(b) := \{a \in [1,n]\backslash I,\ a < b\} := (a_1^{(b)} < \cdots < a_p^{(b)})$ for some p (depending on b). If $I = (b_1 < b_2 < \cdots < b_k)$, i.e., $b_i = a_i^{(b)}$, then we set

$$\overset{\nearrow}{\prod_{b\in I}} \Bigg(\overset{\searrow}{\prod_{\substack{a\notin I\\ a<b}}} h_{ba} \Bigg) = \prod_{j=1}^{k} \Big(u_{b_j,a_s} u_{b_j,a_{s-1}} \cdots u_{b_j,a_1} \Big).$$

For example, let us take $n = 6$ and $I = (1, 3, 5)$, then

$$\widetilde{\mathfrak{D}_I} = h_{32}h_{54}h_{52}(1+z_1)h_{16}h_{14}h_{12}(1+z_3)h_{36}h_{34}(1+z_5)h_{56}.$$

Let us *stress* that the element $\widetilde{\mathfrak{D}_I} \in \overline{3HT_n(\beta)}$ is a linear combination of *square free* monomials and therefore, a computation of the left hand side of the equality stated in Theorem 3.17 can be performed in the "classical case" that is in the case $q_{ij} = 0$, $\forall\, i < j$. This case corresponds to the computation of the classical equivariant cohomology of the type A_{n-1} complete flag variety $\mathcal{F}l_n$ if $h = 1$.

A proof of the $\beta = 0$ case given in [76, Theorem 1], can be immediately extended to the case $\beta \neq 0$.

Exercises 3.27.

(1) Show that

$$\sum_{1\le i_1<\cdots<i_k\le n} \prod_{a=1}^{k} (1+\beta)^{n-k-i_a+a} = \begin{bmatrix} n\\ k\end{bmatrix}_{1+t\beta}.$$

(2) (β,h)-Stirling polynomials of the second type. Define polynomials $S_{n,k}(\beta,h)$ as follows

$$S_{n,k}(\beta,h)=\sum_{\substack{I\subset[1,n]\\ I=\{1\le i_1,\ldots,i_k\le n\}}}\prod_{a=1}^{k}\left(\beta^{n-k-i_a+a}+h\frac{\beta^{n-k-i_a+a}-1}{\beta-1}\right).$$

Show that

$$S_{n,k}(1,1)=\left\{\begin{matrix}n+1\\ k+1\end{matrix}\right\},\qquad S_{n,k}(\beta,0)=\begin{bmatrix}n\\ k\end{bmatrix}_{\beta}.$$

4 Algebra $3T_n^{(0)}(\Gamma)$ and Tutte polynomial of graphs

4.1 Graph and nil-graph subalgebras, and partial flag varieties

Let's consider the set $R_n := \{(i,j)\in\mathbb{Z}\times\mathbb{Z}\,|\,1\le i<j\le n\}$ as the set of edges of the complete graph K_n on n labeled vertices $v_1,\ldots,v_n$. Any subset $S\subset R_n$ is the set of edges of a unique subgraph $\Gamma:=\Gamma_S$ of the complete graph K_n.

Definition 4.1 (graph and nil-graph subalgebras)**.** The graph subalgebra $3T_n(\Gamma)$ (resp. nil-graph subalgebra $3T_n^{(0)}(\Gamma)$) corresponding to a subgraph $\Gamma\subset K_n$ of the complete graph K_n, is defined to be the subalgebra in the algebra $3T_n$ (resp. $3T_n^{(0)}$) generated by the elements $\{u_{ij}\,|\,(i,j)\in\Gamma\}$.

In subsequent Sections 4.1.1 and 4.1.2 we will study some examples of graph subalgebras corresponding to the complete multipartite graphs, cycle graphs and linear graphs.

4.1.1 Nil-Coxeter and affine nil-Coxeter subalgebras in $3T_n^{(0)}$

Our first example is concerned with the case when the graph Γ corresponds to either the set $S:=\{(i,i+1)\,|\,i=1,\ldots,n-1\}$ of *simple roots* of type A_{n-1}, or the set $S^{\mathrm{aff}}:=S\cup\{(1,n)\}$ of *affine simple roots* of type $A_{n-1}^{(1)}$.

Definition 4.2.

(a) Denote by $\widetilde{\mathrm{NC}}_n$ subalgebra in the algebra $3T_n^{(0)}$ generated by the elements $u_{i,i+1}$, $1\le i\le n-1$.

(b) Denote by $\widetilde{\mathrm{ANC}}_n$ subalgebra in the algebra $3T_n^{(0)}$ generated by the elements $u_{i,i+1}$, $1\le i\le n-1$ and $-u_{1,n}$.

Theorem 4.3.

(A) *The subalgebra* $\widetilde{\mathrm{NC}}_n$ *is canonically isomorphic to the nil-Coxeter algebra* NC_n*. In particular,* $\mathrm{Hilb}(\widetilde{\mathrm{NC}}_n,t)=[n]_t!$ (*cf.* [8]).

(B) *The subalgebra* $\widetilde{\mathrm{ANC}}_n$ *has finite dimension and its Hilbert polynomial is equal to*

$$\mathrm{Hilb}(\widetilde{\mathrm{ANC}}_n,t)=[n]_t\prod_{1\le j\le n-1}[j(n-j)]_t=[n]_t!\prod_{1\le j\le n-1}[j]_{t^{n-j}}.$$

In particular, $\dim\widetilde{\mathrm{ANC}}_n=(n-1)!n!$, $\deg_t\mathrm{Hilb}(\widetilde{\mathrm{ANC}}_n,t)=\binom{n+1}{3}$.

(C) *The kernel of the map* $\pi\colon \widetilde{\mathrm{ANC}}_n \longrightarrow \widetilde{\mathrm{NC}}_n$, $\pi(u_{1,n}) = 0$, $\pi(u_{i,i+1}) = u_{i,i+1}$, $1 \le i \le n-1$, *is generated by the following elements:*

$$f_n^{(k)} = \prod_{j=k}^{1} \prod_{a=j}^{n-k+j-1} u_{a,a+1}, \qquad 1 \le k \le n-1.$$

Note that $\deg\ f_n^{(k)} = k(n-k)$.

The statement (C) of Theorem 4.3 means that the element $f_n^{(k)}$ which does not contain the generator $u_{1,n}$, can be written as a linear combination of degree $k(n-k)$ monomials in the algebra $\widetilde{\mathrm{ANC}}_n$, each contains the generator $u_{1,n}$ at least once. By this means we obtain a set of all extra relations (i.e., additional to those in the algebra $\widetilde{\mathrm{NC}}_n$) in the algebra $\widetilde{\mathrm{ANC}}_n$. Moreover, each monomial M in all linear combinations mentioned above, appears with coefficient $(-1)^{\#|u_{1,n} \in M|+1}$. For example,

$$\begin{aligned} f_4^{(1)} &:= u_{1,2}u_{2,3}u_{3,4} = u_{2,3}u_{3,4}u_{1,4} + u_{3,4}u_{1,4}u_{1,2} + u_{1,4}u_{1,2}u_{2,3}, \\ f_4^{(2)} &:= u_{2,3}u_{3,4}u_{1,2}u_{2,3} = u_{1,2}u_{3,4}u_{2,3}u_{1,4} + u_{1,2}u_{2,3}u_{1,4}u_{1,2} + u_{2,3}u_{1,4}u_{1,2}u_{3,4} \\ &\quad + u_{3,4}u_{2,3}u_{1,4}u_{3,4} - u_{1,4}u_{1,2}u_{3,4}u_{1,4}. \end{aligned}$$

Worthy of mention is that $\dim(\widetilde{\mathrm{ANC}}_n) = (n-1)!n!$ is equal to the number of (directed) Hamiltonian cycles in the complete bipartite graph $K_{n,n}$, see, e.g., [131, A010790] for additional information.

Remark 4.4. More generally, let (W,S) be a finite crystallographic Coxeter group of rank l with the set of exponents $1 = m_1 \le m_2 \le \dots \le m_l$.

Let $\mathcal{B}_W$ be the corresponding Nichols–Woronowicz algebra, see, e.g., [8]. Follow [8], denote by $\widetilde{\mathrm{NC}}_W$ the subalgebra in $\mathcal{B}_W$ generated by the elements $[\alpha_s] \in \mathcal{B}_W$ corresponding to simple roots $s \in S$. Denote by $\widetilde{\mathrm{ANWC}}_W$ the subalgebra in $\mathcal{B}_W$ generated by $\widetilde{\mathrm{NC}}_W$ and the element $[a_\theta]$, where $[a_\theta]$ stands for the element in $\mathcal{B}_W$ corresponding to the highest root θ for W. In other words, $\widetilde{\mathrm{ANWC}}_W$ is the image of the algebra $\widetilde{\mathrm{ANC}}_W$ under the natural map $B\mathcal{E}(W) \longrightarrow \mathcal{B}_W$, see, e.g., [8, 73]. It follows from [8, Section 6], that $\mathrm{Hilb}(\widetilde{\mathrm{NC}}_W, t) = \prod_{i=1}^{l} [m_i + 1]_t$.

Conjecture 4.5 (Yu. Bazlov and A.N. Kirillov, 2002)**.**

$$\mathrm{Hilb}\big(\widetilde{\mathrm{ANWC}}_W, t\big) = \prod_{i=1}^{l} \frac{1 - t^{m_i+1}}{1 - t^{m_i}} \prod_{i=1}^{l} \frac{1 - t^{a_i}}{1 - t} = P_{\mathrm{aff}}(W,t) \prod_{i=1}^{l} (1 - t^{a_i}),$$

where

$$P_{\mathrm{aff}}(W,t) := \sum_{w \in W_{\mathrm{aff}}} t^{l(w)} = \prod_{i=1}^{l} \frac{(1 + t + \dots + t^{m_i})}{1 - t^{m_i}}$$

denotes the Poincaré polynomial corresponding to the affine Weyl group W_{aff}*, see* [17, p. 245]*;* $a_i := (2\rho, \alpha_i^\vee)$, $1 \le i \le l$, *denote the coefficients of the decomposition of the sum of positive roots* 2ρ *in terms of the simple roots* α_i.

In particular,

$$\dim \widetilde{\mathrm{ANWC}}_W = |W| \frac{\prod_{i=1}^{l} a_i}{\prod_{i=1}^{l} m_i} \qquad \text{and} \qquad \deg \mathrm{Hilb}\big(\widetilde{ANWC}_W, t\big) = \sum_{1=1}^{l} a_i.$$

It is well-known that the product $\prod_{i=1}^{l} \frac{1-t^{a_i}}{1-t^{m_i}}$ is a symmetric (and *unimodal*?) polynomial with non-negative integer coefficients.

Example 4.6.

$$
\begin{aligned}
(a)\quad & \operatorname{Hilb}\big(\widetilde{\mathrm{ANC}}_3, t\big) = [2]_t^2[3]_t, \qquad \operatorname{Hilb}\big(\widetilde{ANC}_4, t\big) = [3]_t^2[4]_t^2, \\
& \operatorname{Hilb}\big(\widetilde{ANC}_5, t\big) = [4]_t^2[5]_t[6]_t^2, \\
(b)\quad & \operatorname{Hilb}(B\mathcal{E}_2, t) = (1+t)^4\big(1+t^2\big)^2, \\
& \operatorname{Hilb}(\widetilde{\mathrm{ANC}}_{B_2}, t) = (1+t)^3\big(1+t^2\big)^2 = P_{\mathrm{aff}}(B_2, t)\big(1-t^3\big)\big(1-t^4\big). \\
(c)\quad & \operatorname{Hilb}\big(\widetilde{\mathrm{ANC}}_{B_3}, t\big) = (1+t)^3\big(1+t^2\big)^2\big(1+t^3\big)\big(1+t^4\big)\big(1+t+t^2\big)\big(1+t^3+t^6\big) \\
& \qquad = P_{\mathrm{aff}}(B_3, t)\big(1-t^5\big)\big(1-t^8\big)\big(1-t^9\big).
\end{aligned}
$$

Indeed, $m_{B_3} = (1, 3, 5)$, $a_{B_3} = (5, 8, 9)$.

Definition 4.7. Let $\langle\widetilde{\mathrm{ANC}}_n\rangle$ denote the two-sided ideal in $3T_n^{(0)}$ generated by the elements $\{u_{i,i+1}\}$, $1 \le i \le n-1$, and $u_{1,n}$. Denote by U_n the quotient $U_n = 3T_n^0/\langle\widetilde{\mathrm{ANC}}_n\rangle$.

Proposition 4.8.

$$U_4 \cong \langle u_{1,3}, u_{2,4}\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2, \qquad U_5 \cong \langle u_{1,4}, u_{2,4}, u_{2,5}, u_{3,5}, u_{1,3}\rangle \cong \widetilde{\mathrm{ANC}}_5.$$

In particular, $\operatorname{Hilb}\big(3T_5^{(0)}, t\big) = \big[\operatorname{Hilb}(\widetilde{\mathrm{ANC}}_5, t)\big]^2$.

4.1.2 Parabolic 3-term relations algebras and partial flag varieties

In fact one can construct an analogue of the algebra $3HT_n$ and a commutative subalgebra inside it, for *any* graph $\Gamma = (V, E)$ on n vertices, possibly with loops and multiple edges [72]. We denote this algebra by $3T_n(\Gamma)$, and denote by $3T_n^{(0)}(\Gamma)$ its *nil-quotient*, which may be considered as a "classical limit of the algebra $3T_n(\Gamma)$".

The case of the complete graph $\Gamma = K_n$ reproduces the results of the present paper and those of [72], i.e., the case of the full flag variety $\mathcal{F}l_n$. The case of the *complete multipartite graph* $\Gamma = K_{n_1,\dots,n_r}$ reproduces the analogue of results stated in the present paper for the full flag variety $\mathcal{F}l_n$, to the case of the *partial flag* variety $\mathcal{F}_{n_1,\dots,n_r}$, see [72] for details.

We *expect* that in the case of the complete graph with all edges having the same multiplicity m, denoted by either $\Gamma = K_n^{(m)}$, or mK_n in the present paper, the commutative subalgebra generated by the Dunkl elements in the algebra $3T_n^{(0)}(\Gamma)$ is related to the algebra of coinvariants of the diagonal action of the symmetric group $\mathbb{S}_n$ on the ring of polynomials $\mathbb{Q}\big[X_n^{(1)}, \dots, X_n^{(m)}\big]$, where we set $X_n^{(i)} = \big\{x_1^{(i)}, \dots, x_n^{(i)}\big\}$.

Example 4.9. Take $\Gamma = K_{2,2}$. The algebra $3T^{(0)}(\Gamma)$ is generated by four elements $\{a = u_{13}$, $b = u_{14}, c = u_{23}, d = u_{24}\}$ subject to the following set of (defining) relations

- $a^2 = b^2 = c^2 = d^2 = 0$, $cb = bc$, $ad = da$,
- $aba + bab = 0 = aca + cac$, $bdb + dbd = 0 = cdc + dcd$,
  $abd - bdc - cab + dca = 0 = acd - bac - cdb + dba$,
- $abca + adbc + badb + bcad + cadc + dbcd = 0$.

It is not difficult to see that[31]

$$\operatorname{Hilb}\big(3T^{(0)}(K_{2,2}),t\big) = [3]_t^2[4]_t^2, \qquad \operatorname{Hilb}\big(3T^{(0)}(K_{2,2})^{ab},t\big) = (1,4,6,3).$$

Here for any algebra A we denote by A^{ab} its *abelianization*[32].

The commutative subalgebra in $3T^{(0)}(K_{2,2})$, which corresponds to the intersection $3T^{(0)}(K_{2,2}) \cap \mathbb{Z}[\theta_1,\theta_2,\theta_3,\theta_4]$, is generated by the elements $c_1 := \theta_1+\theta_2 = (a+b+c+d)$ and $c_2 := \theta_1\theta_2 = (ac+ca+bd+db+ad+bc)$. The elements c_1 and c_2 commute and satisfy the following relations

$$c_1^3 - 2c_1c_2 = 0, \qquad c_2^2 - c_1^2c_2 = 0.$$

The ring of polynomials $\mathbb{Z}[c_1,c_2]$ is isomorphic to the cohomology ring $H^*(\operatorname{Gr}(2,4),\mathbb{Z})$ of the Grassmannian variety $\operatorname{Gr}(2,4)$.

To continue exposition, let us take $m \le n$, and consider the complete multipartite graph $K_{n,m}$ which corresponds to the Grassmannian variety $\operatorname{Gr}(n,m+n)$. One can show

$$\begin{aligned}\operatorname{Hilb}\big(3T^{(0)}_{n+m}(K_{n,m})^{ab},t\big) &= \sum_{k=0}^{n-1}(-1)^k(1+(n-k)t)^{m-1}\prod_{j=1}^{n-k}(1+jt)\left\{ n \atop n-k \right\} \\ &= t^{n+m-1}\operatorname{Tutte}\big(K_{n,m},1+t^{-1},0\big),\end{aligned}$$

where $\left\{ n \atop k \right\} := S(n,k)$ denotes the Stirling numbers of the second kind, that is the number of ways to partition a set of n labeled objects into k nonempty unlabeled subsets, and for any graph Γ, $\operatorname{Tutte}(\Gamma,x,y)$ denotes the *Tutte polynomial*[33] corresponding to graph Γ.

It is well-known that the Stirling numbers $S(n,k)$ satisfy the following identities

$$\sum_{k=0}^{n-1}(-1)^k\ S(n,n-k)\prod_{j=1}^{n-k}(1+jt) = (1+t)^n, \qquad \sum_{n\ge k}\left\{ n \atop k \right\}\frac{x^n}{n!} = \frac{(e^x-1)^k}{k!}.$$

Let us observe that

$$\begin{aligned}\dim\big(3T^{(0)}(K_{n,n})^{ab}\big) &= \sum_{k=0}^{n-1}(-1)^k(n+1-k)^{n-1}(n+1-k)!\left\{ n \atop n-k \right\} \\ &= \sum_{k=1}^{n+1}((k-1)!)^2\left\{ n+1 \atop k \right\}^2,\end{aligned}$$

see, e.g., [131, $A048163$].

Moreover, if $m \ge 0$, then

$$\sum_{n\ge1}\dim\big(3T^{(0)}(K_{n,n+m})^{ab}\big)t^n = \sum_{k\ge1}\frac{k^{k+m-1}(k-1)!t^k}{\prod_{j=1}^{k-1}(1+kjt)},$$

$$\sum_{n\ge1}\operatorname{Hilb}\big(3T^{(0)}(K_{n,m})^{ab},t\big)z^{n-1} = \sum_{k\ge0}(1+kt)^{m-1}\prod_{j=1}^{k}\frac{z(1+jt)}{1+jz}.$$

[31] Hereinafter we shell use notation $(a_0,a_1,\dots,a_k)_t := a_0+a_1t+\cdots+a_kt^k$.

[32] See http://groupprops.subwiki.org/wiki/Abelianization.

[33] See, e.g., https://en.wikipedia.org/wiki/Tutte_polynomial. It is well-known that

$$\operatorname{Tutte}(\Gamma,1+t,0) = (-1)^{|\Gamma|}t^{-\kappa(\Gamma)}\operatorname{Chrom}(\Gamma,-t),$$

where for any graph Γ, $|\Gamma|$ is equal to the number of vertices and $\kappa(\Gamma)$ is equal to the number of connected components of Γ. Finally $\operatorname{Chrom}(\Gamma,t)$ denotes the *chromatic polynomial* corresponding to graph Γ, see, e.g., [140], or https://en.wikipedia.org/wiki/Chromatic_polynomial.

Comments 4.10 (poly-Bernoulli numbers)**.** Based on listed above identities involving the Stirling numbers $S(n,k)$, one can prove the following *combinatorial* formula

$$\dim\left(3T^{(0)}(K_{n,m})^{ab}\right) = \sum_{j=0}^{\min(n,m)} (j!)^2 \left\{ {n+1 \atop j+1} \right\} \left\{ {m+1 \atop j+1} \right\} = B_n^{(-m)} = B_m^{(-n)}, \tag{4.1}$$

where $B_n^{(k)}$ denotes the *poly-Bernoulli number* introduced by M. Kaneko [64].

On the other hand, it is well-known, see, e.g., footnote 33, that for any graph Γ the specialization $\mathrm{Tutte}(\Gamma;2,0)$ of the Tutte polynomial associated with graph Γ, counts the number of acyclic orientations of Γ. Therefore, the poly-Bernulli number $B_n^{(-m)}$ counts the number of acyclic orientatations of the complete bipartite graph $K_{n,m}$.

For example, $\dim\left(3T^{(0)}(K_{3,3})^{ab}\right) = 230 = 1 + 7^2 + (2!)^2 6^2 + (3!)^2$, cf. Example 4.16(3).

For the reader's convenient, we recall below a definition of *poly-Bernoulli numbers*. To start with, let k be an integer, consider the formal power series

$$\mathrm{Li}_k(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^k}.$$

If $k \geq 1$, $\mathrm{Li}_k(z)$ is the k-th *polylogarithm*, and if $k \leq 0$, then $\mathrm{Li}_k(z)$ is a rational function. Clearly $\mathrm{Li}_1(z) = -\ln(1-z)$. Now define *poly-Bernoulli numbers* through the generating function

$$\frac{\mathrm{Li}_k(1-e^{-z})}{1-e^{-z}} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{z^n}{n!}.$$

Note that a combinatorial formula for the numbers $B_n^{(-k)}$ stated in (4.1) is a consequence of the following identity [64]

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} B_n^{(-k)} \frac{x^n}{n!} \frac{z^k}{k!} = \frac{e^{x+z}}{1-(1-e^x)(1-e^z)}.$$

Note that the poly-Bernoulli numbers $B_n^{(-m)} (= B_m^{(-n)})$ have the following combinatorial interpretation[34], namely, the number $B_n^{(-m)}$, and therefore the dimension of the algebra $3T^{(0)}(K_{n,m})$ is equal to

$$B_n^{(-m)} = T(n-1,m) + T(n,m-1),$$

where [26]

$$T(n,m) := \sum_{j=0}^{\min(n,m)} j!(j+1)! \left\{ {n+1 \atop j+1} \right\} \left\{ {m+1 \atop j+1} \right\}$$

is equal to the number of permutations $w \in \mathbb{S}_{n+m}$ having the *excedance set* $\{1,2,\ldots,m\}$.

Exercise 4.11. Show that polynomial $\mathrm{Hilb}(3T^{(0)}(K_{n,m}),t)$ has degree $n+m-1$, and

$$\mathrm{Coeff}_{t^{n+m-1}}\left(\mathrm{Hilb}\left(3T^{(0)}(K_{n,m}),t\right)\right) = T(n-1,m-1).$$

Problem 4.12. *To find a bijective proof of the identity* (4.1).

[34] See for example, [131, *A*136126], [131, *A*099594] or [26, Theorem 3.1], and the literature quoted therein. Recall, that the *excedance set* of a permutation $\pi \in \mathbb{S}_n$ is the set of indices i, $1 \leq i \leq n$, such that $\pi(i) > i$.

Final remark, the explicit expression for the chromatic polynomial of the complete tripartite graph $K_{n,n,n}$ can be found in [131, *A*212220].

Now let $\theta_i^{(n+m)} = \sum\limits_{j \neq i} u_{ij}$, $1 \le i \le n+m$, be the Dunkl elements in the algebra $3T^{(0)}(K_{n+m})$, define the following elements the in the algebra $3T^{(0)}(K_{n,m})$

$$\begin{aligned} c_k &:= e_k\big(\theta_1^{(n+m)}, \dots, \theta_n^{(n+m)}\big), \qquad 1 \le k \le n, \\ \overline{c}_r &:= e_r\big(\theta_{n+1}^{(n+m)}, \dots, \theta_{n+m}^{(n+m)}\big), \qquad 1 \le r \le m. \end{aligned}$$

Clearly,

$$\left(1 + \sum_{k=1}^{n} c_k t^k\right)\left(1 + \sum_{r=1}^{m} \overline{c}_r t^r\right) = \prod_{j=1}^{n+m} \big(1 + \theta_j^{(n+m)}\big) = 1.$$

Moreover, there exist the natural isomorphisms of algebras

$$\begin{aligned} &H^*(\mathrm{Gr}(n, n+m), \mathbb{Z}) \cong \mathbb{Z}[c_1, \dots, c_n] \Big/ \left\langle \left(1 + \sum_{k=1}^{n} c_k t^k\right)\left(1 + \sum_{r=1}^{m} \overline{c}_r t^r\right) - 1 \right\rangle, \\ &QH^*(\mathrm{Gr}(n, n+m)) \cong \mathbb{Z}[q][c_1, \dots, c_n] \Big/ \left\langle \left(1 + \sum_{k=1}^{n} c_k t^k\right)\left(1 + \sum_{r=1}^{m} \overline{c}_r t^r\right) - 1 - q t^{n+m} \right\rangle. \end{aligned}$$

Let us recall, see Section 2, footnote 26, that for a commutative ring R and a polynomial $p(t) = \sum\limits_{j=1}^{s} g_j t^j \in R[t]$, we denote by $\langle p(t) \rangle$ the ideal in the ring R generated by the coefficients $g_1, \dots, g_s$.

These examples are illustrative of the similar results valid for the *general complete multipartite graphs* $K_{n_1,\dots,n_r}$, i.e., for the partial flag varieties [72].

To state our results for *partial flag varieties* we need a bit of notation. Let $N := n_1 + \dots + n_r$, $n_j > 0$, $\forall\, j$, be a composition of size N. We set $N_j := n_1 + \dots + n_j$, $j = 1, \dots, r$, and $N_0 = 0$. Now, consider the commutative subalgebra in the algebra $3T_N^{(0)}(K_N)$ generated by the set of Dunkl elements $\{\theta_1^{(N)}, \dots, \theta_N^{(N)}\}$, and define elements $\{c_{k_j}^{(j,N)} \in 3T_N^{(0)}(K_{n_1,\dots,n_r})\}$ to be the degree k_j elementary symmetric polynomials of the Dunkl elements $\theta_{N_{j-1}+1}^{(N)}, \dots, \theta_{N_j}^{(N)}$, namely,

$$\begin{aligned} &c_k^{(j)} := c_{k_j}^{(j,N)} = e_k\big(\theta_{N_{j-1}+1}^{(N)}, \dots, \theta_{N_j}^{(N)}\big), \qquad 1 \le k_j \le n_j, \qquad j = 1, \dots, r, \\ &c_0^{(j)} = 1, \qquad \forall\, j. \end{aligned}$$

Clearly

$$\prod_{j=1}^{r} \left(\sum_{a=0}^{n_j} c_a^{(j)} t^a\right) = \prod_{j=1}^{N} \big(1 + \theta_j^{(N)} t^j\big) = 1.$$

Theorem 4.13. *The commutative subalgebra generated by the elements* $\{c_{k_j}^{(j)}, 1 \le k_j \le n_j, 1 \le j \le r-1\}$, *in the algebra* $3T_N^{(0)}(K_{n_1,\dots,n_r})$ *is isomorphic to the cohomology ring* $H^*(\mathcal{F}l_{n_1,\dots,n_r}, \mathbb{Z})$ *of the partial flag variety* $\mathcal{F}l_{n_1,\dots,n_r}$.

In other words, we treat the Dunkl elements $\{\theta_{N_{j-1}+a}^{(N)}, 1 \le a \le n_j\}$, $j = 1, \dots, r$, as the *Chern roots* of the vector bundles $\{\xi_j := \mathcal{F}_j / \mathcal{F}_{j-1}\}$, $j = 1, \dots, r$, over the partial flag variety $\mathcal{F}l_{n_1,\dots,n_r}$.

Recall that a point $\boldsymbol{F}$ of the partial flag variety $\mathcal{F}l_{n_1,\dots,n_r}$, $n_1+\cdots+n_r=N$, is a sequence of embedded subspaces

$$\boldsymbol{F}=\left\{0=F_0\subset F_1\subset F_2\subset\cdots\subset F_r=\mathbb{C}^N\right\}$$

such that

$$\dim(F_i/F_{i-1})=n_i,\qquad i=1,\dots,r.$$

By definition, the fiber of the vector bundle ξ_i over a point $\boldsymbol{F}\in\mathcal{F}l_{n_1,\dots,n_r}$ is the n_i-dimensional vector space F_i/F_{i-1}.

To conclude, let us describe the set of (defining) relations among the elements $\left\{c_a^{(j)}\right\}$, $1\le a\le n_j$, $1\le j\le r-1$. To proceed, let us introduce the set of variables $\left\{x_a^{(j)}\,|\,1\le a\le n_j,\ 1\le j\le r-1\right\}$, and define polynomials $b_0=1, b_k:=b_k\left(\left\{x_a^{(j)}\right\}\right)$, $k\ge 1$ by the use of generating function

$$\frac{1}{\prod\limits_{j=1}^{r-1}\prod\limits_{a=1}^{n_j}\left(1+x_a^{(j)}\right)t^a}=\sum_{k\ge 0}b_kt^k.$$

Now let us introduce matrix $M_m\left(\left\{x_a^{(j)}\right\}\right):=(m_{ij})$, where

$$m_{ij}:=\begin{cases}b_{i+j-1} & \text{if}\ \ j>i,\\ 1 & \text{if}\ \ j=i-1,\ i\ge 2,\\ 0 & \text{if}\ j<i-1.\end{cases}$$

Claim 4.14. $\det M_m\left(\left\{c_a^{(i)}\right\}\right)=0$, $N_{r-1}<m\le N$. *Moreover,*

$$H^*(\mathcal{F}l_{n_1,\dots,n_r},\mathbb{Z})\cong\mathbb{Z}[\{x_a^j\}]/\langle M_{N_{r-1}+1},\dots,M_N\rangle.$$

A meaning of the algebra $3T_n^{(0)}(\Gamma)$ and the corresponding commutative subalgebra inside it for a general graph Γ is still unclear.

Conjecture 4.15. [35]

(1) *Let $\Gamma=(V,E)$ be a connected subgraph of the complete graph K_n on n vertices. Then*

$$\operatorname{Hilb}\left(3T_n^{(0)}(\Gamma)^{ab},t\right)=t^{|V|-1}\operatorname{Tutte}\left(\Gamma;1+t^{-1},0\right).$$

(2) *Let $\Gamma=(V,E,\{m_{ij},(ij)\in E\})$ be a connected subgraph of the complete graph $K_n^{(\boldsymbol{m})}$ with multiple edges such that an edge $(ij)\in K_n$ has the multiplicity m_{ij}. Let $3T_n^{(0)}(\Gamma,\boldsymbol{m})$ denotes the subalgebra in the algebra $3T_n^{(0)}(\boldsymbol{m})$ generated by elements $\{u_{ij}^{(\alpha_{(ij)})}$, $(ij)\in E$, $1\le\alpha_{(ij)}\le m_{ij}\}$, see Section 2.2. Let $\mathcal{A}(\Gamma,\{m_{ij}\})$ denotes the graphic arrangement corresponding to the graph $(\Gamma,\{m_{ij}\})$, that is the set of hyperplanes $\{H_{(ij),a}=(x_i-x_j=a)$, $0\le a\le m_{ij}-1$, $(ij)\in E\}$. Then*

$$3T_n^{(0)}(\Gamma,\boldsymbol{m})^{ab}=\mathrm{OS}^+(\mathcal{A}(\Gamma,\{m_{ij}\})),$$

where for any arrangements of hyperplanes $\mathcal{A}$, $\mathrm{OS}^+(\mathcal{A})$ denotes the even Orlik–Solomon algebra of the arrangement $\mathcal{A}$ [113]. In the case when $m_{ij}=1$, $\forall\,1\le i<j\le n$, $3T_n^{(0)}(\Gamma)^{\mathrm{anti}}=\mathrm{OS}(\mathcal{A}(\Gamma))$.

[35] Part (1) of this conjecture has been proved in [89]. In [89] the author has used notation $\mathrm{OT}(\Gamma)$ for the Orlik–Terao algebra associated with (simple) graph Γ. In our paper we prefer to denote the corresponding Orlik–Terao algebra by $\mathrm{OS}^+(\Gamma)$.

Examples 4.16.

(1) Let $G = K_{2,2}$ be complete bipartite graph of type $(2,2)$. Then

$$\mathrm{Hilb}\big(3T_4^0(2,2)^{ab}, t\big) = (1,4,6,3) = t^2(1+t) + t(1+t)^2 + (1+t)^3,$$

and the Tutte polynomial for the graph $K_{2,2}$ is equal to $x + x^2 + x^3 + y$.

(2) Let $G = K_{3,2}$ be complete bipartite graph of type $(3,2)$. Then

$$\begin{aligned}\mathrm{Hilb}\big(3T_5^0(3,2)^{ab}, t\big) &= (1,6,15,17,7)\\ &= t^3(1+t) + 3t^2\ (1+t)^2 + 2t(1+t)^3 + (1+t)^4,\end{aligned}$$

and the Tutte polynomial for the graph $K_{3,2}$ is equal to

$$x + 3x^2 + 2x^3 + x^4 + y + 3xy + y^2.$$

(3) Let $G = K_{3,3}$ be complete bipartite graph of type $(3,3)$. Then

$$\begin{aligned}\mathrm{Hilb}\big(3T_6^0(3,3)^{ab}, t\big) &= (1,9,36,75,78,31) = (1+t)^5 + 4t(1+t)^4\\ &\quad + 10t^2(1+t)^3 + 11t^3(1+t)^2 + 5t^4(1+t),\end{aligned}$$

and the Tutte polynomial of the bipartite graph $K_{3,3}$ is equal to

$$5x + 11x^2 + 10x^3 + 4x^4 + x^5 + 15xy + 9x^2y + 6xy^2 + 5y + 9y^2 + 5y^3 + y^4.$$

(4) Consider complete multipartite graph $K_{2,2,2}$. One can show that

$$\begin{aligned}\mathrm{Hilb}\big(3T_6^{(0)}(K_{2,2,2})^{ab}, t\big) &= (1,12,58,137,154,64) = 11t^4(1+t) + 25t^3(1+t)^2\\ &\quad + 20t^2(1+t)^3 + 7t(1+t)^4 + (1+t)^5,\end{aligned}$$

and

$$\begin{aligned}\mathrm{Tutte}(K_{2,2,2}, x, y) &= x(11,25,20,7,1)_x + y(11,46,39,8)_x + y^2(32,52,12)_x\\ &\quad + y^3(40,24)_x + y^4(29,6)_x + 15y^5 + 5y^6 + y^7.\end{aligned}$$

The above examples show that the Hilbert polynomial $\mathrm{Hilb}(3T_n^0(G)^{ab}, t)$ appears to be a certain specialization of the Tutte polynomial of the corresponding graph G.

Instead of using the Hilbert polynomial of the algebra $3T_n^0(G)^{ab}$ one can consider the graded Betti numbers (over a field $\boldsymbol{k}$) polynomial $\mathrm{Betti}_{\boldsymbol{k}}(3T_n^0(G)^{ab}, x, y)$. For example,

$$\begin{aligned}
&\mathrm{Betti}_{\mathbb{Q}}\big(3T_3^0(K_3)^{ab}, x, y\big) = 1 + xy(4,2)_x + x^2y^2(3,2)_x,\\
&\mathrm{Betti}_{\mathbb{Q}}\big(3T_4^0(K_{2,2})^{ab}, x, y\big) = 1 + 4xy + xy^2(1,9,3)_x + x^2y^3(1,6,3)_x,\\
&\mathrm{Betti}_{\mathbb{Q}}\big(3T_5^0(K_{3,2})^{ab}, x, y\big) = 1 + 6xy + xy^2(3,25,9) + x^2y^3(6,45,34,7) + x^3y^4(3,25,25,7),\\
&\mathrm{Betti}_{\mathbb{Q}}\big(3T_4^0(K_4)^{ab}, x, y\big) = 1 + xy(10,10) + x^2y^2(25,46,26,6) + x^3y^3(15,36,25,6),\\
&\mathrm{Betti}_{\mathbb{Z}/2\mathbb{Z}}\big(3T_4^0(K_4)^{ab}, x, y\big) = 1 + xy(10,10,\mathbf{1})_x + x^2y^2(25,46,26,6) + x^3y^3(\mathbf{16},36,25,6),\\
&\mathrm{Betti}_{\mathbb{Q}}\big(3T_5^0(K_5)^{ab}, x, y\big) = 1 + xy(20,30) + x^2y^2(109,342,315,72) + x^3y^3(195,852,1470,\\
&\qquad\qquad 1232,639,190,24) + x^4y^4(105,540,1155,1160,639,190,24),\\
&\mathrm{Betti}_{\mathbb{Q}}\big(3T_5^0(K_5)^{ab}, 1, 1\big) = 9304,\\
&\mathrm{Betti}_{\mathbb{Z}/3\mathbb{Z}}\big(3T_5^0(K_5)^{ab}, x, y\big) = 1 + xy(20,30) + x^2y^2(109,342,315,72,\mathbf{1})
\end{aligned}$$

$$
\begin{aligned}
&\qquad + x^3y^3(195, 852, 1471, 1232, 640, 190, 24)\\
&\qquad + x^4y^4(105, 540, \mathbf{1156}, 1160, 639, 190, 24),\\
&\mathrm{Betti}_{\mathbb{Z}/3\mathbb{Z}}\big(3T_5^0(K_5)^{ab}, 1, 1\big) = 9308,\\
&\mathrm{Betti}_{\mathbb{Z}/2\mathbb{Z}}\big(3T_5^0(K_5)^{ab}, x, y\big) = 1 + xy(20, 30, \mathbf{5}) + x^2y^2(\mathbf{114}, 342, \mathbf{340}, \mathbf{131}, \mathbf{10})\\
&\qquad + x^3y^3(\mathbf{220}, \mathbf{911}, \mathbf{1500}, 1291, 649, 190, 24)\\
&\qquad + x^4y^4(\mathbf{125}, \mathbf{599}, \mathbf{1165}, 1160, 639, 190, 24),\\
&\mathrm{Betti}_{\mathbb{Z}/2\mathbb{Z}}\big(3T_5^0(K_5)^{ab}, 1, 1\big) = 9680,\\
&\mathrm{Betti}_{\mathbb{Z}/2\mathbb{Z}}\big(3T_6^0(K_{3,3})^{ab}, x, y\big) = 1 + 9xy + xy^2(9, 69, 27) + x^2y^3(40, 285, 257, \mathbf{52})\\
&\qquad + x^3y^4(59, 526, 866, 563, 201, 31)\\
&\qquad + x^4y^5(28, \mathbf{311}, 636, 520, 201, 31),\\
&\mathrm{Betti}_{\mathbb{Z}/2\mathbb{Z}}\big(3T_6^0(K_{3,3})^{ab}, 1, 1\big) = 4740,\\
&\mathrm{Betti}_{\mathbb{Q}}\big(3T_6^0(K_{3,3})^{ab}, x, y\big) = 1 + 9xy + xy^2(9, 69, 27) + x^2y^3(40, 285, 257, 43)\\
&\qquad + x^3y^4(59, 526, 866, 563, 201, 31)\\
&\qquad + x^4y^5(28, 302, 636, 520, 201, 31),\\
&\mathrm{Betti}_{\mathbb{Q}}\big(3T_6^0(K_{3,3})^{ab}, 1, 1\big) = 4704.
\end{aligned}
$$

Let us observe that in all examples displayed above, the Betti polynomials are divisible by $1+xy$.

It should be emphasize that in the literatute one can find definitions of big variety of (graded) Betti's numbers associated with a given simple graph Γ, depending on choosing an algebra/ideal has been attached to graph Γ. For example, to define Betti's numbers, one can start with *edge graph ideal/algebra* associated with a graph in question, the *Stanley–Reisner* ideal/ring and so on and so far. We refer the reader to carefully written book by E. Miller and B. Sturmfels [105] for definitions and results concerning combinatorial commutative algebra graded Betti's numbers. As far as I'm aware, the graded Betti numbers we are looking for in the present paper, are different from those treated in [105], and more close to those studied in [11].

It is not difficult to see (A.K.) that for a simple connected graph Γ the coefficient just before the (unique!) monomial of the maximal degree in $\mathrm{Betti}_{\boldsymbol{k}}\big(3T^0(\Gamma)^{ab}, x, y\big)$ is equal to $\mathrm{Tutte}(\Gamma; 1, 0)$. It is known [10] that the number $\mathrm{Tutte}(\Gamma; 1, 0)$ counts that of acyclic orientations of the edges of Γ with a unique source at a vertex $v \in \Gamma$, or equivalently [10], the number of maximum Γ-parking functions relative to vertex v.

Claim 4.17. *Let $G = (V, E)$ be a connected graph without loops. Then over any field $\boldsymbol{k}$*

$$
\mathrm{Betti}_{\boldsymbol{k}}\big(3T_n^0(G)^{ab}, -x, x\big) = (1-x)^e \mathrm{Hilb}\big(3T_n^0(G)^{ab}, x\big),
$$

where $n = |V(G)|$ = number of vertices, $e = |E(G)|$ = number of edges.

Question 4.18.

- *Let G be a connected subgraph of the complete graph K_n. Does the graded Betti polynomial $\mathrm{Betti}_{\mathbb{Q}}(3T_n^0(G)^{ab}, x, y)$ is a certain specialization of the Tutte polynomial $T(G, x, y)$? If not, give example of two (simple) graphs such that their Orlik–Terao algebras have the same Tutte polynomial, but different Betti polynomials over $\mathbb{Q}$, and vice versa.*

- *It is clear that for any graph Γ (or matroid) one has $\mathrm{Tutte}(\Gamma, x, y) = a(\Gamma)(x+y) +$ (higher degree terms) for some integer $a(\Gamma) \in \mathbb{N}$. Does the number $a(\Gamma)$ have a simple combinatorial interpretation?*

Proposition 4.19. *Let $\boldsymbol{n} = (n_1, \ldots, n_r)$ be a composition of $n \in \mathbb{Z}_{\geq 1}$, then*

$$\operatorname{Hilb}\big(3T^{(0)}(K_{n_1,\ldots,n_r})^{ab}, t\big) = \sum_{\substack{\boldsymbol{k}=(k_1,\ldots,k_r)\\ 0<k_j\leq n_j}} (-t)^{|\boldsymbol{n}|-|\boldsymbol{k}|} \prod_{j=1}^{r} \left\{ {n_j \atop k_j} \right\} \prod_{j=1}^{|\boldsymbol{k}|-1} (1+jt),$$

where we set $|\boldsymbol{k}| := k_1 + \cdots + k_r$.

Remark 4.20. This proposition is a consequence of Conjecture 4.15(1), which has been proved in [89].

Corollary 4.21. *One has*

$$(a)\quad 1 + t(t-1) \sum_{(n_1,\ldots,n_r)\in\mathbb{Z}^r_{\geq 0}\setminus 0^r} \operatorname{Hilb}\big(3T^{(0)}(K_{n_1,\ldots,n_r})^{ab}, t\big) \frac{x_1^{n_1}}{n_1!} \cdots \frac{x_r^{n_r}}{n_r!} = \left(1 + t\sum_{j=1}^{r}(e^{-x_j}-1)\right)^{1-t},$$

$$(b)\quad \sum_{(n_1,n_2,\ldots,n_r)\in\mathbb{Z}_{\geq 0}\setminus 0^r} \dim\left(3T^{(0)}(K_{n_1,\ldots,n_r})^{ab}\right) \frac{x^{n_1}}{n_1!} \cdots \frac{x^{n_r}}{n_r!} = -\log\left(1 - r + \sum_{j=1}^{r} e^{-x_j}\right),$$

$$(c)\quad \operatorname{Hilb}\big(3T^{(0)}(K_{n_1,\ldots,n_r})^{ab}, t\big) = (-t)^{|\boldsymbol{n}|} \operatorname{Chrom}\big(K_{n_1,\ldots,n_r}, -t^{-1}\big),$$

(d) $\dim\left(3T^{(0)}(\Gamma)^{ab}\right)$ *is equal to the number of acyclic orientations of* Γ,

where Γ stands for a simple graph.

Recall that for any graph Γ we denote by $\operatorname{Chrom}(\Gamma, x)$ the chromatic polynomial of that graph.

Indeed, one can show[36]

Proposition 4.22. *If $r \in \mathbb{Z}_{\geq 1}$, then*

$$\operatorname{Chrom}(K_{n_1,\ldots,n_r}, t) = \sum_{\boldsymbol{k}=(k_1,\ldots,k_r)} \prod_{j=1}^{r} \left\{ {n_j \atop k_j} \right\} (t)_{|\boldsymbol{k}|},$$

where by definition $(t)_m := \prod_{j=1}^{m-1}(t-j)$, $(t)_0 = 1$, $(t)_m = 0$ if $m < 0$.

Finally we describe explicitly the exponential generating function for the *Tutte polynomials* of the weighted complete multipartite graphs. We refer the reader to [98] for a definition and a list of basic properties of the Tutte polynomial of a graph.

[36] If $r = 1$, the complete unipartite graph $K_{(n)}$ consists of n distinct points, and

$$\operatorname{Chrom}(K_{(n)}, x) = x^n = \sum_{k=0}^{n-1} \left\{ {n \atop k} \right\} (x)_k.$$

Let us stress that to abuse of notation the complete unipartite graph $K_{(n)}$ consists of n disjoint points with the Tutte polynomial equals to 1 for all $n \geq 1$, whereas the complete graph K_n is equal to the complete multipartite graph $K_{(1^n)}$.

Definition 4.23. Let $r \ge 2$ be a positive integer and $\{S_1,\ldots,S_r\}$ be a collection of sets of cardinalities $\#|S_j| = n_j$, $j=1,\ldots,r$. Let $\boldsymbol{\ell} := \{\ell_{ij}\}_{1\le i<j\le n}$ be a collection of non-negative integers.

The $\boldsymbol{\ell}$-weighted complete multipartite graph $K_{n_1,\ldots,n_r}^{(\boldsymbol{\ell})}$ is a graph with the set of vertices equals to the disjoint union $\coprod_{j=1}^{r} S_i$ of the sets $S_1,\ldots,S_r$, and the set of edges $\{(\alpha_i,\beta_j),\ \alpha_i\in S_i,\ \beta_j\in S_j\}_{1\le i<j\le r}$ of multiplicity ℓ_{ij} each edge (α_i,β_j).

Theorem 4.24. *Let us fix an integer $r\ge 2$ and a collection of non-negative integers $\boldsymbol{\ell} := \{\ell_{ij}\}_{1\le i<j\le r}$. Then*

$$1+\sum_{\substack{\boldsymbol{n}=(n_1,\ldots,n_r)\in\mathbb{Z}_{\ge 0}^r\\ \boldsymbol{n}\ne\boldsymbol{0}}} (x-1)^{\kappa(\boldsymbol{\ell},\boldsymbol{n})}\operatorname{Tutte}\big(K_{n_1,\ldots,n_r}^{(\boldsymbol{\ell})},x,y\big)\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_r^{n_r}}{n_r!}$$
$$=\left(\sum_{\boldsymbol{m}=(m_1,\ldots,m_r)\in\mathbb{Z}_{\ge 0}^r} y^{\sum_{1\le i<j\le r}\ell_{ij}m_im_j}(y-1)^{-|\boldsymbol{m}|}\frac{t_1^{m_1}}{m_1!}\cdots\frac{t_r^{m_r}}{m_r!}\right)^{(x-1)(y-1)},$$

where $\kappa(\boldsymbol{\ell},\boldsymbol{n})$ denotes the number of connected components of the graph $K_{n_1,\ldots,n_r}^{(\boldsymbol{\ell})}$.

Comments 4.25.

(a) Clearly the condition $\ell_{ij}=0$ means that there are no edges between vertices from the sets S_i and S_j. Therefore Theorem 4.24 allows to compute the Tutte polynomial of *any* (finite) graph. For example,

$$\begin{aligned}\operatorname{Tutte}\big(K_{2,2,2,2}^{(1^6)},x,y\big) = \big\{&(0,362,927,911,451,121,17,1)_x,\\ &(362,2154,2928,1584,374,32)_x,(1589,4731,3744,1072,96)_x,\\ &(3376,6096,2928,448,16)_x,(4828,5736,1764,152)_x,\\ &(5404,4464,900,32)_x,(5140,3040,380)_x,(4340,1840,124)_x,\\ &(3325,984,24)_x,(2331,448)_x,(1492,168)_x,(868,48)_x,(454,8)_x,\\ &210,84,28,7,1\big\}_y.\end{aligned}$$

(b) One can show that a formula for the chromatic polynomials from Proposition 4.19 corresponds to the specialization $y=0$ (*but* not direct substitution!) of the formula for generating function for the Tutte polynomials stated in Theorem 4.24.

(c) The Tutte polynomial $\operatorname{Tutte}\big(K_{n_1,\ldots,n_r}^{(\boldsymbol{\ell})},x,y\big)$ does not symmetric with respect to parameters $\{\ell_{ij}\}_{1\le i<j\le r}$. For example, let us write $\boldsymbol{\ell}=(\ell_{12},\ell_{23},\ell_{13},\ell_{14},\ell_{24},\ell_{34})$, then

$$\operatorname{Tutte}\big(K_{2,2,2,2}^{(6,3,4,5,2,4)},1,1\big) = 2^8\cdot 3\cdot 5\cdot 11^3\cdot 241 = 1231760640.$$

On the other hand,

$$\operatorname{Tutte}\big(K_{2,2,2,2}^{(6,4,3,5,2,4)},1,1\big) = 2^{13}\cdot 3\cdot 7\cdot 11^2\cdot 61 = 1269768192.$$

4.1.3 Universal Tutte polynomials

Let $\boldsymbol{m}=(m_{ij},\ 1\le i<j\le n)$ be a collection of non-negative integers. Define *generalized Tutte polynomial* $\widetilde{T}_n(\boldsymbol{m},x,y)$ as follows

$$(x-1)^{\kappa(n,\boldsymbol{m})}\widetilde{T}_n(\boldsymbol{m},x,y)$$

$$= \mathrm{Coeff}_{[t_1\cdots t_n]}\left[\left(\sum_{\substack{\ell_1,\dots,\ell_n\\ \ell_i\in\{0,1\},\,\forall\, i}} y^{\sum\limits_{1\le i<j\le n} m_{ij}\ell_i\ell_j}(y-1)^{-(\sum_j \ell_j)}t_1^{\ell_1}\cdots t_n^{\ell_n}\right)^{(x-1)(y-1)}\right],$$

where as before, $\kappa(n,\boldsymbol{m})$ denotes the number of connected components of the graph $K_n^{(\boldsymbol{m})}$.

Clearly that if $\Gamma\subset K_n^{(\boldsymbol{\ell})}$ is a subgraph of the weighted complete graph $K_n^{(\boldsymbol{\ell})} \overset{\text{def}}{=} K_{1^n}^{(\boldsymbol{\ell})}$, then the Tutte polynomial of graph Γ multiplied by $(x-1)^{\kappa(\Gamma)}$ is equal to the following specialization

$$m_{ij}=0 \quad \text{if edge} \quad (i,j)\notin\Gamma, \qquad m_{ij}=\ell_{ij} \quad \text{if edge} \quad (i,j)\in\Gamma$$

of the generalized Tutte polynomial

$$(x-1)^{\kappa(\Gamma)}\mathrm{Tutte}(\Gamma,x,y)=\widetilde{T}_n(\boldsymbol{m},x,y)\Big|_{\substack{m_{ij}=0 \ \text{ if } (i,j)\notin\Gamma\\ m_{ij}=\ell_{ij} \ \text{ if } (i,j)\in\Gamma}}.$$

For example,

(a) Take $n=6$ and $\Gamma=K_6\backslash\{15,16,24,25,34,36\}$, then

$$\mathrm{Tutte}(\Gamma,x,y)=\{(0,4,9,8,4,1)_x,(4,13,9)_x,(8,7)_x,5,1\}_y.$$

(b) Take $n=6$ and $\Gamma=K_6\backslash\{15,26,34\}$, then

$$\begin{aligned}\mathrm{Tutte}(\Gamma,x,y)=\{&(0,11,25,20,7,1)_x,(11,46,39,8)_x,(32,52,12)_x,\\ &(40,24)_x,(29,6)_x,15,5,1\}_y.\end{aligned}$$

(c) Take $n=6$ and $\Gamma=K_6\backslash\{12.34.56\}=K_{2,2,2}$. As a result one obtains an expression for the Tutte polynomial of the graph $K_{2,2,2}$ displayed in Example 4.16(4).

Now set us set

$$q_{ij}:=\frac{y^{m_{ij}}-1}{y-1}.$$

Lemma 4.26. *The generalized Tutte polynomial $\widetilde{T}_n(\boldsymbol{m},x,y)$ is a polynomial in the variables $\{q_{ij}\}_{1\le i<j\le n}$, x and y.*

Definition 4.27. The universal Tutte polynomial $T_n(\{q_{ij}\},x,y)$ is defined to be the polynomial in the variables $\{q_{ij}\}$, x, and y defined in Lemma 4.26.

Explicitly,

$$\begin{aligned}&(x-1)T_n(\{q_{ij}\},x,y)\\ &=\mathrm{Coeff}_{[t_1\cdots t_n]}\left[\left(\sum_{\substack{\ell_1,\dots,\ell_n\\ \ell_i\in\{0,1\},\,\forall\, i}}\ \prod_{1\le i<j\le n}(q_{ij}(y-1)+1)^{\ell_i\ell_j}(y-1)^{-(\sum_j\ell_j)}t_1^{\ell_1}\cdots t_n^{\ell_n}\right)^{(x-1)(y-1)}\right].\end{aligned}$$

Corollary 4.28. *Let $\{m_{ij}\}_{1\le i<j\le n}$ be a collection of positive integers. Then the specialization*

$$q_{ij}\longrightarrow [m_{ij}]_y:=\frac{y^{m_{ij}}-1}{y-1}$$

of the universal Tutte polynomial $T_n(\{q_{ij}\},x,y)$ is equal to the Tutte polynomial of the complete graph K_n with each edge (i,j) of the multiplicity m_{ij}.

Further specialization $q_{ij} \longrightarrow 0$ if edge $(i,j) \notin \Gamma$ allows to compute the Tutte polynomial for any graph

$$\begin{aligned}\mathrm{Tutte}_3(\{q_{12},q_{13},q_{23}\},x,y) = (1-q[12])(1-q[13])(1-q[23]) + yq[12]q[13]q[23]) \\ + x(q[12]+q[13]+q[23]-2)+x^2.\end{aligned}$$

It is not difficult to see that the $\mathrm{Tutte}_n(\{q_{ij}\},x,y)$ is a *symmetric polynomial* with respect to parameters $\{q_{ij}\}_{1\le i<j\le n}$.

For more compact expression, it is more convenient to rewrite the universal chromatic polynomial in terms of parameters $p_{ij} := 1-q_{ij},\ 1\le i<j\le n$, and denote it by $Ch_n(\{p_{ij}\},x)$. For example,

$$\begin{aligned}\mathrm{Ch}_4(\{p_{ij}\},x) = -p_{12}p_{13}p_{14}p_{23}p_{24}p_{34} + x\big(2 - p_{12} - p_{13} - p_{14} - p_{23} - p_{24} - p_{34} \\ + p_{12}p_{34} + p_{14}p_{23} + p_{13}p_{24} + p_{12}p_{13}p_{23} + p_{12}p_{14}p_{24} + p_{13}p_{14}p_{34} \\ + p_{23}p_{24}p_{34}\big) + x^2\big(3 - p_{12} - p_{13} - p_{14} - p_{23} - p_{24} - p_{34}\big) + x^3.\end{aligned}$$

Note that $p_{12}p_{34} + p_{14}p_{23} + p_{13}p_{24}$ is a symmetric polynomial of the variables p_{12}, p_{34}, p_{13}, p_{24}, p_{14}, p_{23}. It is important to keep in mind that parameters $\{m_{ij}\}$ and $\{p_{ij}\}$ are connected by relations

$$p_{ij} = \frac{y-y^{m_{ij}}}{y-1}, \qquad 1\le i<j\le n.$$

Therefore, $p_{ij}=1$ if $(i,j)\notin \mathrm{Edge}(\Gamma)$, $p_{ij}=0$ if $m_{ij}=1$. We emphasize that the latter equalities are valid for arbitrary y. It is not difficult to see that

$$\mathrm{Ch}_n(\{q_{ij}=0,\ \forall\, i,j\} = \mathrm{Tutte}(K_n;x,0), \qquad \mathrm{Ch}_n(\{q_{ij}=1,\ \forall\, i,j\} = (x-1)^{n-1}.$$

Define *universal chromatic* polynomial to be $\mathrm{Ch}_n(\{p_{ij}\},x) = \mathrm{Tutte}_n(\{p_{ij}\},x,0)$, where we treat $\{p_{ij}\}_{1\le i<j\le n}$ as a collection of a free parameters.

To state our result concerning the universal chromatic polynomial $\mathrm{Ch}_n(\{p_{ij}\},x)$, first we introduce a bit of notation. Let $n\ge 2$ be an integer, consider a partition $\mathcal{B} = \big\{B_i = \big(b_1^{(i)},\dots,b_{r_i}^{(i)}\big)\big\}_{1\le i\le k}$ of the set $[1,n] := [1,2,\dots,n]$. In other words one has that $[1,n] = \cup_{i=1}^k B_i$ and $B_i\cap B_j = \varnothing$ if $i\ne j$. We assume that $b_1^{(1)} < b_1^{(2)} < \cdots < b_1^{(k)}$. We define $\kappa(\mathcal{B}) := k$. To a given partition $\mathcal{B}$ we associate a monomial $p_{\mathcal{B}} := \prod_{a=1}^{k} p_{B_a}$, where $p_{B_a}=1$ if $\kappa(\mathcal{B})=1$, and

$$p_{B_a} = \prod_{\substack{i,j\in B_a\\ i<j}} p_{ij}.$$

For a given partition $\lambda\vdash n$ denote by $\mathcal{L}_\lambda^{(\beta)}(\{p_{ij}\})$ the sum of all monomials $p_{\mathcal{B}}\beta^{\kappa(\mathcal{B})-2}$ such that $\lambda = \lambda(\mathcal{B}) \overset{\text{def}}{=} (|B_1|,\dots,|B_{\kappa(\mathcal{B})}|)^+$, where for any composition $\alpha\models n$, α^+ denotes a unique partition obtained from α by the reordering of its parts.

Define β-universal chromatic polynomial to be

$$\mathrm{Ch}_n^{(\beta)}(\{p_{ij}\},x) = \beta^{-1}\mathcal{L}_{(n)} + \sum_{\lambda\vdash n} \mathrm{Tutte}(K_{\ell(\lambda)-1};x,0)\mathcal{L}_\lambda^{(\beta)},$$

where summation runs over all partitions λ of n; we set $K_0 := \varnothing$ and $\mathrm{Tutte}(\varnothing;x,y)=0$. For the reader convenience we are reminded that for the complete graph K_n, $n>0$, one has

$$\mathrm{Tutte}(K_n,x,0)=\prod_{j=1}^{n-1}(x+j-1)=\sum_{k=0}^{n-1}s(k,n-1)x^k,$$

where $s(k,n)$ denotes the Stirling number of the first kind[37].

Theorem 4.29 (formula for universal chromatic polynomials)**.**

$$\mathrm{Ch}_n(\{p_{i,j}\},x)=\mathrm{Ch}_n^{(\beta=-1)}(\{p_{ij}\},x).$$

For a given partition $\lambda\vdash n$ denote by $\mathcal{L}_\lambda(\{p_{ij}\})$ the sum of all monomials $p_{\mathcal{B}}$ such that $\lambda=\lambda(\mathcal{B})\overset{\text{def}}{=}(|B_1|,\dots,|B_{\kappa(\mathcal{B})}|)^+$, where for any composition $\alpha\models n$, α^+ denotes a unique partition obtained from α by the reordering of its parts.

It is clear that for a graph $\Gamma\subset K_n$ and partition $\mathcal{B}$ the value of monomial $p_{\mathcal{B}}$ under the specialization $p_{ij}=0$ if $(ij)\in\mathrm{Edge}(\Gamma)$ and $p_{ij}=1$ if $(ij)\notin\mathrm{Edge}(\Gamma)$, is equal to 1 iff the complementary graph $K_n\backslash\Gamma$ contains a subgraph which is isomorphic to the disjoint union of complete graphs $K(\lambda):=\coprod_{i=1}^{k}K_{\lambda_i}$, where $(\lambda_1,\dots,\lambda_k)=\lambda(\mathcal{B})$. Therefore the specialization

$$\mathcal{L}_\lambda\Big|_{\substack{p_{ij}=0,\ (ij)\in\Gamma,\\ p_{ij}=1,\ (ij)\notin\Gamma}}$$

is equal to the number of non isomorphic subgraphs of the complementary graph $K_n\backslash\Gamma$ which are isomorphic to the graph $K(\lambda)$.

Example 4.30. Take $n=6$, then

$$\begin{aligned}\mathrm{Ch}_6^{(\beta)}&=\beta^{-1}\mathcal{L}_{(6)}+x(x+1)(x+2)(x+3)(x+4)\mathcal{L}_{(1^6)}\beta^4\\&\quad+x(x+1)(x+2)(x+3)\mathcal{L}_{(2,1^4)}\beta^3+x(x+1)(x+2)\big(\mathcal{L}_{(2^2,1^2)}+\mathcal{L}_{(3,1^3)}\big)\beta^2\\&\quad+x(x+1)\big(\mathcal{L}_{(2^3)}+\mathcal{L}_{(3,2,1)}+\mathcal{L}_{(4,1^2)}\big)\beta+\mathcal{L}_{(3^2)}+\mathcal{L}_{(4,2)}+\mathcal{L}_{(5,1)}.\end{aligned}$$

Since p_{ij} is equal to either 1 or 0, one can see that $\mathcal{L}_{(n)}=0$ unless graph Γ is a collection of n distinct points and therefore $\mathcal{L}=1$.

> The chromatic polynomial of any graph is a $\mathbb{Z}$-linear combination of the chromatic polynomials corresponding to a set of complete graphs.

Corollary 4.31 (formula for universal β-Tutte polynomials[38])**.**

$$\begin{aligned}&(1-y)^{n-1}\mathrm{Tutte}_n^{(\beta)}(\{p_{i,j}\};x,y)\\&\quad=\prod_{1\le i<j\le n}p_{ij}+\sum_{\lambda\vdash n}\mathrm{Tutte}(K_{\ell(\lambda)-1};x+y+\beta xy,0)\mathcal{L}_\lambda^{(\beta)}(\{p_{ij}\}).\end{aligned}$$

> The polynomial $(1-y)^{|V(\Gamma)|-1}\mathrm{Tutte}(\Gamma;x,y)$ is a $\mathbb{Z}[y]$-linear combination of the chromatic polynomials $\mathrm{Tutte}(K_m;x+y-xy,0)$ corresponding to a family of complete graphs $\{K_m\}$.

Here $V(\Gamma)$ denotes the set of vertices of graph Γ.

[37] See, e.g., [131, A008275] or https://en.wikipedia.org/wiki/Stirling_numbers_of_the_first_kind.

[38] It should be remembered that $\mathrm{Tutte}(K_1;x,y)=1$ and $\mathrm{Tutte}(K_0;x,y)=0$, since the graph $K_1:=\{pt\}$ and graph $K_0=\varnothing$.

Comments 4.32.

(*i*) Let us write

$$\mathrm{Ch}_n^{(\beta)}(\{p_{ij}\},x)=-\mathcal{L}_{(n)}\beta^{-1}+\sum_{k=1}^{n-1}a_n^{(k)}(\{p_{ij})x^k.$$

It follows from Theorem 4.29 that

$$a_n^{(k)}=\sum_{\lambda\vdash n}s(\ell(\lambda)-1,k)\mathcal{L}_\lambda^{(\beta)},$$

where as before, $s(k,n)$ denote the Stirling numbers of the first kind, see, e.g., footnote 37. For example,

$$a(\Gamma)=a_n^{(1)}(\{p_{ij}\})\Big|_{\substack{p_{ij}=0,\,(ij)\in\Gamma,\\ p_{ij}=1,\,(ij)\notin\Gamma}}(\ell(\lambda)-2)!\mathcal{N}_\lambda(\Gamma)\beta^{\ell(\lambda)-2},$$

where $\mathcal{N}_\lambda(\Gamma)$ denotes the number non isomorphic subgraphs in the complementary graph $K_n\backslash\Gamma$, which are isomorphic to the graph $K_n(\lambda)$.

(*ii*) It is clear that for a general set of parameters $\{p_{ij}\}$ the number of different monomials which appear in $\mathcal{L}_\lambda^{(\beta)}(\{p_{ij}\})$, where partition $\lambda=\sum\limits_{j=1}^{n}jm_j$, $\lambda\vdash n$, is equal to

$$\frac{n!}{\prod\limits_{j\geq 1}(j!)^{m_j}m_j!}.$$

(*iii*) For general set of parameters $\{p_{ij}\}$ one can show that the number of different monomials which appear in polynomial $a_n^{(1)}(\{p_{ij}\})$ is equal to $\mathrm{Bell}(n)-1$, where $\mathrm{Bell}(n)$ denotes the n-th Bell number, see, e.g., [131, *A*000110].

(*iv*) In the limit $y\longrightarrow 1$ one has $q_{ij}=m_{ij}$ and $p_{ij}=1-m_{ij}$.

(*v*) Let us introduce a modified universal Tutte polynomial, namely,

$$\begin{aligned}\mathrm{Tutte}(\{q_{ij}\};x,y,z)&:=(-1)^{n-1}\mathrm{Coeff}_{[t_1\cdots t_n]}\\ &\times\Bigg[\Bigg(\sum_{\substack{\ell_1,\dots,\ell_n\\ \ell_i\in\{0,1\},\,\forall\, i}}\prod_{1\leq i<j\leq n}(zq_{ij}y+1)^{\ell_i\ell_j}y^{-(\sum_j\ell_j)}t_1^{\ell_1}\cdots t_n^{\ell_n}\Bigg)^{xy}x^{-1}\Bigg].\end{aligned}$$

We set $\deg(q_{ij})=1$,

Proposition 4.33.

(*a*) $\mathrm{Tutte}(\{q_{ij}\};x,y,z)\in\mathbb{N}[\{q_{ij}][x,y,z]$.

(*b*) *Degree $n-1$ monomials of the polynomial* $\mathrm{Tutte}(\{q_{ij}\};0,y,z)$ *are in one-to-one correspondence with the set of spanning trees of the complete graph K_n. Moreover, the polynomial* $\mathrm{Tutte}(\{q_{ij}=1,\,\forall\, i,j\};x,0,1)$ *is equal to the generating function of forests on n labeled vertices, counting according to the number of connected components, whereas the polynomial* $\mathrm{Tutte}(\{q_{ij}=1,\,\forall\, i,j\};1,0,z)$ *is equal to the Hilbert polynomial of the even Orlik–Solomon* *algebra*[39] $\mathrm{OS}^{+}(\Gamma_n)$ *associated to the type* A_{n-1} *generic hyperplane arrangement* Γ_n, *see* [119, Section 5] *or* [72], *namely,*

$$\mathrm{Tutte}(\{q_{ij}=1,\,\forall\,i,j\};1,0,z)=\mathrm{Hilb}(\mathrm{OS}^{+}(\Gamma_n),z)=\sum_{\mathcal{F}} z^{|\mathcal{F}|},$$

where the sum runs over all forests on the vertices $\{1,\dots,n\}$, *and* $|\mathcal{F}|$ *denotes the number of edges of* $\mathcal{F}$.

(*c*) *More generally, denote by* $F_n(x,t):=\sum_{\mathcal{F}} x^{|\mathcal{F}|}t^{\mathrm{inv}(\mathcal{F})}$ *the generating function of statistics* $|\mathcal{F}|$ *and* $\mathrm{inv}(\mathcal{F})$ *on the set* $F(n)$ *of forests on* n *labeled vertices. Recall that the symbol* $|\mathcal{F}|$ *denotes the number of edges in a forest* $\mathcal{F}\in F(n)$ *and that* $\mathrm{inv}(\mathcal{F})$ *its inversion index*[40].

Lemma 4.34. *One can show that*

$$F_n(x,t)=(xt)^{n-1}\mathrm{Tutte}(K_n;1+(xt)^{-1},t-1),$$
$$\mathrm{Coeff}_{(xt)^{n-1}}[F_n(x,t)]=I_n(t),$$

where $I_n(t):=\sum_{\mathcal{F}\in\mathrm{Tree}(n)}t^{\mathrm{inv}(\mathcal{F})}$ *denotes the tree inversion polynomial, see, e.g.,* [51, 134].

(*d*) *Set*

$$DU_n(x):=(zt)^{n-1}\mathrm{Hilb}\big(K_n;1+(zt)^{-1},z-1\big)\Big|_{\substack{t:=-1\\ z:=-x}}=F_n(-x,-1).$$

One (A.K.) can show that $(n\ge 2)$

$$DU_n(x)\in\mathbb{N}[x],\qquad DU_n(1)=UD_{n+1},\qquad \mathrm{Coeff}_{x^{n-1}}[DU_n(x)]=UD_{n-1},$$

where UD_n *denote the Euler or up/down numbers associated with the exponential generating function* $\sec(x)+\tan(x)$, *see*[41], *e.g.,* [131, A000111].

(*e*) *One has*

$$x^{\binom{n}{2}}\mathrm{Tutte}\big(\{q_{ij}=1,\,\forall\,i,j\};x,x^{-1}-1,1\big)=\mathrm{Hilb}(\mathcal{A}_n,x),$$

where $\mathcal{A}_n$ *denotes the algebra generated by the curvature of* 2*-forms of the standard Hermitian linear bundles over the flag variety* $\mathcal{F}l_n$, *see* [72, 118, 129] *or Section* 4.2.2, *Theorem* 4.56(*B*).

(*f*) *Write* $\mathrm{Tutte}(\{q_{ij}\};0,y,z)=\sum_{k=0}^{n-1}a_n^{(k)}(y,z)$, *then monomials which appear in polynomial* $a_n^{(k)}(y,z)$ *are in one-to-one correspondence with the set of labeled graphs with* n *nodes having exactly* k *connected components.*

(*g*) *One has* $\mathrm{Tutte}((\{q_{ij}\};x,-1,1)=\mathrm{Tutte}(\{q_{ij}\},x+1,0)$.

[39] Known also as Orlik–Terao algebra.

[40] For the readers convenience we recall definitions of statistics $\mathrm{inv}(\mathcal{F})$ and the major index $\mathrm{maj}(\mathcal{F})$. Given a forest $\mathcal{F}$ on n labeled nodes, one can construct a tree $\mathcal{T}$ by adding a new vertex (root) connected with the maximal vertices in the connected components of $\mathcal{F}$.

The inversion index $\mathrm{inv}(\mathcal{F})$ is equal to the number of pairs (i,j) such that $1\le i<j\le n$, and the vertex labeled by j lies on the shortest path in $\mathcal{T}$ from the vertex labeled by i to the root.

The major index $\mathrm{maj}(\mathcal{F})$ is equal to $\sum_{x\in\mathrm{Des}(\mathcal{F})}h(x)$; here for any vertex $x\in\mathcal{F}$, $h(x)$ is the size of the subtree rooted at x; the descent set $\mathrm{Des}(\mathcal{F})$ of $\mathcal{F}$ consists of the vertices $x\in\mathcal{F}$ which have the labeling strictly greater than the labeling of its child's.

[41] The fact that $I_n(-1)=UD_{n-1}$ is due to G. Kreweras [82].

(h) *Recurrence relations for polynomials $F_n(x,t)$, cf.* [82],

$$F_0(x,t) = F_1(x,t) = 1, \qquad F_{n+1}(x,t) = \sum_{k=0}^{n} \binom{n}{k} (xt)^k I_k(t) F_{n-k}(x,t).$$

Example 4.35. Take $n = 5$, then

$$\mathrm{Tutte}(K_5; x, y) = (0,6,11,6,1) + (6,20,10)y + 15(1,1)y^2 + 5(3,1)y^3 + 10y^4 + 4y^5 + y^6,$$
$$F_5(-x,-1) = (1,10,25,20,5).$$

Write $F_n(x,t) = \tilde{F}_n(u,t)\big|_{u=xt}$, then

$$\tilde{F}_5(u,t) = 1 + 10u + u^2(35+10t) + u^3(50,40,15,5)_t + u^4(24,36,30,20,10,4,1)_t,$$
$$\tilde{F}_n(u,0) = \prod_{j=1}^{n-1}(1 + j\ u),$$
$$\mathrm{Hilb}(\mathcal{A}_5, t) = (1,4,10,20,35,51,64,60,35,10,1)_t,$$
$$\mathrm{Hilb}(\mathrm{OS}^+(\Gamma_5), t) = (1,10,45,110,125)_t.$$

Exercises 4.36.

(1) Assume that $\ell_{ij} = \ell$ for all $1 \le i < j \le r$. Based on the above formula for the exponential generating function for the Tutte polynomials of the complete multipartite graphs $K_{n_1,\ldots,n_r}$, *deduce* the following well-known formula

$$\mathrm{Tutte}\big(K^{(\ell)}_{n_1,\ldots,n_r}, 1, 1\big) = \ell^{N-1} N^{r-2} \prod_{j=1}^{r} (N - n_j)^{n_j - 1},$$

where $N := n_1 + \cdots + n_r$. It is well-known that the number $\mathrm{Tutte}(\Gamma, 1, 1)$ is equal to the number of *spanning trees* of a connected graph Γ.

(2) Take $r = 3$ and let n_1, n_2, n_3 and ℓ_{12}, ℓ_{13}, ℓ_{23} be positive integers. Set $N := \ell_{12}\ell_{13}n_1 + \ell_{12}\ell_{23}n_2 + \ell_{13}\ell_{23}n_3$. Show that

$$\mathrm{Tutte}\big(K^{\ell_1,\ell_2,\ell_3}_{n_1,n_2,n_3}, 1, 1\big) = N(\ell_{12}n_2 + \ell_{13}n_3)^{n_1-1}(\ell_{12}n_1 + \ell_{13}n_3)^{n_2-1}(\ell_{13}n_1 + \ell_{23}n_2)^{n_3-1}.$$

(3) Let $r \ge 2$, consider weighted complete multipartite graph $K^{(\boldsymbol{\ell})}_{\underbrace{n,\ldots,n}_{r}}$, where $\boldsymbol{\ell} = (\ell_{ij})$ such that $\ell_{1,j} = \ell$, $j = 1,\ldots,r$ and $\ell_{ij} = k$, $2 \le i < j \le r$. Show that

$$\mathrm{Tutte}\big(K^{(\boldsymbol{\ell})}_{\underbrace{n,\ldots,n}_{r}}, 1, 1\big) = k^n (r-1)^{n-1}((r-1)\ell + k)^{r-2}((r-2)\ell + k)^{(r-1)(n-1)} n^{nr-1}.$$

Let $\Gamma_n(*)$ be a spanning star subgraph of the complete graph K_n. For example, one can take for a graph $\Gamma_n(*)$ the subgraph $K_{1,n-1}$ with the set of vertices $V := \{1,2,\ldots,n\}$ and that of edges $E := \{(i,n),\ i = 1,\ldots,n-1\}$. The algebra $3T_n^{(0)}(K_{1,n-1})$ can be treated as a "noncommutative analog" of the projective space $\mathbb{P}^{n-1}$. We have $\theta_1 = u_{12} + u_{13} + \cdots + u_{1n}$. It is not difficult to see that $\mathrm{Hilb}(3T_n^{(0)}(K_{1,n-1})^{ab}, t) = (1+t)^{n-1}$, and $\theta_1^n = 0$. Let us observe that $\mathrm{Chrom}(\Gamma_n(\star), t) = t(t-1)^{n-1}$.

Problem 4.37. *Compute the Hilbert series of the algebra* $3T_n^{(0)}(K_{n_1,\ldots,n_r})$.

The first non-trivial case is that of *projective space*, i.e., the case $r = 2$, $n_1 = 1$, $n_2 = 5$.

On the other hand, if $\Gamma_n = \{(1,2) \to (2,3) \to \cdots \to (n-1,n)\}$ is the Dynkin graph of type A_{n-1}, then the algebra $3T_n^{(0)}(\Gamma_n)$ is isomorphic to the nil-Coxeter algebra of type A_{n-1}, and if $\Gamma_n^{(\mathrm{aff})} = \{(1,2) \to (2,3) \to \cdots \to (n-1,n) \to -(1,n)\}$ is the Dynkin graph of type $A_{n-1}^{(1)}$, i.e., *a cycle*, then the algebra $3T_n^{(0)}(\Gamma_n^{(\mathrm{aff})})$ is isomorphic to a certain quotient of the affine nil-Coxeter algebra of type $A_{n-1}^{(1)}$ by the two-sided ideal which can be described explicitly [72]. Moreover [72],

$$\mathrm{Hilb}\big(3T_n^{0)}(\Gamma^{(\mathrm{aff})}), t\big) = [n]_t \prod_{j=1}^{n-1} [j(n-j)]_t,$$

see Theorem 4.3. Therefore, the dimension $\dim(3T^{(0)}(\Gamma^{\mathrm{aff}}))$ is equal to $n!(n-1)!$ and is equal also, as it was pointed out in Section 4.1.1, to the number of (directed) Hamiltonian cycles in the complete bipartite graph $K_{n,n}$, see [131, A010790].

It is not difficult to see that

$$\mathrm{Hilb}\big(3T_n^{(0)}(\Gamma_n)^{ab}, t\big) = (t+1)^{n-1}, \qquad \mathrm{Hilb}\big(3T^{(0)}\big(\Gamma_n^{\mathrm{aff}}\big)^{ab}, t\big) = t^{-1}\big((t+1)^n - t - 1\big),$$

whereas

$$\mathrm{Chrom}(\Gamma_n, t) = t(t-1)^{n-1}, \qquad \mathrm{Chrom}\big(\Gamma_n^{\mathrm{aff}}, t\big) = (t-1)^n + (-1)^n(t-1).$$

Exercise 4.38. Let $K_{n_1,\dots,n_r}$ be complete multipartite graph, $N := n_1 + \cdots + n_r$. Show that[42]

$$\mathrm{Hilb}(3T_N(K_{n_1,\dots,n_r}), t) = \frac{\prod\limits_{j=1}^{r} \prod\limits_{a=1}^{n_j-1} (1-at)}{\prod\limits_{j=1}^{N-1} (1-jt)}.$$

4.1.4 Quasi-classical and associative classical Yang–Baxter algebras of type B_n

In this section we introduce an analogue of the algebra $3T_n(\beta)$ for the classical root systems.

Definition 4.39.

(A) *The quasi-classical Yang–Baxter algebra* $\widehat{\mathrm{ACYB}(B_n)}$ *of type* B_n is an associative algebra with the set of generators $\{x_{ij}, y_{ij}, z_i, 1 \le i \ne j \le n\}$ subject to the set of defining relations

(1) $x_{ij} + x_{ij} = 0$, $y_{ij} = y_{ji}$ if $i \ne j$,
(2) $z_i z_j = z_j z_i$,
(3) $x_{ij}x_{kl} = x_{kl}x_{ij}$, $x_{ij}y_{kl} = y_{kl}x_{ij}$, $y_{ij}y_{kl} = y_{kl}y_{ij}$ if i, j, k, l are distinct,
(4) $z_i x_{kl} = x_{kl} z_i$, $z_i y_{kl} = y_{kl} z_i$ if $i \ne k, l$,
(5) *three term relations*:

$$x_{ij}x_{jk} = x_{ik}x_{ij} + x_{jk}x_{ik} - \beta x_{ik}, \qquad x_{ij}y_{jk} = y_{ik}x_{ij} + y_{jk}y_{ik} - \beta y_{ik},$$
$$x_{ik}y_{jk} = y_{jk}y_{ij} + y_{ij}x_{ik} + \beta y_{ij}, \qquad y_{ik}x_{jk} = x_{jk}y_{ij} + y_{ij}y_{ik} + \beta y_{ij}$$

if $1 \le i < j < k \le n$,

[42] It should be remembered that to abuse of notation, the complete graph K_n, by definition, is equal to the complete multipartite graph $K(\underbrace{(1,\dots,1)}_{n})$, whereas the graph $K_{(n)}$ is a collection of n distinct points.

(6) *four term relations*:

$$x_{ij}z_j = z_ix_{ij} + y_{ij}z_i + z_jy_{ij} - \beta z_i$$

if $i < j$.

(B) *The associative classical Yang–Baxter algebra* $\mathrm{ACYB}(B_n)$ *of type* B_n is the special case $\beta = 0$ of the algebra $\widehat{\mathrm{ACYB}(B_n)}$.

Comments 4.40.

- In the case $\beta = 0$ the algebra $\mathrm{ACYB}(B_n)$ has a rational representation

$$x_{ij} \longrightarrow (x_i - x_j)^{-1}, \qquad y_{ij} \longrightarrow (x_i + x_j)^{-1}, \qquad z_i \longrightarrow x_i^{-1}.$$

- In the case $\beta = 1$ the algebra $\widehat{\mathrm{ACYB}(B_n)}$ has a "trigonometric" representation

$$x_{ij} \longrightarrow \left(1 - q^{x_i - x_j}\right)^{-1}, \qquad y_{ij} \longrightarrow \left(1 - q^{x_i + x_j}\right)^{-1}, \qquad z_i \longrightarrow \left(1 + q^{x_i}\right)\left(1 - q^{x_i}\right)^{-1}.$$

Definition 4.41. *The bracket algebra* $\mathcal{E}(B_n)$ *of type* B_n is an associative algebra with the set of generators $\{x_{ij}, y_{ij}, z_i, 1 \le i \ne j \le n\}$ subject to the set of relations (1)–(6) listed in Definition 4.39, and the additional relations

$$(5a) \quad x_{jk}x_{ij} = x_{ij}x_{ik} + x_{ik}x_{jk} - \beta x_{ik}, \qquad y_{jk}x_{ij} = x_{ij}y_{ik} + y_{ik}y_{jk} - \beta y_{ik},$$
$$y_{jk}x_{ik} = y_{ij}y_{jk} + x_{ik}y_{ij} + \beta y_{ij}, \qquad x_{jk}y_{ik} = y_{ij}x_{jk} + y_{ik}y_{ij} + \beta y_{ij}$$

if $1 \le i < j < k \le n$,

$$(6a) \quad z_jx_{ij} = x_{ij}z_i + z_iy_{ij} + y_{ij}z_j - \beta z_i$$

if $i < j$.

Definition 4.42. The quasi-classical Yang–Baxter algebra $\widehat{\mathrm{ACYB}(D_n)}$ of type D_n, as well as the algebras $\mathrm{ACYB}(D_n)$ and $\mathcal{E}(D_n)$ are defined by putting $z_i = 0$, $i = 1, \ldots, n$, in the corresponding B_n-versions of algebras in question.

Conjecture 4.43. *The both algebras* $\mathcal{E}(B_n)$ *and* $\mathcal{E}(D_n)$ *are Koszul, and*

$$\mathrm{Hilb}(\mathcal{E}(B_n), t) = \left(\prod_{j=1}^{n}(1 - (2j-1)t)\right)^{-1};$$

if $n \ge 4$

$$\mathrm{Hilb}(\mathcal{E}(D_n), t) = \left(\prod_{j=1}^{n-1}(1 - 2jt)\right)^{-1}.$$

Example 4.44.

$$\mathrm{Hilb}(\mathrm{ACYB}(B_2), t) = \left(1 - 4t + 2t^2\right)^{-1},$$
$$\mathrm{Hilb}(\mathrm{ACYB}(B_3), t) = \left(1 - 9t + 16t^2 - 4t^3\right)^{-1},$$
$$\mathrm{Hilb}(\mathrm{ACYB}(B_4), t) = \left(1 - 16t + 64t^2 - 60t^3 + 9t^4\right)^{-1},$$
$$\mathrm{Hilb}(\mathrm{ACYB}(D_4), t) = \left(1 - 12t + 18t^2 - 4t^3\right)^{-1}.$$

However,

$$\mathrm{Hilb}(\mathrm{ACYB}(B_5), t) = \left(1 - 25t + 180t^2 - 400t^3 + 221t^4 - 31t^5\right)^{-1}.$$

Let us introduce the following Coxeter type elements

$$h_{B_n} := \prod_{a=1}^{n-1} x_{a,a+1} z_n \in \mathcal{E}(B_n) \qquad \text{and} \qquad h_{D_n} := \prod_{a=1}^{n-1} x_{a,a+1} y_{n-1,n} \in \mathcal{E}(D_n). \tag{4.2}$$

Let us bring the element h_{B_n} (resp. h_{D_n}) to the reduced form in the algebra $\mathcal{E}(B_n)$ that is, let us consecutively apply the defining relations (1)–(6), (5a), (6a) to the element h_{B_n} (resp. apply to h_{D_n} the defining relations for algebra $\mathcal{E}(D_n)$) in any order until unable to do so. Denote the resulting (noncommutative) polynomial by $P_{B_n}(x_{ij}, y_{ij}, z)$ (resp. $P_{D_n}(x_{ij}, y_{ij})$). In principal, this polynomial itself can depend on the order in which the relations (1)–(6), (5a), (6a) are applied.

Conjecture 4.45 (cf. [133, Exercise 8.C5(c)])**.**

(1) *Apart from applying the commutativity relations (1)–(4), the polynomial $P_{B_n}(x_{ij}, y_{ij}, z)$ (resp. $P_{D_n}(x_{ij}, y_{ij})$) does not depend on the order in which the defining relations have been applied.*

(2) *Define polynomial $P_{B_n}(s, r, t)$ (resp. $P_{D_n}(s, r)$) to be the the image of that $P_{B_n}(x_{ij}, y_{ij}, z)$ (resp. $P_{D_n}(x_{ij}, y_{ij})$) under the specialization*

$$x_{ij} \longrightarrow s, \qquad y_{ij} \longrightarrow r, \qquad z_i \longrightarrow t.$$

Then $P_{B_n}(1,1,1) = \frac{1}{2}\binom{2n}{n} = \frac{1}{2}\mathrm{Cat}_{B_n}$.

Note that $P_{B_n}(1,0,1) = \mathrm{Cat}_{A_{n-1}}$.

Problem 4.46. *Investigate the B_n and D_n types reduced polynomials corresponding to the Coxeter elements (4.2), and the reduced polynomials corresponding to the longest elements*

$$w_{B_n} := \prod_{J=1}^{n} z_j \left(\prod_{1 \le i < j \le n} x_{ij} y_{ij} \right), \qquad w_{D_n} = \prod_{1 \le i < j \le n} x_{ij} y_{ij}.$$

4.2 Super analogue of 6-term relations and classical Yang–Baxter algebras

4.2.1 Six term relations algebra $6T_n$, its quadratic dual $(6T_n)^!$, and algebra $6HT_n$

Definition 4.47. *The* 6 *term relations algebra* $6T_n$ is an associative algebra (say over $\mathbb{Q}$) with the set of generators $\{r_{i,j}, 1 \le i \ne j < n\}$, subject to the following relations:

1) $r_{i,j}$ and $r_{k,l}$ commute if $\{i,j\} \cap \{k,l\} = \varnothing$,
2) *unitarity condition*: $r_{ij} + r_{ji} = 0$,
3) *classical Yang–Baxter relations*: $[r_{ij}, r_{ik} + r_{jk}] + [r_{ik}, r_{jk}] = 0$ if i, j, k are distinct.

We denote by CYB_n, named by *classical Yang–Baxter algebra*, an associative algebra over $\mathbb{Q}$ generated by elements $\{r_{ij}, 1 \le i \ne j \le n\}$ subject to relations 1) and 3).

Note that the algebra $6T_n$ is given by $\binom{n}{2}$ generators and $\binom{n}{3} + 3\binom{n}{4}$ quadratic relations.

Definition 4.48. Define *Dunkl elements* in the algebra $6T_n$ to be

$$\theta_i = \sum_{j \ne i} r_{ij}, \qquad i = 1, \ldots, n.$$

It easy to see that the Dunkl elements $\{\theta_i\}_{1 \le i \le n}$ generate a commutative subalgebra in the algebra $6T_n$.

Example 4.49 (some "rational and trigonometric" representations of the algebra $6T_n$)**.** Let $A = U(\mathfrak{sl}(2))$ be the universal enveloping algebra of the Lie algebra $\mathfrak{sl}(2)$. Recall that the algebra $\mathfrak{sl}(2)$ is spanned by the elements e, f, h, such that $[h,e] = 2e$, $[h,f] = -2f$, $[e,f] = h$.

Let's search for solutions to the CYBE in the form

$$r_{i,j} = a(u_i,u_j)h \otimes h + b(u_i,u_j)e \otimes f + c(u_i,u_j)f \otimes e,$$

where $a(u,v), b(u,v) \neq 0, c(u,v) \neq 0$ are meromorphic functions of the variables $(u,v) \in \mathbb{C}^2$, defined in a neighborhood of $(0,0)$, taking values in $A \otimes A$. Let $a_{ij} := a(u_i,u_j)$ (resp. $b_{ij} := b(u_i,u_j)$, $c_{ij} := c(u_i,u_j)$).

Lemma 4.50. *The elements* $r_{i,j} := a_{ij}h \otimes h + b_{ij}e \otimes f + c_{ij}f \otimes e$ *satisfy CYBE iff*

$$b_{ij}b_{jk}c_{ik} = c_{ij}c_{jk}b_{ik} \qquad \textit{and} \qquad 4a_{ik} = b_{ij}b_{jk}/b_{ik} - b_{ik}c_{jk}/b_{ij} - b_{ik}c_{ij}/b_{jk}$$

for $1 \le i < j < k \le n$.

It is not hard to see that

- there are three rational solutions:

$$r_1(u,v) = \frac{1/2h \otimes h + e \otimes f + f \otimes e}{u - v},$$
$$r_2(u,v) = \frac{u+v}{4(u-v)}h \otimes h + \frac{u}{u-v}e \otimes f + \frac{v}{u-v}f \otimes e,$$

  and $r_3(u,v) := -r_2(v,u)$,
- there is a trigonometric solution

$$r_{\text{trig}}(u,v) = \frac{1}{4}\frac{q^{2u}+q^{2v}}{q^{2u}-q^{2v}}h \otimes h + \frac{q^{u+v}}{q^{2u}-q^{2v}}\big(e \otimes f + f \otimes e\big).$$

Notice that the *Dunkl element* $\theta_j := \sum\limits_{a \neq j} r_{\text{trig}}(u_a,u_j)$ corresponds to the truncated (or level 0) *trigonometric Knizhnik–Zamolodchikov operator*.

In fact, the "$\mathfrak{sl}_n$-Casimir element"

$$\Omega = \frac{1}{2}\left(\sum_{i=1}^{n} E_{ii} \otimes E_{ii}\right) + \sum_{1 \le i < j \le n} E_{ij} \otimes E_{ji}$$

satisfies the 4-term relations

$$[\Omega_{12}, \Omega_{13} + \Omega_{23}] = 0 = [\Omega_{12} + \Omega_{13}, \Omega_{23}],$$

and the elements $r_{ij} := \frac{\Omega_{ij}}{u_i - u_j}$, $1 \le i < j \le n$, satisfy the classical Yang–Baxter relations.

Recall that the set $\{E_{ij} := (\delta_{ik}\delta_{jl})_{1 \le k,l \le n},\ 1 \le i,j \le n\}$, stands for the standard basis of the algebra $\operatorname{Mat}(n,\mathbb{R})$.

Definition 4.51. Denote by $6T_n^{(0)}$ the quotient of the algebra $6T_n$ by the (two-sided) ideal generated by the set of elements $\{r_{i,j}^2,\ 1 \le i < j \le n\}$.

More generally, let $\{\beta, q_{ij},\ 1 \le i < j \le n\}$ be a set of parameters. Let $R := \mathbb{Q}[\beta][q_{ij}^{\pm 1}]$.

Definition 4.52. Denote by $6HT_n$ the quotient of the algebra $6T_n \otimes R$ by the (two-sided) ideal generated by the set of elements $\{r_{i,j}^2 - \beta r_{i,j} - q_{ij},\ 1 \le i < j \le n\}$.

All these algebras are naturally graded, with $\deg(r_{i,j}) = 1$, $\deg(\beta) = 1$, $\deg(q_{ij}) = 2$. It is clear that the algebra $6T_n^{(0)}$ can be considered as the infinitesimal deformation $R_{i,j} := 1 + \epsilon r_{i,j}$, $\epsilon \longrightarrow 0$, of the Yang–Baxter group YB_n.

For the reader convenience we recall the definition of the *Yang–Baxter* group.

Definition 4.53. The Yang–Baxter group YB_n is a group generated by elements $\{R_{ij}^{\pm 1}, 1 \leq i < j \leq n\}$, subject to the set of defining relations

- $R_{ij}R_{kl} = R_{kl}R_{ij}$ if i, j, k, l are distinct,
- *quantum Yang–Baxter relations*:

$$R_{ij}R_{ik}R_{jk} = R_{jk}R_{ik}R_{ij} \qquad \text{if} \quad 1 \leq i < j < k \leq n.$$

Corollary 4.54. *Define $h_{ij} = 1 + r_{ij} \in 6HT_n$. Then the following relations in the algebra $6HT_n$ are satisfied:*

(1) $r_{ij}r_{ik}r_{jk} = r_{jk}r_{ik}r_{ij}$ *for all pairwise distinct i, j and k;*

(2) *Yang–Baxter relations: $h_{ij}h_{ik}h_{jk} = h_{jk}h_{ik}h_{ij}$ if $1 \leq i < j < k \leq n$.*

Note, the item (1) includes three relations in fact.

Proposition 4.55.

(1) *The quadratic dual $(6T_n)^!$ of the algebra $6T_n$ is a quadratic algebra generated by the elements $\{t_{i,j}, 1 \leq i < j \leq n\}$ subject to the set of relations*

  (*i*) $t_{i,j}^2 = 0$ *for all $i \neq j$;*

  (*ii*) *anticommutativity: $t_{ij}t_{k,l} + t_{k,l}t_{i,j} = 0$ for all $i \neq j$ and $k \neq l$;*

  (*iii*) $t_{i,j}t_{i,k} = t_{i,k}t_{j,k} = t_{i,j}t_{j,k}$ *if i, j, k are distinct.*

(2) *The quadratic dual $(6T_n^{(0)})^!$ of the algebra $6T_n^{(0)}$ is a quadratic algebra with generators $\{t_{i,j}, 1 \leq i < j \leq n\}$ subject to the relations (ii)–(iii) above only.*

4.2.2 Algebras $6T_n^{(0)}$ and $6T_n^{\star}$

We are reminded that the algebra $6T_n^{(0)}$ is the quotient of the six term relation algebra $6T_n$ by the two-sided ideal generated by the elements $\{r_{ij}\}_{1\leq i<j\leq n}$. Important *consequence* of the classical Yang–Baxter relations and relations $r_{ij}^2 = 0$, $\forall\, i \neq j$, is that the both additive Dunkl elements $\{\theta_i\}_{1\leq i\leq n}$ and multiplicative ones

$$\left\{\Theta_i = \prod_{a=i-1}^{1} h_{ai}^{-1} \prod_{a=i+1}^{n} h_{ia}\right\}_{1\leq i\leq n}$$

generate commutative subalgebras in the algebra $6T_n^{(0)}$ (and in the algebra $6T_n$ as well), see Corollary 4.54. The problem we are interested in, is to describe commutative subalgebras generated by additive (resp. multiplicative) Dunkl elements in the algebra $6T_n^{(0)}$. Notice that the subalgebra generated by additive Dunkl elements in the abelianization[43] of the algebra $6T_n(0)$ has been studied in [118, 129]. In order to state the result from [118] we need, let us introduce a bit of notation. As before, let $\mathcal{F}l_n$ denotes the complete flag variety, and denote by $\mathcal{A}_n$ the algebra generated by the curvature of 2-forms of the standard Hermitian linear bundles over

[43]See, e.g., http://mathworld.wolfram.com/Abelianization.html.

the flag variety $\mathcal{F}l_n$, see, e.g., [118]. Finally, denote by I_n the ideal in the ring of polynomials $\mathbb{Z}[t_1,\dots,t_n]$ generated by the set of elements

$$(t_{i_1}+\cdots+t_{i_k})^{k(n-k)+1}$$

for all sequences of indices $1\le i_1<i_2<\cdots<i_k\le n$, $k=1,\dots,n$.

Theorem 4.56 ([118, 129])**.**

(A) *There exists a natural isomorphism*

$$\mathcal{A}_n \longrightarrow \mathbb{Z}[t_1,\dots,t_n]/I_n,$$

(B)

$$\mathrm{Hilb}(\mathcal{A}_n,t)=t^{\binom{n}{2}}\mathrm{Tutte}\big(K_n,1+t,t^{-1}\big).$$

Therefore the dimension of $\mathcal{A}_n$ (as a $\mathbb{Z}$-vector space) is equal to the number $\mathcal{F}(n)$ of forests on n labeled vertices. It is well-known that

$$\sum_{n\ge1}\mathcal{F}(n)\frac{x^n}{n!}=\exp\left(\sum_{n\ge1}n^{n-1}\frac{x^n}{n!}\right)-1.$$

For example,

$$\mathrm{Hilb}(\mathcal{A}_3,t)=(1,2,3,1),\qquad \mathrm{Hilb}(\mathcal{A}_4,t)=(1,3,6,10,11,6,1),$$
$$\mathrm{Hilb}(\mathcal{A}_5,t)=(1,4,10,20,35,51,64,60,35,10,1),$$
$$\mathrm{Hilb}(\mathcal{A}_6,t)=(1,5,15,35,70,126,204,300,405,490,511,424,245,85,15,1).$$

Problem 4.57. *Describe subalgebra in $(6T_n^{(0)})^{ab}$ generated by the multiplicative Dunkl elements $\{\Theta_i\}_{1\le i\le n}$.*

On the other hand, the commutative subalgebra $\mathcal{B}_n$ generated by the additive Dunkl elements in the algebra $6T_n^{(0)}$, $n\ge3$, has *infinite* dimension. For example,

$$\mathcal{B}_3\cong\mathbb{Z}[x,y]/\langle xy(x+y)\rangle,$$

and the Dunkl elements $\theta_j^{(3)}$, $j=1,2,3$, have infinite order.

Definition 4.58. Define algebra $6T_n^{\bigstar}$ to be the quotient of that $6T_n^{(0)}$ by the two-sided ideal generated by the set of "cyclic relations"

$$\sum_{j=2}^{m}\prod_{a=j}^{m}r_{i_1,i_a}\prod_{a=2}^{j}r_{i_1,i_a}=0$$

for all sequences $\{1\le i_1,i_2,\dots,i_m\le n\}$ of pairwise distinct integers, and all integers $2\le m\le n$.

For example,

- $\mathrm{Hilb}(6T_3^{\bigstar},t)=(1,3,5,4,1)=(1+t)(1,2,3,1)$,
- subalgebra (over $\mathbb{Z}$) in the algebra $6T_3^{\bigstar}$ generated by Dunkl elements θ_1 and θ_2 has the Hilbert polynomial equal to $(1,2,3,1)$, and the following presentation: $\mathbb{Z}[x,y]/I_3$, where I_3 denotes the ideal in $\mathbb{Z}[x,y]$ generated by x^3, y^3, and $(x+y)^3$,

- $\mathrm{Hilb}(6T_4^{\bigstar}, t) = (1, 6, 23, 65, 134, 164, 111, 43, 11, 1)_t$.

As a consequence of the cyclic relations, one can check that for any integer $n \geq 2$ the n-th power of the additive Dunkl element θ_i is equal to zero in the algebra $6T_n^{\bigstar}$ for all $i = 1, \dots, n$. Therefore, the Dunkl elements generate a finite-dimensional commutative subalgebra in the algebra $6T_n^{\bigstar}$. There exist natural homomorphisms

$$6T_n^{\bigstar} \longrightarrow 3T_n^{(0)}, \qquad \mathcal{B}_n \xrightarrow{\tilde{\pi}} \mathcal{A}_n \longrightarrow H^*(\mathcal{F}l_n, \mathbb{Z}) \tag{4.3}$$

The first and third arrows in (4.3) are epimorphism. We expect that the map $\tilde{\pi}$ is also epimorphism[44], and looking for a description of the kernel $\ker(\tilde{\pi})$.

Comments 4.59.

- Let us denote by $\mathcal{B}_n^{\mathrm{mult}}$ and $\mathcal{A}_n^{\mathrm{mult}}$ the subalgebras generated by *multiplicative* Dunkl elements in the algebras $6T_n^{(0)}$ and $\left(6T_n^{(0)}\right)^{ab}$ correspondingly. One can define a sequence of maps

$$\mathcal{B}_n^{\mathrm{mult}} \longrightarrow \mathcal{A}_n^{\mathrm{mult}} \xrightarrow{\tilde{\phi}} K^*(\mathcal{F}l_n), \tag{4.4}$$

  which is a K-theoretic analog of that (4.3). It is an interesting problem to find a geometric interpretation of the algebra $\mathcal{A}_n^{\mathrm{mult}}$ and the map $\tilde{\phi}$.

- "Quantization". Let β and $\{q_{ij} = q_{ji}, 1 \leq i, j \leq n\}$ be parameters.

Definition 4.60. Define algebra $6HT_n$ to be the quotient of the algebra $6T_n$ by the two sided ideal generated by the elements $\{r_{ij}^2 - \beta r_{ij} - q_{ij}\}_{1 \leq i,j \leq n}$.

Lemma 4.61. *The both additive $\{\theta_i\}_{1\leq i\leq n}$ and multiplicative $\{\Theta_i\}_{1\leq i\leq n}$ Dunkl elements generate commutative subalgebras in the algebra $6HT_n$.*

Therefore one can define algebras $6\mathcal{HB}_n$ and $6\mathcal{HA}_n$ which are a "quantum deformation" of algebras $\mathcal{B}_n$ and $\mathcal{A}_n$ respectively. We *expect* that in the case $\beta = 0$ and a special choice of "arithmetic parameters" $\{q_{ij}\}$, the algebra $\mathcal{HA}_n$ is connected with the arithmetic Schubert and Grothendieck calculi, cf. [129, 137]. Moreover, for a "general" set of parameters $\{q_{ij}\}_{1\leq i,j\leq n}$ and $\beta = 0$, we *expect* an existence of a natural homomorphism

$$\mathcal{HA}_n^{\mathrm{mult}} \longrightarrow \mathcal{QK}^*(\mathcal{F}l_n),$$

where $\mathcal{QK}^*(\mathcal{F}l_n)$ denotes a *multiparameter quantum deformation* of the K-theory ring $K^*(\mathcal{F}l_n)$ [72, 76]; see also Section 3.1. Thus, we treat the algebra $\mathcal{HA}_n^{\mathrm{mult}}$ as the K-theory version of a multiparameter quantum deformation of the algebra $\mathcal{A}_n^{\mathrm{mult}}$ which is generated by the curvature of 2-forms of the Hermitian linear bundles over the flag variety $\mathcal{F}l_n$.

- One can define an analogue of the algebras $6T_n^{(0)}$, $6HT_n$ etc., denoted by $6T(\Gamma)$ etc., for any subgraph $\Gamma \subset K_n$ of the complete graph K_n, and in fact for any oriented matroid. It is known that $\mathrm{Hilb}((6T_n(\Gamma)^{ab}, t) = t^{e(\Gamma)}\mathrm{Tutte}(\Gamma, 1+t, t^{-1})$, see, e.g., [11] and the literature quoted therein.

[44] Contrary to the case of the map $\mathrm{pr}_n \colon \mathbb{Z}[\theta_1, \dots, \theta_n] \longrightarrow (3T_n(0))^{ab}$, where the image $\mathrm{Im}(\mathrm{pr}_n)$ has dimension equals to the number of permutations in $\mathbb{S}_n$ with $(n-1)$ inversions see [131, A001892].

4.2.3 Hilbert series of algebras CYB_n and $6T_n$[45]

Examples 4.62.

$$\mathrm{Hilb}(6T_3,t)=\big(1-3t+t^2\big)^{-1},\qquad \mathrm{Hilb}(6T_4,t)=\big(1-6t+7t^2-t^3\big)^{-1},$$
$$\mathrm{Hilb}(6T_5,t)=\big(1-10t+25t^2-15t^3+t^4\big)^{-1},$$
$$\mathrm{Hilb}(6T_6,t)=\big(1-15t+65t^2-90t^3+31t^4-t^5\big)^{-1},$$
$$\mathrm{Hilb}\big(6T_3^{(0)},t\big)=[2][3](1-t)^{-1},\qquad \mathrm{Hilb}\big(6T_4^{(0)},t\big)=[4](1-t)^{-2}\big(1-3t+t^2\big)^{-1}.$$

In fact, the following statements are true.

Proposition 4.63 (cf. [7])**.** *Let $n\geq 2$, then*

- *The algebras $6T_n$ and CYB_n are Koszul.*
- *We have*

$$\mathrm{Hilb}(6T_n,t)=\left(\sum_{k=0}^{n-1}(-1)^k\left\{{n \atop n-k}\right\}t^k\right)^{-1},$$

where $\left\{{n \atop k}\right\}$ stands for the Stirling numbers of the second kind, i.e., the number of ways to partition a set of n things into k nonempty subsets.

- $$\mathrm{Hilb}(\mathrm{CYB}_n,t)=\left(\sum_{k=0}^{n-1}(-1)^k(k+1)!N(k,n)t^k\right)^{-1},$$

where $N(k,n)=\frac{1}{n}\binom{n}{k}\binom{n}{k+1}$ denotes the Narayana number, i.e., the number of Dyck n-paths with exactly k peaks.

Corollary 4.64.

(*A*) *The Hilbert polynomial of the quadratic dual of the algebra $6T_n$ is equal to*

$$\mathrm{Hilb}\big(6T_n^!,t\big)=\sum_{k=0}^{n-1}\left\{{n \atop n-k}\right\}t^k.$$

It is well-known that

$$\sum_{n\geq 0}\left(\sum_{k=0}^{n-1}\left\{{n \atop n-k}\right\}t^k\right)\frac{z^n}{n!}=\exp\left(\frac{\exp(zt)-1}{t}\right).$$

Therefore,

$$\dim(6T_n)^!=\mathrm{Bell}_n,$$

where Bell_n denotes the n-th Bell number, i.e., the number of ways to partition n things into subsets, see [131]*. Recall, that*

$$\sum_{n\geq 0}\mathrm{Bell}_n\frac{z^n}{n!}=\exp(\exp(z)-1)).$$

[45] Results of this subsection have been obtained independently in [7]. This paper contains, among other things, a description of a basis in the algebra $6T_n$, and much more.

(*B*) *The Hilbert polynomial of the quadratic dual of the algebra* CYB_n *is equal to*

$$\mathrm{Hilb}\big((\mathrm{CYB}_n)^!, t\big) = \sum_{k=0}^{n-1} (k+1)! N(k,n) t^k = (n-1)! L_{n-1}^{(\alpha=1)}\big(-t^{-1}\big) t^{n-1},$$

where

$$L_n^{(\alpha)}(x) = \frac{x^{-\alpha} e^x}{n!} \frac{d^n}{dx^n}\big(e^{-x} x^{n+\alpha}\big)$$

denotes the generalized Laguerre polynomial. The numbers $(k+1)!N(n,k) := L(n,n-k)$ *are known as Lah numbers, see, e.g.,* [131, *A*008297], *moreover* [131],

$$\dim(\mathrm{CYB}_n)^! = A000262.$$

It is well-known that

$$\sum_{n\geq 0} \left(\sum_{k\geq 0}^{n-1} (k+1)! N(k,n) t^k \right) \frac{z^n}{n!} = \exp\big(z(1-zt)^{-1}\big).$$

Comments 4.65. Let $\mathcal{E}_n(u)$, $u \neq 0, 1$, be the *Yokonuma–Hecke* algebra, see, e.g., [124] and the literature quoted therein. It is known that the dimension of the Yokonuma–Hecke algebra $\mathcal{E}_n(u)$ is equal to $n! B_n$, where B_n denotes as before the n-th Bell number. Therefore, $\dim(\mathcal{E}_n(u)) = \dim((6T_n)^! \rtimes \mathbb{S}_n)$, where $(6T_n)^! \rtimes \mathbb{S}_n$ denotes the semi-direct product of the algebra $(6T_n)^!$ and the symmetric group $\mathbb{S}_n$. It seems an interesting task to check whether or not the algebras $(6T_n)^! \rtimes \mathbb{S}_n$ and $\mathcal{E}_n(u)$ are isomorphic.

Remark 4.66. Denote by $\mathcal{M}\mathrm{YB}_n$ the group algebra over $\mathbb{Q}$ of the *monoid* corresponding to the Yang–Baxter group YB_n, see, e.g., Definition 4.48. Let $P(\mathcal{M}\mathrm{YB}_n, s, t)$ denotes the Poincaré polynomial of the algebra $\mathcal{M}\mathrm{YB}_n$. One can show that

$$\mathrm{Hilb}(6T_n, s) = P(\mathcal{M}\mathrm{YB}_n, -s, 1)^{-1}.$$

For example,

$$\begin{aligned}
&P(\mathcal{M}\mathrm{YB}_3, s, t) = 1 + 3st + s^2 t^3,\\
&P(\mathcal{M}\mathrm{YB}_4, s, t) = 1 + 6st + s^2\big(3t^2 + 4t^3\big) + s^3 t^6,\\
&P(\mathcal{M}\mathrm{YB}_5, s, t) = 1 + 10st + s^2\big(15t^2 + 10t^3\big) + s^3\big(10t^4 + 5t^6\big) + s^4 t^{10}.
\end{aligned}$$

Note that $\mathrm{Hilb}(\mathcal{M}\mathrm{YB}_n, t) = P(\mathcal{M}\mathrm{YB}_n, -1, t)^{-1}$ and $P(\mathcal{M}\mathrm{YB}_n, 1, 1) = \mathrm{Bell}_n$, the n-th Bell number.

Conjecture 4.67.

$$P(\mathcal{M}\mathrm{YB}_n, s, t) = \sum_{\pi} s^{\#(\pi)} t^{n(\pi)},$$

where the sum runs over all partitions $\pi = (I_1, \dots, I_k)$ *of the set* $[n] := [1, \dots, n]$ *into nonempty subsets* $I_1, \dots, I_k$, *and we set by definition,* $\#(\pi) := n - k$, $n(\pi) := \sum_{a=1}^{k} \binom{|I_a|}{2}$.

Remark 4.68. For any finite Coxeter group (W, S) one can define the algebra $\mathrm{CYB}(W) := \mathrm{CYB}(W, S)$ which is an analog of the algebra $\mathrm{CYB}_n = \mathrm{CYB}(A_{n-1})$ for other root systems.

Conjecture 4.69 (A.N. Kirillov, Yu. Bazlov). *Let (W,S) be a finite Coxeter group with the root system Φ. Then*

- *the algebra* $\mathrm{CYB}(W)$ *is Koszul;*
- $\mathrm{Hilb}(\mathrm{CYB}(W),t)=\left\{\sum_{k=0}^{|S|} r_k(\Phi)(-t)^k\right\}^{-1}$,

where $r_k(\Phi)$ is equal to the number of subsets in Φ^+ which constitute the positive part of a root subsystem of rank k. For example, $r_1(\Phi)=|\Phi^+|$, and $r_2(\Phi)$ is equal to the number of defining relations in a representation of the algebra $\mathrm{CYB}(W)$.

Example 4.70.

$$\mathrm{Hilb}\big(\mathrm{CYB}(B_2)^!,t\big)=(1,4,3),\qquad \mathrm{Hilb}\big(\mathrm{CYB}(B_3)^!,t\big)=(1,9,13,2),$$
$$\mathrm{Hilb}\big(\mathrm{CYB}(B_4)^!,t\big)=(1,16,46,28,5),\qquad \mathrm{Hilb}\big(\mathrm{CYB}(B_5)^!,t\big)=(1,25,130,200,101,12),$$
$$\mathrm{Hilb}\big(\mathrm{CYB}(D_4)^!,t\big)=(1,12,34,24,4),\qquad \mathrm{Hilb}\big(\mathrm{CYB}(D_5)^!,t\big)=(1,20,110,190,96,11).$$

Definition 4.71. The even generic Orlik–Solomon algebra $\mathrm{OS}^+(\Gamma_n)$ is defined to be an associative algebra (say over $\mathbb{Z}$) generated by the set of *mutually commuting* elements $y_{i,j}$, $1\le i\ne j\le n$, subject to the set of cyclic relations

$$y_{i,j}=y_{j,i},\qquad y_{i_1,i_2}y_{i_2,i_3}\cdots y_{i_{k-1},i_k}y_{i_1,i_k}=0\qquad \text{for}\quad k=2,\dots,n,$$

and all sequences of pairwise distinct integers $1\le i_1,\dots,i_k\le n$.

Exercises 4.72.

(1) Show that

$$\exp\big(z(1-zt)^{-q}\big)=1+\sum_{n\ge1}\left(1+\sum_{k=1}^{n-1}\binom{n-1}{k}\prod_{a=0}^{k-1}(a+(n-k)q)t^k\right)\frac{z^n}{n!}.$$

(2) *The even generic Orlik–Solomon algebra.* Show that the number of degree k, $k\ge 3$, relations in the definition of the Orlik–Solomon algebra $\mathrm{OS}^+(\Gamma_n)$ is equal to $\frac12(k-1)!\binom{n}{k}$ and also is equal to the maximal number of k-cycles in the complete graph K_n.

Note that if one replaces the commutativity condition in the above definition on the condition that $y_{i,j}$'s pairwise *anticommute*, then the resulting algebra appears to be isomorphic to the Orlik–Solomon algebra $\mathrm{OS}(\Gamma_n)$ corresponding to the generic hyperplane arrangement Γ_n, see [119]. It is known [119, Corollary 5.3], that

$$\mathrm{Hilb}(\mathrm{OS}(\Gamma_n),t)=\sum_F t^{|F|},$$

where the sum runs over all forests F on the vertices $1,\dots,n$, and $|F|$ denotes the number of edges in a forest F.

It follows from Corollary 4.64, that

$$\sum_{n\ge1}\mathrm{Hilb}(\mathrm{OS}(\Gamma_n),t)\frac{z^n}{n!}=\exp\left(\sum_{n\ge1}n^{n-2}t^{n-1}\frac{z^n}{n!}\right).$$

It is not difficult to see that $\mathrm{Hilb}(\mathrm{OS}^+(\Gamma_n),t)=\mathrm{Hilb}(\mathrm{OS}(\Gamma_n),t)$. In particular, $\dim\mathrm{OS}^+(\Gamma_n)=\mathcal{F}(n)$. Note also that a sequence $\{\mathrm{Hilb}(\mathrm{OS}(\Gamma_n),-1)\}_{n\ge2}$ appears in [131, A057817]. The polynomials $\mathrm{Hilb}(\mathcal{A}_n,t)$, $F_n(x,t)$ and $\mathrm{Hilb}(\mathrm{OS}^+(\Gamma_n),t)$ can be expressed, see, e.g., [118], as certain *specializations* of the Tutte polynomial $T(G;x,y)$ corresponding to the complete graph $G:=K_n$. Namely,

$$\mathrm{Hilb}(\mathcal{A}_n,t)=t^{\binom{n}{2}}T\big(K_n;1+t,t^{-1}\big),\qquad \mathrm{Hilb}\big(\mathrm{OS}^+(\Gamma_n),t\big)=t^{n-1}T\big(K_n;1+t^{-1},1\big).$$

4.2.4 Super analogue of 6-term relations algebra

Let n, m be non-negative integers.

Definition 4.73. *The super 6-term relations algebra* $6T_{n,m}$ is an associative algebra over $\mathbb{Q}$ generated by the elements $\{x_{i,j},\ 1 \le i \ne j \le n\}$ and $\{y_{\alpha,\beta},\ 1 \le \alpha \ne \beta \le m\}$ subject to the set of relations

(0) $x_{i,j} + x_{j,i} = 0$, $y_{\alpha,\beta} = y_{\beta,\alpha}$;

(1) $x_{i,j}x_{k,l} = x_{k,l}x_{i,j}$, $x_{i,j}y_{\alpha,\beta} = y_{\alpha,\beta}x_{i,j}$, $y_{\alpha,\beta}y_{\gamma,\delta} + y_{\gamma,\delta}y_{\alpha,\beta} = 0$, if tuples (i,j,k,l), (i,j,α,β), as well as $(\alpha,\beta,\gamma,\delta)$ consist of pair-wise distinct integers;

(2) classical Yang–Baxter relations and theirs super analogue: $[x_{i,k}, x_{j,i} + x_{j,k}] + [x_{i,j}, x_{j,k}] = 0$ if $1 \le i,j,k \le n$ are distinct, $[x_{i,k}, y_{j,i} + y_{j,k}] + [x_{i,j}, y_{j,k}] = 0$ if $1 \le i,j,k \le \min(n,m)$ are distinct, $[y_{\alpha,\gamma}, y_{\beta,\alpha} + y_{\beta,\gamma}]_+ + [y_{\alpha,\beta}, y_{\beta,\gamma}]_+ = 0$ if $1 \le \alpha,\beta,\gamma \le m$ are distinct.

Recall that $[a,b]_+ := ab + ba$ denotes the anticommutator of elements a and b.

Conjecture 4.74. *The algebra $6T_{n,m}$ is Koszul.*

Theorem 4.75. *Let $n, m \in \mathbb{Z}_{\ge 1}$, one has*

$$\mathrm{Hilb}\big((6T_n)^!, t\big)\mathrm{Hilb}\big((6T_m)^!, t\big) = \sum_{k=0}^{\min(n,m)-1} \left\{ \begin{matrix} \min(n,m) \\ \min(n,m)-k \end{matrix} \right\} \mathrm{Hilb}\big((6T_{n-k,m-k})^!, t\big) t^{2k},$$

where as before $\left\{ \begin{smallmatrix} n \\ n-k \end{smallmatrix} \right\}$ denotes the Stirling numbers of the second kind, see, e.g., [131, *A*008278].

Corollary 4.76. *Let $n, m \in \mathbb{Z}_{\ge 1}$. One has*

(*a*) *Symmetry:* $\mathrm{Hilb}(6T_{n,m}, t) = \mathrm{Hilb}(6T_{m,n}, t)$.

(*b*) *Let $n \le m$, then*

$$\mathrm{Hilb}\big((6T_{n,m})^!, t\big) = \sum_{k=0}^{n-1} s(n-1, n-k)\mathrm{Hilb}\big((6T_{n-k})^!, t\big)\mathrm{Hilb}\big((6T_{m-k})^!, t\big) t^{2k},$$

where $s(n-1, n-k)$ denotes the Stirling numbers of the first kind, i.e.,

$$\sum_{k=0}^{n-1} s(n-1, n-k) t^k = \prod_{j=1}^{n-1} (1 - jt).$$

(*c*) $\dim(6T_{n,n})^!$ *is equal to the number of pairs of partitions of the set $\{1, 2, \ldots, n\}$ whose meet is the partition $\{\{1\}, \{2\}, \ldots, \{n\}\}$, see, e.g.,* [131, *A*059849].

Example 4.77.

$$\mathrm{Hilb}\big((6T_{3,2})^!, t\big) = \mathrm{Hilb}\big((6T_{2,3})^!, t\big) = (1,4,3),$$
$$\mathrm{Hilb}\big((6T_{2,4})^!, t\big) = \mathrm{Hilb}\big((6T_{4,2})^!, t\big) = (1,7,12,5), \qquad \mathrm{Hilb}\big((6T_{3,3})^!, t\big) = (1,6,8),$$
$$\mathrm{Hilb}\big((6T_{2,5})^!, t\big) = \mathrm{Hilb}\big((6T_{5,2})^!, t\big) = (1,11,34,34,9),$$
$$\mathrm{Hilb}\big((6T_{3,4})^!, t\big) = \mathrm{Hilb}\big((6T_{4,3})^!, t\big) = (1,9,23,16),$$
$$\mathrm{Hilb}\big((6T_{4,4})^!, t\big) = (1,12,44,50,6),$$
$$\mathrm{Hilb}\big((6T_{3,5})^!, t\big) = \mathrm{Hilb}\big((6T_{5,3})^!, t\big) = (1,13,53,79,34),$$
$$\mathrm{Hilb}\big((6T_{4,5})^!, t\big) = \mathrm{Hilb}\big((6T_{5,4})^!, t\big) = (1,16,86,182,131,12),$$
$$\mathrm{Hilb}\big((6T_{5,5})^!, t\big) = (1,20,140,410,462,120).$$

Now let us define in the algebra $6T_{n,m}$ the Dunkl elements $\theta_i := \sum_{j \neq i} x_{i,j}$, $1 \le i \le n$, and $\bar{\theta}_\alpha := \sum_{\beta \neq \alpha} y_{\alpha,\beta}$, $1 \le \alpha \le m$.

Lemma 4.78. *One has*

- $[\theta_i, \theta_j] = 0$,
- $[\theta_i, \bar{\theta}_\alpha] = [x_{i,\alpha}, y_{i,\alpha}]$,
- $[\bar{\theta}_\alpha, \bar{\theta}_\beta]_+ = 2y_{\alpha,\beta}^2$ *if* $\alpha \neq \beta$.

Remark 4.79 ("odd" six-term relations algebra)**.** In particular, one can define an "odd" analog $6T_n^{(-)} = 6T_{0,n}$ of the six term relations algebra $6T_n$. Namely, the algebra $6T_n^{(-)}$ is given by the set of generators $\{y_{ij},\ 1 \le i < j \le n\}$, and that of relations:

1) $y_{i,j}$ and $y_{k,l}$ *anticommute* if i, j, k, l are pairwise distinct;
2) $[y_{i,j}, y_{i,k} + y_{j,k}]_+ + [y_{i,k}, y_{j,k}]_+ = 0$, if $1 \le i < j \le k \le n$, where $[x,y]_+ = xy + yx$ denotes the anticommutator of x and y.

The "odd" three term relations algebra $3T_n^-$ can be obtained as the quotient of the algebra $6T_n^-$ by the two-sided ideal generated by the three term relations $y_{ij}y_{jk} + y_{jk}y_{ki} + y_{ki}y_{ij} = 0$ if i, j, k are pairwise distinct.

One can show that the Dunkl elements θ_i and θ_j, $i \neq j$, given by formula

$$\theta_i = \sum_{j \neq i} y_{ij}, \qquad i = 1, \dots, n,$$

form an *anticommutative* family of elements in the algebra $6T_n^{(-)}$.

In a similar fashion one can define an "odd" analogue of the dynamical six term relations algebra $6DT_n$, see Definition 2.3 and Section 2.1.1, as well as define an "odd' analogues of the algebra $3MT_n(\beta, \mathbf{0})$, see Definition 3.7, the Kohno–Drinfeld algebra, the Hecke algebra and few others considered in the present paper. Details are omitted in the present paper.

More generally, one can ask what are natural q-analogues of the six term and three term relations algebras? In other words to describe relations which ensure the q-commutativity of Dunkl elements defined above. First of all it would appear natural that the "q-locality and q-symmetry conditions" hold among the set of generators $\{y_{ij},\ 1 \le i \neq j \le n\}$, that is $y_{ij} + qy_{ji} = 0$, $y_{ij}y_{kl} = qy_{kl}y_{ij}$ if $i < j$, $k < l$, and $\{i,j\} \cap \{k,l\} = \varnothing$.

Another natural condition is the fulfillment of q-analogue of the classical Yang–Baxter relations, namely, $[y_{ik}, y_{jk}]_q + [y_{ik}, y_{ji}]_q + [y_{ij}, y_{jk}]_q = 0$ if $i < j < k$, where $[x,y]_q := xy - qyx$ denotes the q-commutator. However we are not able to find the q-analogue of the classical Yang–Baxter relation listed above in the mathematical and physical literature yet. Only cases $q = 1$ and $q = -1$ have been extensively studied.

4.3 Four term relations algebras / Kohno–Drinfeld algebras

4.3.1 Kohno–Drinfeld algebra $4T_n$ and algebra CYB_n

Definition 4.80. *The* 4*-term relations algebra* (or the Kohno–Drinfeld algebra, or infinitesimal pure braids algebra) $4T_n$ is an associative algebra (say over $\mathbb{Q}$) with the set of generators $y_{i,j}$, $1 \le i < j \le n$, subject to the following relations

1) $y_{i,j}$ and $y_{k,l}$ are commute, if i, j, k, l are all distinct;
2) $[y_{i,j}, y_{i,k} + y_{j,k}] = 0$, $[y_{i,j} + y_{i,k}, y_{j,k}] = 0$ if $1 \le i < j \le k \le n$.

Note that the algebra $4T_n$ is given by $\binom{n}{2}$ generators and $2\binom{n}{3}+3\binom{n}{4}$ quadratic relations, and the element

$$c := \sum_{1\le i<j\le n} y_{i,j}$$

belongs to the center of the Kohno–Drinfeld algebra.

Definition 4.81. Denote by $4T_n^{(0)}$ the quotient of the algebra $4T_n$ by the (two-sided) ideal generated by by the set of elements $\{y_{i,j}^2,\ 1\le i<j\le n\}$.

More generally, let β, $\{q_{ij},\ 1\le i<j\le n\}$ be the set of parameters, denote by $4HT_n$ the quotient of the algebra $4T_n$ by the two-sided ideal generated by the set of elements $\{y_{ij}^2-\beta y_{ij}-q_{ij},\ 1\le i<j\le n\}$.

These algebras are naturally graded, with $\deg(y_{i,j})=1$, $\deg(\beta)=1$, $\deg(q_{ij})=2$, as well as each of that algebras has a natural filtration by setting $\deg(y_{i,j})=1$, $\deg(\beta)=0$, $\deg(q_{ij})=0$, $\forall\, i\neq j$.

It is clear that the algebra $4T_n$ can be considered as the infinitesimal deformation $g_{i,j}:=1+\epsilon y_{i,j}$, $\epsilon\longrightarrow 0$, of the pure braid group P_n.

There is a natural action of the symmetric group $\mathbb{S}_n$ on the algebra $4T_n$ (and also on $4T_n^0$) which preserves the grading: it is defined by $w\cdot y_{i,j}=y_{w(i),w(j)}$ for $w\in\mathbb{S}_n$. The semi-direct product $\mathbb{Q}\mathbb{S}_n\ltimes 4T_n$ (and also that $\mathbb{Q}\mathbb{S}_n\ltimes 4T_n^0$) is a Hopf algebra denoted by $\mathcal{B}_n$ (respectively $\mathcal{B}_n^{(0)}$).

Remark 4.82. There exists the natural map

$$\mathrm{CYB}_n\longrightarrow 4T_n \qquad \text{given by}\quad y_{i,j}:=u_{i,j}+u_{j,i}.$$

Indeed, one can easily check that

$$[y_{ij},y_{ik}+y_{jk}]=w_{ijk}+w_{jik}-w_{kij}-w_{kji},$$

see Section 2.3.1, Definition 2.21 for a definition of the classical Yang–Baxter algebra CYB_n, and Section 2, equation (2.3), for a definition of the element w_{ijk}.

Remark 4.83.

- Much as the relations in the algebra $6T_n$ are chosen in a way to imply (and "essentially"[46] equivalent) the pair-wise commutativity of the Dunkl elements $\{\theta_i\}_{1\le i\le n}$, the relations in the Kohno–Drinfeld algebra imply (and "essentially" equivalent) to pair-wise commutativity of the Jucys–Murphy elements (or, equivalently, *dual* JM-elements) $d_j:=\sum_{1\le a<j} y_{aj}$, $2\le j\le n$ (resp. $\overline{d}_i=\sum_{1\le a\le i} y_{n-i,n-a+1}$, $1\le i\le n-1$).
- It follows from the classical 3-term identity ("Jacobi identity")

$$\frac{1}{(a-b)(a-c)}-\frac{1}{(a-b)(b-c)}+\frac{1}{(a-c)(b-c)}=0,$$

that if elements $\{y_{i,j}\,|\,1\le i<j\le n\}$ satisfy the 4-term algebra relations, see Definition 4.80, and $t_1,\dots,t_n$, a set of (pair-wise) commuting parameters, then the elements

$$r_{i,j}:=\frac{y_{i,j}}{t_i-t_j}$$

[46]Together with locality and factorization conditions a set of defining relations in the algebra $6T_n$ is *equivalent* to the commutativity property of Dunkl's elements.

satisfy the set of defining relations of the 6-term relations algebra $6T_n$, see Section 4.2.1, Definition 4.47. In particular, the Knizhnik–Zamolodchikov elements

$$\mathrm{KZ}_j := \sum_{i\neq j} \frac{y_{i,j}}{t_i - t_j}, \qquad 1 \le j \le n,$$

form a pair-wise commuting family (by definition, we put $y_{i,j} = y_{j,i}$ if $i > j$).

Example 4.84.

(1) *Yang representation of the* $4T_n$. Let $\mathbb{S}_n$ be the symmetric group acting identically on the set of variables $\{x_1, \dots, x_n\}$. Clearly that the elements $\{y_{i,j} := s_{ij}\}_{1\le i<j\le n}$, $y_{i,j} := y_{j,i}$ if $i > j$, satisfy the Kohno–Drinfeld relations listed in Definition 4.80. Therefore the operators u_{ij} defined by

$$u_{ij} = (x_i - x_j)^{-1} s_{ij}$$

give rise to a representation of the algebra $3T_n$ on the field of rational functions $\mathbb{Q}(x_1, \dots, x_n)$. The Dunkl–Gaudin elements

$$\theta_i = \sum_{j, j\neq i} y_{ij}, \qquad i = 1, \dots, n$$

correspond to the truncated Gaudin operators acting in the tensor space $(\mathbb{C})^{\otimes n}$. Cf. Section 3.3.

(2) Let $A = U(\mathfrak{sl}(2))$ be the universal enveloping algebra of the Lie algebra $\mathfrak{sl}(2)$. Recall that the algebra $\mathfrak{sl}(2)$ is spanned by the elements e, f, h, so that $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$. Consider the element $\Omega = \frac{1}{2} h\otimes h + e\otimes f + f\otimes e$. Then the map $y_{i,j} \longrightarrow \Omega_{i,j} \in A^{\otimes n}$ defines a representation of the Kohno–Drinfeld algebra $4T_n$ on that $A^{\otimes n}$. The element KZ_j defined above, corresponds to the truncated (or at critical level) rational Knizhnik–Zamolodchikov operator. Cf. Section 4.2.1, Example 4.49.

Proposition 4.85 (T. Kohno, V. Drinfeld)**.**

$$\mathrm{Hilb}(4T_n, t) = \prod_{j=1}^{n-1} (1 - jt)^{-1} = \sum_{k\ge 0} \left\{ {n+k-1 \atop n-1} \right\} t^k,$$

where $\left\{ {n \atop k} \right\}$ *stands for the Stirling numbers of the second kind, i.e., the number of ways to partition a set of* n *things into* k *nonempty subsets.*

Remark 4.86. It follows from [6] that $\mathrm{Hilb}(4T_n, t)$ is equal to the generating function

$$1 + \sum_{d\ge 1} v_d^{(n)} t^d$$

for the number $v_d^{(n)}$ of *Vassiliev invariants of order* d *for* n*-strand braids*. Therefore, one has the following equality:

$$v_d^{(n)} = \left\{ {n+d-1 \atop n-1} \right\},$$

i.e., the number of Vassiliev invariants of order d for n-strand braids is equal to the Stirling number of the second kind $\left\{ {n+d-1 \atop n-1} \right\}$.

We *expect* that the generating function

$$1+\sum_{d\geq 1}\widehat{v}_d^{(n)}t^d$$

for the number $\widehat{v}_d^{(n)}$ of *Vassiliev invariants of order* d for n*-strand virtual braids* is equal to the Hilbert series $\operatorname{Hilb}(4NT_n,t)$ of the *nonsymmetric* Kohno–Drinfeld algebra $4NT_n$, see Section 4.3.2.

Proposition 4.87 (cf. [7])**.** *The algebra $4NT_n,t)$ is Koszul, and*

$$\operatorname{Hilb}(4NT_n,t)=\left(\sum_{k=0}^{n-1}(k+1)!N(k,n)(-t)^k\right)^{-1},$$
$$\operatorname{Hilb}\big((4NT_n)^!,t\big)=(n-1)!L_{n-1}^{(\alpha=1)}\big(-t^{-1}\big)t^{n-1},$$

where $N(k,n):=\frac{1}{n}\binom{n}{k}\binom{n}{k+1}$ denotes the Narayana number, i.e., the number of Dyck n-paths with exactly k peaks,

$$L_n^{(\alpha)}(x)=\frac{x^{\alpha}e^x}{n!}\frac{d^n}{dx^n}\big(e_x x^{n+\alpha}\big)$$

denotes the generalized Laguerre polynomial.

See also Theorem 4.91 below.

It is well-known that the quadratic dual $4T_n^!$ of the Kohno–Drinfeld algebra $4T_n$ is isomorphic to the Orlik–Solomon algebra of type A_{n-1}, as well as the algebra $3T_n^{\mathrm{anti}}$. However the algebra $4T_n^0$ is *failed* to be Koszul.

Examples 4.88.

$$\operatorname{Hilb}\big(4T_3^0,t\big)=[2]^2[3],\qquad \operatorname{Hilb}\big(4T_4^0,t\big)=(1,6,19,42,70,90,87,57,23,6,1),$$
$$\operatorname{Hilb}\big(\big(4T_3^0\big)^!,t\big)(1-t)=(1,2,2,1),\qquad \operatorname{Hilb}\big(\big(4T_4^0\big)^!,t\big)(1-t)^2=(1,4,6,2,-4,-3),$$
$$\operatorname{Hilb}\big(\big(4T_5^0\big)^!,t\big)(1-t)^2=(1,8,26,40,24,-3,-6).$$

We *expect* that $\operatorname{Hilb}\big((4T_n^0)^!,t\big)$ is a rational function with the only pole at $t=1$ of order $[n/2]$, cf. Examples 4.77.

Remark 4.89. One can show that if $n\geq 4$, then $\operatorname{Hilb}(4T_n^0,t)<\operatorname{Hilb}(3T_n^0,t)$ *contrary* to the statement of Conjecture 9.6 from [67].

4.3.2 Nonsymmetric Kohno–Drinfeld algebra $4NT_n$, and McCool algebras $\mathcal{P}\Sigma_n$ and $\mathcal{P}\Sigma_n^+$

Definition 4.90. *The nonsymmetric* 4*-term relations algebra* (or the nonsymmetric Kohno–Drinfeld algebra) $4NT_n$ is an associative algebra (say over $\mathbb{Q}$) with the set of generators $y_{i,j}$, $1\leq i\neq j\leq n$, subject to the following relations

1) $y_{i,j}$ and $y_{k,l}$ are commute if i, j, k, l are all distinct;
2) $[y_{i,j},y_{i,k}+y_{j,k}]=0$ if i, j, k are all distinct.

We denote by $4NT_n^+$ the quotient of the algebra $4NT_n$ by the two-sided ideal generated by the elements $\{y_{ij}+y_{ji}=0,\ 1\leq i\neq j\leq n\}$.

Theorem 4.91. *One has*

$$\mathrm{Hilb}(4NT_n,t)=\mathrm{Hilb}(\mathrm{CYB}_n,t),\qquad \mathrm{Hilb}\big(4NT_n^+,t\big)=\mathrm{Hilb}(6T_n,t)$$

for all $n\geq 2$.

We expect that the both algebras $4NT_n$ and $4NT_n^+$ are *Koszul*.

Definition 4.92.

(1) Define *the McCool algebra* $\mathcal{P}\Sigma_n$ to be the quotient of the nonsymmetric Kohno–Drinfeld algebra $4NT_n$ by the two-sided ideal generated by the elements $\{y_{ik}y_{jk}-y_{jk}y_{ik}\}$ for all pairwise distinct i, j and k.

(2) Define *the upper triangular McCool algebra* $\mathcal{P}\Sigma_n^+$ to be the quotient of the McCool algebra $\mathcal{P}\Sigma_n$ by the two-sided ideal generated by the elements $\{y_{ij}+y_{ji}\}$, $1\leq i\neq j\leq n$.

Theorem 4.93. *The quadratic duals of the algebras* $\mathcal{P}\Sigma_n$ *and* $\mathcal{P}\Sigma_n^+$ *have the following Hilbert polynomials*

$$\mathrm{Hilb}\big(\mathcal{P}\Sigma_n^!,t\big)=(1+nt)^{n-1},\qquad \mathrm{Hilb}\big(\big(\mathcal{P}\Sigma_n^+\big)^!,t\big)=\prod_{j=1}^{n-1}(1+jt).$$

Proposition 4.94.

(1) *The quadratic dual* $\mathcal{P}\Sigma_n^!$ *of the algebra* $\mathcal{P}\Sigma_n$ *admits the following description. It is generated over* $\mathbb{Z}$ *by the set of pairwise anticommuting elements* $\{y_{ij},\ 1\leq i\neq j\leq n\}$*, subject to the set of relations*

  (*a*) $y_{ij}^2=0$, $y_{ij}y_{ji}=0$, $1\leq i\neq j\leq n$,
  (*b*) $y_{ik}y_{jk}=0$ *for all distinct* i, j, k,
  (*c*) $y_{ij}y_{jk}+y_{ik}y_{ij}+y_{kj}y_{ik}=0$ *for all distinct* i, j, k.

(2) *The quadratic dual* $(\mathcal{P}\Sigma_n^+)^!$ *of the algebra* $\mathcal{P}\Sigma_n^+$ *admits the following description. It is generated over* $\mathbb{Z}$ *by the set of pairwise anticommuting elements* $\{z_{ij},\ 1\leq i<j\leq n\}$*, subject to the set of relations*

  (*a*) $z_{ij}^2=0$ *for all* $i<j$,
  (*b*) $z_{ij}z_{jk}=z_{ij}z_{ik}$ *for all* $1\leq i<j<k\leq n$.

Comments 4.95 (the McCool groups and algebras)**.** *The McCool group* $P\Sigma_n$ is, by definition, the group of pure symmetric automorphisms of the free group F_n consisting of all automorphism that, for a fixed basis $\{x_1,\dots,x_n\}$, send each x_i to a conjugate of itself. This group is generated by automorphisms α_{ij}, $1\leq i\neq j\leq n$, defined by

$$\alpha_{ij}(x_k)=\begin{cases}x_jx_ix_j^{-1}, & k=i,\\ x_k, & k\neq i.\end{cases}$$

McCool have proved that the relations

$$\begin{cases}[\alpha_{ij},\alpha_{kl}]=1, & i,\ j,\ k,\ l\ \text{ are distinct},\\ [\alpha_{ij},\alpha_{ji}]=1, & i\neq j,\\ [\alpha_{ij},\alpha_{ik}\alpha_{jk}]=1, & i,\ j,\ k\ \text{ are distinct}.\end{cases}$$

form the set of defining relations for the group $P\Sigma_n$ The subgroup of $P\Sigma_n$ generated by the α_{ij} for $1\leq i<j\leq n$ is denoted by $P\Sigma_n^+$ and is called by *upper triangular McCool group*. It is easy to see that the McCool algebras $\mathcal{P}\Sigma_n$ and $\mathcal{P}\Sigma_n^+$ are the "infinitesimal deformations" of the McCool groups $P\Sigma_n$ and $P\Sigma_n^+$ respectively.

Theorem 4.96.

(1) *There exists a natural isomorphism*

$$H^*(P\Sigma_n,\mathbb{Z}) \simeq \mathcal{P}\Sigma_n^!$$

of the quadratic dual $\mathcal{P}\Sigma_n^!$ of the McCool algebra $\mathcal{P}\Sigma_n$ and the cohomology ring $H^*(P\Sigma_n,\mathbb{Z})$ of the McCool group $P\Sigma_n$, see [61].

(2) *There exists a natural isomorphism*

$$H^*(P\Sigma_n^+,\mathbb{Z}) \simeq (\mathcal{P}\Sigma_n^+)^!$$

of the quadratic dual $(\mathcal{P}\Sigma_n^+)^!$ of the upper triangular McCool algebra $\mathcal{P}\Sigma_n^+$ and the cohomology ring $H^*(P\Sigma_n^+,\mathbb{Z})$ of the upper triangular McCool group $P\Sigma_n^+$, see [27].

4.3.3 Algebras $4TT_n$ and $4ST_n$

Definition 4.97.

(I) *Algebra* $4TT_n$ is generated over $\mathbb{Z}$ by the set of elements $\{x_{ij},\ 1 \le i \ne j \le n\}$, subject to the set of relations

(1) $x_{ij}x_{kl} = x_{kl}x_{ij}$ if all i, j, k, l are distinct,
(2) $[x_{ij}+x_{jk}, x_{ik}] = 0$, $[x_{ji}+x_{kj}, x_{ki}] = 0$ if $i<j<k$.

(II) *Algebra* $4ST_n$ is generated over $\mathbb{Z}$ by the set of elements $\{x_{ij},\ 1 \le i \ne j \le n\}$, subject to the set of relations

(1) $[x_{ij}, x_{kl}] = 0$, $[x_{ij}, x_{ji}] = 0$ if i, j, k, l are distinct,
(2) $[x_{ij}, x_{ik}] = [x_{ik}, x_{jk}] = [x_{jk}, x_{ij}]$, $[x_{ji}, x_{ki}] = [x_{ki}, x_{kj}] = [x_{kj}, x_{ii}]$,
(3) $[x_{ij}, x_{ki}] = [x_{kj}, x_{ij}] = [x_{ji}, x_{ik}] = [x_{ik}, x_{kj}] = [x_{ki}, x_{jk}] = [x_{jk}, x_{ji}]$ if $i<j<k$.

Proposition 4.98. *One has*

$$t\sum_{n\ge 2}\operatorname{Hilb}\big((4TT_n)^!,t\big)\frac{z^n}{n!} = \frac{\exp(-tz)}{(1-z)^{2t}} - 1 - tz.$$

Therefore, $\dim(4TT_n)^!$ *is equal to the number of permutations of the set* $[1,\dots,n+1]$ *having no substring* $[k,k+1]$*; also, for* $n \ge 1$ *equals to the maximal permanent of a nonsingular* $n\times n$ $(0,1)$*-matrix, see [131, A000255][47]. Moreover, one has*

$$\operatorname{Hilb}\big((4ST_n)^!,t\big) = (1+t)^n(1+nt)^{n-2},$$

cf. Conjecture 4.112.

We expect that The both algebras $4TT_n$ and $4ST_n$ are *Koszul*.

Problem 4.99. *Give a combinatorial interpretation of polynomials* $\operatorname{Hilb}((4TT_n)^!,t)$ *and construct a monomial basis in the algebras* $(4TT_n)^!$ *and* $4ST_n$.

[47] See also a paper by F. Hivert, J.-C. Novelli and J.-Y. Thibon [57, Section 3.8.4] for yet another combinatorial interpretation of the dimension of the algebra $(4TT_n)^!$.

4.4 Subalgebra generated by Jucys–Murphy elements in $4T_n^0$

Definition 4.100. *The Jucys–Murphy elements* d_j, $2 \le j \le n$, in the quadratic algebra $4T_n$ are defined as follows

$$d_j = \sum_{1 \le i < j} y_{i,j}, \qquad j = 2, \dots, n.$$

It is clear that Jucys–Murphy's elements d_j are the infinitesimal deformation of the elements $D_{1,j} \in P_n$.

Theorem 4.101.

(1) *The Jucys–Murphy elements d_j, $2 \le j \le n$, commute pairwise in the algebra $4T_n$.*

(2) *In the algebra $4T_n^0$ the Jucys–Murphy elements d_j, $2 \le j \le n$, satisfy the following relations*

$$(d_2 + \cdots + d_j) d_j^{2j-3} = 0, \qquad 2 \le j \le n.$$

(3) *Subalgebra (over $\mathbb{Z}$) in $4T_n^0$ generated by the Jucys–Murphy elements $d_2, \dots, d_n$ has the following Hilbert polynomial $\prod_{j=1}^{n-1} [2j]$.*

(4) *There exists an (birational) isomorphism $\mathbb{Z}[x_1, \dots, x_{n-1}]/J_{n-1} \longrightarrow \mathbb{Z}[d_2, \dots, d_n]$ defined by $d_j := \prod_{i=1}^{n-j} x_i$, $2 \le j \le n$, where J_{n-1} is a (two-sided) ideal generated by $e_i(x_1^2, \dots, x_{n-1}^2)$, $1 \le i \le n-1$, and $e_i(x_1, \dots, x_{n-1})$ stands for the i-th elementary symmetric polynomial in the variables $x_1, \dots, x_{n-1}$.*

Remark 4.102.

(1) It is clearly seen that the commutativity of the Jucys–Murphy elements is equivalent to the validity of the Kohno–Drinfeld relations and the locality relations among the generators $\{y_{i,j}\}_{1 \le i < j \le n}$.

(2) Let's stress that $d_j^{2j-2} \ne 0$ in the algebra $4T_n^0$ for $j = 3, \dots, n$. For example, $d_3^4 = y_{13}y_{23}y_{13}y_{23} + y_{23}y_{13}y_{23}y_{13} \ne 0$ since $\dim(4T_3^0)_4 = 1$ and it is generated by the element d_3^4.

(3) The map $\iota\colon y_{i,j} \longrightarrow y_{n+1-j,n+1-i}$ preserves the relations 1) and 2) in the definition of the algebra $4T_n$, and therefore defines an involution of the Kohno–Drinfeld algebra. Hence the elements

$$\widehat{d}_j := \sum_{k=j+1}^{n} y_{j,k} = \iota(d_{n+1-j}), \qquad 1 \le j \le n-1$$

also form a pairwise commuting family.

Problems 4.103.

(*a*) *Compute Hilbert series of the algebra $4T_n^0$ and its quadratic dual algebra $(4T_n^0)^!$.*

(*b*) *Describe subalgebra in the algebra $4HT_n$ generated by the Jucys–Murphy elements d_j, $2 \le j \le n$.*

It is well-known that the Kohno–Drinfeld algebra $4T_n$ is *Koszul*, and its quadratic dual $4T_n^!$ is isomorphic to the anticommutative quotient $3T_n^{0,\mathrm{anti}}$ of the algebra $3T_n^{(-),0}$.

On the other hand, if $n \ge 3$ the algebra $4T_n^0$ is not *Koszul*, and its quadratic dual is isomorphic to the quotient of the ring of polynomials in the set of anticommutative variables $\{t_{i,j} \,|\, 1 \le i < j \le n\}$, where *we do not impose conditions* $t_{ij}^2 = 0$, modulo the ideal generated by Arnold's relations $\{t_{i,j}t_{j,k} + t_{i,k}(t_{i,j} - t_{j,k}) = 0\}$ for all pairwise distinct i, j and k.

4.5 Nonlocal Kohno–Drinfeld algebra $NL4T_n$

Definition 4.104. *Nonlocal Kohno–Drinfeld algebra* $NL4T_n$ is an associative algebra over $\mathbb{Z}$ with the set of generators $\{y_{ij},\ 1 \le i < j \le n\}$ subject to the set of relations

(1) $y_{ij}y_{kl} = y_{kl}y_{ij}$ if $(i-k)(i-l)(j-k)(j-l) > 0$,

(2) $\big[y_{ij}, \sum_{a=i}^{j} y_{ak}\big] = 0$ if $i < j < k$,

(3) $\big[y_{jk}, \sum_{a=j}^{k} y_{ia}\big] = 0$ if $i < j < k$.

It's not difficult to see that relations (1)–(3) imply the following relations

(4) $\big[x_{ij}, \sum_{a=i+1}^{j-1} (y_{ia} + y_{aj})\big] = 0$ if $i < j$.

Let's introduce in the nonlocal Kohno–Drinfeld algebra $NL4T_n$ the *Jucys–Murphy elements* (JM-elements for short) d_j and the *dual JM-elements* $\hat{d}_j$ as follows

$$d_j = \sum_{a=1}^{j-1} y_{aj}, \qquad \hat{d}_j = \sum_{a=n-j+2}^{n} y_{n-j+1,a}, \qquad j = 2, \dots, n. \tag{4.5}$$

It follows from relations (1) and (2) (resp. (1) and (3)) that the Jucys–Murphy elements $d_2, \dots, d_n$ (resp. $\hat{d}_2, \dots, \hat{d}_n$) form a commutative subalgebra in the algebra $NL4T_n$. Moreover, it follows from relations (1)–(3) that the element $c_1 := \sum_{j=2}^{n} d_j = \sum_{j=2}^{n} \hat{d}_j$ belongs to the center of the algebra $NL4T_n$.

Theorem 4.105.

(1) *The algebra $NL4T_n$ is Koszul, and*

$$\operatorname{Hilb}\big((NL4T_n)^!, t\big) = \sum_{k=0}^{n-1} C_k \binom{n+k-1}{2k} t^k,$$

where $C_k = \frac{1}{k+1}\binom{2k}{k}$ stands for the k-th Catalan number.

(2) *The quadratic dual $(NL4T_n)^!$ of the nonlocal Kohno–Drinfeld algebra $NL4T_n$ is an associative algebra generated by the set of mutually anticommuting elements $\{t_{ij},\ 1 \le i < j \le n\}$ subject to the set of relations*

- $t_{ij}^2 = 0$ *if* $1 \le i < j \le n$,
- *Arnold's relations:* $t_{ij}t_{jk} + t_{ik}t_{ij} + t_{jk}t_{ik} = 0$ *if* $i < j < k$,
- *disentanglement relations:* $t_{ik}t_{jl} + t_{il}t_{ik} + t_{jl}t_{il} = 0$ *if* $i < j < k < l$.

Therefore the algebra $(NL4T_n)^!$ is the quotient of the Orlik–Solomon algebra OS_n by the ideal generated by Disentanglement relations, and $\dim((NL4T_{n+1})^!)$ is equal to the number of Schröder paths, i.e., paths from $(0,0)$ to $(2n,0)$ consisting of steps $U = (1,1)$, $D = (1,-1)$, $H = (2,0)$ and never going below the x-axis. The Hilbert polynomial $\operatorname{Hilb}((NL4T_n)^!, t)$ is the generating function of such paths with respect to the number of $U's$, see [131, $A088617$].

Remark 4.106. Denote by $H_n(q)$ "the normalized" Hecke algebra of type A_n, i.e., an associative algebra generated over $\mathbb{Z}[q, q^{-1}]$ by elements $T_1, \dots, T_{n-1}$ subject to the set of relations

(a) $T_iT_j = T_jT_i$ if $|i-j|>1$, $T_iT_jT_i = T_jT_iT_j$ if $|i-j|=1$,

(b) $T_i^2 = (q-q^{-1})\ T_i + 1$ for $i=1,\dots,n-1$.

If $1 \le i < j \le n-1$, let's consider elements $T_{(ij)} := T_iT_{i+1}\cdots T_{j-1}T_jT_{j-1}\cdots T_{i+1}T_i$.

Lemma 4.107. *The elements $\{T_{(ij)},\ 1 \le i < j < n-1\}$ satisfy the defining relations of the non-local Kohno–Drinfeld algebra $NL4T_{n-1}$, see Definition 4.104.*

Therefore the map $y_{ij} \to H_{(ij)}$ defines a epimorphism $\iota_n \colon NL4T_n \longrightarrow H_{n+1}(q)$.

Definition 4.108. Denote by $\mathcal{NL}4T_n$ the quotient of the non-local Kohno–Drinfeld algebra $NL4T_n$ by the two-sided ideal $\mathcal{I}_n$ generated by the following set of degree three elements:

(1) $z_{ij} := y_{i,j+1}y_{ij}y_{j,j+1} - y_{j,j+1}y_{ij}y_{i,j+1}$ if $1 \le i < j \le n$,

(2) $$u_i := y_{i,i+1}\left(\sum_{a=1}^{i-1}\sum_{b=1,\, b\neq a}^{i-1} y_{ai}y_{b,i+1}\right) - \left(\sum_{a=1}^{i-1}\sum_{b=1,\, b\neq a}^{i-1} y_{b,i+1}y_{ai}\right) y_{i,i+1}$$
if $1 \le i \le n-1$,

(3) $$v_i := y_{i,i+1}\left(\sum_{a=i+1}^{n}\sum_{b=i+1,\, b\neq a}^{n} y_{i+1,a}y_{i,b}\right) - \left(\sum_{a=i+1}^{n}\sum_{b=i+1,\, b\neq a}^{n} y_{i+1,a}y_{i,b}\right) y_{i,i+1},$$
if $1 \le i \le n-1$.

Proposition 4.109.

(1) *The ideal $\mathcal{T}_n$ belongs to the kernel of the epimorphism $\iota_n \colon \mathcal{I}_n \subset \mathrm{Ker}(\iota_n)$,*

(2) *Let $d_2,\dots,d_n$ (resp. $\hat{d}_2,\dots,\hat{d}_n$) be the Jucys–Murphy elements (resp. dual JM-elements) in the algebra $\mathcal{NL}4T_n$ given by the formula (4.5).*

Then the all elementary symmetric polynomials $e_k(d_2,\dots,d_n)$ (resp. $e_k(\hat{d}_2,\dots,\hat{d}_n)$) of degree k, $1 \le k < n$, in the Jucys–Murphy elements $d_2,\dots,d_n$, (resp. in the dual JM-elements $\hat{d}_2,\dots,\hat{d}_n$), commute in the algebra $\mathcal{NL}4T_n$ with the all elements $y_{i,i+1}$, $i=1,\dots,n-1$.

Therefore, there exists an epimorphism of algebras $\mathcal{NL}4T_n \longrightarrow H_n(q)$, and images of the elements $e_k(d_2,\dots,d_n)$ (resp. $e_k(\hat{d}_2,\dots,\hat{d}_n)$, $1 \le k < n$, belongs to the center of the "normalized" Hecke algebra $H_n(q)$, and in fact *generate* the center of algebra $H_n(q)$.

Few comments in order:

(A) Let $N\ell 4T_n$ be an associative algebra over $\mathbb{Z}$ with the set of generators $\{y_{ij},\ 1 \le i < j \le n\}$ subject to the set of relations

(1) $y_{ij}y_{kl} = y_{kl}y_{ij}$ if $(i-k)(i-l)(j-k)(j-l) > 0$,

(2) $\left[y_{ij}, \sum\limits_{a=i}^{j} y_{ak}\right] = 0$ if $i<j<k$.

Proposition 4.110.

(1) *The algebra $N\ell 4T_n$ is Koszul and has the Hilbert series equals to*

$$\mathrm{Hilb}(N\ell 4T_n, t) = \left(\sum_{k=0}^{n-1}(-1)^k N(k,n)t^k\right)^{-1},$$

where $N(k,n) := \frac{1}{n}\binom{n}{k}\binom{n}{k+1}$ denotes the Narayana number, i.e., the number of Dyck n-paths with exactly k peaks, see, e.g., [131, A001263]. Therefore, $\dim(N\ell 4T_n)^! = \frac{1}{n+1}\binom{2n}{n}$, *the n-th Catalan number.*

(2) *Elementary symmetric polynomials* $e_k(d_2,\ldots,d_n)$ *of degree* k, $1 \le k < n$, *in the Jucys–Murphy elements* $d_2,\ldots,d_n$, *commute in the algebra* $N\ell 4T_n$ *with the all elements* $y_{i,i+1}$, $i=1,\ldots,n-1$.

(B) The kernel of the epimorphism $\mathcal{NL}4T_n \longrightarrow H_n(q)$ contains the elements

$$\{y_{i,i+1}y_{i+1,i+2}y_{i,i+1} - y_{i+1,i+2}y_{i,i+1}y_{i+1,i+2},\ i=1,\ldots,n-2\},$$
$$\{T_{i,i+1}^2 - (q-q^{-1})T_{i,i+1} - 1\},$$

as well as the following set of commutators

$$[y_{ij}, e_k(d_i,\ldots,d_j)], \qquad 1 \le k \le j-i+1.$$

It is an interesting *task* to find defining relations among the Jucys–Murphy elements $\{d_j, j=2,\ldots,n\}$ in the algebra $NL4T_n$ or that $N\ell 4T_n$. We *expect* that the Jucys–Murphy element d_k satisfies the following relation (= minimal polynomial) in the *Hecke algebra* $H_n(q)$, $n \ge k$,

$$\prod_{a=1}^{k-1}\left(d_k - \frac{q-q^{2a+1}}{1-q^2}\right)\left(d_k + \frac{q^{-1}-q^{-2a-1}}{1-q^{-2}}\right) = 0.$$

4.5.1 On relations among JM-elements in Hecke algebras

Let $H_n(q)$ be the "normalized" Hecke algebra of type A_n, see Remark 4.106. Let $\lambda \vdash n$ be a partition of n. For a box $x=(i,j)\in\lambda$ define

$$c_\lambda(x;q) := q\frac{1-q^{2(j-i)}}{1-q^2}.$$

It is clear that if $q=1$, $c_{q=1}(x)$ is equal to the content $c(x)$ of a box $x\in\lambda$. Denote by

$$\Lambda_q^{(n)} = \mathbb{Z}[q,q^{-1}][z_1,\ldots,z_n]^{\mathbb{S}_n}$$

the space of symmetric polynomials over the ring $\mathbb{Z}[q,q^{-1}]$ in variables $\{z_1,\ldots,z_n\}$.

Definition 4.111. Denote by $J_q^{(n)}$ the set of symmetric polynomials $f\in\Lambda_q^{(n)}$ such that for any *partition* $\lambda\vdash n$ one has

$$f(c_\lambda(x;q)|x\in\lambda) = 0.$$

For example, one can check that symmetric polynomial

$$e_1^2 - (q^2+1+q^{-2})e_2 - 2(q-q^{-1})e_1 - 3$$

belongs to the set $J_q^{(3)}$.

Finally, denote by $\mathbb{J}_q^{(n)}$ the ideal in the ring $\mathbb{Z}[q,q^{-1}][z_1,\ldots,z_n]$ generated by the set $J_q^{(n)}$.

Conjecture 4.112. *The algebra over* $\mathbb{Z}[q,q^{-1}]$ *generated by the Jucys–Murphy elements* d_2, $\ldots, d_n$ *corresponding to the Hecke algebra* $H_n(q)$ *of type* A_{n-1}, *is isomorphic to the quotient of the algebra* $\mathbb{Z}[q,q^{-1}][z_1,\ldots,z_n]$ *by the ideal* $\mathbb{J}_q^{(n)}$.

It seems an interesting *problem* to find a minimal set of generators for the ideal $\mathbb{J}_q^{(n)}$.

Comments 4.113. Denote by $JM(n)$ the algebra over $\mathbb{Z}$ generated by the JM-elements $d_2, \ldots, d_n$, $\deg(d_i) = 1$, $\forall\, i$, corresponding to the symmetric group $\mathbb{S}_n$. In this case one can check Conjecture 4.112 for $n < 8$, and compute the Hilbert polynomial(s) of the associated graded algebra(s) $\operatorname{gr}(JM(n))$. For example[48]

$$\operatorname{Hilb}(\operatorname{gr}(JM(2), t) = (1,1), \qquad \operatorname{Hilb}(\operatorname{gr}(JM(3), t) = (1,2,1),$$
$$\operatorname{Hilb}(\operatorname{gr}(JM(4), t) = (1,3,4,2), \qquad \operatorname{Hilb}(\operatorname{gr}(JM(5), t) = (1,4,8,9,4),$$
$$\operatorname{Hilb}(\operatorname{gr}(JM(6), t) = (1,5,13,21,21,12,3),$$
$$\operatorname{Hilb}(\operatorname{gr}(JM(7), t) = (1,6,19,40,59,60,37,10).$$

It seems an interesting *task* to find a combinatorial interpretation of the polynomials $\operatorname{Hilb}(\operatorname{gr}(JM(n)), t)$ in terms of standard Young tableaux of size n.

Let $\{\chi^\lambda,\ \lambda \vdash n\}$ be the characters of the irreducible representations of the symmetric group $\mathbb{S}_n$, which form a basis of the center $\mathcal{Z}_n$ of the group ring $\mathbb{Z}[\mathbb{S}_n]$. The famous result by A. Jucys [62] states that for any symmetric polynomial $f(z_1, \ldots, z_n)$ the character expansion of $f(d_2, \ldots, d_n, 0)$ $\in \mathcal{Z}_n$ is

$$f(d_2, \ldots, d_n, 0) = \sum_{\lambda \vdash n} \frac{f(C_\lambda)}{H_\lambda} \chi^\lambda,$$

where $H_\lambda = \prod_{x \in \lambda} h_x$ denotes the product of all *hook-lengths* of λ, and $C_\lambda := \{c(x)\}_{x \in \lambda}$ denotes the set of *contents* of all boxes of λ.

Recall that the Jucys–Murphy elements $\{d_j^H\}_{2 \le j \le n}$ in the (normalized) Hecke algebra $H_n(q)$ are defined as follows: $d_j^H := \sum_{i<j} T_{(ij)}$, where $T_{(ij)} := T_i \cdots T_{j-1} T_j T_{j-1} \cdots T_i$. Finally denote by $H_\lambda(q)$ and $C_\lambda^{(q)}$ the hook polynomial and the set $\{c_\lambda x; q)\}_x \in \lambda$. Then for any symmetric polynomial $f(z_1, \ldots, z_n)$ one has

$$f\big(d_2^H, \ldots, d_n^H, 0\big) = \sum_{\lambda \vdash n} \frac{f(C_\lambda^{(q)})}{H_\lambda(q)} \chi_q^\lambda,$$

where χ_q^λ denotes the q-character of the algebra $H_{n(q)}$.

Therefore, if $f \in J_q^{(n)}$, then $f\big(d_2^H, \ldots, d_n^H, 0\big) = 0$. It is an open *problem* to prove/disprove that if $f\big(d_2^H, \ldots, d_n^H, 0\big) = 0$, then $f\big(C_\lambda^{(q)}\big) = 0$ for all partitions of size n (even in the case $q = 1$).

4.6 Extended nil-three term relations algebra and DAHA, cf. [24]

Let $A := \{q, t, a, b, c, h, e, f, \ldots\}$ be a set of parameters.

Definition 4.114. Extended nil-three term relations algebra $3\mathfrak{T}_n$ is an associative algebra over $\mathbb{Z}[q^{\pm 1}, t^{\pm 1}, a, b, c, h, e, \ldots]$ with the set of generators $\{u_{i,j},\ 1 \le i \ne j \le n, x_i,\ 1 \le i \le n, \pi\}$ subject to the set of relations

(0) $u_{i,j} + u_{j,i} = 0$, $u_{i,j}^2 = 0$,

(1) $x_i x_j = x_j x_i$, $u_{i,j} u_{k,l} = u_{k,l} u_{i,j}$ if i, j, k, l are distinct,

(2) $x_i u_{kl} = u_{k,l} x_i$ if $i \ne k, l$,

[48] I would like to thank Dr. S. Tsuchioka for computation the Hilbert polynomials $\operatorname{Hilb}(JM(n), t)$, as well as the sets of defining relations among the Jucys–Murphy elements in the symmetric group $\mathbb{S}_n$ for $n \le 7$.

(3) $x_i u_{i,j} = u_{i,j} x_j + 1$, $x_j u_{i,j} = u_{i,j} x_i - 1$,

(4) $u_{i,j} u_{j,k} + u_{k,i} u_{i,j} + u_{j,k} u_{k,i} = 0$ if i, j, k are distinct,

(5) $\pi x_i = x_{i+1}\pi$ if $1 \le i < n$, $\pi x_n = t^{-1} x_1 \pi$,

(6) $\pi u_{ij} = u_{i+1,j+1}$ if $1 \le i < j < n$, $\pi^j u_{n-j+1,n} = t u_{1,j} \pi^j$, $2 \le j \le n$.

Note that the algebra $3\mathfrak{T}_n$ contains also the set of elements $\{\pi^a u_{jn},\ 1 \le a \le n-j\}$.

Definition 4.115 (cf. [87])**.** Let $1 \le i < j \le n$, define

$$T_{i,j} = a + (bx_i + cx_j + h + e x_i x_j) u_{i,j}.$$

Lemma 4.116.

(1) *$T_{i,j}^2 = (2a + b - c) T_{i,j} - a(a + b - c)$ if $a = 0$, then $T_{ij}^2 = (b - c) T_{ij}$.*

(2) *Coxeter relations*

$$T_{i,j} T_{j,k} T_{i,j} = T_{j,k} T_{i,j} T_{j,k},$$

are valid, if and only if the following relation holds

$$(a + b)(a - c) + he = 0. \tag{4.6}$$

(3) *Yang–Baxter relations*

$$T_{i,j} T_{i,k} T_{j,k} = T_{j,k} T_{i,k} T_{i,j}$$

are valid if and only if $b = c = e = 0$, i.e., $T_{ij} = a + d u_{ij}$.

(4) *$T_{ij}^2 = 1$ if and only if $a = \pm 1$, $c = b \pm 2$, $he = (b \pm 1)^2$.*

(5) *Assume that parameters a, b, c, h, e satisfy the conditions (4.6) and that $bc + 1 = he$. Then*

$$T_{ij} x_i T_{ij} = x_j + (h + (a + b)(x_i + x_j) + e x_i x_j) T_{ij}.$$

(6) *Quantum Yang–Baxterization. Assume that parameters a, b, c, h, e satisfy the conditions (4.6) and that $\beta := 2a + b - c \ne 0$. Then (cf. [59, 85] and the literature quoted therein) the elements $R_{ij}(u,v) := 1 + \frac{\lambda-\mu}{\beta\mu} T_{ij}$ satisfy the twisted quantum Yang–Baxter relations*

$$R_{ij}(\lambda_i, \mu_j) R_{jk}(\lambda_i, \nu_k) R_{ij}(\mu_j, \nu_k) = R_{jk}(\mu_j, \nu_k) R_{ij}(\lambda_i, \nu_k) R_{jk}(\lambda_i, \mu_j), \qquad i < j < k,$$

where $\{\lambda_i, \mu_i, \nu_i\}_{1 \le i \le n}$ are parameters.

Corollary 4.117. *If $(a + b)(a - c) + he = 0$, then for any permutation $w \in \mathbb{S}_n$ the element*

$$T_w := T_{i_1} \cdots T_{i_l} \in 3\mathfrak{T}_n,$$

where $w = s_{i_1} \cdots s_{i_l}$ is any reduced decomposition of w, is well-defined.

Example 4.118. Each of the set of elements

$$s_i^{(h)} = 1 + (x_{i+1} - x_i + h) u_{i,i+1}$$

and

$$t_i^{(h)} = -1 + (x_i - x_{i+1} + h(1 + x_i)(1 + x_{i+1}) u_{ij}, \qquad i = 1, \dots, n-1,$$

by itself generate the symmetric group $\mathbb{S}_n$.

Comments 4.119. Let $A = (a, b, c, h, e)$ be a sequence of integers satisfying the conditions (4.6). Denote by ∂_i^A the divided difference operator

$$\partial_i^A = (a + (bx_i + cx_{i+1} + h + ex_i x_{i+1})\partial_i, \qquad i = 1, \dots, n-1.$$

It follows from Lemma 4.107 that the operators $\{\partial_i^A\}_{1 \le i \le n}$ satisfy the Coxeter relations

$$\partial_i^A \partial_{i+1}^A \partial_i^A = \partial_{i+1}^A \partial_i^A \partial_{i+1}^A, \qquad i = 1, \dots, n-1.$$

Definition 4.120.

(1) Let $w \in \mathbb{S}_n$ be a permutation. Define the generalized Schubert polynomial corresponding to permutation w as follows

$$\mathfrak{S}_w^A(X_n) = \partial_{w^{-1}w_0}^A x^{\delta_n},$$

where

$$x^{\delta_n} := x_1^{n-1} x_2^{n-2} \cdots x_{n-1},$$

and w_0 denotes the longest element in the symmetric group $\mathbb{S}_n$.

(2) Let α be a composition with at most n parts, denote by $w_\alpha \in \mathbb{S}_n$ the permutation such that $w_\alpha(\alpha) = \overline{\alpha}$, where $\overline{\alpha}$ denotes a unique partition corresponding to composition α.

Proposition 4.121 ([71]). *Let $w \in \mathbb{S}_n$ be a permutation.*

- *If $A = (0, 0, 0, 1, 0)$, then $\mathfrak{S}_w^A(X_n)$ is equal to the Schubert polynomial $\mathfrak{S}_w(X_n)$.*
- *If $A = (-\beta, \beta, 0, 1, 0)$, then $\mathfrak{S}_w^A(X_n)$ is equal to the β-Grothendieck polynomial $\mathfrak{G}_w^{(\beta)}(X_n)$ introduced in [42].*
- *If $A = (0, 1, 0, 1, 0)$ then $\mathfrak{S}_w^A(X_n)$ is equal to the dual Grothendieck polynomial [71, 84].*
- *If $A = (-1, 2, 0, 1, 1)$, then $\mathfrak{S}_w^A(X_n)$ is equal to the Di Francesco–Zinn-Justin polynomials and studied in [32, 33, 34] and [71].*

In all cases listed above the polynomials $\mathfrak{S}_w^A(X_n)$ have non-negative integer coefficients.

- *If $A = (1, -1, 1, -h, 0)$, then $\mathfrak{S}_w^A(X_n)$ is equal to the h-Schubert polynomials introduced in [71].*

Define the generalized key or Demazure polynomial corresponding to a composition α as follows

$$K_\alpha^A(X_n) = \partial_{w_\alpha} x^{\overline{\alpha}}.$$

- *If $A = (1, 0, 1, 0, 0)$, then $K_\alpha^A(X_n)$ is equal to key (or Demazure) polynomial corresponding to α.*
- *If $A = (0, 0, 1, 0, 0)$, then $K_\alpha^A(X_n)$ is equal to the reduced key polynomial introduced in [71].*
- *If $A = (1, 0, 1, 0, \beta)$, then $K_\alpha^A(X_n)$ is equal to the key Grothendieck polynomial $KG_\alpha(X_n)$ introduced in [71].*
- *If $A = (0, 0, 1, 0, \beta)$, then $K_\alpha^A(X_n)$ is equal to the reduced key Grothendieck polynomial [71].*

In all cases listed above the polynomials $\mathfrak{S}_w^A(X_n)$ have non-negative integer coefficients.

Exercises 4.122.

(1) Let b, c, h, e be a collection of integers, define elements $P_{ij} := f_{ij}u_{ij} \in 3\mathfrak{T}$, where $f_{ij} := bx_i + cx_j + h + ex_ix_j$. Show that

- $P_{ij}^2 = (b-c)P_{ij}$,
- $P_{ij}P_{jk}P_{ij} = f_{ij}f_{ik}f_{jk}u_{ij}u_{jk}u_{ij} + (bc-eh)P_{ij}$,
  $P_{jk}P_{ij}P_{jk} = f_{ij}f_{ik}f_{jk}u_{ij}u_{jk}u_{ij} - (bc-eh)P_{jk}$.

(2) Assume that $a = q$, $b = -q$, $c = q^{-1}$, $h = e = 0$, and introduce elements

$$e_{ij} := \big(qx_i - q^{-1}x_j\big)u_{ij}, \qquad 1 \le i < j < k \le n.$$

(a) Show that if i, j, k are distinct, then

$$e_{ij}e_{jk}e_{ij} = e_{ij} + \big(qx_i - q^{-1}x_j\big)\big(qx_i - q^{-1}x_k\big)\big(qx_j - q^{-1}x_k\big)u_{ij}u_{jk}u_{ij},$$
$$e_{ij}^2 = \big(q+q^{-1}\big)e_{ij}.$$

(b) Assume additionally that

$$u_{ij}u_{jk}u_{ij} = 0, \qquad \text{if } i,\ j,\ k \text{ are distinct.}$$

Show that the elements $\{e_i := e_{i,i+1},\ i = 1,\dots,n-1\}$, generate a subalgebra in $3\mathfrak{L}_n$ which is isomorphic to the Temperley–Lieb algebra $TL_n(q+q^{-1})$.

(3) Let us set $T_i := T_{i,i+1}$, $i = 1,\dots,n-1$, and define

$$T_0 := \pi T_{n-1}\pi^{-1}.$$

Show that if $(a+b)(a-c) + eh = 0$, then

$$T_1T_0T_1 = T_1T_0T_1, T_{n-1}T_0T_{n-1} = T_0T_{n-1}T_0,$$

Recall that $T_i^2 = (2a+b-c)T_i - a(a+b-c)$, $0 \le i \le n-1$.

In what follows we take $a = q$, $b = -q$, $c = q^{-1}$, $h = e = 0$. Therefore, $T_{i,j}^2 = (q-q^{-1})T_{i,j} + 1$. We denote by $\mathcal{H}_n(q)$ a subalgebra in $3\mathfrak{T}_n$ generated by the elements $T_i := T_{i,i+1}$, $i = 1,\dots,n-1$.

Remark 4.123. Let us stress on a difference between elements T_{ij} as a part of generators of the algebra $3\mathfrak{T}_n$, and the elements

$$T_{(ij)} := T_i \cdots T_{j-1}T_jT_{j-1}\cdots T_i \in \mathcal{H}_n(q).$$

Whereas one has $[T_{ij}, T_{kl}] = 0$ if i, j, k, l are distinct, the relation $[T_{(ij)}, T_{(kl)}] = 0$ in the algebra $\mathcal{H}_n(q)$ holds (for general q and $i \le k$) if and only if either one has $i < j < k < l$ or $i < k < l < j$.

Lemma 4.124.

(1) $T_{ij}T_{kl} = T_{kl}T_{ij}$ *if* i, j, k, l *are distinct,*

(2) $T_{i,j}x_iT_{i,j} = x_j$ *if* $1 \le i < j \le n$,

(3) $\pi T_{i,j} = T_{i+1,j+1}$ *if* $1 \le i < j < n$, $\pi^jT_{n-j+1,n} = T_{1,j}\pi^j$.

Definition 4.125. Let $1 \le i < j \le n$, set

$$Y_{i,j} = T_{i-1,j-1}^{-1}T_{i-2,j-2}^{-1}\cdots T_{1,j-i+1}^{-1}\pi^{j-i}\ T_{n-j+i,n}\cdots T_{i+1,j+1}T_{i,j}, \qquad 1 \le i < j \le n,$$

and $Y_n = T_{n-1,n}^{-1}\cdots T_{1,2}^{-1}\pi$.

For example,

$$Y_{1,j} = \pi^{j-1} T_{n-j+1,n} \cdots T_{1,j}, \qquad j \geq 2,$$
$$Y_{2,j} = T_{1,j-1}^{-1} \pi^{j-2} T_{n-j+2,n} \cdots T_{2,j}, \qquad \text{and so on},$$
$$Y_{j-1,j} = T_{j-2,j-1}^{-1} \cdots T_{1,2}^{-1} \pi T_{n-1,n} \cdots T_{j-1,j}.$$

Proposition 4.126.

(1) $x_j x_j T_{ij} = T_{ij} x_i x_j$,

(2) $Y_{i,j} = T_{i,j} Y_{i+1,j+1} T_{i,j}$ *if* $1 \leq i < j < n$,

(3) $Y_{i,j} Y_{i+k,j+k} = Y_{i+k,j+k} Y_{i,j}$ *if* $1 \leq i < j \leq n-k$,

(4) *one has* $x_{i-1} Y_{i,j}^{-1} = Y_{i,j}^{-1} x_{i-1} T_{i-1,j-1}^2$, $2 \leq i < j \leq n$,

(5) $Y_{i,j} x_1 x_2 \cdots x_n = t x_1 x_2 \cdots x_n Y_{i,j}$,

(6) $x_i Y_1 Y_2 \cdots Y_n = t^{-1} Y_1 Y_2 \cdots Y_n x_i$,

where we set $Y_i := Y_{i,i+1}$, $1 \leq i < j < n$.

Conjecture 4.127. *Subalgebra of* $3\mathfrak{T}_n$ *generated by the elements* $\{T_i := T_{i,i+1}, 1 \leq i < n, Y_1, \ldots, Y_n$, *and* $x_1, \ldots, x_n\}$, *is isomorphic to the double affine Hecke algebra* $\mathrm{DAHA}_{q,t}(n)$.

Note that the algebra $3\mathfrak{T}_n$ contains also two additional commutative subalgebras generated by *additive* $\{\theta_i = \sum_{j \neq i} u_{ij}\}_{1 \leq i \leq n}$ and *multiplicative*

$$\left\{ \Theta_i = \prod_{a=1}^{i-1} (1 - u_{ai}) \prod_{a=i+1}^{n} (1 + u_{ia}) \right\}_{1 \leq i \leq n}$$

Dunkl elements correspondingly.

Finally we introduce (cf. [24]) a (projective) representation of the modular group $SL(2,\mathbb{Z})$ on the *extended affine Hecke algebra* $\widehat{\mathcal{H}}_n$ over the ring $\mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$ generated by elements

$$\{T_1, \ldots, T_{n-1}\}, \quad \pi, \quad \text{and} \quad \{x_1, \ldots, x_n\}.$$

It is well-known that the group $\mathrm{SL}(2,\mathbb{Z})$ can be generated by two matrices

$$\tau_+ = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \tau_- = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

which satisfy the following relations

$$\tau_+ \tau_-^{-1} \tau_+ = \tau_-^{-1} \tau_+ \tau_-^{-1}, \qquad \left(\tau_+ \tau_-^{-1} \tau_+\right)^6 = I_{2\times 2}.$$

Let us introduce operators τ_+ and τ_- acting on the extended affine algebra $\widehat{\mathcal{H}}_n$. Namely,

$$\tau_+(\pi) = x_1 \pi, \tau_+(T_i) = T_i, \tau_+(x_i) = x_i, \qquad \forall\, i,$$
$$\tau_-(\pi) = \pi, \qquad \tau_-(T_i) = T_i, \qquad \tau_-(x_i) = \left(\prod_{a=i-1}^{1} T_a\right) \pi \left(\prod_{a=n}^{i} T_a\right) x_i.$$

Lemma 4.128.

$$\tau_{+}(Y_i)=\left(\prod_{a=i-1}^{1}T_a^{-1}\right)\left(\prod_{a=1}^{i-1}T_a^{-1}\right)x_iY_i,$$

$$\tau_{-}(x_i)=\left(\prod_{a=i-1}^{1}T_a\right)\left(\prod_{a=1}^{i-1}T_a\right)Y_ix_i,$$

$$\left(\tau_{+}\tau_{-}^{-1}\tau_{+}\right)(x_i)=Y_i^{-1}=\left(\tau_{-}^{-1}\tau_{+}\tau_{-}^{-1}\right)(x_i),$$

$$\left(\tau_{+}\tau_{-}^{-1}\tau_{+}\right)(Y_i)=tx_i\left(\prod_{a=i-1}^{1}T_a\right)(T_1\cdots T_{n-1})\left(\prod_{a=n-1}^{i}T_a\right),\qquad i=1,\dots,n.$$

In the last formula we set $T_n=1$ for convenience.

4.7 Braid, affine braid and virtual braid groups

The main objective of this section is to describe the distinguish abelian subgroup in the braid group B_n, see Proposition 4.132$\big(2^{(0)}\big)$, and that in the Yang–Baxter groups $\widehat{\mathrm{YB}}_n$ and YB_n, see Proposition 4.132$\big(5^{(0)}\big)$ and $\big(6^{(0)}\big)$ correspondingly. As far as I'm aware, these constructions go back to E. Artin in the case of braid groups, and to C.N. Yang in the case of Yang–Baxter group, and nowadays are widely use in the representation theory of Hecke's type algebras and that of integrable systems. In a few words, by choosing a suitable representation (finite-dimensional or birational) of either B_n or YB_n, or $\widehat{\mathrm{YB}}_n$, one gives rise to a family of mutually commuting operators acting in the space of a representation selected. In the case of braid groups one comes to Jucys–Murphy's type operators/elements, and in the case of Yang–Baxter groups one comes to Dunkl's type operators/elements. See, e.g., [59, 60], where it was used the so-called R-matrix representation of the affine braid group of type $C_n^{(1)}$ to construct the (two boundary) quantum Knizhnik–Zamolodchikov connections with values in the affine Birman–Murakami–Wenzl algebras.

To start with, let $n\geq 2$ be an integer.

Definition 4.129.

- Denote by $\mathbb{S}_n$ *the symmetric group* on n letters, and by s_i the simple transposition $(i,i+1)$ for $1\leq i\leq n-1$. The well-known Moore–Coxeter presentation of the symmetric group has the form

$$\langle s_1,\dots,s_{n-1}\,|\,s_i^2=1,\ s_is_{i+1}s_i=s_{i+1}s_is_{i+1},\ s_is_j=s_js_i\ \text{if}\ |i-j|\geq 2\rangle.$$

  Transpositions $s_{ij}:=s_is_{i+1}\cdots s_{j-2}s_{j-1}s_{j-2}\cdots s_{i+1}s_i$, $1\leq i<j<j\leq n$, satisfy the following set of (defining) relations

$$s_{ij}^2=1,\qquad s_{ij}s_{kl}=s_{kl}s_{ij}\qquad \text{if}\quad \{i,j\}\cap\{k,l\}=\varnothing,$$
$$s_{ij}s_{ik}=s_{jk}s_{ij}=s_{ik}s_{jk},\qquad s_{ik}s_{ij}=s_{ij}s_{jk}=s_{jk}s_{ik},\qquad i<j<k.$$

- *The Artin braid group on n strands B_n* is defined by generators $\sigma_1,\dots,\sigma_{n-1}$ and relations

$$\sigma_i\sigma_{i+1}\sigma_i=\sigma_{i+1}\sigma_i\sigma_{i+1},\qquad 1\leq i\leq n-2,\qquad \sigma_i\sigma_j=\sigma_j\sigma_i\qquad \text{if}\quad |i-j|\geq 2. \tag{4.7}$$

- *The monoid of positive braids on n strands B_n^{+}* is a monoid generated by the elements $\sigma_1,\dots,\sigma_{n-1}$ subject to the set of relations (4.7).

- *A new representation of the braid group* [15]. The Birman–Ko–Lee representation of the braid group B_n has the set of generators $\{\sigma_{i,j} \,|\, 1 \le i < j \le n\}$ subject to the Birman–Ko–Lee (defining) relations

$$\sigma_{i,j}\sigma_{k,l} = \sigma_{k,l}\sigma_{i,j} \qquad \text{if} \quad (j-l)(j-k)(i-l)(i-k) > 0,$$
$$\sigma_{i,j}\sigma_{i,k} = \sigma_{j,k}\sigma_{i,j} = \sigma_{i,k}\sigma_{j,k} \qquad \text{if} \quad 1 \le i < j < k \le n.$$

One can take $\sigma_{i,j} := (\sigma_{j-1}\cdots\sigma_{i+1})\sigma_i(\sigma_{i+1}^{-1}\cdots\sigma_{j-1}^{-1})$, see [14] for details. It would be well to note that as a corollary of the Birman–Ko–Lee relations one can deduce the $2D$ *Coxeter relations* among the Birman–Ko–Lee generators

$$\sigma_{i,j}\sigma_{j,k}\sigma_{i,j} = \sigma_{j,k}\sigma_{i,j}\sigma_{j,k}, \qquad 1 \le i < j < k \le n.$$

- *The Birman–Ko–Lee monoid* BKL_n is a monoid generated by the elements $\sigma_{i,j}$, $1 \le i < j \le n$, subject to the Birman–Ko–Lee relations. We denote by $\mathrm{BKL}(n)$ and called it as *the Birman–Ko–Lee algebra*, the group algebra $\mathbb{Q}[\mathrm{BKL}_n]$ of the Birman–Ko–Lee monoid. The Hilbert series of the Birman–Ko–Lee algebra $\mathrm{BKL}(n)$ will be computed in Section 4.7.3, Theorem 4.134.
- *The pure braid group* PB_n is defined to be the kernel of the natural (non-split) projection $p\colon B_n \longrightarrow \mathbb{S}_n$ given by $p(\sigma_i) = s_i$. It is well-known that the pure braid group PB_n is generated by the elements

$$g_{i,j} := \sigma_{i,j}^2 = \sigma_{j-1}\sigma_{j-2}\cdots\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}\cdots\sigma_{j-2}^{-1}\sigma_{j-1}^{-1} \qquad \text{for} \quad 1 \le i < j \le n,$$

subject to the following defining relations

$$g_{i,j}g_{k,l} = g_{k,l}g_{i,j} \qquad \text{if} \quad (i-k)(i-l)(j-k)(j-l) > 0,$$
$$g_{i,j}g_{i,k}g_{j,k} = g_{i,k}g_{j,k}g_{i,j} = g_{j,k}g_{i,j}g_{i,k} \qquad \text{if} \quad 1 \le i < j < k \le n,$$
$$g_{i,k}g_{i,l}g_{j,l}g_{k,l} = g_{i,l}g_{j,l}g_{k,l}g_{i,k} \qquad \text{if} \quad 1 \le i < j < k < l \le n.$$

Comments 4.130. It is easy to see that the defining relations for the pure braid group PB_n listed above are equivalent to the following list of defining relations

$$g_{i,j}^{-1}g_{k,l}g_{i,j} = \begin{cases} g_{k,l} & \text{if } (i-k)(i-l)(j-k)(j-l) > 0, \\ g_{i,l}g_{k,l}g_{i,l}^{-1} & \text{if } i < k = j < l, \\ g_{i,l}g_{j,l}g_{k,l}g_{i,l}^{-1}g_{j,l}^{-1} & \text{if } i = k < j < l, \\ g_{i,l}g_{j,l}g_{i,l}^{-1}g_{j,l}^{-1}g_{k,l}g_{j,l}g_{i,l}g_{j,l}^{-1}g_{i,l}^{-1} & \text{if } i < k < j < l, \end{cases}$$

commonly used in the literature to describe the defining relations among the generators $\{g_{ij}\}$ of the pure braid group P_n, see, e.g., [14].

- *The affine Artin braid group* B_n^{aff}, cf. [112], is an extension of the Artin braid group on n strands B_n by the element τ subject to the set of crossing relations

$$\sigma_1\tau\sigma_1\tau = \tau\sigma_1\tau\sigma_1, \qquad \sigma_i\tau = \tau\sigma_i \qquad \text{for} \quad 2 \le i \le n-1.$$

- *The virtual braid group* VB_n is a group generated by $\sigma_1,\dots,\sigma_{n-1}$ and $s_1,\dots,s_{n-1}$ subject to the relations:
  (1) *braid relations* $\sigma_1,\dots,\sigma_{n-1}$ generate the Artin braid group B_n;
  (2) *Moore–Coxeter relations* $s_1,\dots,s_{n-1}$ generate the symmetric group $\mathbb{S}_n$;
  (3) *crossing relations* $\sigma_i s_j = s_j\sigma_i$ if $|i-j| \ge 2$, $s_is_{i+1}\sigma_i = \sigma_{i+1}s_is_{i+1}$ if $1 \le i \le n-2$.
- *The virtual pure braid group* VP_n is defined to be the kernel of the natural map

$$\eta\colon\ \mathrm{VB}_n \longrightarrow \mathbb{S}_n, \qquad \eta(\sigma_i) = \eta(s_i) = s_i, \qquad i = 1,\dots,n-1.$$

4.7.1 Yang–Baxter groups

Definition 4.131.

- *The quasitriangular Yang–Baxter group* $\widehat{\mathrm{YB}}_n$, cf. [7], is a group generated by the set of elements $\{Q_{i,j},\ 1 \le i \ne j \le n\}$, subject to the set of defining relations

  (1) $[Q_{i,j}, Q_{k,l}] = 0$ if i, j, k and l are all distinct,

  (2) *Yang–Baxter relations* $Q_{i,j}Q_{i,k}Q_{j,k} = Q_{j,k}Q_{i,k}Q_{i,j}$ if i, j, k are distinct.

  According to [5, Theorem 1], the *quasitriangular Yang–Baxter group* $\widehat{\mathrm{YB}}_n$ is isomorphic to the *virtual pure braid group* VP_n.

- *The Yang–Baxter monoid* $\widetilde{\mathrm{YB}}_n$ is a monoid generated by the elements $Q_{i,j}$, $1 \le i \ne j \le n$. Important particular case corresponds to the case when $Q_{i,j}Q_{j,i} = 1$ for all $1 \le i \ne j \le n$.

- *The Yang–Baxter group* YB_n is defined by the set of generators $R_{i,j}$, $1 \le i < j \le n$, subject to the set of defining relations

  (1) $R_{i,j}R_{k,l} = R_{k,l}R_{i,j}$ if i, j, k and l are pairwise distinct,

  (2) $R_{i,j}R_{i,k}R_{j,k} = R_{j,k}R_{i,k}R_{i,j}$ if $1 \le i < j < k \le n$.

4.7.2 Some properties of braid and Yang–Baxter groups

For the sake of convenience and future references, below we state some basic properties of the groups P_n, YB_n and B_n^{aff}.

Proposition 4.132. *Let F_m denotes the free group with m generators.*

(1^0) *The elements $g_{1,n}, g_{2,n}, \ldots, g_{n-1,n}$ generate a free normal subgroup F_{n-1} in P_n, and $P_n = P_{n-1} \ltimes \langle g_{1,n}, g_{2,n}, \ldots, g_{n-1,n}\rangle$. Hence P_n is an iterated extension of free groups.*

(2^0) *Let us consider the following elements in the group B_n^{aff}:*

$$\gamma_1 = \tau, \qquad \gamma_i = \prod_{j=i-1}^{1} \sigma_j \tau \prod_{j=1}^{i-1} \sigma_j, \qquad 2 \le i \le n.$$

Then

(a) *commutativity, $\gamma_i\gamma_j = \gamma_j\gamma_i$ for all $1 \le i, j \le n$;*

(b) *the elements $\gamma_1, \ldots, \gamma_n$ generate a free abelian subgroup of rang n in B_n^{aff}.*[49]

(3^0) *Let us introduce elements*

$$D_{i,j} := \sigma_{j-1}\sigma_{j-2}\cdots\sigma_{i+1}\ \sigma_i^2\ \sigma_{i+1}\cdots\sigma_{j-2}\sigma_{j-1} = \prod_{i \le a < j} g_{a,j} \in P_n,$$

$$F_{i,j} := \sigma_{n-j}\sigma_{n-j+1}\cdots\sigma_{n-i-1}\sigma_{n-i}^2\sigma_{n-i-1}\cdots\sigma_{n-j+1}\sigma_{n-j} = \prod_{a=j}^{i+1} g_{n-a,n-i} \in P_n,$$

where $1 \le i < j \le n$. For example,

$$D_{i,i+1} = \sigma_i^2, \qquad D_{i,i+2} = \sigma_{i+1}\sigma_i^2\sigma_{i+1}, \qquad F_{i,i+1} = \sigma_{n-i}^2,$$
$$F_{i,i+2} = \sigma_{n-i-1}\sigma_{n-i}^2\sigma_{n-i-1}$$

and etc. Then

[49] We refer the reader to [112] for more details about affine braid groups. Here we only remark that the type A affine Weyl groups $\widehat{\mathbb{S}}_n$, the Hecke algebras $H_{n,q}$, the affine Hecke algebras $\widehat{H}_{n,q}$, the Ariki–Koike, or cyclotomic Hecke, algebras $H_{r,1,n}$, the affine and cyclotomic Birman–Murakami–Wenzl algebras $\mathcal{Z}_{r,1,n}$, all can be obtained as certain quotients of the group algebra $\mathbb{C}B_n^{\mathrm{aff}}$ of the affine braid group.

- *For each $j = 3, \ldots, n$, the element $D_{1,j}$ commutes with $\sigma_1, \ldots, \sigma_{j-2}$.*
- *The elements $D_{1,2}, D_{1,3}, \ldots, D_{1,n}$ (resp. $F_{1,2}, F_{1,3}, \ldots, F_{1,n}$) generate a free abelian subgroup in P_n.*
- *If $n \geq 3$, the element*

$$\prod_{2 \leq j \leq n} D_{1,j} = \prod_{2 \leq j \leq n} F_{1,j} = (\sigma_1 \cdots \sigma_{n-1})^n$$

generates the center of the braid group B_n and that of the pure braid group P_n.
- $D_{i,j} D_{i,j+1} D_{j,j+1} = D_{j,j+1} D_{i,j+1} D_{i,j}$ *if $i < j$.*
- *Consider the elements $s := \sigma_1\sigma_2\sigma_1$, $t := \sigma_1\sigma_2$ in the braid group B_3. Then $s^2 = t^3$ and the element $c := s^2$ generates the center of the group B_3. Moreover,*

$$B_3/\langle c\rangle \cong \mathrm{PSL}_2(\mathbb{Z}), \qquad B_3/\langle c^2\rangle \cong \mathrm{SL}_2(\mathbb{Z}).$$

(4^0) *Let us introduce the following elements in the quasitriangular Yang–Baxter group $\widehat{\mathrm{YB}_n}$:*

$$C_{i,j} = \left(\prod_{a=j-1}^{i} Q_{a,j}\right)\left(\prod_{b=i}^{j-1} Q_{j,b}\right), \qquad f_{i,j} = \left(\prod_{a=j-1}^{i+1} Q_{a,j}\right) Q_{i,j} Q_{j,i} \left(\prod_{b=i+1}^{j-1} Q_{b,j}\right)^{-1}.$$

Then

- *The elements $C_{1,2}, C_{1,3}, \ldots, C_{1,n}$ generate a free abelian subgroup in $\widehat{\mathrm{YB}_n}$.*
- *The elements $f_{i,j}$, $1 \leq i < j \leq n$, generate a subgroup in $\widehat{\mathrm{YB}_n}$, which is isomorphic to the pure braid group P_n.*[50]

(5^0) *Assume that the following additional relations in $\widehat{\mathrm{YB}_n}$ are satisfied*

$$Q_{i,j} Q_{j,i} = Q_{j,i} Q_{i,j}, \qquad Q_{k,l} Q_{i,j} Q_{j,i} = Q_{j,i} Q_{i,j} Q_{k,l}$$

if $i \neq j$ and $k \neq l$. In other words, the elements $Q_{i,j}$ and $Q_{j,i}$ commute, and the elements $Q_{i,j}Q_{j,i} = Q_{j,i}Q_{i,j}$ are central. Under these assumptions, we have that the elements

$$\Theta_i := \prod_{j=i-1}^{1} Q_{j,i} \prod_{j=n}^{i+1} Q_{i,j}, \qquad \bar{\Theta}_i := \prod_{j=i+1}^{n} Q_{j,i} \prod_{j=1}^{i-1} Q_{i,j}, \qquad 1 \leq i \leq n,$$

[50] It is enough to check that the elements $\{f_{i,j},\ 1 \leq i < j \leq n\}$ satisfy the defining relations for the pure braid group P_n only in the case $n = 4$. Let us prove that

$$f_{1,4} f_{2,4} f_{3,4} f_{1,3} = f_{1,3} f_{1,4} f_{2,4} f_{3,4}.$$

Other relations are simple and can be checked in a similar fashion.

Let

$$\text{l.h.s.} = f_{1,4} f_{2,4} f_{3,4} f_{1,3} = Q_{34}Q_{24}Q_{14}Q_{41}Q_{42}Q_{43}Q_{23}Q_{13}Q_{31}Q_{23}^{-1},$$
$$\text{r.h.s.} = f_{1,3} f_{1,4} f_{2,4} f_{3,4} = Q_{23}Q_{13}Q_{31}\boldsymbol{Q_{23}^{-1}Q_{34}Q_{24}}Q_{14}Q_{41}Q_{42}Q_{43}.$$

Now we are going to apply the Yang–Baxter relations

$$Q_{23}^{-1}Q_{34}Q_{24} = Q_{24}Q_{34}Q_{23}^{-1}, \qquad Q_{23}^{-1}Q_{42}Q_{43} = Q_{43}Q_{42}Q_{23}^{-1}, \qquad Q_{31}Q_{34}Q_{14} = Q_{14}Q_{34}Q_{31}.$$

Therefore,

$$\begin{aligned}\text{r.h.s.} &= Q_{23}Q_{13}Q_{31}\boldsymbol{Q_{23}^{-1}Q_{34}Q_{24}}Q_{14}Q_{41}Q_{42}Q_{23}^{-1} = \boldsymbol{Q_{23}Q_{24}Q_{34}}Q_{14}Q_{13}\boldsymbol{Q_{31}Q_{41}Q_{43}}Q_{42}Q_{23}^{-1}\\ &= Q_{34}Q_{24}Q_{14}Q_{23}\boldsymbol{Q_{13}Q_{43}Q_{41}}Q_{42}Q_{31}Q_{23}^{-1} = Q_{34}Q_{24}Q_{14}Q_{41}Q_{23}Q_{43}Q_{42}Q_{13}Q_{31}Q_{23}^{-1}\\ &= Q_{34}Q_{24}Q_{14}Q_{41}Q_{42}Q_{43}Q_{23}Q_{13}Q_{31}Q_{23}^{-1} = \text{l.h.s.}\end{aligned}$$

satisfy the following relations

$$[\Theta_i,\Theta_j]=0=[\bar{\Theta}_i,\bar{\Theta}_j],$$

$$\Theta_i\bar{\Theta}_i=\prod_{j\neq i}Q_{i,j}Q_{j,i}=\bar{\Theta}_i\Theta_i,\qquad \prod_{i=1}^{n}\Theta_i=\prod_{1\le i\neq j\le n}Q_{i,j}Q_{j,i}=\prod_{i=1}^{n}\bar{\Theta}_i.$$

(6^0) *In the special case $Q_{i,j}Q_{j,i}=1$ for all $i\neq j$, the following statement holds: the elements*

$$\Theta_j=\prod_{a=j-1}^{1}R_{a,j}^{-1}\prod_{b=n}^{j+1}R_{j,b},\qquad 1\le j\le n-1,$$

generate a subgroup in the Yang–Baxter group YB_n*, which is isomorphic to the free abelian group of rang $n-1$.*

4.7.3 Artin and Birman–Ko–Lee monoids

Let (W,S) be a finite Coxeter group, $B(W)$ and $B^+(W)$ be the corresponding braid group and monoid of positive braids. Denote by $P_W(s,t)=\sum\limits_{i\ge 0,\,j\ge 0}B_{\mathbb{Q}[B^+(W)]}(i,j)s^it^j$ the Poincaré polynomial of the group algebra over $\mathbb{Q}$ of the monoid $B^+(W)$.

Conjecture 4.133. $P_W(s,1)=(s+1)^{|S|}$.

It is known [30, 125] that the Hilbert series of the group algebra of the monoid $B^+(W)$ is a *rational* function of the form $\frac{1}{P(t)}$ for a some polynomial $P(t):=P_W(t)\in\mathbb{Z}[t]$.

Theorem 4.134.

(1) *Some Betti numbers of the group algebra over $\mathbb{Q}$ of the monoid $B^+(A_{n-1})$:*

$$B_{\mathbb{Q}[B^+(A_{n-1})]}(k,k)=\binom{n-k}{k},$$

$$B_{\mathbb{Q}[B^+(A_{n-1})]}\left(k,\binom{k+1}{2}\right)=n-k,\qquad 1\le k\le n-1,$$

$$B_{\mathbb{Q}[B^+(A_{n-1})]}(k,k+1)=(k-1)\binom{n-k}{k-1},$$

$$B_{\mathbb{Q}[B^+(A_{n-1})]}(k,k+2)=\binom{k-2}{2}\binom{n-k}{k-2},$$

$$B_{\mathbb{Q}[B^+(A_{n-1})]}(k,k+3)=(k-2)\binom{n-k}{k-2}+\max(3k-17,0)\binom{n-k}{k-3}\quad \text{if } k\ge 3.$$

(2) *The Birman–Ko–Lee algebra* $\mathrm{BKL}(n)$ *is Koszul, and the Hilbert polynomial of its quadratic dual is equal to*

$$\mathrm{Hilb}\big(\mathrm{BKL}(n)^!,t\big)=\sum_{k=0}^{n-1}\frac{1}{k+1}\binom{n-1}{k}\binom{n+k-1}{k}t^k.$$

Conjecture 4.135 (type A_{n-1} case)**.** *Let $I\subset[1,n-1]$ be a subset of vertices in the Dynkin diagram of type A_{n-1}, and R_I denotes the root system generated by the positive roots $\{\alpha_{ij}=\epsilon_i-\epsilon_j,\ (i,j)\in I\times I\}$. Assume that*

$$R_I\cong A_{n_1}\coprod\cdots\coprod A_{n_k},\qquad n_1+\cdots+n_k=n-1$$

stands for the decomposition of the root system R_I into the disjoint union of irreducible root subsystems of type A. The numbers $n_1, \ldots, n_k$ are defined uniquely up to a permutation. Let us set $n(I) = \sum_{a=1}^{k} \binom{n_a}{2}$. Then

$$P_{A_{n-1}}(s,t) = \sum_{I} s^{|I|} t^{n(I)},$$

where the sum runs over the all subsets of vertices I in the Dynkin diagram of type A_{n-1}, including the empty set, and $|I|$ denotes the cardinality of the set I.

Comments 4.136.

(A) The Hilbert polynomial of the Birman–Ko–Li algebra BKL(n) has been computed also by M. Albenque and P. Nadeau, see [2].

(B) Let's consider the *truncated theta function* $\theta^{+}(z,t) = \sum_{n\geq 0} t^{n(n+1)/2} z^n$. Then

$$\sum_{n\geq 1} P_{A_{n-1}}(s,t) z^{n-1} = \theta^{+}(t,sz)/(1 - z(\theta^{+}(t,sz))).$$

(C) It is well known that the number

$$T(n,k) = \frac{1}{k+1}\binom{n}{k}\binom{n+k}{k}$$

counts the number of Schröder paths (i.e., consisting of steps $(1,1)$, $(1,-1)$ and $(2,0)$ and never going below x-axis) from $(0,0)$ to $(2n,0)$, having exactly k $(1,1)$ steps. In particular,

$$\dim(\mathrm{BKL}(n)^{!}) = \mathrm{Sch}(n),$$

is the n-th (large) Schröder number, see [131, A006318]. It is a classical result that

$$\sum_{k=0}^{n} T(n,k) x^k (1-x)^{n-k} = \sum_{k=0}^{n-1} N(n,k) x^k,$$

where $N(n,k) := \frac{1}{n}\binom{n}{k}\binom{n}{k+1}$ denotes the Narayana number. Some explicit combinatorial interpretations of the values of the above polynomials for $x = 0,1,2,4$ can be found in [131, A088617]. Note that $\mathrm{Hilb}(\mathrm{BKL}^{!}, t) = (1+t)\mathrm{Ass}_{n-2}(t)$, where

$$\mathrm{Ass}_n(t) := \sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k}\binom{n+k}{k} t^k$$

denotes the f-vector polynomial corresponding to the associahedron of type A_n.

(D) The polynomials $F(n,t) := \sum_{k\geq 0} B_{\mathbb{Q}[B^{+}(A_{n-1})]}(k,k) t^k$ appear to be equal to the so-called *Fibonacci* polynomials, see, e.g., [131, A011973]. It is well-known that

$$\sum_{n\geq 0} F(n,t) y^n = \frac{1+ty}{1-y-ty^2}.$$

Moreover, the coefficient $B_{\mathbb{Q}[B^{+}(A_{n-1})]}(k,k)$ is equal to the number of compositions of $n+2$ into $k+1$ parts, all ≥ 2, see [131, A011973].

(*E*) *Monoid of positive pure braids.* The monoid of positive pure braids PB_n^+ (of the type A_{n-1}) is the monoid generated by the set $\{g_{i,j},\ 1 \le i < j \le n\}$ of the Artin generators of the pure braid group PB_n.

Conjecture 4.137. *The following list of relations is the defining set of relations in the monoid* PB_n*:*

$$(a)\quad [g_{i,j}, g_{k,l}] = 0, \qquad [g_{i,l}, g_{j,k}] = 0, \qquad \text{if} \quad 1 \le i < j < k < l \le n,$$

$$(b)\quad \left[g_{\overline{j_1+m}, \overline{j_{k-1}+m}}, \prod_{a=1}^{k-1} g_{\overline{j_a+m}, \overline{j_k+m}} \right] = 0,$$

for all sequences of integers $1 \le j_1 < j_2 < \cdots < j_k \le n$ *of the length* $k \ge 4$ *and* $m = 0, \dots, n-1$*. Here we assume that* $g_{i,j} = g_{j,i}$ *for all* $i \ne j$*, and for any non-negative integer* a *we denote by* $\overline{a}$ *a unique integer* $1 \le \overline{a} \le n$ *such that* $a \equiv \overline{a} \pmod{n+1}$*.*

It is worth noting that the defining relations in the pure braid group P_n are that listed in (a) and the part of that listed in (b) corresponding to $k = 3$, $m = 0$ and 1, and that for $k = 4$, $m = 0$.

5 Combinatorics of associative Yang–Baxter algebras

Let α and β be parameters.

Definition 5.1 ([66, 70, 72], cf. [1, 115])**.**

(1) The associative quasi-classical Yang–Baxter algebra of weight (α, β), denoted by $\widehat{\mathrm{ACYB}}_n(\alpha, \beta)$, is an associative algebra, over the ring of polynomials $\mathbb{Z}[\alpha, \beta]$, generated by the set of elements $\{x_{ij},\ 1 \le i < j \le n\}$, subject to the set of relations

   (a) $x_{ij}x_{kl} = x_{kl}x_{ij}$ if $\{i, j\} \cap \{k, l\} = \varnothing$,
   (b) $x_{ij}x_{jk} = x_{ik}x_{ij} + x_{jk}x_{ik} + \beta x_{ik} + \alpha$ if $1 \le 1 < i < j \le n$.

(2) Define *associative quasi-classical Yang–Baxter algebra of weight* β, denoted by $\widehat{\mathrm{ACYB}}_n(\beta)$, to be $\widehat{\mathrm{ACYB}}_n(0, \beta)$.

Comments 5.2. The algebra $3T_n(\beta)$, see Definition 3.1, is the quotient of the algebra $\widehat{\mathrm{ACYB}}_n(-\beta)$, by the "dual relations"

$$x_{jk}x_{ij} - x_{ij}x_{ik} - x_{ik}x_{jk} + \beta x_{ik} = 0, \qquad i < j < k.$$

The (truncated) Dunkl elements $\theta_i = \sum_{j \ne i} x_{ij}$, $i = 1, \dots, n$, do not commute in the algebra $\widehat{\mathrm{ACYB}}_n(\beta)$. However a certain version of noncommutative elementary polynomial of degree $k \ge 1$, still is equal to zero after the substitution of Dunkl elements instead of variables [72]. We state here the corresponding result only "in classical case", i.e., if $\beta = 0$ and $q_{ij} = 0$ for all i, j.

Lemma 5.3 ([72])**.** *Define noncommutative elementary polynomial* $L_k(x_1, \dots, x_n)$ *as follows*

$$L_k(x_1, \dots, x_n) = \sum_{I = (i_1 < i_2 < \cdots < i_k) \subset [1,n]} x_{i_1} x_{i_2} \cdots x_{i_k}.$$

Then $L_k(\theta_1, \theta_2, \dots, \theta_n) = 0$.

Moreover, if $1 \le k \le m \le n$*, then one can show that the value of the noncommutative polynomial* $L_k(\theta_1^{(n)}, \dots, \theta_m^{(n)})$ *in the algebra* $\widehat{\mathrm{ACYB}}_n(\beta)$ *is given by the Pieri formula, see* [45, 117]*.*

5.1 Combinatorics of Coxeter element

Consider the "Coxeter element" $w \in \widehat{\mathrm{ACYB}}_n(\alpha,\beta)$ which is equal to the ordered product of "simple generators":

$$w := w_n = \prod_{a=1}^{n-1} x_{a,a+1}.$$

Let us bring the element w to the reduced form in the algebra $\widehat{\mathrm{ACYB}}_n(\alpha,\beta)$, that is, let us consecutively apply the defining relations (a) and (b) to the element w in any order until unable to do so. Denote the resulting (noncommutative) polynomial by $P_n(x_{ij};\alpha,\beta)$. In principal, the polynomial itself can depend on the order in which the relations (a) and (b) are applied. We set $P_n(x_{ij};\beta) := P_n(x_{ij};0,\beta)$.

Proposition 5.4 (cf. [133, Exercise 6.C5(c)], [99, 100])**.**

(1) *Apart from applying the relation (a) (commutativity), the polynomial $P_n(x_{ij};\beta)$ does not depend on the order in which relations (a) and (b) have been applied, and can be written in a unique way as a linear combination:*

$$P_n(x_{ij};\beta) = \sum_{s=1}^{n-1} \beta^{n-s-1} \sum_{\{i_a\}} \prod_{a=1}^{s} x_{i_a,j_a},$$

where the second summation runs over all sequences of integers $\{i_a\}_{a=1}^{s}$ such that $n-1 \geq i_1 \geq i_2 \geq \cdots \geq i_s = 1$, and $i_a \leq n-a$ for $a = 1,\ldots,s-1$; moreover, the corresponding sequence $\{j_a\}_{a=1}^{n-1}$ can be defined uniquely by that $\{i_a\}_{a=1}^{n-1}$.

- *It is clear that the polynomial $P(x_{ij};\beta)$ also can be written in a unique way as a linear combination of monomials $\prod_{a=1}^{s} x_{i_a,j_a}$ such that $j_1 \geq j_2 \cdots \geq j_s$.*

(2) *Let us set $\deg(x_{ij}) = 1$, $\deg(\beta) = 0$. Denote by $T_n(k,r)$ the number of degree k monomials in the polynomial $P(x_{ij};\beta)$ which contain exactly r factors of the form $x_{*,n}$. (Note that $1 \leq r \leq k \leq n-1$.) Then*

$$T_n(k,r) = \frac{r}{k}\binom{n+k-r-2}{n-2}\binom{n-2}{k-1}.$$

In other words,

$$P_n(t,\beta) = \sum_{1\leq r\leq k<n} T_n(k,r)t^r\beta^{n-1-k},$$

where $P_n(t,\beta)$ denotes the following specialization

$$x_{ij} \longrightarrow 1 \quad \text{if} \quad j<n, \qquad x_{in} \longrightarrow t, \quad \forall\, i = 1,\ldots,n-1$$

of the polynomial $P_n(x_{ij};\beta)$.

In particular, $T_n(k,k) = \binom{n-2}{k-1}$ and $T_n(k,1) = T(n-2,k-1)$, where

$$T(n,k) := \frac{1}{k+1}\binom{n+k}{k}\binom{n}{k}$$

is equal to the number of Schröder paths (i.e., consisting of steps $U=(1,1)$, $D=(1,-1)$, $H=(2,0)$ and never going below the x-axis) from $(0,0)$ to $(2n,0)$, having k U's, see [131, A088617].

Moreover, $T_n(n-1,r)=\operatorname{Tab}(n-2,r-1)$, *where*

$$\operatorname{Tab}(n,k):=\frac{k+1}{n+1}\binom{2n-k}{n}=F_{n-k}^{(2)}(k)$$

is equal to the number of standard Young tableaux of the shape $(n,n-k)$, see [131, A009766]. *Recall that* $F_n^{(p)}(b)=\frac{1+b}{n}\binom{np+b}{n-1}$ *stands for the generalized Fuss–Catalan number.*

(3) *After the specialization $x_{ij}\longrightarrow 1$ the polynomial $P(x_{ij})$ is transformed to the polynomial*

$$P_n(\beta):=\sum_{k=0}^{n-1}N(n,k)(1+\beta)^k,$$

where $N(n,k):=\frac{1}{n}\binom{n}{k}\binom{n}{k+1}$, $k=0,\dots,n-1$, *stand for the Narayana numbers.*

Furthermore, $P_n(\beta)=\sum\limits_{d=0}^{n-1}s_n(d)\beta^d$, *where*

$$s_n(d)=\frac{1}{n+1}\binom{2n-d}{n}\binom{n-1}{d}$$

is the number of ways to draw $n-1-d$ diagonals in a convex $(n+2)$-gon, such that no two diagonals intersect their interior.

Therefore, the number of (nonzero) terms in the polynomial $P(x_{ij};\beta)$ is equal to the n-th little Schröder number $s_n:=\sum\limits_{d=0}^{n-1}s_n(d)$, *also known as the n-th super-Catalan number, see, e.g.,* [131, A001003].

(4) *Upon the specialization $x_{1j}\longrightarrow t$, $1\le j\le n$, and that $x_{ij}\longrightarrow 1$ if $2\le i<j\le n$, the polynomial $P(x_{ij};\beta)$ is transformed to the polynomial*

$$P_n(\beta,t)=t\sum_{k=1}^{n}(1+\beta)^{n-k}\sum_{\pi}t^{p(\pi)},$$

where the second summation runs over the set of Dick paths π of length $2n$ with exactly k picks (UD-steps), and $p(\pi)$ denotes the number of valleys (DU-steps) that touch upon the line $x=0$.

(5) *The polynomial $P(x_{ij};\beta)$ is invariant under the action of anti-involution $\phi\circ\tau$, see* [72, Section 5.1.1] *for definitions of ϕ and τ.*

(6) *Follow* [133, Exercise 6.C8(c)] *consider the specialization*

$$x_{ij}\longrightarrow t_i,\qquad 1\le i<j\le n,$$

and define $P_n(t_1,\dots,t_{n-1};\beta)=P_n(x_{ij}=t_i;\beta)$.

One can show, cf. [133], *that*

$$P_n(t_1,\dots,t_{n-1};\beta)=\sum\beta^{n-k}t_{i_1}\cdots t_{i_k},$$

where the sum runs over all pairs $\{(a_1,\dots,a_k),(i_1,\dots,i_k)\in\mathbb{Z}_{\ge 1}\times\mathbb{Z}_{\ge 1}\}$ such that $1\le a_1<a_2<\cdots<a_k$, $1\le i_1\le i_2\cdots\le i_k\le n$ and $i_j\le a_j$ for all j.

Now we are ready to state our main result about polynomials $P_n(t_1,\dots,t_n;\beta)$. Let $\pi := \pi_n \in \mathbb{S}_n$ be the permutation

$$\pi = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ 1 & n & n-1 & \cdots & 2 \end{pmatrix}.$$

Then

$$P_n(t_1,\dots,t_{n-1};\beta) = \left(\prod_{i=1}^{n-1} t_i^{n-i}\right) \mathfrak{G}_\pi^{(\beta)}\big(t_1^{-1},\dots,t_{n-1}^{-1}\big) = \sum_{\mathcal{T}} wt(\mathcal{T}), \tag{5.1}$$

where $\mathfrak{G}_w^{(\beta)}(x_1,\dots,x_{n-1})$ denotes the β-Grothendieck polynomial corresponding to a permutation $w \in \mathbb{S}_n$, see [42] or Appendix A.1; summation in the right hand side of the second equality runs over the set of all *dissections* $\mathcal{T}$ of a convex $(n+2)$-gon, and $wt(\mathcal{T})$ denotes *weight* of a dissection $\mathcal{T}$, namely,

$$wt(\mathcal{T}) = \prod_{d \in \mathcal{T}} x_d \beta^{n-3-|\mathcal{T}|},$$

where the product runs over diagonals in $\mathcal{T}$, $x_d = x_{ij}$, if diagonal d connects vertices i and j, $i<j$, and $|\mathcal{T}|$ denotes the number of diagonals in dissection $\mathcal{T}$.

In particular,

$$\mathfrak{G}_\pi^{(\beta)}(x_1=1,\dots,x_{n-1}=1) = \sum_{k=0}^{n-1} N(n,k)(1+\beta)^k,$$

where $N(n,k)$ denotes the Narayana numbers, see item (3) of Proposition 5.4.

More generally, write $P_n(t,\beta) = \sum_k P_n^{(k)}(\beta) t^k$. Then

$$\mathfrak{G}_\pi^{(\beta)}(x_1=t,\, x_i=1,\, \forall\, i \geq 2) = \sum_{k=0}^{n-1} P_{n-1}^{(k)}\big(\beta^{-1}\big)\beta^k t^{n-1-k}.$$

Comments 5.5.

- Note that if $\beta=0$, then one has $\mathfrak{G}_w^{(\beta=0)}(x_1,\dots,x_{n-1}) = \mathfrak{S}_w(x_1,\dots,x_{n-1})$, that is the β-Grothendieck polynomial at $\beta=0$, is equal to the Schubert polynomial corresponding to the same permutation w. Therefore, if

  $$\pi = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ 1 & n & n-1 & \dots & 2 \end{pmatrix},$$

  then

  $$\mathfrak{S}_\pi(x_1=1,\dots,t_{n-1}=1) = C_{n-1}, \tag{5.2}$$

  where C_m denotes the m-th Catalan number. Using the formula (5.2) it is not difficult to check that the following formula for the principal specialization of the Schubert polynomial $\mathfrak{S}_\pi(X_n)$ is true

  $$\mathfrak{S}_\pi\big(1,q,\dots,q^{n-1}\big) = q^{\binom{n-1}{3}} C_{n-1}(q), \tag{5.3}$$

  where $C_m(q)$ denotes the Carlitz–Riordan q-analogue of the Catalan numbers, see, e.g., [134]. The formula (5.3) has been proved in [44] using the observation that π is a *vexillary* permutation, see [92] for the a definition of the latter. A combinatorial/bijective proof of the formula (5.2) is *due* to A. Woo [142].

- The Grothendieck polynomials, had been defined originally by A. Lascoux and M.-P. Schützenberger, see, e.g., [86], correspond to the case $\beta = -1$. In this case $P_n(-1) = 1$ if $n \geq 0$, and therefore the specialization $\mathfrak{G}_w^{(-1)}(x_1 = 1, \ldots, x_{n-1} = 1) = 1$ for all $w \in \mathbb{S}_n$.
- In Section 5.2.2, Theorems 5.28 and 5.29, we state a generalization of the second equality in the formula (5.1) to the case of Richardson's permutations of the form $1^k \times w_0^{(n-k)} := \pi_k^{(n)}$, and relate monomials which appear in a combinatorial formula[51] for the corresponding β-Grothendieck polynomial, and/with the set of k-dissections and k-triangulations of a convex $(n+k+1)$-gon, and the Lagrange inversion formula, see Section 5.4.2 for more details.

Clearly, the Richardson permutations $\pi_k^{(0)}$ are special subset of permutations of the form $1^k \times w_\lambda := w_k^{(\lambda)}$, where w_λ stands for the dominant permutation of shape λ. An analogue and extension of the first equality in the formula (5.1) for permutations of the form $w_1^{(\lambda)}$ has been proved in [39, Theorem 5.4]. We state here a particular case of that result related with the Fuss–Catalan numbers obtained independently by the author of the present paper as a generalization of [133, Exercise 8C5(c)] and [142] to the case of Fuss–Catalan numbers. Namely, let $\lambda = (\lambda_1, \ldots, \lambda_k = 1)$ be a Young diagram such that $\lambda_i - \lambda_{i+1} \leq 1$. Therefore, the boundary $\partial(\lambda)$ of λ, that is the set of the last boxes in each row of λ, is a disjoint union of vertical intervals. To the last box of the lowermost interval we attach the generator x_{23}. To the next box of that interval, if exists, we attach the generator u_{24} and so on, up to the top box of that interval is equipped with the generator, say x_{2,k_1}. It is clear that $k_1 = \lambda'_1 - \lambda'_2 + 2$. Now let us consider the next vertical interval. To the bottom box of that interval we attach the variable x_{k_1,k_1+1}, to the next box we attach the variable x_{k_1,k_1+2} and so on. Let the top of that vertical interval is equipped with the generator x_{k_1,k_2}; it is clear that $k_2 = \lambda'_1 - \lambda'_3 + 2$. Applying this procedure successively step by step to each vertical interval, we attach the variable u_b to each box b in the boundary of Young diagram λ. Finally we attach the monomial

$$M_\lambda = x_{12} \prod_{b \in \partial(\lambda)} x_b.$$

Theorem 5.6 ([39]). *Let λ be a partition such that $\lambda_i - \lambda_{i+1} \leq 1$, $\forall\, i \geq 1$, and set $N := \lambda'_1 + 2$. Let $w_\lambda \in \mathbb{S}_N$ be a unique dominant partition of shape λ, and $M_\lambda \in \widehat{\mathrm{ACYB}}_N(\beta)$ be the monomial associated with the boundary $\partial(\lambda)$ of partition λ. Then*

$$P_{M_\lambda}(x_{ij} = t_i, \forall\, i, j; \beta) = t^\lambda \mathfrak{G}_{1 \times w_\lambda}^{(\beta)}\big(t_1^{-1}, \ldots, t_N^{-1}\big),$$

where $t^\lambda := t_1^{\lambda'_1} \cdots t_N^{\lambda'_N}$. In other words, after the specialization $x_{ij} \longrightarrow t_i^{-1}$, $\forall\, i, j$, the specialized reduced polynomial corresponding to the monomial M_λ is equal to $t^{-\lambda}$ multiplied by the β-Grothendieck polynomial associated with the permutation $1 \times w_\lambda$.

Let us illustrate the above theorem by example. We take $\lambda = 43221$. In this case $N = 7 = 5{+}2$ and $w := w_\lambda = [1, 6, 5, 4, 7, 3, 2]$. The monomial corresponding to the boundary of λ is equal to

$$x_{12}x_{23}x_{34}x_{35}x_{56}x_{67} \in \widehat{\mathrm{ACYB}}_7.$$

Since the both reduced and β-Grothendieck polynomials appearing in this example are huge, we display only its specialized values at $x_{ij} = 1$, $\forall\, i, j$ and $t_i = 1$, $\forall\, i$. We set also $d := \beta - 1$. It is not difficult to check that the reduced polynomial corresponding to monomial $x_{12}x_{23}x_{34}x_{35}x_{56}$ after the specialization $x_{ij} = 1$, $\forall\, 1 \leq i < j \leq 5$, and the identification $x_{i,6} = x_{1,6}$, $1 \leq i \leq 5$, is equal to

$$(9, 20, 14, 3)_\beta x_{16} + (9, 15, 6)_\beta x_{16}^2 + (4, 4)_\beta x_{16}^3 + x_{16}^4.$$

[51] See [13, 44, 77] for example.

Finally after multiplication of the above expression by x_{67}, applying 3-term relations (b) in the algebra $\widehat{\mathrm{ACYB}}_7$ to the result obtained,and and taking the specialization $x_{i,7}=1$, $\forall\, i$, we will come to the following expression

$$(9,20,14,3)_\beta(2+\beta)+(9,15,6)_\beta(3+2\beta)^2+(4,4)_\beta(4+3\beta)+(5+4\beta)$$
$$=(66,144,108,32,3)_\beta.$$

One can check that the latter polynomial is equal to $\mathfrak{G}_w^{\beta}(1)$.

Corollary 5.7 (monomials and Fuss–Catalan numbers $\mathrm{FC}_n^{(p+1)}$)**.** *Let p, n, b be integers, consider diagram $\lambda=(n^b,(n-1)^p,(n-2)^p,\dots,2^p,1^p)$ and dominant permutation $w\in\mathbb{S}_{(n-1)p+b+2}$ of shape λ. Let us define monomial*

$$M_{n,p,b}=x_{12}\prod_{j=0}^{n-2}\left(\prod_{a=3}^{p+2}x_{jp+2,jp+a}\right)\prod_{a=3}^{b+2}x_{(n-1)p+2,(n-1)p+a}.$$

Then

$$P_{M_{n,p,b}}(x_{ij}=1,\forall\, i,j)(\beta)=\sum_{k=1}^{n}\frac{1}{k}\binom{n-1}{k-1}\binom{pn-\overline{b}}{k-1}(\beta+1)^{k-1}.$$

Moreover,

$$P_{M_{n,p,b}}(x_{ij}=1,\forall\, i,j)(\beta=0)=\frac{1}{np-\overline{b}+1}\binom{n(p+1)-\overline{b}}{n}=\frac{1}{n}\binom{n(p+1)-\overline{b}}{n-1},$$

where $\overline{b}:=b-\frac{1-(-1)^b}{2}$.

For $b=0$ the right hand side of the above equality is equal to the Fuss–Narayana polynomial, see Theorem 5.46 and Proposition 5.47; a combinatorial interpretation of the value $P_{M_{n,p,b}}(x_{ij}=1,\forall\, i,j)(\beta=1)$ one can find in [110]. Note that reduced expressions for monomial $M_{n,p,b}$ in the (noncommutative) algebra $\widehat{\mathrm{ACYB}}_n(\beta)$ up to applying the commutativity rules (a), Definition 5.1, is unique.

It seems an interesting *problem* to construct a natural bijection between the set of monomials in the (noncommutative) reduced expression associated with monomials $M_{n,p,0}$ and the set of $(p+1)$-gulations[52] Finally we remark that there are certain connections of the β-Grothendieck polynomials corresponding to shifted dominance permutations (i.e., permutations of the form $1^k\times w_\lambda$) and some generating functions for the set of bounded by k plane partitions of shape λ, see, e.g., [44]. In the case of a staircase shape partition $\lambda=(n-1,\dots,1)$ one can envision (cf. [128, 135]) a connection/bijection between the set of k-bounded plane partitions of that shape and k-dissections of a convex $(n+k+1)$-gon. However in the case $k\geq 2$ it is not clear does there exist a monomial M in the algebra $\widehat{\mathrm{ACYB}}_n$ such that the value of the corresponding reduced polynomial at $x_{ij}=1$, $\forall\, i,j$ is equal to the number of k-dissections $(k\geq 2)$ of a convex $(n+k+1)$-gon.

Exercises 5.8.

(1)

(a) Let as before,

$$\pi=\begin{pmatrix}1&2&3&\dots&n\\1&n&n-1&\dots&2\end{pmatrix}.$$

[52] That is the set of dissections of a convex pk-gon by (maximal) collection of non-crossing diagonals such that the all regions obtained are a convex $(p+2)$-gons of a convex kp-gon.

Show that

$$\mathfrak{S}_\pi(x_1=q,\, x_j=1,\, \forall\, j\neq i)=\sum_{a=0}^{n-2}\frac{n-a-1}{n-1}\binom{n+a-2}{a}q^a.$$

Note that the number

$$\frac{n-k+1}{n+1}\binom{n+k}{k}$$

is equal to the dimension of irreducible representation of the symmetric group $\mathbb{S}_{n+k}$ that corresponds to partition $(n+k,k)$.

(*b*) Big Schröder numbers, paths and polynomials $\mathfrak{G}^{(\beta)}_{1\times w_0^{(n-1)}}(x_1=q,\, x_i=1,\, \forall\, i\geq 2)$. Let $n\geq 1$ and $k\geq 0$ be integers, denote by $S_{k,n}$ the number of *big Schröder paths of type* (k,n), that is lattice paths from the point $(0,0)$ and ending at point $(2n+k,k)$, using only the steps $U=(1,1)$, $H=(2,0)$ and $D=(1,-1)$ and never going below the line $x=0$. The numbers $S(n):=S_{0,n}$ commonly known as *big Schröder numbers*, see, e.g., [131, *A*001003]. It is well-known that

$$S_{k,n}=\frac{k+1}{n}\sum_{a=0}^{n}\binom{n}{a}\binom{n+k+a}{n-1}.$$

Show that

$$\mathfrak{G}^{(\beta)}_{1\times w_0^{(n-1)}}(x_1=q,\, x_i=1,\, \forall\, i\geq 2)=\sum_{k=0}^{n-2}S_{k,n-2-k}(\beta)q^{n-k-2},$$

where $S_{k,n}(\beta)$ is the generating functions of the big Schröder paths of type (k,n) according to the number of horizontal steps H.

(*c*) *Show* that the polynomial $\mathfrak{G}^{(\beta)}_{1\times w_0^{(n-1-1)}}(x_1=q,\, x_i=1,\, \forall\, i\geq 2)$ belongs to the ring $\mathbb{N}[q,\beta+1]$. For example, for $n=8$ one has

$$\begin{aligned}\mathfrak{G}^{(\beta)}_{1\times w_0^{(7)}}(x_1=q,\, x_i=1,\, \forall\, i\geq 2)&=(0,1,15,50,50,15,1)_{\beta+1}t^6+(0,2,24,60,40,6)_{\beta+1}t^5\\&+(0,3,27,45,15)_{\beta+1}t^4+(0,4,24,20)_{\beta+1}t^3+(0,5,15)_{\beta+1}t^2+6(\beta+1)t+1.\end{aligned}$$

Show that

$$S_{k,n}(\beta)=\frac{k+1}{n}\sum_{a=0}^{n}\binom{n}{a}\binom{n+k+a}{n-1}\beta^{n-a}=\frac{k+1}{k+1+n}\binom{2n+k}{n}+\cdots+\binom{n+k}{n}\beta^n.$$

(*d*) Write

$$\mathfrak{G}^{(\beta)}_{1^k\times w_0^{(n-k)}}(x_1=q,\, x_i=1,\, \forall\, i\geq 2)=A_{k,n}(\beta)q^{n-k-1}+\cdots+B_{n,k}(\beta).$$

Show that

$$A_{k,n}=(1+\beta)^k\mathfrak{G}^{(\beta)}_{k,n-1}(x_i=1,\, \forall\, i\geq 1),\qquad B_{k,n}=\mathfrak{G}^{(\beta)}_{k-1,n-1}(x_i=1,\, \forall\, i\geq 1).$$

(2) Consider the commutative quotient $\widetilde{\mathrm{ACYB}}^{ab}_n(\alpha,\beta)$ of the algebra $\widetilde{\mathrm{ACYB}}_n(\alpha,\beta)$, i.e., assume that the all generators $\{x_{ij}\,|\, 1\leq i<j\leq n$ are mutually commute. Denote by $\overline{P}_n(x_{ij};\alpha,\beta)$

the image of polynomial the $P_n(x_{ij};\alpha,\beta) \in \widetilde{\mathrm{ACYB}}_n(\alpha,\beta)$ in the algebra $\widetilde{\mathrm{ACYB}}_n^{ab}(\alpha,\beta)$. Finally, define polynomials $P_n(t,\alpha,\beta)$ to be the specialization

$$x_{ij} \longrightarrow 1 \quad \text{if} \quad j < n, \qquad x_{in} \longrightarrow t \quad \text{if} \quad 1 \le i < n.$$

Show that

(a) Polynomial $P_n(t,\alpha,\beta)$ does not depend on on order in which relations (a) and (b), see Definition 5.1, have been applied.

(b)

$$P_n(1,\alpha=1,\beta=0) = \sum_{k\ge 0} \frac{(2n-2k)!}{k!(n+1-k)!(n-2k)!},$$

see [131, $A052709(n)$] for combinatorial interpretations of these numbers.

For example,

$$\begin{aligned} P_7(t,\alpha,\beta) = {} & t^7 + 6(1+\beta)t^6 + [(0,5,15)_{\beta+1} + 6\alpha]t^5 + [(0,4,24,20)_{\beta+1} + \alpha(5,29)_{\beta+1}]t^4 \\ & + [(0,3,27,45,15)_{\beta+1} + \alpha(4,45,55)_{\beta+1} + 14\alpha^2]t^3 \\ & + [(0,2,24,60,40,6)_{\beta+1} + \alpha(3,48,115,50)_{\beta+1} + \alpha^2(21,49)_{\beta+1}]t^2 \\ & + [(0,1,15,50,50,15,1)_{\beta+1} + \alpha(2,38,130,110,20)_{\beta+1} + \alpha^2(21,91,56)_{\beta+1} \\ & + 14\alpha^3]t + \alpha(1,15,50,50,15,1)_{\beta+1} + \alpha^2(14,70,70,14)_{\beta+1} + \alpha^3(21,21)_{\beta+1}. \end{aligned}$$

(c) *Show* that in fact

$$P_n(1,\alpha,0) = \sum_{k\ge 0} \frac{1}{n+1}\binom{2n-2k}{n}\binom{n+1}{k}\alpha^k = \sum_{k\ge 0} \frac{T_{n+2}(n-k,k+1)}{2n-1-2k}\alpha^k,$$

see Proposition 5.4(2), for definition of numbers $T_n(k,r)$. As for a combinatorial interpretation of the polynomials $P_n(1,\alpha,0)$, see [131, $A117434$, $A085880$].

(3) Consider polynomials $P_n(t,\beta)$ as it has been defined in Proposition 5.4(2). *Show* that

$$P_n(t,\beta) = P_n(t,\alpha=0,\beta) = t^n + \sum_{r=1}^{n-1} t^r \left(\sum_{k=0}^{n-1-r} \frac{r}{n}\binom{n}{k+r}\binom{n-r-1}{k}(1+\beta)^{n-r-k} \right),$$

cf., e.g., [131, $A033877$].

A few comments in order. Several combinatorial interpretations of the integer numbers

$$U_n(r,k) := \frac{r}{n+1}\binom{n+1}{k+r}\binom{n-r}{k}$$

are well-known. For example, if $r=1$, the numbers $U_n(1,k) = \frac{1}{n}\binom{n}{k+1}\binom{n}{k}$ are equal to the Narayana numbers, see, e.g., [131, $A001263$]; if $r=2$, the number $U_n(2,k)$ counts the number of Dyck $(n+1)$-paths whose last descent has length 2 and which contain $n-k$ peaks, see [131, $A108838$] for details.

Finally, it's easily seen, that $P_n(1,\beta) = A127529(n)$, and $P_n(t,1) = A033184(n)$, see [131].

(4) *Show* that

$$P_n(t,\alpha,\beta) \in \mathbb{N}[t,\alpha][\beta+1],$$

that is the polynomial $P_n(t,\alpha,\beta)$ is a polynomial of $\beta+1$ with coefficients from the ring $\mathbb{N}[t,\alpha]$.

Show that

$$P_n(0,1,\beta) \in \mathbb{N}[\beta+2].$$

For example,

$$P_7(0,1,\beta) = (0,3,8,14,10,1)_{\beta+2}, \qquad P_8(0,1,\beta) = (1,3,11,25,35,15,1)_{\beta+2}.$$

Show that [131]

$$P_n(1,1,0) = A052709(n+1),$$

that is the number of underdiagonal lattice paths from $(0,0)$ to $(n-1,n-1)$ and such that each step is either $(1,0)$, $(0,1)$, or $(2,1)$. For example, $P_7(1,1,0) = 1697 = A052709(8)$. Cf. with the next exercise.

Show that [131]

$$P_n(0,1,0) = A052705(n),$$

namely, the number of underdiagonal paths from (0,0) to the line $x = n-2$, using only steps $(1,0)$, $(0,1)$ and $NE = (2,1)$. For example,

$$P_7(0,1,0) = 36 + 106 + 120 + 64 + 15 + 1 = 342 = A052705(7).$$

Show that [131]

$$\frac{\partial}{\partial a} P_n(a, \boldsymbol{b} = \boldsymbol{1}, \boldsymbol{\beta} = \boldsymbol{0}, \boldsymbol{\alpha} = \boldsymbol{1}, \boldsymbol{y} = \boldsymbol{z} = \boldsymbol{1}) = A005775,$$

that is the number number of paths in the half-plane $x \geq 0$ from $(0,0)$ to $(n-1,2)$ or $(n-1,-3)$, and consisting of steps $U = (1,1)$, $D = (1,-1)$ and $H = (1,0)$ that contain at least one UUU but avoid $UUU's$ starting above level 0.

5.1.1 Multiparameter deformation of Catalan, Narayana and Schröder numbers

Let $\mathfrak{b} = (\beta_1, \ldots, \beta_{n-1})$ be a set of mutually commuting parameters. We define a multiparameter analogue of the associative quasi-classical Yang–Baxter algebra $\widehat{\mathrm{MACYB}}_n(\mathfrak{b})$ as follows.

Definition 5.9 (cf. Definition 2.20)**.** The multiparameter associative quasi- classical Yang–Baxter algebra of weight $\mathfrak{b}$, denoted by $\widehat{\mathrm{MACYB}}_n(\mathfrak{b})$, is an associative algebra, over the ring of polynomials $\mathbb{Z}[\beta_1, \ldots, \beta_{n-1}]$, generated by the set of elements $\{x_{ij},\, 1 \leq i < j \leq n\}$, subject to the set of relations

(a) $x_{ij}x_{kl} = x_{kl}x_{ij}$ if $\{i,j\} \cap \{k,l\} = \varnothing$,

(b) $x_{ij}x_{jk} = x_{ik}x_{ij} + x_{jk}x_{ik} + \beta_i x_{ik}$ if $1 \leq 1 < i < j \leq n$.

Consider the "Coxeter element" $w_n \in \widehat{\mathrm{MACYB}}_n(\mathfrak{b})$ which is equal to the ordered product of "simple generators":

$$w_n := \prod_{a=1}^{n-1} x_{a,a+1}.$$

Now we can use the same method as in [133, Exercise 8.C5(c)], see Section 5.1, to define the *reduced form* of the Coxeter element w_n. Namely, let us bring the element w_n to the reduced form in the algebra $\widehat{\mathrm{MACYB}}_n(\mathfrak{b})$, that is, let us consecutively apply the defining relations (a) and (b) to the element w_n in any order until unable to do so. Denote the resulting (noncommutative) polynomial by $P(x_{ij};\mathfrak{b})$. In principal, the polynomial itself can depend on the order in which the relations (a) and (b) are applied.

Proposition 5.10 (cf. [133, Exercise 8.C5(c)], [99, 100])**.** *The specialized polynomial $P(x_{ij}=1, \forall\, i,j,\, \mathfrak{b})$ does not depend on the order in which relations* (*a*) *and* (*b*) *have been applied.*

To state our main result of this subsection, let us define polynomials

$$Q(\beta_1,\dots,\beta_{n-1}) := P(x_{ij}=1,\, \forall\, i,j;\ \beta_1-1,\beta_2-1,\dots,\beta_{n-1}-1).$$

Example 5.11.

$$\begin{aligned} Q(\beta_1,\beta_2) &= 1+2\beta_1+\beta_2+\beta_1^2,\\ Q(\beta_1,\beta_2,\beta_3) &= 1+3\beta_1+2\beta_2+\beta_3+3\beta_1^2+\beta_1\beta_2+\beta_1\beta_3+\beta_2^2+\beta_1^3,\\ Q(\beta_1,\beta_2,\beta_3,\beta_4) &= 1+4\beta_1+3\beta_2+2\beta_3+\beta_4+\beta_1(6\beta_1+3\beta_2+3\beta_3+2\beta_4)\\ &\quad +\beta_2(3\beta_2+\beta_3+\beta_4)+\beta_3^2+\beta_1^2(4\beta_1+\beta_2+\beta_3+\beta_4)\\ &\quad +\beta_1(\beta_2^2+\beta_3^2)+\beta_2^3+\beta_1^4. \end{aligned}$$

Theorem 5.12. *Polynomial $Q(\beta_1,\dots,\beta_{n-1})$ has non-negative integer coefficients.*

It follows from [133] and Proposition 5.4, that

$$Q(\beta_1,\dots,\beta_{n-1})\big|_{\beta_1=1,\dots,\beta_{n-1}=1} = \mathrm{Cat}_n.$$

Polynomials $Q(\beta_1,\dots,\beta_{n-1})$ and $Q(\beta_1+1,\dots,\beta_{n-1}+1)$ can be considered as a multiparameter deformation of the Catalan and (small) Schröder numbers correspondingly, and the homogeneous degree k part of $Q(\beta_1,\dots,\beta_{n-1})$ as a multiparameter analogue of Narayana numbers.

5.2 Grothendieck and q-Schröder polynomials

5.2.1 Schröder paths and polynomials

Definition 5.13. A Schröder path of the length n is an over diagonal path from $(0,0)$ to (n,n) with steps $(1,0)$, $(0,1)$ and steps $D=(1,1)$ *without* steps of type D on the diagonal $x=y$.

If p is a Schröder path, we denote by $d(p)$ the number of the diagonal steps resting on the path p, and by $a(p)$ the number of unit squares located between the path p and the diagonal $x=y$. For each (unit) diagonal step D of a path p we denote by $i(D)$ the x-coordinate of the column which contains the diagonal step D. Finally, define the index $i(p)$ of a path p as the some of the numbers $i(D)$ for all diagonal steps of the path p.

Definition 5.14. Define q-Schröder polynomial $S_n(q;\beta)$ as follows

$$S_n(q;\beta) = \sum_p q^{a(p)+i(p)}\beta^{d(p)}, \tag{5.4}$$

where the sum runs over the set of all Schröder paths of length n.

Example 5.15.

$$\begin{aligned} &S_1(q;\beta)=1, \qquad S_2(q;\beta)=1+q+\beta q,\\ &S_3(q;\beta)=1+2q+q^2+q^3+\beta\big(q+2q^2+2q^3\big)+\beta^2q^3,\\ &S_4(q;\beta)=1+3q+3q^2+3q^3+2q^4+q^5+q^6+\beta\big(q+3q^2+5q^3+6q^4+3q^5+3q^6\big)\\ &\qquad\qquad +\beta^2\big(q^3+2q^4+3q^5+3q^6\big)+\beta^3q^6. \end{aligned}$$

Comments 5.16. The q-Schröder polynomials defined by the formula (5.4) are *different* from the q-analogue of Schröder polynomials which has been considered in [19]. It seems that there are no simple connections between the both.

Proposition 5.17 (recurrence relations for q-Schröder polynomials)**.** *The q-Schröder polynomials satisfy the following relations*

$$S_{n+1}(q;\beta) = \big(1+q^n+\beta q^n\big)S_n(q;\beta) + \sum_{k=1}^{k=n-1} \big(q^k+\beta q^{n-k}\big)S_k(q;q^{n-k}\beta)S_{n-k}(q;\beta),$$

and the initial condition $S_1(q;\beta)=1$.

Note that $P_n(\beta)=S_n(1;\beta)$ and in particular, the polynomials $P_n(\beta)$ satisfy the following recurrence relations

$$P_{n+1}(\beta) = (2+\beta)\ P_n(\beta) + (1+\beta)\sum_{k=1}^{n-1} P_k(\beta)\ P_{n-k}(\beta). \tag{5.5}$$

Theorem 5.18 (evaluation of the Schröder–Hankel determinant)**.** *Consider permutation*

$$\pi_k^{(n)} = \begin{pmatrix} 1 & 2 & \dots & k & k+1 & k+2 & \dots & n \\ 1 & 2 & \dots & k & n & n-1 & \dots & k+1 \end{pmatrix}.$$

Let as before

$$P_n(\beta) = \sum_{j=0}^{n-1} N(n,j)(1+\beta)^j, \qquad n\geq 1, \tag{5.6}$$

be Schröder polynomials. Then

$$(1+\beta)^{\binom{k}{2}}\mathfrak{G}^{(\beta)}_{\pi_k^{(n)}}(x_1=1,\dots,x_{n-k}=1) = \operatorname{Det}|P_{n+k-i-j}(\beta)|_{1\leq i,j\leq k}.$$

Proof is based on an observation that the permutation $\pi_k^{(n)}$ is a *vexillary* one and the recurrence relations (5.5).

Comments 5.19. **(1)** In the case $\beta=0$, i.e., in the case of *Schubert polynomials*, Theorem 5.18 has been proved in [44].

(2) In the cases when $\beta=1$ and $0\leq n-k\leq 2$, the value of the determinant in the r.h.s. of (5.6) is known[53]. One can check that in the all cases mentioned above, the formula (5.6) gives the same results.

(3) *Grothendieck and Narayana polynomials.* It follows from the expression (5.5) for the Narayana–Schröder polynomials that $P_n(\beta-1)=\mathfrak{N}_n(\beta)$, where

$$\mathfrak{N}_n(\beta) := \sum_{j=0}^{n-1}\frac{1}{n}\binom{n}{j}\binom{n}{j+1}\beta^j,$$

denotes the n-th Narayana polynomial. Therefore, $P_n(\beta-1)=\mathfrak{N}_n(\beta)$ is a symmetric polynomial in β with non-negative integer coefficients. Moreover, the value of the polynomial $P_n(\beta-1)$ at $\beta=1$ is equal to the n-th Catalan number $C_n := \frac{1}{n+1}\binom{2n}{n}$.

[53] See, e.g., [19], or M. Ichikawa talk "Hankel determinants of Catalan, Motzkin and Schröder numbers and its q-analogue", http://www.uec.tottori-u.ac.jp/~mi/talks/kyoto07.pdf.

It is well-known, see, e.g., [136], that the Narayana polynomial $\mathfrak{N}_n(\beta)$ is equal to the generating function of the statistics $\pi(\mathfrak{p})$ = (number of peaks of a Dick path $\mathfrak{p}$) -1 on the set Dick_n of Dick paths of the length $2n$

$$\mathfrak{N}_n(\beta) = \sum_{\mathfrak{p}} \beta^{\pi(\mathfrak{p})}.$$

Moreover, using the Lindström–Gessel–Viennot lemma[54], one can see that

$$\mathrm{DET}\,|\mathfrak{N}_{n+k-i-j}(\beta)|_{1\le i,j\le k} = \beta^{\binom{k}{2}} \sum_{(\mathfrak{p}_1,\ldots,\mathfrak{p}_k)} \beta^{\pi(\mathfrak{p}_1)+\cdots+\pi(\mathfrak{p}_k)}, \tag{5.7}$$

where the sum runs over k-tuple of non-crossing Dick paths $(\mathfrak{p}_1,\ldots,\mathfrak{p}_k)$ such that the path $\mathfrak{p}_i$ starts from the point $(i-1,0)$ and has length $2(n-i+1)$, $i=1,\ldots,k$.

We *denote* the sum in the r.h.s. of (5.7) by $\mathfrak{N}_n^{(k)}(\beta)$. Note that $\mathfrak{N}_{k-1}^{(k)}(\beta)=1$ for all $k\ge 2$.

Thus, $\mathfrak{N}_n^{(k)}(\beta)$ is a symmetric polynomial in β with non-negative integer coefficients, and

$$\mathfrak{N}_n^{(k)}(\beta=1) = C_n^{(k)} = \prod_{1\le i\le j\le n-k} \frac{2k+i+j}{i+j} = \prod_{2a\le n-k-1} \frac{\binom{2n-2a}{2k}}{\binom{2k+2a+1}{2k}}.$$

As a corollary we obtain the following statement

Proposition 5.20. *Let $n\ge k$, then*

$$\mathfrak{G}_{\pi_k^{(n)}}^{(\beta-1)}(x_1=1,\ldots,x_n=1) = \mathfrak{N}_n^{(k)}(\beta).$$

Summarizing, the specialization $\mathfrak{G}_{\pi_k^{(n)}}^{(\beta-1)}(x_1=1,\ldots,x_n=1)$ is a symmetric polynomial in β with non-negative integer coefficients, and coincides with the generating function of the statistics $\sum_{i=1}^{k}\pi(\mathfrak{p}_i)$ on the set k-Dick_n of k-tuple of non-crossing Dick paths $(\mathfrak{p}_1,\ldots,\mathfrak{p}_k)$.

Example 5.21. Take $n=5$, $k=1$. Then $\pi_1^{(5)}=(15432)$ and one has

$$\mathfrak{G}_{\pi_1^{(5)}}^{(\beta)}\left(1,q,q^2,q^3\right) = q^4(1,3,3,3,2,1,1) + q^5(1,3,5,6,3,3)\beta + q^7(1,2,3,3)\beta^2 + q^{10}\beta^3.$$

It is easy to compute the Carlitz–Riordan q-analogue of the Catalan number C_5, namely,

$$C_5(q) = (1,3,3,3,2,1,1).$$

Remark 5.22. The value $\mathfrak{N}_n(4)$ of the Narayana polynomial at $\beta=4$ has the following combinatorial interpretation: $\mathfrak{N}_n(4)$ is equal to the number of different lattice paths from the point $(0,0)$ to that $(n,0)$ using steps from the set $\Sigma=\{(k,k)\,\mathrm{or}\,(k,-k),\ k\in\mathbb{Z}_{>0}\}$, that never go below the x-axis, see [131, *A*059231].

Exercises 5.23.

(a) Show that

$$\gamma_{k,n} := \frac{C_n^{(k+1)}}{C_n^{(k)}} = \frac{(2n-2k)!(2k+1)!}{(n-k)!(n+k+1)!}.$$

[54] See, e.g., https://en.wikipedia.org/wiki/Lindstrom-Gessel-Viennot_lemma.

(*b*) Show that $\gamma_{k,n} \le 1$ if $k \le n \le 3k+1$, and $\gamma_{k,n} \ge 2^{n-3k-1}$ if $n > 3k+1$.

(4) *Polynomials $\mathfrak{F}_w(\beta)$, $\mathfrak{H}_w(\beta)$, $\mathfrak{H}_w(q,t;\beta)$ and $\mathfrak{R}_w(q;\beta)$.* Let $w \in \mathbb{S}_n$ be a permutation, $\mathfrak{G}_w^{(\beta)}(X_n)$ and $\mathfrak{G}_w^{(\beta)}(X_n,Y_n)$ be the corresponding β-Grothendieck and double β-Grothendieck polynomials. We denote by $\mathfrak{G}_w^{(\beta)}(1)$ and by $\mathfrak{G}_w^{(\beta)}(1;1)$ the specializations $X_n := (x_1 = 1, \dots, x_n = 1)$, $Y_n := (y_1 = 1, \dots, y_n = 1)$ of the β-Grothendieck polynomials introduced above.

Theorem 5.24. *Let $w \in \mathbb{S}_n$ be a permutation. Then*

(*i*) *The polynomials $\mathfrak{F}_w(\beta) := \mathfrak{G}_w^{(\beta-1)}(1)$ and $\mathfrak{H}_w(\beta) := \mathfrak{G}_w^{(\beta-1)}(1;1)$ have both non-negative integer coefficients.*

(*ii*) *One has*

$$\mathfrak{H}_w(\beta) = (1+\beta)^{\ell(w)} \mathfrak{F}_w\big(\beta^2\big).$$

(*iii*) *Let $w \in \mathbb{S}_n$ be a permutation, define polynomials*

$$\mathfrak{H}_w(q,t;\beta) := \mathfrak{G}_w^{(\beta)}(x_1 = q, x_2 = q, \dots, x_n = q, y_1 = t, y_2 = t, \dots, y_n = t)$$

to be the specialization $\{x_i = q,\ y_i = t,\ \forall\, i\}$ of the double β-Grothendieck polynomial $\mathfrak{G}_w^{(\beta)}(X_n,Y_n)$. Then

$$\mathfrak{H}_w(q,t;\beta) = (q+t+\beta qt)^{\ell(w)} \mathfrak{F}_w((1+\beta q)(1+\beta t)).$$

In particular, $\mathfrak{H}_w(1,1;\beta) = (2+\beta)^{\ell(w)} \mathfrak{F}_w((1+\beta)^2)$.

(*iv*) *Let $w \in \mathbb{S}_n$ be a permutation, define polynomial*

$$\mathcal{R}_w(q;\beta) := \mathfrak{G}_w^{(\beta-1)}(x_1 = q, x_2 = 1, x_3 = 1, \dots)$$

to be the specialization $\{x_1 = q,\ x_i = 1,\ \forall\, i \ge 2\}$, of the $(\beta-1)$-Grothendieck polynomial $\mathfrak{G}_w^{(\beta-1)}(X_n)$. Then

$$\mathcal{R}_w(q;\beta) = q^{w(1)-1} \mathfrak{R}_w(q;\beta),$$

where $\mathfrak{R}_w(q;\beta)$ is a polynomial in q and β with non-negative integer coefficients, and $\mathfrak{R}_w(0;\beta=0) = 1$.

(*v*) *Consider permutation $w_n^{(1)} := [1, n, n-1, n-2, \dots, 3, 2] \in \mathbb{S}_n$. Then $\mathfrak{H}_{w_n^{(1)}}(1,1;1) = 3^{\binom{n-1}{2}} \mathfrak{N}_n(4)$.*

In particular, if $w_n^{(k)} = (1, 2, \dots, k, n, n-1, \dots, k+1) \in \mathbb{S}_n$, then

$$\mathfrak{G}_{w_n^{(k)}}^{(\beta-1)}(1;1) = (1+\beta)^{\binom{n-k}{2}} \mathfrak{G}_{w_n^{(k)}}^{(\beta-1)}\big(\beta^2\big).$$

See Remark 5.22 for a combinatorial interpretation of the number $\mathfrak{N}_n(4)$.

Example 5.25. Consider permutation $v = [2,3,5,6,8,9,1,4,7] \in \mathbb{S}_9$ of the length 12, and set $x := (1+\beta q)(1+\beta t)$. One can check that

$$\mathfrak{H}_v(q,t;\beta) = x^{12}(1+2x)\big(1+6x+19x^2+24x^3+13x^4\big),$$

and $\mathfrak{F}_v(\beta) = (1+2\beta)(1+6\beta+19\beta^2+24\beta^3+13\beta^4)$.

Note that $\mathfrak{F}_v(\beta = 1) = 27 \times 7$, and $7 = \mathrm{AMS}(3)$, $26 = \mathrm{CSTCTPP}(3)$, cf. Conjecture 5.52, Section 5.2.4.

Remark 5.26. One can show, cf. [92, p. 89], that if $w \in \mathbb{S}_n$, then $\mathcal{R}_w(1,\beta) = \mathcal{R}_{w^{-1}}(1,\beta)$. However, the equality $\mathfrak{R}_w(q,\beta) = \mathfrak{R}_{w^{-1}}(q,\beta)$ can be violated, and it seems that in general, there are no simple connections between polynomials $\mathfrak{R}_w(q,\beta)$ and $\mathfrak{R}_{w^{-1}}(q,\beta)$, if so.

From this point we shell use the notation $(a_0,a_1,\dots,a_r)_\beta := \sum_{j=0}^{r} a_j\beta^j$, etc.

Example 5.27. Let us take $w = [1,3,4,6,7,9,10,2,5,8]$. Then

$$\begin{aligned}\mathfrak{R}_w(q,\beta) = {}&(1,6,21,36,51,48,26)_\beta + q\beta(6,36,126,216,306,288,156)_\beta \\ &+ q^2\beta^3(20,125,242,403,460,289)_\beta + q^3\beta^5(6,46,114,204,170)_\beta.\end{aligned}$$

Moreover, $\mathfrak{R}_w(q,1) = (189,1134,1539,540)_q$. On the other hand, $w^{-1} = [1,8,2,3,9,4,5,10,6,7]$, and

$$\begin{aligned}\mathfrak{R}_{w^{-1}}(q,\beta) = {}&(1,6,21,36,51,48,26)_\beta + q\beta(1,6,31,56,96,110,78)_\beta \\ &+ q^2\beta(1,6,27,58,92,122,120,78)_\beta + q^3\beta(1,6,24,58,92,126,132,102,26)_\beta \\ &+ q^4\beta(1,6,22,57,92,127,134,105,44)_\beta \\ &+ q^5\beta(1,6,21,56,91,126,133,104,50)_\beta \\ &+ q^6\beta(1,6,21,56,91,126,133,104,50)_\beta.\end{aligned}$$

Moreover, $\mathfrak{R}_{w^{-1}}(q,1) = (189,378,504,567,588,588,588)_q$.

Notice that $w = 1 \times u$, where $u = [2,3,5,6,8,9,1,4,7]$. One can show that

$$\mathfrak{R}_u(q,\beta) = (1,6,11,16,11)_\beta + q\beta^2(10,20,35,34)_\beta + q^2\beta^4(5,14,26)_\beta.$$

On the other hand, $u^{-1} = [7,1,2,8,3,4,9,5,6]$ and

$$\mathfrak{R}_{u^{-1}}(1,\beta) = (1,6,21,36,51,48,26)_\beta = \mathfrak{R}_u(1,\beta).$$

Recall that by our definition $(a_0,a_1,\dots,a_r)_\beta := \sum_{j=0}^{r} a_j\beta^j$.

5.2.2 Grothendieck polynomials and k-dissections

Let $k \in \mathbb{N}$ and $n \geq k-1$, be a integer, define *a k-dissection* of a convex $(n+k+1)$-gon to be a collection $\mathcal{E}$ of diagonals in $(n+k+1)$-gon not containing $(k+1)$-subset of pairwise crossing diagonals and such that at least $2(k-1)$ diagonals are coming from each vertex of the $(n+k+1)$-gon in question. One can show that the number of diagonals in any k-dissection $\mathcal{E}$ of a convex $(n+k+1)$-gon contains at least $(n+k+1)(k-1)$ and at most $n(2k-1)-1$ diagonals. We define the *index* of a k-dissection $\mathcal{E}$ to be $i(\mathcal{E}) = n(2k-1)-1-\#|\mathcal{E}|$. Denote by

$$\mathcal{T}_n^{(k)}(\beta) = \sum_{\mathcal{E}} \beta^{i(\mathcal{E})}$$

the generating function for the number of k-dissections with a fixed index, where the above sum runs over the set of all k-dissections of a convex $(n+k+1)$-gon.

Theorem 5.28.

$$\mathfrak{G}^{(\beta)}_{\pi_k^{(n)}}(x_1=1,\dots,x_n=1) = \mathcal{T}_n^{(k)}(\beta).$$

Mopre generally, let $n \geq k > 0$ be integers, consider a convex $(n+k+1)$-gon P_{n+k+1} and a vertex $v_0 \in P_{n+k+1}$. Let us label clockwise the vertices of P_{n+k+1} by the numbers $1, 2, \dots, n+k+1$ starting from the vertex v_0. Let $\mathrm{Dis}(P_{n+k+1})$ denotes the set of all k-dissections of the $(n+k+1)$-gon P_{n+k+1}. We denote by $D_0 := \mathrm{Dis}_0(P_{n+k+1})$ the "minimal" k-dissection of the $(n+k+1)$-gon P_{n+k+1} in question consisting of the set of diagonals connecting vertices v_a and $v_{\overline{a+r}}$, where $2 \leq r \leq k$, $1 \leq a \leq n+k+1$, and for any positive integer a we denote by $\overline{a}$ a unique integer such that $1 \leq \overline{a} \leq n+k+1$ and $a \equiv \overline{a} (\operatorname{mod} (n+k+1))$. For example, if $k=1$, then $\mathrm{Dis}_0(P_{n+2}) = \varnothing$; if $k=3$ and $n=4$, in other words, P_8 is a octagon, the minimal 3-dissection consists of 16 diagonals connecting vertices with the following labels

$$\begin{aligned}
&1 \to 3 \to 5 \to 7 \to \overline{9} = 1, \qquad 2 \to 4 \to 6 \to 8 \to \overline{10} = 2, \\
&1 \to 4 \to 7 \to \overline{10} = 2 \to 5 \to 8 \to \overline{11} = 3 \to 6 \to \overline{9} = 1.
\end{aligned}$$

Now let $D \in \mathrm{Dis}(P_{n+k+1})$ be a dissection. Consider a diagonal $d_{ij} \in (D \backslash D_0)$, $i < j$ which connects vertex v_i with that v_j. We attach variable x_i to the diagonal d_{ij} in question and consider the following expression

$$\mathcal{T}_{P_{n+k+1}}(X_{n+k+1}) = \sum_{D \in \mathrm{Dis}(P_{n+k+1})} \beta^{\#|D \backslash D_0|} \sum_{\substack{d_{ij} \in (D \backslash D_0) \\ i<j}} \prod x_i.$$

Theorem 5.29. *One has*

$$\mathcal{T}_{P_{n+n+1}}(X_{n+k+1}) = \beta^{k(n-k)} \prod_{a=1}^{n} x_a^{\min(n-a+1, n-k)} \mathfrak{G}_{w_k^n}^{\beta^{-1}}\left(x_1^{-1}, \dots, x_n^{-1}\right).$$

Exercises 5.30. It is not difficult to check that

$$\begin{aligned}
\mathfrak{G}_{15432}^{\beta}(X_5) = {} & \beta^3 x_1^3 x_2^3 x_3^2 x_4 + \beta^2 (x_1^3 x_2^3 x_3 + 2 x_1^3 x_2^3 x_3 x_4 + 3 x_1^3 x_2^2 x_3^2 x_4 + 3 x_1^2 x_2^3 x_3^2 x_4) \\
& + \beta (x_1^3 x_2^3 x_3 + x_1^3 x_2^3 x_4 + 2 x_1^3 x_2^2 x_3 + 2 x_1^2 x_2^3 x_3^2 + 3 x_1^3 x_2^2 x_3 x_4 + 3 x_1^3 x_2 x_3^2 x_4 \\
& + 3 x_1^2 x_2^3 x_3 x_4 + 3 x_1^2 x_2^2 x_3^2 x_4 + 3 x_1 x_2^3 x_3^2 x_4) + x_1^3 x_2^2 x_3 + x_1^3 x_2^2 x_4 + x_1^3 x_2 x_3^2 \\
& + x_1^3 x_2 x_3 x_4 + x_1^3 x_3^2 x_4 + x_1^2 x_2^3 x_3 + x_1^2 x_2^3 x_4 + x_1^2 x_2^2 x_3^2 + x_1^2 x_2^2 x_3 x_4 + x_1^2 x_2 x_3^2 x_4 \\
& + x_1 x_2^3 x_3^2 + x_1 x_2^3 x_3 x_4 + x_1 x_2^2 x_3^2 x_4 + x_2^3 x_3^2 x_4.
\end{aligned}$$

Describe bijection between dissections of hexagon P_6 (the case $k=1$, $n=4$) and the above listed monomials involved in the β-Grothendieck polynomial $\mathfrak{G}_{15432}^{\beta}(x_1, x_2, x_3, x_4)$.

A k-dissection of a convex $(n+k+1)$-gon with the maximal number of diagonals (which is equal to $n(2k-1)-1$) is called *k-triangulation*. It is well-known that the number of k-triangulations of a convex $(n+k+1)$-gon is equal to the Catalan–Hankel number $C_{n-1}^{(k)}$. Explicit bijection between the set of k-triangulations of a convex $(n+k+1)$-gon and the set of k-tuple of non-crossing Dick paths $(\gamma_1, \dots, \gamma_k)$ such that the Dick path γ_i connects points $(i-1, 0)$ and $(2n-i-1, 0)$, has been constructed in [128, 135].

5.2.3 Grothendieck polynomials and q-Schröder polynomials

Let $\pi_k^{(n)} = 1^k \times w_0^{(n-k)} \in \mathbb{S}_n$ be the vexillary permutation as before, see Theorem 5.18. Recall that

$$\pi_k^{(n)} = \begin{pmatrix} 1 & 2 & \dots & k & k+1 & k+2 & \dots & n \\ 1 & 2 & \dots & k & n & n-1 & \dots & k+1 \end{pmatrix}.$$

(**A**) **Principal specialization of the Schubert polynomial $\mathfrak{S}_{\pi_k^{(n)}}$.** Note that $\pi_k^{(n)}$ is a vexillary permutation of the staircase shape $\lambda = (n-k-1, \dots, 2, 1)$ and has the staircase flag $\phi = (k+1, k+2, \dots, n-1)$. It is known, see, e.g., [92, 139], that for a vexillary permutation $w \in \mathbb{S}_n$ of the shape λ and flag $\phi = (\phi_1, \dots, \phi_r)$, $r = \ell(\lambda)$, the corresponding Schubert polynomial $\mathfrak{S}_w(X_n)$ is equal to the multi-Schur polynomial $s_\lambda(X_\phi)$, where X_ϕ denotes the flagged set of variables, namely, $X_\phi = (X_{\phi_1}, \dots, X_{\phi_r})$ and $X_m = (x_1, \dots, x_m)$. Therefore we can write the following determinantal formula for the principal specialization of the Schubert polynomial corresponding to the vexillary permutation $\pi_k^{(n)}$

$$\mathfrak{S}_{\pi_k^{(n)}}(1, q, q^2, \dots) = \mathrm{DET}\left(\begin{bmatrix} n-i+j-1 \\ k+i-1 \end{bmatrix}_q \right)_{1 \le i,j \le n-k},$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_q$ denotes the q-binomial coefficient.

Let us observe that the Carlitz–Riordan q-analogue $C_n(q)$ of the Catalan number C_n is equal to the value of the q-Schröder polynomial at $\beta = 0$, namely, $C_n(q) = S_n(q, 0)$.

Lemma 5.31. *Let k, n be integers and $n > k$, then*

$$\text{(1)} \quad \mathrm{DET}\left(\begin{bmatrix} n-i+j-1 \\ k+i-1 \end{bmatrix}_q \right)_{1 \le i,j \le n-k} = q^{\binom{n-k}{3}} C_n^{(k)}(q),$$

$$\text{(2)} \quad \mathrm{DET}\left(C_{n+k-i-j}(q) \right)_{1 \le i,j \le k} = q^{k(k-1)(6n-2k-5)/6}\, C_n^{(k)}(q).$$

(**B**) **Principal specialization of the Grothendieck polynomial $\mathfrak{G}^{(\beta)}_{\pi_k^{(n)}}$.**

Theorem 5.32.

$$q^{\binom{n-k+1}{3} - (k-1)\binom{n-k}{2}} \,\mathrm{DET}\left| S_{n+k-i-j}(q; q^{i-1}\beta) \right|_{1 \le i,j \le k}$$
$$= q^{k(k-1)(4k+1)/6} \prod_{a=1}^{k-1} (1 + q^{a-1}\beta)\, \mathfrak{G}_{\pi_k^{(n)}}(1, q, q^2, \dots).$$

Corollary 5.33.

(1) *If $k = n-1$, then*

$$\mathrm{DET}\, |S_{2n-1-i-j}(q; q^{i-1}\beta)|_{1 \le i,j \le n-1} = q^{(n-1)(n-2)(4n-3)/6} \prod_{a=1}^{n-2} (1 + q^{a-1}\beta)^{n-a-1},$$

(2) *If $k = n-2$, then*

$$q^{n-2}\, \mathrm{DET} \left| S_{2n-2-i-j}(q; q^{i-1}\beta) \right|_{1 \le i,j \le n-2}$$
$$= q^{(n-2)(n-3)(4n-7)/6} \prod_{a=1}^{n-3} (1 + q^{a-1}\beta)^{n-a-2} \left\{ \frac{(1+\beta)^{n-1} - 1}{\beta} \right\}.$$

Generalization. Let $\boldsymbol{n} = (n_1, \dots, n_p) \in \mathbb{N}^p$ be a composition of n so that $n = n_1 + \dots + n_p$. We set $n^{(j)} = n_1 + \dots + n_j$, $j = 1, \dots, p$, $n^{(0)} = 0$.

Now consider the permutation $w^{(\boldsymbol{n})} = w_0^{(n_1)} \times w_0^{(n_2)} \times \dots \times w_0^{(n_p)} \in \mathbb{S}_n$, where $w_0^{(m)} \in \mathbb{S}_m$ denotes the longest permutation in the symmetric group $\mathbb{S}_m$. In other words,

$$w^{(\boldsymbol{n})} = \begin{pmatrix} 1 & 2 & \dots & n_1 & n^{(2)} & \dots & n_1+1 & \dots & n^{(p-1)} & \dots n \\ n_1 & n_1-1 & \dots & 1 & n_1+1 & \dots & n^{(2)} & \dots & n & \dots n^{(p-1)+1} \end{pmatrix}.$$

For the permutation $w^{(\boldsymbol{n})}$ defined above, one has the following factorization formula for the Grothendieck polynomial corresponding to $w^{(\boldsymbol{n})}$ [92]

$$\mathfrak{G}_{w^{(\boldsymbol{n})}}^{(\beta)} = \mathfrak{G}_{w_0^{(n_1)}}^{(\beta)} \times \mathfrak{G}_{1^{n_1} \times w_0^{(n_2)}}^{(\beta)} \times \mathfrak{G}_{1^{n_1+n_2} \times w_0^{(n_3)}}^{(\beta)} \times \cdots \times \mathfrak{G}_{1^{n_1+\dots n_{p-1}} \times w_0^{(n_p)}}^{(\beta)}.$$

In particular, if

$$w^{(\boldsymbol{n})} = w_0^{(n_1)} \times w_0^{(n_2)} \times \cdots \times w_0^{(n_p)} \in \mathbb{S}_n, \tag{5.8}$$

then the principal specialization $\mathfrak{G}_{w^{(\boldsymbol{n})}}^{(\beta)}$ of the Grothendieck polynomial corresponding to the permutation w, is the product of q-Schröder–Hankel polynomials. Finally, we observe that from discussions in Section 5.2.1(3), *Grothendieck and Narayana polynomials*, one can deduce that

$$\mathfrak{G}_{w^{(\boldsymbol{n})}}^{(\beta-1)}(x_1 = 1, \dots, x_n = 1) = \prod_{j=1}^{p-1} \mathfrak{N}_{n^{(j+1)}}^{(n^{(j)})}(\beta).$$

In particular, the polynomial $\mathfrak{G}_{w^{(\boldsymbol{n})}}^{(\beta-1)}(x_1, \dots, x_n)$ is a symmetric polynomial in β with non-negative integer coefficients.

Example 5.34.

(1) Let us take (non vexillary) permutation $w = 2143 = s_1 s_3$. One can check that

$$\mathfrak{G}_w^{(\beta)}(1,1,1,1) = 3 + 3\beta + \beta^2 = 1 + (\beta+1) + (\beta+1)^2,$$

and

$$\mathfrak{N}_4(\beta) = (1,6,6,1), \qquad \mathfrak{N}_3(\beta) = (1,3,1), \qquad \mathfrak{N}_2(\beta) = (1,1).$$

It is easy to see that

$$\beta \mathfrak{G}_w^{(\beta)}(1,1,1,1) = \mathrm{DET} \begin{vmatrix} \mathfrak{N}_4(\beta) & \mathfrak{N}_3(\beta) \\ \mathfrak{N}_3(\beta) & \mathfrak{N}_2(\beta) \end{vmatrix}.$$

On the other hand,

$$\mathrm{DET} \begin{vmatrix} P_4(\beta) & P_3(\beta) \\ P_3(\beta) & P_2(\beta) \end{vmatrix} = (3,6,4,1) = \big(3 + 3\beta + \beta^2\big)(1+\beta).$$

It is more involved to check that

$$q^5(1+\beta)\ \mathfrak{G}_w^{(\beta)}\big(1,q,q^2,q^3\big) = \mathrm{DET} \begin{vmatrix} S_4(q;\beta) & S_3(q;\beta) \\ S_3(q;q\beta) & S_2(q;q\beta) \end{vmatrix}.$$

(2) Let us illustrate Theorem 5.32 by a few examples. For the sake of simplicity, we consider the case $\beta = 0$, i.e., the case of *Schubert polynomials*. In this case $P_n(q;\beta=0) = C_n(q)$ is equal to the Carlitz–Riordan q-analogue of Catalan numbers. We are reminded that the q-Catalan–Hankel polynomials are defined as follows

$$C_n^{(k)}(q) = q^{k(1-k)(4k-1)/6}\,\mathrm{DET}\,|C_{n+k-i-j}(q)|_{1 \le i,j \le n}.$$

In the case $\beta = 0$ the Theorem 5.32 states that if $\boldsymbol{n} = (n_1, \dots, n_p) \in \mathbb{N}^p$ and the permutation $w_{(\boldsymbol{n})} \in \mathbb{S}_n$ is defined by the use of (5.7), then

$$\mathfrak{S}_{w^{(\boldsymbol{n})}}\big(1,q,q^2,\dots\big) = q^{\sum \binom{n_i}{3}} C_{n_1+n_2}^{(n_1)}(q) \times C_{n_1+n_2+n_3}^{(n_1+n_2)}(q) \times C_n^{(n-n_p)}(q).$$

Now let us consider a few examples for $n = 6$.

- $\boldsymbol{n} = (1,5) \Longrightarrow \mathfrak{S}_{w(\boldsymbol{n})}(1,q,\dots) = q^{10}C_6^{(1)}(q) = C_5(q)$.
- $\boldsymbol{n} = (2,4) \Longrightarrow \mathfrak{S}_{w(\boldsymbol{n})}(1,q,\dots) = q^4C_6^{(2)}(q) = \mathrm{DET}\begin{vmatrix} C_6(q) & C_5(q) \\ C_5(q) & C_4(q)\end{vmatrix}$.

Note that $\mathfrak{S}_{w^{(2,4)}}(1,q,\dots) = \mathfrak{S}_{w^{(1,1,4)}}(1,q,\dots)$.

- $\boldsymbol{n} = (2,2,2) \Longrightarrow \mathfrak{S}_{w(\boldsymbol{n})}(1,q,\dots) = C_4^{(2)}(q)C_6^{(4)}(q)$.
- $\boldsymbol{n} = (1,1,4) \Longrightarrow \mathfrak{S}_{w(\boldsymbol{n})}(1,q,\dots) = q^4C_2^{(1)}(q)C_4^{(2)}(q) = q^4C_4^{(2)}(q)$, the last equality follows from that $C_{k+1}^{(k)}(q) = 1$ for all $k \geq 1$.
- $\boldsymbol{n} = (1,2,3) \Longrightarrow \mathfrak{S}_{w(\boldsymbol{n})}(1,q,\dots) = qC_3^{(1)}(q)C_6^{(3)}(q)$.
- $\boldsymbol{n} = (3,2,1) \Longrightarrow \mathfrak{S}_{w(\boldsymbol{n})}(1,q,\dots) = qC_5^{(3)}(q)C_6^{(5)}(q) = qC_5^{(3)}(q) = q(1,1,1,1)$. Note that $C_{k+2}^{(k)}(q) = \begin{bmatrix} k+1 \\ 1 \end{bmatrix}_q$.

Exercises 5.35. Let $1 \leq k \leq m \leq n$ be integers, $n \geq 2k+1$. Consider permutation

$$w = \begin{pmatrix} 1 & 2 & \dots & k & k+1 & \dots & & n \\ m & m-1 & \dots & m-k+1 & n & \dots & \dots & 1 \end{pmatrix} \in \mathbb{S}_n.$$

Show that

$$\mathfrak{S}_w(1,q,\dots) = q^{n(D(w))}C_{n-m+k}^{(m)}(q),$$

where for any permutation w, $n(D(w)) = \sum \binom{d_i(w)}{2}$ and $d_i(w)$ denotes the number of boxes in the i-th column of the (*Rothe*) diagram $D(w)$ of the permutation w, see [92, p. 8].

(**C**) **A determinantal formula for the Grothendieck polynomials $\mathfrak{G}^{(\beta)}_{\pi_k^{(n)}}$.** Define polynomials

$$\Phi_n^{(m)}(X_n) = \sum_{a=m}^{n} e_a(X_n)\beta^{a-m},$$

$$A_{i,j}(X_{n+k-1}) = \frac{1}{(i-j)!}\left(\frac{\partial}{\partial\beta}\right)^{j-1} \Phi_{k+n-i}^{(n+1-i)}(X_{k+n-i}) \qquad \text{if} \quad 1 \leq i \leq j \leq n,$$

and

$$A_{i,j}(X_{k+n-1}) = \sum_{a=0}^{i-j-1} e_{n-i-a}(X_{n+k-i}) \binom{i-j-1}{a} \qquad \text{if} \quad 1 \leq j < i \leq n.$$

Theorem 5.36.

$$\mathrm{DET}\,|A_{i,j}|_{1\leq i,j\leq n} = \mathfrak{G}^{(\beta)}_{\pi_{k+n}^{(k)}}(X_{k+n-1}).$$

Comments 5.37.

(*a*) One can compute the Grothendieck polynomials for yet another interesting family of permutations. namely, *grassmannian* permutations

$$\begin{aligned}\sigma_k^{(n)} &= \begin{pmatrix} 1 & 2 & \dots & k-1 & k & k+1 & k+2 & \dots & n+k \\ 1 & 2 & \dots & k-1 & n+k & k & k+1 & \dots & n+k-1 \end{pmatrix} \\ &= s_k s_{k+1} \cdots s_{n+k-1} \in \mathbb{S}_{n+k}.\end{aligned}$$

Then

$$\mathfrak{G}^{(\beta)}_{\sigma_k(n)}(x_1,\ldots,x_{n+k}) = \sum_{j=0}^{k-1} s_{(n,1^j)}(X_k)\beta^j,$$

where $s_{(n,1^j)}(X_k)$ denotes the Schur polynomial corresponding to the hook shape partition $(n,1^j)$ and the set of variables $X_k := (x_1,\ldots,x_k)$. In particular,

$$\mathfrak{G}^{(\beta)}_{\sigma_k(n)}(x_j=1,\,\forall\, j) = \binom{n+k-1}{k}\left(\sum_{j=0}^{k-1}\frac{k}{n+j}\binom{k-1}{j}\beta^j\right) = \sum_{j=0}^{k-1}\binom{n+j-1}{j}(1+\beta)^j.$$

(*b*) *Grothendieck polynomials for grassmannian permutations.* In the case of a *grassmannian* permutation $w := \sigma_\lambda \in \mathbb{S}_\infty$ of the shape $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n)$ where n is a unique descent of w, one can prove the following formulas for the β-Grothendieck polynomial

$$\mathfrak{G}^{(\beta)}_{\sigma_\lambda}(X_n) = \frac{\mathrm{DET}\left|x_i^{\lambda_j+n-j}(1+\beta x_i)^{j-1}\right|_{1\leq i,j\leq n}}{\prod\limits_{1\leq i<j\leq n}(x_i-x_j)}, \tag{5.9}$$
$$\mathrm{DET}\left|h^{(\beta)}_{\lambda_j+i,j}(X_{[i,n]})\right|_{1\leq i,j\leq n} = \mathrm{DET}\left|h^{(\beta)}_{\lambda_j+i,j}(X_n)\right|_{1\leq i,j\leq n},$$

where $X_{[i,n]} = (x_i, x_{i+1},\ldots,x_n)$, and for any set of variables X

$$h^{(\beta)}_{n,k}(X) = \sum_{a=0}^{k-1}\binom{k-1}{a}h_{n-k+a}(X)\beta^a,$$

and $h_k(X)$ denotes the complete symmetric polynomial of degree k in the variables from the set X.

A proof is a straightforward adaptation of the proof of special case $\beta = 0$ (the case of *Schur* polynomials) given by I. Macdonald [92, Section 2, equation (2.10) and Section 4, equation (4.9)].

Indeed, consider β-divided difference operators $\pi^{(\beta)}_j$, $j = 1,\ldots,n-1$, and $\pi^{(\beta)}_w$, $w \in \mathbb{S}_n$, introduced in [42]. For example,

$$\pi^{(\beta)}_j(f) = \frac{1}{x_j - x_{j+1}}\big((1+\beta x_{j+1})f(X_n) - (1+\beta x_j)f(s_j(X_n))\big).$$

Now let $w_0 := w_0^{(n)}$ be the longest element in the symmetric group $\mathbb{S}_n$. The same proves of the Statements 2.10, 2.16 from [92] show that

$$\pi^{(\beta)}_{w_0} = a_\delta^{-1}w_0\left(\sum_{\sigma\in\mathbb{S}_n}(-1)^{\ell(\sigma)}\prod_{j=1}^{n-1}(1+\beta x_j)^{n-j}\sigma\right),$$

where $a_\delta = \prod\limits_{1\leq i<j\leq n}(x_i - x_j)$.

On the other hand, the same arguments as in the proof of Statement 4.8 from [92] show that

$$\mathfrak{G}^{(\beta)}_{\sigma_\lambda}(X_n) = \pi^{(\beta)}_{w^{(0)}}\left(x^{\lambda+\delta_n}\right).$$

Application of the formula for operator $\pi^{(\beta)}_{w_n^{(0)}}$ displayed above to the monomial $x^{\lambda+\delta_n}$ finishes the proof of the first equality in (5.8). The statement that the right hand side of the equality (5.9) coincides with determinants displayed in the identity (5.9) can be checked by means of simple transformations.

Problems 5.38.

(1) *Give a bijective prove of Theorem 5.28, i.e., construct a bijection between*

- *the set of k-tuple of mutually non-crossing Schröder paths $(\mathfrak{p}_1, \ldots, \mathfrak{p}_k)$ of lengths $(n, n-1, \ldots, n-k+1)$ correspondingly, and*
- *the set of pairs $(\mathfrak{m}, \mathcal{T})$, where $\mathcal{T}$ is a k-dissection of a convex $(n+k+1)$-gon, and $\mathfrak{m}$ is a upper triangle $(0,1)$-matrix of size $(k-1) \times (k-1)$, which is compatible with natural statistics on the both sets.*

(2) *Let $w \in \mathbb{S}_n$ be a permutation, and $\mathrm{CS}(w)$ be the set of compatible sequences corresponding to w, see, e.g., [13]. Define statistics $c(\bullet)$ on the set $\mathrm{CS}(w)$ such that*

$$\mathfrak{G}_w^{(\beta-1)}(x_1 = 1, x_2 = 1, \ldots) = \sum_{a \in \mathrm{CS}(w)} \beta^{c(a)}.$$

(3) *Let w be a vexillary permutation. Find a determinantal formula for the β-Grothendieck polynomial $\mathfrak{G}_w^{(\beta)}(X)$.*

(4) *Let w be a permutation. Find a geometric interpretation of coefficients of the polynomials $\mathfrak{S}_w^{(\beta)}(x_i = 1)$ and $\mathfrak{S}_w^{(\beta)}(x_i = q,\, x_j = 1,\, \forall\, j \neq i)$.*

For example, let $w \in \mathbb{S}_n$ be an involution, i.e., $w^2 = 1$, and $w' \in \mathbb{S}_{n+1}$ be the image of w under the natural embedding $\mathbb{S}_n \hookrightarrow \mathbb{S}_{n+1}$ given by $w \in \mathbb{S}_n \longrightarrow (w, n+1) \in \mathbb{S}_{n+1}$. It is well-known, see, e.g., [77, 142], that the multiplicity $m_{e,w}$ of the 0-dimensional Schubert cell $\{pt\} = Y_{w_0^{(n+1)}}$ in the Schubert variety $\overline{Y}_{w'}$ is equal to the specialization $\mathfrak{S}_w(x_i = 1)$ of the Schubert polynomial $\mathfrak{S}_w(X_n)$. Therefore one can consider the polynomial $\mathfrak{S}_w^{(\beta)}(x_i = 1)$ as a β-deformation of the multiplicity $m_{e,w}$.

Question 5.39. *What is a geometrical meaning of the coefficients of the polynomial $\mathfrak{S}_w^{(\beta)}(x_i = 1) \in \mathbb{N}[\beta]$?*

Conjecture 5.40. *The polynomial $\mathfrak{S}_w^{(\beta)}(x_i = 1)$ is a unimodal polynomial for any permutation w.*

5.2.4 Specialization of Schubert polynomials

Let n, k, r be positive integers and p, b be non-negative integers such that $r \leq p+1$. It is well-known [92] that in this case there exists a unique *vexillary* permutation $\varpi := \varpi_{\lambda,\phi} \in \mathbb{S}_\infty$ which has the *shape* $\lambda = (\lambda_1, \ldots, \lambda_{n+1})$ and the *flag* $\phi = (\phi_1, \ldots, \phi_{n+1})$, where

$$\lambda_i = (n-i+1)p + b, \qquad \phi_i = k + 1 + r(i-1), \qquad 1 \leq i \leq n+1-\delta_{b,0}.$$

According to a theorem by M. Wachs [139], the Schubert polynomial $\mathfrak{S}_\varpi(X)$ admits the following determinantal representation

$$\mathfrak{S}_\varpi(X) = \mathrm{DET}\left(h_{\lambda_i - i + j}(X_{\phi_i})\right)_{1 \leq i,j \leq n+1}.$$

Therefore we have

$$\begin{aligned}\mathfrak{S}_\varpi(1) &:= \mathfrak{S}_\varpi(x_1 = 1, x_2 = 1, \ldots) \\ &= \mathrm{DET}\left(\binom{(n-i+1)p + b - i + j + k + (i-1)r}{k + (i-1)r}\right)_{1 \leq i,j \leq n+1}.\end{aligned}$$

We denote the above determinant by $D(n,k,r,b,p)$.

Theorem 5.41.

$$D(n,k,r,b,p) = \prod_{(i,j)\in\mathcal{A}_{n,k,r}} \frac{i+b+jp}{i} \prod_{(i,j)\in\mathcal{B}_{n,k,r}} \frac{(k-i+1)(p+1)+(i+j-1)r+r(b+np)}{k-i+1+(i+j-1)r},$$

where

$$\mathcal{A}_{n,k,r} = \big\{(i,j)\in\mathbb{Z}^2_{\geq 0}\,|\, j\leq n, j<i\leq k+(r-1)(n-j)\big\},$$
$$\mathcal{B}_{n,k,r} = \big\{(i,j)\in\mathbb{Z}^2_{\geq 1}\,|\, i+j\leq n+1,\ i\neq k+1+rs,\ s\in\mathbb{Z}_{\geq 0}\big\}.$$

It is convenient to re-write the above formula for $D(n,k,r,b,p)$ in the following form

$$D(n,k,r,b,p) = \prod_{j=1}^{n+1} \frac{((n-j+1)p+b+k+(j-1)(r-1))!(n-j+1)!}{(k+(j-1)r)!((n-j+1)(p+1)+b)!}$$
$$\times \prod_{1\leq i\leq j\leq n} ((k-i+1)(p+1)+jr+(np+b)r).$$

Corollary 5.42 (some special cases)**.** *(A) The case $r=1$.*

We consider below some special cases of Theorem 5.41 in the case $r=1$. To simplify notation, we set $D(n,k,b,p) := D(n,k,r=1,b,p)$. Then we can rewrite the above formula for $D(n,k,r,b,p)$ as follows

$$D(n,k,b,p) = \prod_{j=1}^{n+1} \frac{((n+k-j+1)(p+1)+b)!((n-j+1)p+b+k)!(j-1)!}{((n-j+1)(p+1)+b)!((k+n-j+1)p+b+k)!(k+j-1)!}.$$

(1) *If $k\leq n+1$, then*

$$D(n,k,b,p) = \prod_{j=1}^{k} \binom{(n+k+1-j)(p+1)+b}{n-j+1} \binom{(k-j)p+b+k}{j} \frac{j!(k-j)!(n-j+1)!}{(n+k-j+1)!}.$$

In particular,

- *if $k=1$, then*

$$D(n,1,b,p) = \frac{1+b}{1+b+(n+1)p} \binom{(p+1)(n+1)+b}{n+1} := F_{n+1}^{(p+1)}(b),$$

  where $F_n^p(b) := \frac{1+b}{1+b+(p-1)n}\binom{pn+b}{n}$ denotes the generalized Fuss–Catalan number,

- *if $k=2$, then*

$$D(n,2,b,p) = \frac{(2+b)(2+b+p)}{(1+b)(2+b+(n+1)p)(2+b+(n+2)p)} F_{n+1}^{(p+1)}(b) F_{n+2}^{(p+1)}(b),$$

  in particular,

$$D(n,2,0,1) = \frac{6}{(n+3)(n+4)} \mathrm{Cat}_{n+1}\mathrm{Cat}_{n+2}.$$

  See [131, A005700] *for several combinatorial interpretations of these numbers.*

(2) *Consider the Young diagram (see R.A. Proctor* [122])

$$\lambda := \lambda_{n,p,b} = \{(i,j) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1} \,|\, 1 \leq i \leq n+1,\ 1 \leq j \leq (n+1-i)p+b\}.$$

For each box $(i,j) \in \lambda$ define the numbers $c(i,j) := n+1-i+j$, and

$$l_{(i,j)}(k) = \begin{cases} \dfrac{k+c(p,j)}{c(i,j)} & \text{if } j \leq (n+1-i)(p-1)+b, \\ \dfrac{(p+1)k+c(i,j)}{c(i,j)} & \text{if } (n+1-i)(p-1) < j-b \leq (n+1-i)p. \end{cases}$$

Then

$$D(n,k,b,p) = \prod_{(i,j)\in\lambda} l_{(i,j)}(k). \tag{5.10}$$

Therefore, $D(n,k,b,p)$ is a polynomial in k with rational coefficients.

(3) *If $p=0$, then*

$$D(n,k,b,0) = \dim V_{(n+1)^k}^{\mathfrak{gl}(b+k)} = \prod_{j=1}^{n+k} \left(\frac{j+b}{j}\right)^{\min(j,n+k+1-j)},$$

where for any partition μ, $\ell(\mu) \leq m$, $V_\mu^{\mathfrak{gl}(m)}$ denotes the irreducible $\mathfrak{gl}(m)$-module with the highest weight μ. In particular,

$$D(n,2,b,0) = \frac{1}{n+2+b}\binom{n+2+b}{b}\binom{n+2+b}{b+1}$$

is equal to the Narayana number $N(n+b+2,b)$,

$$D(1,k,b,0) = \frac{(b+k)!(b+k+1)!}{k!b!(k+1)!(b+1)!} := N(b+k+1,k),$$

and therefore the number $D(1,k,b,0)$ counts the number of pairs of non-crossing lattice paths inside a rectangular of size $(b+1)\times(k+1)$, which go from the point $(1,0)$ (resp. from that $(0,1)$) to the point $(b+1,k)$ (resp. to that $(b,k+1)$), consisting of steps $U=(1,0)$ and $R=(0,1)$, see [131, *A001263*], *for some list of combinatorial interpretations of the Narayana numbers.*

(4) *If $p=b=1$, then*

$$D(n,k,1,1) = C_{n+k+1}^{(k)} := \prod_{1\leq i\leq j\leq n+1} \frac{2k+i+j}{i+j}.$$

(5) *If $p=1$ and b is odd integer, then $D(n,k,b,1)$ is equal to the dimension of the irreducible representation of the symplectic Lie algebra $\mathrm{Sp}(b+2n+1)$ with the highest weight $k\omega_{n+1}$ (R.A. Proctor* [120, 121]*).*

(6) *If $p=1$ and $b=0$, then*

$$D(n,k,1,0) = D(n-1,k,1,1) = \prod_{1\leq i\leq j\leq n} \frac{2k+i+j}{i+j} = C_{n+k}^{(k)},$$

see section on Grothendieck and Narayana polynomials.

(7) *Let ϖ_λ be a unique dominant permutation of shape $\lambda := \lambda_{n,p,b}$ and $\ell := \ell_{n,p,b} = \frac{1}{2}(n+1)(np+2b)$ be its length (cf.* [44]*). Then*

$$\sum_{\boldsymbol{a}\in R(\varpi_\lambda)} \prod_{i=1}^{\ell} (x+a_i) = \ell! B(n,x,p,b).$$

Here for any permutation w of length l, we denote by $R(w)$ the set $\{\boldsymbol{a} = (a_1,\ldots,a_l)\}$ of all reduced decompositions of w.

Exercises 5.43. Show that

$$\mathrm{DET}\,\left|F^{(2)}_{n+i+j-2}(0)\right|_{1\le i,j\le k} = \prod_{j=1}^{k} F^{(2)}_{n+j-1}(0)\frac{\binom{k+1}{2}!}{\prod\limits_{\substack{1\le i\le k-1\\ 1\le j\le k}}(n+i+j)},$$

$$D(n,k,b,1) = \prod_{j=1}^{k} F^{(2)}_{n+j}(b)\frac{\prod\limits_{1\le i\le j\le k}(b+i+j-1)}{\prod\limits_{\substack{1\le i\le k-1\\ 1\le j\le k}}(n+b+i+j+1)}.$$

Clearly that if $b=0$, then $F^{(2)}_n(0) = C_n$, and $D(n,k,0,1)$ is equal to the Catalan–Hankel determinant $C^{(k)}_n$.

Finally we recall that the generalized Fuss–Catalan number $F^{(p+1)}_{n+1}(b)$ counts the number of lattice paths from $(0,0)$ to $(b+np,n)$ that do not go above the line $x=py$, see, e.g., [81].

Comments 5.44. It is well-known, see, e.g., [122] or [134, Vol. 2, Exercise 7.101.b], that the number $D(n,k,b,p)$ is equal to the total number $pp^{\lambda_{n,p,b}}(k)$ of plane partitions[55] bounded by k and contained in the shape $\lambda_{n,b,p}$.

More generally, see, e.g., [44], for any partition λ denote by $w_\lambda \in \mathfrak{S}_\infty$ a unique *dominant* permutation of shape λ, that is a unique permutation with the code $c(w)=\lambda$. Now for any non-negative integer k consider the so-called *shifted dominant* permutation $w^{(k)}_\lambda$ which has the shape λ and the flag $\phi = (\phi_i = k+i-1,\ i=1,\dots,\ell(\lambda))$. Then

$$\mathfrak{S}_{w^{(k)}_\lambda}(1) = pp^{\lambda}(\le k),$$

where $pp^\lambda(\le k)$ denotes the number of all plane partitions bounded by k and contained in λ. Moreover,

$$\sum_{\pi\in PP^\lambda(\le k)} q^{|\pi|} = q^{n(\lambda)}\mathfrak{S}_{w^{(k)}_\lambda}\big(1,q^{-1},q^{-2},\dots\big),$$

where $PP^\lambda(\le k)$ denotes the set of all plane partitions bounded by k and contained in λ.

Exercises 5.45.

(1) Show that

$$\lim_{k\to\infty} \mathfrak{S}_{w^{(k)}_\lambda}\big(1,q,q^2,\dots\big) = \frac{q^{n(\lambda)}}{H_\lambda(q)},$$

where $H_\lambda(q) = \prod\limits_{x\in\lambda}(1-q^{h(x)})$ denotes the *hook* polynomial corresponding to a given partition λ.

(2) Let $\lambda = ((n+\ell)^\ell,\ell^n)$ be a *fat hook*. Show that

$$\lim_{k\to\infty} q^{n(\lambda)}\mathfrak{S}_{w^{(k)}_\lambda}\big(1,q^{-1},q^{-2},\dots\big) = q^{s(\ell,n)}\frac{K_\lambda(q)}{M_\ell(2n+2\ell-1;q)},$$

where $a(\ell,n)$ is a certain integer we don't need to specify in what follows,

$$M_\ell(N;q) = \prod_{j=1}^{N}\left(\frac{1}{1-q^j}\right)^{\min(j,N+1-j,\ell)}$$

[55] Let λ be a partition. A plane (ordinary) partition bounded by d and shape λ is a filling of the shape λ by the numbers from the set $\{0,1,\dots,d\}$ in such a way that the numbers along columns and rows are weakly *decreasing*. A *reverse* plane partition bounded by d and shape λ is a filling of the shape λ by the numbers from the set $\{0,1,\dots,d\}$ in such a way that the numbers along columns and rows are weakly *increasing*.

denotes the MacMahon generating function for the number of plane partitions fit inside the box $N \times N \times \ell$, $K_\lambda(q)$ is a polynomial in q such that $K_\lambda(0)=1$.

(a) Show that

$$(1-q)^{|\lambda|} \frac{K_\lambda(q)}{M_\ell(2n+2\ell-1;q)}\bigg|_{q=1} = \frac{1}{\prod\limits_{x\in\lambda} h(x)}.$$

(b) Show that

$$K_\lambda(q) \in \mathbb{N}[q] \qquad \text{and} \qquad K_\lambda(1) = M(n,n,\ell),$$

where $M(a,b,c)$ denotes the number of plane partitions fit inside the box $a\times b\times c$. It is well-known, see, e.g., [93, p. 81], that

$$M(a,b,c) = \prod_{\substack{1\le i\le a\\ 1\le j\le b\\ 1\le k\le c}} \frac{i+j+k-1}{i+j+k-2} = \prod_{i=1}^{c} \frac{(a+b+i-1)!(i-1)!}{(a+i-1)!(b+1-1)!} = \dim V_{(a^c)}^{\mathfrak{gl}_{b+c}}.$$

Show that

$$K_\lambda(q) = \sum_{\pi\in B_{n,n,\ell}} q^{wt_\ell(\pi)},$$

where the sum runs over the set of plane partitions $\pi=(\pi_{ij})_{1\le i,j\le n}$ fit inside the box $B_{n,n,\ell} := n\times n\times \ell$, and

$$wt_\ell(\pi) = \sum_{i,j} \pi_{ij} + \ell \sum_i \pi_{ii}.$$

(c) Assume as before that $\lambda := ((n+\ell)^\ell, \ell^n)$. Show that

$$\lim_{n\to\infty} K_\lambda(q) = M_\ell(q) \sum_{\substack{\mu\\ \ell(\mu)\le\ell}} q^{|\mu|} \left(\frac{q^{n(\mu)}}{\prod_{x\in\mu}(1-q^{h(x)})} \right)^2,$$

where the sum runs over the set of partitions μ with the number of parts at most ℓ, and $n(\mu) = \sum_i (i-1)\mu_i$,

$$M_\ell(q) := \prod_{j\ge 1} \left(1-q^j\right)^{\min(j,\ell)}.$$

Therefore the generating function $PP^{(\ell,0)}(q) := \sum\limits_{\pi\in PP^{(\ell,0)}} q^{|\pi|}$ is equal to

$$\sum_{\substack{\mu\\ \ell(\mu)\le\ell}} q^{|\mu|} \left(\frac{q^{n(\mu)}}{\prod\limits_{x\in\mu}(1-q^{h(x)})} \right)^2,$$

where $PP^{(\ell,k)} := \{\pi=(\pi_{ij})_{i,j\ge 1} \mid \pi_{ij}\ge 0, \pi_{\ell+1,\ell+1}\le k\}$, $|\pi| = \sum\limits_{i,j}\pi_{ij}$.

(d) Show that

$$PP^{(\ell,0)}(q) = \frac{1}{M_\ell(q)^2} \sum_{\substack{\mu,\\ \ell(\mu)\le\ell}} (-q)^{|\mu|} q^{n(\mu)+n(\mu')} \left(\dim_q V_\mu^{\mathfrak{gl}(\ell)}\right)^2,$$

where μ' denotes the *conjugate* partition of μ, therefore $n(\mu') = \sum\limits_{i\ge 1} \binom{\mu_i}{2}$.

The formula (5.10) is the special case $n = m$ of [109, Theorem 1.2]. In particular, if $\ell = 1$ then one come to following identity

$$\frac{1}{(q;q)_\infty^2} \sum_{k\ge 0} (-1)^k q^{\binom{k+1}{2}} = \sum_{k\ge 0} q^k \left(\frac{1}{(q;q)_k}\right)^2 .$$

(e) Let $k \ge 0$, $\ell \ge 1$ be integers. Show that the (*fermionic*) generating function for the number of plane partitions $\pi = (\pi_{ij}) \in PP^{(\ell,k)}$ is equal to

$$\sum_{\pi \in PP^{(\ell,k)}} q^{|\pi|} = \sum_{\substack{\mu \\ \mu_{\ell+1} \le k}} q^{|\mu|} \left(\frac{q^{n(\mu)}}{\prod\limits_{x\in\mu} (1 - q^{h(x)})} \right)^2 .$$

(B) *The case $k = 0$.*

(1) $D(n,0,1,p,b) = 1$ for all nonnegative n, p, b.

(2) $D(n,0,2,2,2) = \mathrm{VSASM}(n)$, i.e., the number of alternating sign $(2n+1)\times(2n+1)$ matrices symmetric about the vertical axis, see, e.g., [131, A005156].

(3) $D(n,0,2,1,2) = \mathrm{CSTCPP}(n)$, i.e., the number of cyclically symmetric transpose complement plane partitions, see, e.g., [131, A051255].

Theorem 5.46. *Let $\varpi_{n,k,p}$ be a unique vexillary permutation of the shape $\lambda_{n.p} := (n, n-1, \ldots, 2, 1)p$ and flag $\phi_{n,k} := (k+1, k+2, \ldots, k+n-1, k+n)$. Then*

$$\mathfrak{G}^{(\beta-1)}_{\varpi_{n,1,p}}(1) = \sum_{j=1}^{n+1} \frac{1}{n+1} \binom{n+1}{j} \binom{(n+1)p}{j-1} \beta^{j-1}.$$

If $k \ge 2$, then $G_{n,k,p}(\beta) := \mathfrak{G}^{(\beta-1)}_{\varpi_{n,k,p}}(1)$ is a polynomial of degree nk in β, and

$$\mathrm{Coeff}_{[\beta^{nk}]}(G_{n,k,p}(\beta)) = D(n,k,1,p-1,0).$$

The polynomial

$$\sum_{j=1}^{n} \frac{1}{n} \binom{n}{j} \binom{pn}{j-1} t^{j-1} := \mathfrak{F}\mathfrak{N}_n(t)$$

is known as the Fuss–Narayana polynomial and can be considered as a t-deformation of the Fuss–Catalan number $\mathrm{FC}_n^p(0)$.

Recall that the number $\frac{1}{n}\binom{n}{j}\binom{pn}{j-1}$ counts paths from $(0,0)$ to $(np,0)$ in the first quadrant, consisting of steps $U = (1,1)$ and $D = (1,-p)$ and have j peaks (i.e., UD's), cf. [131, A108767].

For example, take $n = 3$, $k = 2$, $p = 3$, $r = 1$, $b = 0$. Then

$$\varpi_{3,2,3} = [1,2,12,9,6,3,4,5,7,8,10,11] \in \mathbb{S}_{12},$$
$$G_{3,2,3}(\beta) = (1,18,171,747,1767,1995,1001).$$

Therefore,

$$G_{3,2,3}(1) = 5700 = D(3,2,3,0) \qquad \text{and} \qquad \mathrm{Coeff}_{[\beta^6]}(G_{3,2,3}(\beta)) = 1001 = D(3,2,2,0).$$

Proposition 5.47 ([110]). *The value of the Fuss–Catalan polynomial at $t=2$, that is the number*

$$\sum_{j=1}^{n} \frac{1}{n}\binom{n}{j}\binom{pn}{j-1} 2^{j-1}$$

is equal to the number of hyperplactic classes of p-parking functions of length n, see [110] for definition of p-parking functions, its properties and connections with some combinatorial Hopf algebras.

Therefore, the value of the *Grothendieck polynomial* $\mathfrak{G}_{\varpi_{n,1,p}}^{(\beta=1)}(1)$ at $\beta=1$ and $x_i=1$, $\forall\, i$, is equal to the number of p-parking functions of length $n+1$. It is an open problem to find combinatorial interpretations of the polynomials $\mathfrak{G}_{\varpi_{n,k,p}}^{(\beta)}(1)$ in the case $k \geq 2$. Note finally, that in the case $p=2$, $k=1$ the values of the Fuss–Catalan polynomials at $t=2$ one can find in [131, $A034015$].

Comments 5.48. ($\Longrightarrow$) *The case* $r=0$. It follows from Theorem 5.32 that in the case $r=0$ and $k \geq n$, one has

$$D(n,k,0,p,b) = \dim V_{\lambda_{n,p,b}}^{\mathfrak{gl}(k+1)} = (1+p)^{\binom{n+1}{2}} \prod_{j=1}^{n+1} \frac{\binom{(n-j+1)p+b+k-j+1}{k-j+1}}{\binom{(n-j+1)(p+1)+b}{n-j+1}}.$$

Now consider the conjugate $\nu := \nu_{n,p,b} := ((n+1)^b, n^p, (n-1)^p, \ldots, 1^p)$ of the partition $\lambda_{n,p,b}$, and a rectangular shape partition $\psi = (\underbrace{k,\ldots,k}_{np+b})$. If $k \geq np+b$, then there exists a unique *grassmannian* permutation $\sigma := \sigma_{n,k,p,b}$ of the *shape* ν and the *flag* ψ [92]. It is easy to see from the above formula for $D(n,k,0,p,b)$, that

$$\begin{aligned}\mathfrak{S}_{\sigma_{n,k,p,b}}(1) &= \dim V_{\nu_{n,p,b}}^{\mathfrak{gl}(k-1)} \\ &= (1+p)^{\binom{n}{2}} \binom{k+n-1}{b} \prod_{j=1}^{n} \frac{(p+1)(n-j+1)}{(n-j+1)(p+1)+b} \prod_{j=1}^{n} \frac{\binom{k+j-2}{(n-j+1)p+b}}{\binom{(n-j+1)(p+1)+b-1}{n-j}}.\end{aligned}$$

After the substitution $k := np+b+1$ in the above formula we will have

$$\mathfrak{S}_{\sigma_{n,np+b+1,p,b}}(1) = (1+p)^{\binom{n}{2}} \prod_{j=1}^{n} \frac{\binom{np+b+j-1}{(n-j+1)p}}{\binom{j(p+1)-1}{j-1}}.$$

In the case $b=0$ some simplifications are happened, namely,

$$\mathfrak{S}_{\sigma_{n,k,p,0}}(1) = (1+p)^{\binom{n}{2}} \prod_{j=1}^{n} \frac{\binom{k+j-2}{(n-j+1)p}}{\binom{(n-j+1)p+n-j}{n-j}}.$$

Finally we observe that if $k=np+1$, then

$$\prod_{j=1}^{n} \frac{\binom{np+j-1}{(n-j+1)p}}{\binom{(n-j+1)p+n-j}{n-j}} = \prod_{j=2}^{n} \frac{\binom{np+j-1}{(p+1)(j-1)}}{\binom{j(p+1)-1}{j-1}} = \prod_{j=1}^{n-1} \frac{j!(n(p+1)-j-1)!}{((n-j)(p+1))!((n-j)(p+1)-1)!} := A_n^{(p)},$$

where the numbers $A_n^{(p)}$ are *integers* that generalize the numbers of *alternating sign matrices* (ASM) of size $n \times n$, recovered in the case $p=2$, see [33, 111] for details.

Examples 5.49.

(1) Let us consider polynomials $\mathfrak{G}_n(\beta) := \mathfrak{G}^{(\beta-1)}_{\sigma_{n,2n,2,0}}(1)$.

If $n=2$, then

$$\sigma_{2,4,2,0} = 235614 \in \mathbb{S}_6, \qquad \mathfrak{G}_2(\beta) = (1,2,\mathbf{3}) := 1+2\beta+\mathbf{3}\beta^2.$$

Moreover,

$$\mathfrak{R}_{\sigma_{2,4,2,0}}(q;\beta) = (1,\mathbf{2})_\beta + \mathbf{3}q\beta^2.$$

If $n=3$, then

$$\sigma_{3,6,2,0} = 235689147 \in \mathbb{S}_9, \qquad \mathfrak{G}_3(\beta) = (1,6,21,36,51,48,\mathbf{26}).$$

Moreover,

$$\mathfrak{R}_{\sigma_{3,6,2,0}}(q;\beta) = (1,6,11,16,\mathbf{11})_\beta + q\beta^2(10,20,35,34)_\beta + q^2\beta^4(5,14,\mathbf{26})_\beta,$$
$$\mathfrak{R}_{\sigma_{3,6,2,0}}(q;1) = (45,99,45)_q.$$

If $n=4$, then

$$\sigma_{4,8,2,0} = [2,3,5,6,8,9,11,12,1,4,7,10] \in \mathbb{S}_{12},$$
$$\mathfrak{G}_4(\beta) = (1,12,78,308,903,2016,3528,4944,5886,5696,4320,2280,\mathbf{646}).$$

Moreover,

$$\begin{aligned}\mathfrak{R}_{\sigma_{4,8,2,0}}(q;\beta) = {} & (1,12,57,182,392,602,763,730,493,\mathbf{170})_\beta \\ & + q\beta^2(21,126,476,1190,1925,2626,2713,2026,804)_\beta \\ & + q^2\beta^4(35,224,833,1534,2446,2974,2607,1254)_\beta \\ & + q^3\beta^6(7,54,234,526,909,1026,\mathbf{646})_\beta,\end{aligned}$$
$$\mathfrak{R}_{\sigma_{4,8,2,0}}(q;1) = (3402,11907,11907,3402)_q = 1701\ (2,7,7,2)_q.$$

• If $n=5$, then

$$\sigma_{5,10,2} = [2,3,5,6,8,9,11,12,14,15,1,4,7,10,13] \in \mathbb{S}_{15},$$
$$\begin{aligned}\mathfrak{G}_5(\beta) = {} & (1,20,210,1420,7085,27636,87430,230240,516375,997790,1676587,2466840, \\ & 3204065,3695650,3778095,3371612,2569795,1610910,782175,262200,\mathbf{45885}).\end{aligned}$$

Moreover,

$$\begin{aligned}\mathfrak{R}_{\sigma_{5,10,2,0}}(q;\beta) = {} & (1,20,174,988,4025,12516,31402,64760,111510,162170, \\ & 202957,220200,202493,153106,89355,35972,\mathbf{7429})_\beta \\ & + q\beta^2(36,432,2934,13608,45990,123516,269703,487908,738927, \\ & 956430,1076265,1028808,813177,499374,213597,47538)_\beta \\ & + q^2\beta^4(126,1512,9954,40860,127359,314172,627831,1029726,1421253, \\ & 1711728,1753893,1492974,991809,461322,112860)_\beta \\ & + q^3\beta^6(84,1104,7794,33408,105840,255492,486324,753984,1019538, \\ & 1169520,1112340,825930,428895,117990)_\beta \\ & + q^4\beta^8(9,132,1032,4992,17730,48024,102132,173772,244620,276120,\end{aligned}$$

$$240420, 144210, \mathbf{45885})_\beta,$$

$$\begin{aligned}\mathfrak{R}_{\sigma_{5,10,2,0}}(q;1) &= (1299078, 6318243, 10097379, 6318243, 1299078)_q \\ &= 59049(22, 107, 171, 107, 22)_q.\end{aligned}$$

We are reminded that over the paper we have used the notation

$$(a_0, a_1, \ldots, a_r)_\beta := \sum_{j=0}^{r} a_j \beta^j,$$

etc.

One can show that $\deg_{[\beta]} \mathfrak{G}_n(\beta) = n(n-1)$, $\deg_{[q]} \mathfrak{R}_{\sigma_{n,2n,2,0}}(q,1) = n-1$, and looking on the numbers 3, 26, 646, 45885 we made

Conjecture 5.50. *Let* $a(n) := \mathrm{Coeff}[\beta^{n(n-1)}]\ (\mathfrak{G}_n(\beta))$. *Then*

$$a(n) = \mathrm{VSASM}(n) = \mathrm{OSASM}(n) = \prod_{j=1}^{n-1} \frac{(3j+2)(6j+3)!(2j+1)!}{(4j+2)!(4j+3)!},$$

where $\mathrm{VSASM}(n)$ *is the number of alternating sign* $(2n+1)\times(2n+1)$ *matrices symmetric about the vertical axis,* $\mathrm{OSASM}(n)$ *is the number of* $2n \times 2n$ *off-diagonal symmetric alternating sign matrices. See* [131, *A*005156], [111] *and references therein, for details.*

Conjecture 5.51. *Polynomial* $\mathfrak{R}_{\sigma_{n,2n,2,0}}(q;1)$ *is symmetric and*

$$\mathfrak{R}_{\sigma_{n,2n,2,0}}(0;1) = A20342(2n-1),$$

see [131].

(2) Let us consider polynomials $\mathfrak{F}_n(\beta) := \mathfrak{G}^{(\beta-1)}_{\sigma_{n,2n+1,2,0}}(1)$.

If $n=1$, then

$$\sigma_{1,3,2,0} = 1342 \in \mathbb{S}_4, \qquad \mathfrak{F}_2(\beta) = (1, \mathbf{2}) := 1 + \mathbf{2}\beta.$$

If $n=2$, then

$$\sigma_{2,5,2,0} = 1346725 \in \mathbb{S}_7, \qquad \mathfrak{F}_3(\beta) = (1, 6, 11, 16, \mathbf{11}).$$

Moreover,

$$\mathfrak{R}_{\sigma_{2,5,2,0}}(q;\beta) = (1, 2, \mathbf{3})_\beta + q\beta(4, 8, 12)_\beta + q^2\beta^3(4, \mathbf{11})_\beta.$$

If $n=3$, then

$$\begin{aligned}&\sigma_{3,7,2,0} = [1, 3, 4, 6, 7, 9, 10, 2, 5, 8] \in \mathbb{S}_{10}, \\ &\mathfrak{F}_4(\beta) = (1, 12, 57, 182, 392, 602, 763, 730, 493, \mathbf{170}).\end{aligned}$$

Moreover,

$$\begin{aligned}\mathfrak{R}_{\sigma_{3,7,2,0}}(q;\beta) &= (1, 6, 21, 36, 51, 48, \mathbf{26})_\beta + q\beta(6, 36, 126, 216, 306, 288, 156)_\beta \\ &\quad + q^2\beta^3(20, 125, 242, 403, 460, 289)_\beta + q^3\beta^5(6, 46, 114, 204, \mathbf{170})_\beta, \\ \mathfrak{R}_{\sigma_{3,7,2,0}}(q;1) &= (189, 1134, 1539, 540)_q = 27(7, 42, 57, 20)_q.\end{aligned}$$

If $n = 4$, then

$$\sigma_{4,9,2,0} = [1,3,4,6,7,9,10,12,13,2,5,8,11] \in \mathbb{S}_{13},$$
$$\mathfrak{F}_5(\beta) = (1,20,174,988,4025,12516,31402,64760,111510,162170,202957, 220200,202493,153106,89355,35972,\mathbf{7429}).$$

Moreover,

$$\begin{aligned}\mathfrak{R}_{\sigma_{4,9,2,0}}(q;\beta) = {} & (1,12,78,308,903,2016,3528,4944,5886,5696,4320,2280,\mathbf{646})_\beta\\ & + q\beta(8,96,624,2464,7224,16128,28224,39552,47088,45568,\\ & \ 34560,18240,5168)_\beta\\ & + q^2\beta^3(56,658,3220,11018,27848,53135,78902,100109,103436,\\ & \ 84201,47830,14467)_\beta\\ & + q^3\beta^5(56,728,3736,12820,29788,50236,72652,85444,78868,\\ & \ 50876,17204)_\beta\\ & + q^4\beta^7(8,117,696,2724,7272,13962,21240,24012,18768,\mathbf{7429})_\beta,\end{aligned}$$
$$\mathfrak{R}_{\sigma_{4,9,2,0}}(q;1) = (30618,244944,524880,402408,96228)_q = 4374(7,56,120,92,22)_q.$$

One can show that $\mathfrak{F}_n(\beta)$ is a polynomial in β of degree n^2, and looking on the numbers 2, 11, 170, 7429 we made

Conjecture 5.52. *Let* $b(n) := \mathrm{Coeff}_{[\beta^{(n-1)^2}]}(\mathfrak{F}_n(\beta))$. *Then* $b(n) = \mathrm{CSTCPP}(n)$. *In other words,* $b(n)$ *is equal to the number of cyclically symmetric transpose complement plane partitions in an* $2n \times 2n \times 2n$ *box. This number is known to be*

$$\prod_{j}^{n-1} \frac{(3j+1)(6j)!(2j)!}{(4j+1)!(4j)!},$$

see [131, *A*051255], [18, p. 199].

It ease to see that polynomial $\mathfrak{R}_{\sigma_{n,2n+1,2,0}}(q;1)$ has degree n.

Conjecture 5.53.

$$\mathrm{Coeff}_{[\beta^n]}\big(\mathfrak{R}_{\sigma_{n,2n+1,2,0}}(q;1)\big) = A20342(2n),$$

see [131]*;*

$$\mathfrak{R}_{\sigma_{n,2n+1,2,0}}(0;1) = A^{(1)}_{\mathrm{QT}}(4n;3) = 3^{n(n-1)/2}\mathrm{ASM}(n),$$

see [83, Theorem 5] *or* [131, *A*059491].

Proposition 5.54. *One has*

$$\mathfrak{R}_{\sigma_{4,2n+1,2,0}}(0;\beta) = \mathfrak{G}_n(\beta) = \mathfrak{G}^{(\beta-1)}_{\sigma_{n,2n,2,0}}(1), \qquad \mathfrak{R}_{\sigma_{n,2n,2,0}}(0,\beta) = \mathfrak{F}_n(\beta) = \mathfrak{G}^{(\beta-1)}_{\sigma_{n,2n+1,2,0}}(1).$$

Finally we define (β,q)-deformations of the numbers $\mathrm{VSASM}(n)$ and $\mathrm{CSCTPP}(n)$. To accomplish these ends, let us consider permutations

$$w_k^- = (2,4,\dots,2k,2k-1,2k-3,\dots,3,1),$$
$$w_k^+ = (2,4,\dots,2k,2k+1,2k-1,\dots,3,1).$$

Proposition 5.55. *One has*

$$\mathfrak{S}_{w_k^-}(1) = \mathrm{VSAM}(k), \qquad \mathfrak{S}_{w_k^+}(1) = \mathrm{CSTCPP}(k).$$

Therefore the polynomials $\mathfrak{S}^{(\beta-1)}_{w_k^-}(x_1 = q,\, x_j = 1,\, \forall\, j \geq 2)$ and $\mathfrak{S}^{(\beta-1)}_{w_k^+}(x_1 = q,\, x_j = 1,\, \forall\, j \geq 2)$ define (β, q)-deformations of the numbers VSAM(k) and CSTCPP(k) respectively. Note that the inverse permutations

$$(w_k^-)^{-1} = (\underbrace{2k, 1}, \dots, \underbrace{2k+1-i, i}, \dots, \underbrace{k+1, k}),$$
$$(w_k^+)^{-1} = (\underbrace{2k+1, 1}, \dots, \underbrace{2k+2-j, j}, \dots, \underbrace{k+2, k}, k+1)$$

also define a (β, q)-deformation of the numbers considered above.

Problem 5.56. *It is well-known, see, e.g.,* [37, p. 43], *that the set* $\mathcal{VSASM}(n)$ *of alternating sign* $(2n+1) \times (2n+1)$ *matrices symmetric about the vertical axis has the same cardinality as the set* $\mathrm{SYT}_2(\lambda(n), \leq n)$ *of semistandard Young tableaux of the shape* $\lambda(n) := (2n-1, 2n-3, \dots, 3, 1)$ *filled by the numbers from the set* $\{1, 2, \dots, n\}$*, and such that the entries are weakly increasing down the anti-diagonals.*

On the other hand, consider the set $\mathcal{CS}(w_k^-)$ *of compatible sequences, see, e.g.,* [13, 42]*, corresponding to the permutation* $w_k^- \in \mathbb{S}_{2k}$*.*

Challenge 5.57. Construct bijections between the sets $\mathcal{CS}(w_k^-)$, $\mathrm{SYT}_2(\lambda(k), \leq\ k)$ and $\mathcal{VSASM}(k)$.

Remark 5.58. One can compute the principal specialization of the Schubert polynomial corresponding to the transposition $t_{k,n} := (k, n-k) \in \mathbb{S}_n$ that interchanges k and $n-k$, and fixes all other elements of $[1, n]$.

Proposition 5.59.

$$q^{(n-1)(k-1)} \mathfrak{S}_{t_{k,n-k}}(1, q^{-1}, q^{-2}, q^{-3}, \dots)$$
$$= \sum_{j=1}^{k} (-1)^{j-1} q^{\binom{j}{2}} \begin{bmatrix} n-1 \\ k-j \end{bmatrix}_q \begin{bmatrix} n-2+j \\ k+j-1 \end{bmatrix}_q = \sum_{j=1}^{n-2} q^j \left(\begin{bmatrix} j+k-2 \\ k-1 \end{bmatrix}_q \right)^2.$$

Exercises 5.60.

(1) Show that if $k \geq 1$, then

$$\mathrm{Coeff}_{[q^k \beta^{2k}]}(\mathfrak{R}_{\sigma_{n,2n,2,0}}(q; t)) = \binom{2n-1}{2k},$$
$$\mathrm{Coeff}_{[q^k \beta^{2k-1}]}(\mathfrak{R}_{\sigma_{n,2n+1,2,0}}(q; t)) = \binom{2n}{2k-1}.$$

(2) Let $n \geq 1$ be a positive integer, consider "zig-zag" permutation

$$w = \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & 2k+1 & 2k+2 & \dots & 2n-1 & 2n \\ 2 & 1 & 4 & 3 & \dots & 2k+2 & 2k+1 & \dots & 2n & 2n-1 \end{pmatrix} \in \mathbb{S}_{2n}.$$

Show that

$$\mathfrak{R}_w(q, \beta) = \prod_{k=0}^{n-1} \left(\frac{1-\beta^{2k}}{1-\beta} + q\beta^{2k} \right).$$

(3) Let $\sigma_{k,n,m}$ be grassmannian permutation with shape $\lambda = (n^m)$ and flag $\phi = (k+1)^m$, i.e.,

$$\sigma_{k,n,m} = \begin{pmatrix} 1 & 2 & \dots & k & k+1 & \dots & k+n & k+n+1 & \dots & k+n+m \\ 1 & 2 & \dots & k & k+m+1 & \dots & k+m+n & k+1 & \dots & k+m \end{pmatrix}.$$

Clearly $\sigma_{k+1,n,m} = 1 \times \sigma_{k,n,m}$.

Show that the coefficient $\mathrm{Coeff}_{\beta^m}(\mathfrak{R}_{\sigma_{k,n,m}}(1,\beta))$ is equal to the Narayana number $N(k+n+m,k)$.

(4) Consider permutation $w := w^{(n)} = (w_1,\dots,w_{2n+1})$, where $w_{2k-1} = 2k+1$ for $k = 1,\dots,n$, $w_{2n+1} = 2n$, $w_2 = 1$ and $w_{2k} = 2k-2$ for $k = 2,\dots,n$. For example, $w^{(3)} = (3152746)$. We set $w^{(0)} = 1$. Show that the polynomial $\mathfrak{S}_w^{(\beta)}(x_i = 1, \forall i)$ has degree $n(n-1)$ and the coefficient $\mathrm{Coeff}_{\beta^{n(n-1)}}(\mathfrak{S}_w^{(\beta)}(x_i = 1, \forall\, i))$ is equal to the n-th Catalan number C_n.

Note that the specialization $\mathfrak{S}_w^{(\beta)}(x_i = 1)|_{\beta=1}$ is equal to the $2n$-th Euler (or up/down) number, see [131, $A000111$].

More generally, consider permutation $w_k^{(n)} := 1^k \times w^{(n)} \in \mathbb{S}_{k+2n+1}$, and polynomials

$$P_k(z) = \sum_{j \geq 0} (-1)^j \mathfrak{S}_{w_{k-2j}^{(j)}}(x_i = 1) z^{k-2j}, \qquad k \geq 0.$$

Show that

$$\sum_{k \geq 0} P_k(z) \frac{t^k}{k!} = \exp(tz)\,\mathrm{sech}(t).$$

The polynomials $P_k(z)$ are well-known as *Swiss–Knife* polynomials, see [131, $A153641$], where one can find an overview of some properties of the Swiss–Knife polynomials.

(5) Assume that $n = 2k+3$, $k \geq 1$, and consider permutation $v_n = (v_1,\dots,v_n) \in \mathbb{S}_n$, where $v_{2a+1} = 2a+3$, $a = 0,\dots,n-1$, $w_2 = 1$ and $w_{2a} = 2a-2$, $a = 2,\dots,k+1$. For example, $v_4 = [31527496, 11, 8, 10]$ and $\mathfrak{S}_{v_4}(1) = 50521 = E_{10}$.

Show that

$$\mathfrak{S}_{v_n}(q,\, x_i = 1,\, \forall\, i \geq 2) = (n-2)E_{n-3}q^2 + \cdots + 2^{k-1}(k-1)!q^{k+2},$$
$$\mathfrak{S}_{v_n}(x_i = 1,\, \forall\, i \geq 1) = E_{n-1}.$$

Set $\beta = d-1$, consider polynomials $\mathcal{E}_n(q,d) = \mathfrak{G}_{v_n}^{(\beta)}(x_1 = q,\, x_i = 1,\, \forall\, i \geq 2)$. Clearly, see the latter formula, $\mathcal{E}_n(1,1) = E_{n-1}$. Give a combinatorial prove that $\mathcal{E}_n(q,d) \in \mathbb{N}[q,d]$, that is to give combinatorial interpretation(s) of coefficients of the polynomial $\mathcal{E}_n(q,d)$.

Show that $\deg_d \mathcal{E}_n(1,d) = n(n+1)$ and the leading coefficient is equal to the Catalan number C_{n+1}.

(6) Consider permutation $u := u_n = (u_1,\dots,u_{2n}) \in \mathbb{S}_{2n}$, $n \geq 2$, where $u_1 = 2$, $u_{2k+1} = 2k-1$, $k = 1,\dots,n$, $u_{2k} = 2k+2$, $k = 1,\dots,n-1$, $u_{2n} = 2n-1$. For example, $u_4 = (24163857)$.

Now consider polynomial

$$R_n^{(k)}(q) = \mathfrak{S}_{1^k \times u_n}(x_1 = q,\, x_i = 1,\, \forall\, i \geq 2).$$

Show that $R_n^{(k)}(1) = \binom{2n+k-1}{k} E_{2n-1}$, where E_{2k-1}, $k \geq 1$, denotes the Euler number, see [131, $A00111$]. In particular, $R_n^{(1)}(1) = 2^{2n-1}G_n$, where G_n denotes the *unsigned Genocchi* number, see [131, $A110501$].

Show that $\deg_q R_n^{(k)}(q) = n$ and $\mathrm{Coeff}_{q^n}\big(R_n^{(0)}(q)\big) = (2n-3)!!$.

(7) Consider permutation $w_n \in \mathbb{S}_{2n+2}$, where $w_2 = 1$, $w_4 = 2$, and

$$w_{2k-1} = 2k+2, \quad 1 \leq k \leq n, \qquad w_{2k} = 2k-3, \quad 3 \leq k \leq n,$$

$$w_{2n+1} = 2n-3, \qquad w_{2n+2} = 2n-1.$$

For example, $w_5 = [4,1,6,2,8,3,10,5,12,7,9,11]$.
Show that

$$\mathfrak{S}_{w_n}(x_i = 1, \forall\, i) = (2n+1)!!\big(2^{2n}-2\big)|B_{2n}|,$$

where B_{2n} denotes the *Bernoulli* numbers[56].

(8) Consider permutation $w_k := (2k+1, 2k-1, \ldots, 3, 1, 2k, 2k-2, \ldots, 4, 2) \in \mathbb{S}_{2k+1}$. Show that

$$\mathfrak{S}^{(\beta-1)}_{w_k}(x_1 = q,\, x_j = 1, \forall\, j \geq 2) = q^{2k}(1+\beta)^{\binom{n}{2}}.$$

(9) Consider permutations $\sigma_k^+ = (1,3,5,\ldots,2k+1,2k+2,2k,\ldots,4,2)$ and $\sigma_k^- = (1,3,5,\ldots,$ $2k+1, 2k, 2k-2, \ldots, 4, 2)$, and define polynomials

$$S_k^{\pm}(q) = \mathfrak{S}_{\sigma_k^{\pm}}(x_1 = q,\, x_j = 1, \forall\, j \geq 2).$$

Show that

$$\begin{aligned}
&S_k^+(0) = \mathrm{VSASM}(k), \qquad S_k^+(1) = \mathrm{VSASM}(k+1),\\
&\frac{\partial}{\partial q} S_k^+(q)|_{q=0} = 2k S_k^+(0), \qquad \mathrm{Coeff}_{q^k}(S_k^+(q)) = \mathrm{CSTCPP}(k+1),\\
&S_k^-(0) = \mathrm{CSTCPP}(k), \qquad S_k^-(1) = \mathrm{CSTCPP}(k+1),\\
&\frac{\partial}{\partial q} S_k^-(q)|_{q=0} = (2k-1) S_k^-(0), \qquad \mathrm{Coeff}_{q^k}(S_k^-(q)) = \mathrm{VSASM}(k).
\end{aligned}$$

Let's observe that $\sigma_k^{\pm} = 1 \times \tau_{k-1}^{\pm}$, where $\tau_k^+ = (2,4,\ldots,2k,2k+1,2k-1,\ldots,3,1)$ and $\tau_k^- = (2,4,\ldots,2k,2k-1,2k-3,\ldots,3,1)$. Therefore,

$$\mathfrak{S}_{\tau_k^{\pm}}(x_1 = q,\, x_j = 1, \forall j \geq 2) = q S_{k-1}^{\pm}(q).$$

Recall that $\mathrm{CSTCPP}(n)$ denotes the number of cyclically symmetric transpose compliment plane partitions in a $2n \times 2n$ box, see, e.g., [131, A051255], and $\mathrm{VSASM}(n)$ denotes the number of alternating sign $(2n+1)\times(2n+1)$ matrices symmetric the vertical axis, see, e.g., [131, A005156].

It might be well to point out that

$$\begin{aligned}
&\mathfrak{S}_{\sigma_{n-1}^+}(x_1 = x,\, x_i = 1, \forall\, i \geq 2) = G_{2n-1,n-1}(x, y=1),\\
&\mathfrak{S}_{\sigma_n^-}(x_1 = x,\, x_i = 1, \forall\, i \geq 2) = F_{2n,n-1}(x, y=1),
\end{aligned}$$

where (homogeneous) polynomials $G_{m,n}(x,y)$ and $F_{m,n}(x,y)$ are defined in [123], and related with integral solutions to Pascal's hexagon relations

$$f_{m-1,n} f_{m+1,n} + f_{m,n-1} f_{m,n+1} = f_{m-1,n-1} f_{m+1,n+1}, \qquad (m,n) \in \mathbb{Z}^2.$$

(10) Consider permutation

$$u_n = \begin{pmatrix} 1 & 2 & \ldots & n & n+1 & n+2 & n+3 & \ldots & 2n \\ 2 & 4 & \ldots & 2n & 1 & 3 & 5 & \ldots & 2n-1 \end{pmatrix},$$

[56] See, e.g., https://en.wikipedia.org/wiki/Bernoulli_number.

and set $u_n^{(k)} := 1^{2k+1} \times u_n$. Show that

$$\mathfrak{G}^{(\beta-1)}_{u_n^{(k)}}(x_i = 1, \forall\, i \geq 1) = (1+\beta)^{\binom{n+1}{2}} \mathfrak{G}^{((\beta)^2-1)}_{1^k \times w_0^{(n+1)}}(x_i = 1, \forall\, i \geq 1),$$

where $w_0^{(n+)}$ denotes the permutation $(n+1, n, n-1, \ldots, 2, 1)$.

(11) Let $n \geq 0$ be an integer. Consider permutation $u_n = 1^n \times 321 \in \mathbb{S}_{3+n}$. Show that

$$\mathfrak{S}_{u_n}(x_1 = t,\, x_i = 1, \forall\, i \geq 2) = \frac{1}{4}\binom{2n+2}{3} + \frac{n}{2}\binom{2n+2}{1} t + \frac{1}{2}\binom{2n+2}{1} t^2.$$

Consider permutation $v_n := 1^n \times 4321 \in \mathbb{S}_{n+4}$. Show that

$$\mathfrak{S}_{v_n}(x_1 = t,\, x_i = 1, \forall\, i \geq 2)$$
$$= \frac{1}{24}\binom{2n+4}{5}\binom{2n+2}{1} + \frac{1}{2}\binom{2n+4}{5} t + \frac{n}{4}\binom{2n+4}{3} t^2 + \frac{1}{4}\binom{2n+4}{3} t^3.$$

(12) Show that

$$\sum_{(a,b,c) \in (\mathbb{Z}_{\geq 0})^3} q^{a+b+c} \begin{bmatrix} a+b \\ b \end{bmatrix}_q \begin{bmatrix} a+c \\ c \end{bmatrix}_q \begin{bmatrix} b+c \\ b \end{bmatrix}_q = \frac{1}{(q;q)_\infty^3} \left(\sum_{k \geq 2} (-1)^k \binom{k}{2} q^{\binom{k}{2}-1} \right).$$

It is not difficult to see that the left hand side sum of the above identity counts the weighted number of plane partitions $\pi = (\pi_{ij})$ such that

$$\pi_{i,j} \geq 0, \qquad \pi_{ij} \geq \max(\pi_{i+1,j}, \pi_{i,j+1}), \qquad \pi_{ij} \leq 1 \quad \text{if} \quad i \geq 2 \quad \text{and} \quad j \geq 2,$$

and the weight $wt(\pi) := \sum_{i,j} \pi_{ij}$.

(13) Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p > 0)$ be a partition of size n. For an integer k such that $1 \leq k \leq n-p$ define a grassmannian permutation

$$w_\lambda^{(k)} = [1, \ldots, k, \lambda_p + k + 1, \lambda_{p-1} + k + 2, \ldots, \lambda_1 + k + p, a_1, \ldots, a_{n-p-k}],$$

where we denote by $(a_1 < a_2 < \cdots < a_{n-k-p})$ the complement $[1,n] \backslash (1, \ldots, k, \lambda_p + k + 1, \lambda_{p-1} + k + 2, \ldots, \lambda_1 + k + p)]$.

Show that the Grothendieck polynomial

$$G_\lambda(\beta) := \mathfrak{G}^{\beta-1}_{w_\lambda{}^k}(1^n)$$

is a polynomial of β with *nonnegative* coefficients. Clearly, $G_\lambda(1) = \dim V_\lambda^{\mathfrak{Gl}(k+\ell(\lambda))}$.

Find a combinatorial interpretations of polynomial $G_\lambda(\beta)$.

Final remark, it follows from the seventh exercise listed above, that the polynomials $\mathfrak{S}^{(\beta)}_{\sigma_k^{\pm}}(x_1 = q,\, x_j = 1, \forall\, j \geq 2)$ define a (q,β)-deformation of the number VSASM(k) (the case σ_k^+) and the number CSTCPP(k) (the case σ_k^-), respectively.

5.2.5 Specialization of Grothendieck polynomials

Let p, b, n and i, $2i < n$ be positive integers. Denote by $\mathcal{T}^{(i)}_{p,b,n}$ the *trapezoid*, i.e., a convex quadrangle having vertices at the points

$$(ip, i), \quad (ip, n-i), \quad (b + ip, i) \quad \text{and} \quad (b + (n-i)p, n-i).$$

Definition 5.61. Denote by $\mathrm{FC}_{b,p,n}^{(i)}$ the set of lattice path from the point (ip, i) to that $(b + (n-i)p, n-i)$ with east steps $E = (0,1)$ and north steps $N = (1,0)$, which are located inside of the trapezoid $\mathcal{T}_{p,b,n}^{(i)}$.

If $\mathfrak{p} \in \mathrm{FC}_{b,p,n}^{(i)}$ is a path, we denote by $p(\mathfrak{p})$ the number of *peaks*, i.e.,

$$p(\mathfrak{p}) = NE(\mathfrak{p}) + E_{\mathrm{in}}(\mathfrak{p}) + N_{\mathrm{end}}(\mathfrak{p}),$$

where $NE(\mathfrak{p})$ is equal to the number of steps NE resting on path $\mathfrak{p}$; $E_{\mathrm{in}}(\mathfrak{p})$ is equal to 1, if the path $\mathfrak{p}$ *starts* with step E and 0 otherwise; $N_{\mathrm{end}}(\mathfrak{p})$ is equal to 1, if the path $\mathfrak{p}$ *ends* by the step N and 0 otherwise.

Note that the equality $N_{\mathrm{end}}(\mathfrak{p}) = 1$ may happened only in the case $b = 0$.

Definition 5.62. Denote by $\mathrm{FC}_{b,p,n}^{(k)}$ the set of k-tuples $\mathfrak{P} = (\mathfrak{p}_1, \ldots, \mathfrak{p}_k)$ of *non-crossing* lattice paths, where for each $i = 1, \ldots, k$, $\mathfrak{p}_i \in \mathrm{FC}_{b,p,n}^{(i)}$.

Let

$$\mathrm{FC}_{b,p,n}^{(k)}(\beta) := \sum_{\mathfrak{P} \in \mathrm{FC}_{b,p,n}^{(k)}} \beta^{p(\mathfrak{P})}$$

denotes the generating function of the statistics $p(\mathfrak{P}) := \sum_{i=1}^{k} p(\mathfrak{p}_{\mathfrak{i}}) - k$.

Theorem 5.63. *The following equality holds*

$$\mathfrak{G}_{\sigma_{n,k,p,b}}^{(\beta)}(x_1 = 1, x_2 = 1, \ldots) = \mathrm{FC}_{p,b,n+k}^{(k)}(\beta + 1),$$

where $\sigma_{n,k,p,b}$ is a unique grassmannian permutation with shape $((n+1)^b, n^p, (n-1)^p, \ldots, 1^p)$ and flag $(\underbrace{k, \ldots, k}_{np+b})$.

5.3 The "longest element" and Chan–Robbins–Yuen polytope[57]

5.3.1 The Chan–Robbins–Yuen polytope $\mathcal{CRY}_n$

Assume additionally, cf. [133, Exercise 6.C8(d)], that the condition (a) in Definition 5.1 is replaced by that

(a') x_{ij} and x_{kl} *commute* for all i, j, k and l.

Consider the element $w_0^{(n)} := \prod_{1 \le i < j \le n} x_{ij}$. Let us bring the element $w_0^{(n)}$ to the reduced form, that is, let us consecutively apply the defining relations (a') and (b) to the element $w_0^{(n)}$ in *any* order until unable to do so. Denote the resulting polynomial by $Q_n(x_{ij}; \alpha, \beta)$. Note that the polynomial itself *depends* on the order in which the relations (a') and (b) are applied.

We denote by $Q_n(\beta)$ the specialization

$$x_{ij} = 1 \qquad \text{for all } i \text{ and } j,$$

of the polynomial $Q_n(x_{ij}; \alpha = 0, \beta)$.

[57]Some results of this section, e.g., Theorems 5.63 and 5.65, has been proved independently and in greater generality in [102].

Example 5.64.

$$
\begin{aligned}
&Q_3(\beta) = (2,1) = 1 + (\beta+1), \qquad Q_4(\beta) = (10,13,4) = 1 + 5(\beta+1) + 4(\beta+1)^2,\\
&Q_5(\beta) = (140,336,280,92,9) = 1 + 16(\beta+1) + 58(\beta+1)^2 + 56(\beta+1)^3 + 9(\beta+1)^4,\\
&Q_6(\beta) = 1 + 42(\beta+1) + 448(\beta+1)^2 + 1674(\beta+1)^3 + 2364(\beta+1)^4\\
&\qquad\quad + 1182(\beta+1)^5 + 169(\beta+1)^6,\\
&Q_7(\beta) = (1,99,2569,25587,114005,242415,248817,118587,22924,1156)_{\beta+1},\\
&Q_8(\beta) = (1,219,12444,279616,2990335,16804401,52421688,93221276,94803125,\\
&\qquad\quad 53910939,16163947,2255749,108900)_{\beta+1}.
\end{aligned}
$$

What one can say about the polynomial $Q_n(\beta) := Q_n(x_{ij};\beta)|_{x_{ij}=1,\,\forall\, i,j}$?

It is known, [133, Exercise 6.C8(d)], that the constant term of the polynomial $Q_n(\beta)$ is equal to the product of Catalan numbers $\prod\limits_{j=1}^{n-1} C_j$. It is not difficult to see that if $n \geq 3$, then $\mathrm{Coeff}_{[\beta+1]}(Q_n(\beta)) = 2^n - 1 - \binom{n+1}{2}$, see [131, $A002662$], for a number of combinatorial interpretations of the numbers $2^n - 1 - \binom{n+1}{2}$.

Theorem 5.65. *One has*

$$
Q_n(\beta-1) = \left(\sum_{m\geq 0} \iota(\mathcal{CRY}_{n+1}, m)\beta^m\right)(1-\beta)^{\binom{n+1}{2}+1},
$$

where $\mathcal{CRY}_m$ denotes the Chan–Robbins–Yuen polytope [20, 21]*, i.e., the convex polytope given by the following conditions:*

$$
\mathcal{CRY}_m = \left\{(a_{ij}) \in \mathrm{Mat}_{m\times m}(\mathbb{Z}_{\geq 0})\right\}
$$

such that

(1) $\sum_i a_{ij} = 1$, $\sum_j a_{ij} = 1$,

(2) $a_{ij} = 0$ *if* $j > i+1$.

Here for any integral convex polytope $\mathcal{P} \subset \mathbb{Z}^d$, $\iota(\mathcal{P}, n)$ denotes the number of integer points in the set $n\mathcal{P} \cap \mathbb{Z}^d$.

In particular, the polynomial $Q_n(\beta)$ does not depend on the order in which the relations (a') and (b) have been applied.

Now let us denote by $\widehat{Q}_n(q,t;\alpha,\beta)$ the specialization

$$
x_{ij} = 1, \quad i<j<n, \qquad \text{and} \qquad x_{i,n} = q \quad \text{if} \quad i = 2,\dots,n-1, \qquad x_{1,n} = t
$$

of the (reduced) polynomial $Q_n(x_{ij};\alpha,\beta)$ obtained by applying the relations (a') and (b) in a *certain* order. The polynomial $Q_n(x_{ij};\alpha,\beta)$ itself *depends* on the order selected. To define polynomials which are frequently appear in Section 5, we apply the rules (a) and (b) stated in Definition 5.1 to a given monomial $x_{i_1,j_1}\cdots x_{i_p,j_p} \in \widehat{\mathrm{ACYB}}_n(\alpha,\beta)$ consequently according to the order in which the monomial taken has been written. We set $Q_n(t,\alpha,\beta) := \widehat{Q}_n(q=t,t;\alpha,\beta)$.

Conjecture 5.66. *Let $n \geq 3$ and write*

$$
Q_n(t=1;\alpha,\beta) = \sum_{k\geq 0}(1+\beta)^k c_{k,n}(\alpha),
$$

then $c_{k,n}(\alpha) \in \mathbb{Z}_{\geq 0}[\alpha]$.

The polynomial $Q_n(t,\beta,\alpha=0)$ *has degree* $d_n := [\frac{(n-1)^2}{4}]$ *with respect to* β. *Write*

$$Q_n(t,\beta) := Q_n(t;\alpha=0,\beta) = t^{n-2} \sum_{k=0}^{d_n} c_n^{(k)}(t)\beta^k.$$

Then $c_n^{(d_n)}(1) = a_n^2$ *for some non-negative integer* a_n. *Moreover, there exists a polynomial* $a_n(t) \in \mathbb{N}[t]$ *such that*

$$c_n^{(d_n)}(t) = a_n(1)a_n(t), \qquad a_n(0) = a_{n-1}.$$

The all roots of the polynomial $Q_n(\beta)$ *belong to the set* $\mathbb{R}_{<-1}$.

For example,

$$\begin{aligned}
&Q_4(t=1;\alpha,\beta) = (1,5,4)_{\beta+1} + \alpha(5,7)_{\beta+1} + 3\alpha^2,\\
&Q_5(t=1;\alpha,\beta) = (1,16,58,56,9)_{\beta+1} + \alpha(16,109,146,29)_{\beta+1}\\
&\qquad\qquad\qquad\quad + \alpha^2(51,125,34)_{\beta+1} + \alpha^3(35,17)_{\beta+1},\\
&c_6^{(6)} = 13(2,3,3,3,2), \qquad c_7^{(9)}(t) = 34(3,5,6,6,6,5,3),\\
&c_8^{(12)}(t) = 330(13,27,37,43,45,45,43,37,27,13),\\
&Q_4(t,\beta,\alpha=0)t^{-1} = t^2 + (\beta+1)\big(3t+2t^2\big) + (\beta+1)^2(t+1)^2,\\
&\widehat{Q}_4(q,t;\alpha=0,\beta) = \big(qt^2+t^3+2qt^3+q^2t^3+q^3t^3+t^4+2qt^4+q^2t^4\big)\\
&\qquad + \big(2qt^2+2t^3+3qt^3+2q^2t^3+2t^4+2qt^4\big)\beta + \big(t^2+t^3\big)(q+t)\beta^2,\\
&\widehat{Q}_5(q,t;\alpha=0,\beta) = \big(3q^2t+q^3t+5qt^2+6q^2t^2+2q^3t^2+2t^3+10qt^3+10q^2t^3+6q^3t^3\\
&\qquad + 3q^4t^3+3q^5t^3+2q^6t^3+3t^4+11qt^4+11q^2t^4+8q^3t^4+5q^4t^4+3q^5t^4\\
&\qquad + 3t^5+9qt^5+9q^2t^5+6q^3t^5+3q^4t^5+2t^6+6qt^6+6q^2t^6+2q^3t^6\big)\\
&\qquad + \big(9q^2t+2q^3t+17qt^2+18q^2t^2+4q^3t^2+7t^3+31qt^3+29q^2t^3\\
&\qquad + 15q^3t^3+10q^4t^3+7q^5t^3+10t^4+31qt^4+29q^2t^4+18q^3t^4\\
&\qquad + 10q^4t^4+10t^5+24qt^5+21q^2t^5+10q^3t^5+6t^6+12qt^6+6q^2t^6\big)\beta\\
&\qquad + \big(9q^2t+q^3t+21qt^2+18q^2t^2+2q^3t^2+9t^3+34qt^3+28q^2t^3\\
&\qquad + 14q^3t^3+9q^4t^3+12t^4+30qt^4+24q^2t^4+12q^3t^4+12t^5+21qt^5\\
&\qquad + 12q^2t^5+6t^6+6qt^6\big)\beta^2 + \big(3q^2t+11qt^2+6q^2t^2+5t^3+15qt^3\\
&\qquad + 10q^2t^3+5q^3t^3+6t^4+11qt^4+6q^2t^4+6t^5+6qt^5+2t^6\big)\beta^3\\
&\qquad + \big(2qt^2+t^3+2qt^3+q^2t^3+t^4+qt^4+t^5\big)\beta^4.
\end{aligned}$$

Note that polynomials $\widehat{Q}_n(q,t;\alpha=0,\beta=0)$ give rise to a two parameters deformation of the product of Catalan's numbers $C_1C_2\cdots C_{n-1}$. Are there combinatorial interpretations of these polynomials and polynomials $\widehat{Q}_n(q,t;\alpha=0,\beta)$?

Comments 5.67. We *expect* that for each integer $n \geq 2$ the set

$$\Psi_{n+1} := \left\{ w \in \mathbb{S}_{2n-1} \,|\, \mathfrak{S}_w(1) = \prod_{j=1}^{n} \mathrm{Cat}_j \right\}$$

is *non empty*, whereas the set

$$\left\{ w \in \mathbb{S}_{2n-2} \,|\, \mathfrak{S}_w(1) = \prod_{j=1}^{n} \mathrm{Cat}_j \right\}$$

is *empty*. For example,

$$\Psi_4 = \{[1,5,3,4,2]\}, \qquad \Psi_5 = \{[1,5,7,3,2,6,4],\ [1,5,4,7,2,6,3]\},$$
$$\Psi_6 = \big\{w := [1,3,2,8,6,9,4,5,7], w^{-1}, \dots\big\}, \qquad \Psi_7 = \{???\},$$

but one can check that for $w = [2358, 10, 549, 12, 11] \in \mathbb{S}_{12}$

$$\mathfrak{S}_w(1) = 776160 = \prod_{j=2}^{6} \mathrm{Cat}_j.$$

More generally, for any positive integer N define

$$\kappa(N) = \min\{n \,|\, \exists\, w \in \mathbb{S}_n \text{ such that } \mathfrak{S}_w(1) = N\}.$$

It is clear that $\kappa(N) \le N + 1$.

Problem 5.68. *Compute the following numbers*

$$\kappa(n!), \quad \kappa\left(\prod_{j=1}^{n} \mathrm{Cat}_j\right), \quad \kappa(\mathrm{ASM}(n)), \quad \kappa\big((n+1)^{n-1}\big).$$

For example, $10 \le \kappa(\mathrm{ASM}(6) = 7436) \le 12$. Indeed, take $w = [716983254, 10, 12, 11] \in \mathbb{S}_{12}$. One can show that

$$\mathfrak{S}_w(x_1 = t,\, x_i = 1,\, \forall\, i \ge 2) = 13t^6(t+10)(15t+37),$$

so that $\mathfrak{S}_w(1) = ASM(6)$; $\kappa(6^4) = 9$, namely, one can take $w = [157364298]$.

Question 5.69. *Let N be a positive integer. Does there exist a vexillary (grassmannian?) permutation $w \in \mathbb{S}_n$ such that $n \le 2\kappa(N)$ and $\mathfrak{S}_w(1) = N$?*

For example, $w = [1,4,5,6,8,3,5,7] \in \mathbb{S}_8$ is a grassmannian permutation such that $\mathfrak{S}_w(1) = 140$, and $\mathfrak{R}_w(1,\beta) = (1,9,27,43,38,18,4)$.

Remark 5.70. We expect that for $n \ge 5$ there are *no* permutations $w \in \mathbb{S}_\infty$ such that $Q_n(\beta) = \mathfrak{S}_w^{(\beta)}(1)$.

The numbers $\mathfrak{C}_n := \prod_{j=1}^{n} \mathrm{Cat}_j$ appear also as the values of the *Kostant partition function* of the type A_{n-1} on some special vectors. Namely,

$$\mathfrak{C}_n = K_{\Phi(1^n)}(\gamma_n), \qquad \text{where} \quad \gamma_n = \left(1,2,3,\dots,n-1,-\binom{n}{2}\right),$$

see, e.g., [133, Exercise 6.C10], and [69, pp. 173–178]. More generally [69, Exercise g, p. 177, (7.25)], one has

$$K_{\Phi(1^n)}(\gamma_{n,d}) = pp^{\delta_n}(d)\mathfrak{C}_{n-1} = \prod_{j=d}^{n+d-2} \frac{1}{2j+1} \binom{n+d+j}{2j},$$

where $\gamma_{n,d} = (d+1, d+2, \ldots, d+n-1, -n(2d+n-1)/2)$, $pp^{\delta_n}(d)$ denotes the set of reversed (weak) plane partitions bounded by d and contained in the shape $\delta_n = (n-1, n-2, \ldots, 1)$. Clearly, $pp^{\delta_n}(1) = \prod_{1 \le i < j \le n} \frac{i+j+1}{i+j-1} = C_n$, where C_n is the n-th Catalan number[58].

Conjecture 5.71. *For any permutation $w \in \mathbb{S}_n$ there exists a graph $\Gamma_w = (V, E)$, possibly with multiple edges, such that the reduced volume $\widetilde{\mathrm{vol}}(\mathcal{F}_{\Gamma_w})$ of the flow polytope $\mathcal{F}_{\Gamma_w}$, see, e.g., [132] for a definition of the former, is equal to $\mathfrak{S}_w(1)$.*

For a family of vexillary permutations $w_{n,p}$ of the shape $\lambda = p\delta_{n+1}$ and flag $\phi = (1, 2, \ldots, n-1, n)$ the corresponding graphs $\Gamma_{n,p}$ have been constructed in [101, Section 6]. In this case the reduced volume of the flow polytope $\mathcal{F}_{\Gamma_{n,p}}$ is equal to the Fuss–Catalan number

$$\frac{1}{1+(n+1)p}\binom{(n+1)(p+1)}{n+1} = \mathfrak{S}_{w_{n,p}}(1),$$

cf. Corollary 5.33.

Exercises 5.72.

(*a*) Show that the polynomial $R_n(t) := t^{1-n} Q_n(t;0,0)$ is symmetric (*unimodal?*), and $R_n(0) = \prod_{k=1}^{n-2} \mathrm{Cat}_k$. For example,

$$R_4(t) = (1+t)\big(2+t+2t^2\big), \qquad R_5(t) = 2(5,10,13,14,13,10,5)_t,$$
$$R_6(t) = 10(2,3,2)_t(7,7,10,13,10,13,10,7,7)_t, \qquad R_7(t) = 30\big(196 + \cdots + 196t^{15}\big).$$

Note that $R_n(1) = \prod_{k=1}^{n-1} \mathrm{Cat}_k$.

(*b*) More generally, write as before,

$$Q_n(t;0,\beta) = t^{n-2} \sum_{k \ge 0} c_n^{(k)}(t) \beta^k.$$

Show that the polynomials $c_n^{(k)}(t)$ are symmetric (unimodal?) for all k and n.

(*c*) Consider a reduced polynomial $\overline{R}_n(\{x_{ij}\})$ of the element

$$\prod_{\substack{1 \le i < j \le n \\ (i,j) \ne (n-1,n)}} x_{ij} \in \widehat{\mathrm{ACYB}}(\alpha = \beta = 0)^{ab},$$

see Definition 5.1. Here we assume additionally, that all elements $\{x_{ij}\}$ are mutually commute. Define polynomial $\widetilde{R}_n(q,t)$ to be the following specialization

$$x_{ij} \longrightarrow 1 \quad \text{if} \quad i < j < n-1, \qquad x_{i,n-1} \longrightarrow q, \qquad x_{i,n} \longrightarrow t, \quad \forall\, i$$

of the polynomial $\overline{R}_n(\{x_{ij}\})$ in question. Show that polynomials $\widetilde{R}_n(q,t)$ are well-defined, and

$$\widetilde{R}_n(q,t) = \widetilde{R}_n(t,q).$$

[58] For example, if $n = 3$, there exist 5 reverse (weak) plane partitions of shape $\delta_3 = (2,1)$ bounded by 1, namely reverse plane partitions $\left\{ \begin{pmatrix} 0 & 0 \\ 0 & \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & \end{pmatrix} \right\}$.

Examples 5.73.

$$\begin{aligned}
R_4(t,\beta) &= (2,3,3,2)_t + (4,5,4)_t\beta + (2,2)_t\beta^2,\\
R_5(t,\beta) &= (10,20,26,28,26,20,10)_t + (33,61,74,74,61,33)_t\beta + (39,65,72,65,39)_t\beta^2\\
&\quad + (19,27,27,19)_t\beta^3 + (3,3,3)_t\beta^4,\\
R_6(t,\beta) &= (140,350,550,700,790,820,790,700,550,350,140)_t\\
&\quad + (686,1640,2478,3044,3322,3322,3044,2478,1640,686)_t\beta\\
&\quad + (1370,3106,4480,5280,5537,5280,4480,3106,1370)_t\beta^2\\
&\quad + (1420,3017,4113,4615,4615,4113,3017,1420)_t\beta^3\\
&\quad + (800,1565,1987,2105,1987,1565,800)_t\beta^4 + (230,403,465,465,403,230)_t\beta^5\\
&\quad + (26,39,39,39,26)_t\beta^6,\\
R_6(1,\beta) &= (5880,22340,34009,26330,10809,2196,169)_\beta,\\
R_7(t,\beta) &= (5880,17640,32340,47040,59790,69630,76230,79530,79530,76230,69630,\\
&\quad\ 59790,47040,32340,17640,5880)_t + (39980,116510,208196,295954,368410,\\
&\quad\ 420850,452226,462648,452226,420850,368410,295954,208196,116510,\\
&\quad\ 39980)_t\beta + (118179,333345,578812,802004,975555,1090913,1147982,\\
&\quad\ 1147982,1090913,975555,802004,578812,333345,118179)_t\beta^2\\
&\quad + (198519,539551,906940,1221060,1447565,1580835,1624550,1580835,\\
&\quad\ 1447565,1221060,906940,539551,198519)_t\beta^3\\
&\quad + (207712,540840,875969,1141589,1314942,1398556,1398556,1314942,\\
&\quad\ 1141589,875969,540840,207712)_t\beta^4\\
&\quad + (139320,344910,535107,671897,749338,773900,749338,671897,535107,\\
&\quad\ 344910,139320)_t\beta^5 + (59235,137985,203527,244815,263389,263389,244815,\\
&\quad\ 203527,137985,59235)_t\beta^6 + (15119,32635,45333,51865,53691,51865,45333,\\
&\quad\ 32635,15119)_t\beta^7 + (2034,3966,5132,5532,5532,5132,3966,2034)\beta^8\\
&\quad + (102,170,204,204,204,170,102)_t\beta^9,\\
R_7(1,\beta) &= (776160,4266900,10093580,13413490,10959216,5655044,1817902,\\
&\quad\ 343595,33328,1156)_\beta.
\end{aligned}$$

5.3.2 The Chan–Robbins–Mészáros polytope $\mathcal{P}_{n,m}$

Let $m \geq 0$ and $n \geq 2$ be integers, consider the reduced polynomial $Q_{n,m}(t,\beta)$ corresponding to the element

$$M_{n.m} := \left(\prod_{j=2}^{n} x_{1j}\right)^{m+1} \prod_{j=2}^{n-2}\prod_{k=j+2}^{n} x_{jk}.$$

For example,

$$\begin{aligned}
Q_{2,4}(t,\beta) &= (4,7,9,10,10,9,7,4)_t + (10,17,21,22,21,17,10)_t\beta\\
&\quad + (8,13,15,15,13,8)_t\beta^2 + (2,3,3,3,2)_t\beta^3,\\
Q_{2,4}(1,\beta) &= (60,118,72,13)_\beta,\\
Q_{2,5}(t,\beta) &= (60,144,228,298,348,378,388,378,348,298,228,144,60)_t
\end{aligned}$$

$$
\begin{aligned}
&+(262,614,948,1208,1378,1462,1462,1378,1208,948,614,262)_t\beta\\
&+(458,1042,1560,1930,2142,2211,2142,1930,1560,1042,458)_t\beta^2\\
&+(405,887,1278,1526,1640,1640,1526,1278,887,405)_t\beta^4\\
&+(187,389,534,610,632,610,534,389,187)_t\beta^4\\
&+(41,79,102,110,110,102,79,41)_t\beta^5+(3,5,6,6,6,5,3)_t\beta^6,
\end{aligned}
$$

$$
\begin{aligned}
Q_{2,5}(1,\beta) &= (3300,11744,16475,11472,4072,664,34)_\beta,\\
Q_{2,6}(1,\beta) &= (660660,3626584,8574762,11407812,9355194,4866708,1589799,\\
&\quad 310172,32182,1320)_\beta,\\
Q_{2,7}(\beta) &= (1,213,12145,279189,3102220,18400252,61726264,120846096,139463706,\\
&\quad 93866194,5567810,7053370,626730,16290)_{\beta+1}.
\end{aligned}
$$

Theorem 5.74. *One has*

$$
\begin{aligned}
&Q_{m,n}(1,1)=\prod_{k=1}^{n-2}\mathrm{Cat}_k\prod_{1\le i<j\le n-1}\frac{2(m+1)+i+j-1}{i+j-1},\\
&\sum_{k\ge 0}\iota(\mathcal{P}_{n,m};k)\beta^k=\frac{Q_{m,n}(1,\beta-1)}{(1-\beta)^{\binom{n+1}{2}+1}},
\end{aligned}
$$

where $\mathcal{P}_{n,m}$ denotes the generalized Chan–Robbins–Yuen polytope defined in [101], and for any integral convex polytope $\mathcal{P}$, $\iota(\mathcal{P},k)$ denotes the Ehrhart polynomial of polytope $\mathcal{P}$.

Conjecture 5.75. *Let $n\ge 3$, $m\ge 0$ be integers, , write*

$$
Q_{m,n}(t,\beta)=\sum_{k\ge 0}c_{m,n}^{(k)}(t)\beta^k,\qquad \text{and set}\qquad b(m,n):=\max\big(k\,|\,c_{m,n}^{(k)}(t)\ne 0\big).
$$

Denote by $\tilde{c}_{m,n}(t)$ the polynomial obtained from that $c_{m,n}^{(b(m,n))}(t)$ by dividing the all coefficients of the latter on their GCD. Then

$$
\tilde{c}_{n,m}(t)=a_{n+m}(t),
$$

where the polynomials $a_n(t):=c_{0,n}(t)$ have been defined in Conjecture 5.66.

For example,

$$
\begin{aligned}
&c_{2,5}(t)=4a_7(t),\qquad c_{2,6}(t)=10a_8(t),\qquad c_{3,5}(t)=a_8(t),\\
&c_{2,7}(t)=10(34,78,118,148,168,178,181,178,168,148,118,78,34)\overset{?}{=}10a_9(t).
\end{aligned}
$$

It is known [69, 99, 100] that

$$
\begin{aligned}
\prod_{k=1}^{n-2}\mathrm{Cat}_k\prod_{1\le i<j\le n-1}\frac{2(m+1)+i+j-1}{i+j-1}&=\prod_{j=m+1}^{m+n-2}\frac{1}{2j+1}\binom{n+m+j}{2j}\\
&=K_{A_{n+1}}\left(m+1,m+2,\dots,n+m,-mn-\binom{n}{2}\right).
\end{aligned}
$$

Conjecture 5.76. *Let $\boldsymbol{a}=(a_2,a_3,\dots,a_n)$ be a sequence of non-negative integers, consider the following element*

$$
M_{(\boldsymbol{a})}=\left(\prod_{j=2}^{n}x_{1j}^{a_j}\right)\prod_{j=2}^{n-1}\left(\prod_{k=j+1}^{n}x_{jk}\right).
$$

Let $R_{\boldsymbol{a}}(t_1,\dots,t_{n-1},\alpha,\beta)$ be the following specialization $x_{ij} \longrightarrow t_{j-1}$ for all $1 \le i < j \le n$ of the reduced polynomial $R_{\boldsymbol{a}}(x_{ij})$ of monomial $M_{\boldsymbol{a}} \in \widehat{\mathrm{ACYB}}_n(\alpha,\beta)$. Then the polynomial $R_{\boldsymbol{a}}(t_1,\dots,t_{n-1},\alpha,\beta)$ is well-defined, i.e., does not depend on an order in which relations (a') and (b), Definition 5.1, have been applied.

$$Q_{M_{\boldsymbol{a}}}(1,\beta=0) = K_{A_{n+1}}\left(a_2+1, a_3+2,\dots,a_n+n-1,-\binom{n}{2}-\sum_{j=2}^{n} a_j\right).$$

Write

$$Q_{M_{\boldsymbol{a}}}(t,\beta) = \sum_{k\ge 0} c_{\boldsymbol{a}}^{(k)}(t)\beta^k.$$

The polynomials $c_{\boldsymbol{a}}^{(k)}(t)$ are symmetric (unimodal?) for all k.

Example 5.77. Let's take $n=5$, $\boldsymbol{a}=(2,1,1,0)$. One can show that the value of the Kostant partition function $K_{A_5}(3,3,4,4,-14)$ is equal to 1967. On the other hand, one has

$$\begin{aligned}
Q_{(2,1,1,0)}(t,\beta)t^{-3} &= (50,118,183,233,263,273,263,233,183,118,50)_t \\
&\quad + (214,491,738,908,992,992,908,738,491,214)_t\beta \\
&\quad + (365,808,1167,1379,1448,1379,1167,808,365)_t\beta^2 \\
&\quad + (313,661,906,1020,1020,906,661,313)_t\beta^3 \\
&\quad + (139,275,351,373,351,275,139)_t\beta^4 \\
&\quad + (29,52,60,60,52,29)_t\beta^5 + (2,3,3,3,2)_t\beta^6, \\
Q_{(2,1,1,0)}(1,\beta) &= (1967,6686,8886,5800,1903,282,13) = (1,34,279,748,688,204,13)_{\beta+1}.
\end{aligned}$$

It might be well to point out that since we know, see Theorem 5.63, that polynomials $Q_{M_{\boldsymbol{a}}}(1,\beta)$ in face are polynomials of $\beta+1$ with non-negative integer coefficients, we can treat the polynomial $\widetilde{Q}_{M_{\boldsymbol{a}}}(\beta) := Q_{M_{\boldsymbol{a}}}(1,\beta-1)$ as a β-analogue of the Kostant partition function in the dominant chamber. It seems an interesting *problem* to find an interpretation of polynomials $\widetilde{Q}_{M_{\boldsymbol{a}}}(\beta)$ in the framework of the representation theory of Lie algebras. For example,

$$\begin{aligned}
&\widetilde{Q}_{(2,1,1,0)}(\beta) = (1,34,279,748,688,204,13)_\beta, \\
&\widetilde{Q}_{(2,1,1,0)}(\beta=1) = 1967 = K_{A_5}(3,3,4,4,-14).
\end{aligned}$$

Exercises 5.78.

(1) Show that

$$R_n(t,-1) = t^{2(n-2)} R_{n-1}\big(-t^{-1},1\big).$$

(2) Show that the ratio

$$\frac{R_n(0,\beta)}{(1+\beta)^{n-2}}$$

is a polynomial in $(\beta+1)$ with non-negative coefficients.

(3) Show that polynomial $R_n(t,1)$ has degree $e_n := (n+1)(n-2)/2$, and

$$\mathrm{Coeff}[t^{e_n}]R_n(t,1) = \prod_{k=1}^{n-1} \mathrm{Cat}_k.$$

(4) Show that

$$\widetilde{Q}_{(n,2,3,0)}(\beta) = \left(1, 3n+2, \binom{n+1}{2} + n, \binom{n+1}{3} + \binom{n}{2}\right)_{\beta},$$

and

$$K_{A_4}(n,3,4,-n-7) = \frac{(n+2)(n+3)(n+9)}{6}.$$

Problems 5.79.

(1) *Assume additionally to the conditions* (a') *and* (b) *above that*

$$x_{ij}^2 = \beta x_{ij} + 1 \qquad \textit{if} \quad 1 \le i < j \le n.$$

What one can say about a reduced form of the element w_0 *in this case?*

(2) *According to a result by S. Matsumoto and J. Novak [97], if* $\pi \in \mathbb{S}_n$ *is a permutation of the cyclic type* $\lambda \vdash n$*, then the total number of primitive factorizations (see definition in [97]) of* π *into product of* $n-\ell(\lambda)$ *transpositions, denoted by* $\mathrm{Prim}_{n-\ell(\lambda)}(\lambda)$*, is equal to the product of Catalan numbers:*

$$\mathrm{Prim}_{n-\ell(\lambda)}(\lambda) = \prod_{i=1}^{\ell(\lambda)} \mathrm{Cat}_{\lambda_i - 1}.$$

Recall that the Catalan number $\mathrm{Cat}_n := C_n = \frac{1}{n}\binom{2n}{n}$*. Now take* $\lambda = (2,3,\ldots,n+1)$*. Then*

$$Q_n(1) = \prod_{a=1}^{n} \mathrm{Cat}_a = \mathrm{Prim}_{\binom{n}{2}}(\lambda).$$

Does there exist "a natural" bijection between the primitive factorizations and monomials which appear in the polynomial $Q_n(x_{ij};\beta)$*?*

(3) *Compute in the algebra* $\widehat{\mathrm{ACYB}}_n(\alpha,\beta)$ *the specialization*

$$x_{ij} \longrightarrow 1, \quad j < n, \qquad x_{ij} \longrightarrow t, \quad 1 \le i < n,$$

denoted by $P_{w_n}(t,\alpha,\beta)$*, of the reduced polynomial* $P_{s_{ij}}(\{x_{ij}\},\alpha,\beta)$ *corresponding to the transposition*

$$s_{ij} := \left(\prod_{k=i}^{j-2} x_{k,k+1}\right) x_{j-1,j} \left(\prod_{k=j-2}^{i} x_{k,k+1}\right) \in \widehat{\mathrm{ACYB}}_n(\alpha,\beta).$$

For example,

$$P_{s_{14}}(t,\alpha,\beta) = t^5 + 3(1+\beta)t^4 + ((3,5,2)_\beta + 3\alpha)t^3 + (2(1+\beta)^2 + \alpha(5+4\beta))t^2$$
$$+ ((1+\beta((1+3\alpha)+2\alpha^2)t + \alpha + \alpha^2.$$

5.4 Reduced polynomials of certain monomials

In this subsection we compute the reduced polynomials corresponding to *dominant* monomials of the form

$$x_{\boldsymbol{m}} := x_{1,2}^{m_1} x_{23}^{m_2} \cdots x_{n-1,n}^{m_{n-1}} \in \left(\widehat{\mathrm{ACYB}}_n(\beta)\right)^{ab},$$

where $\boldsymbol{m} = (m_1 \geq m_2 \geq \cdots \geq m_{n-1} \geq 0)$ is a *partition*, and we apply the relations (a') and (b) in the algebra $(\widehat{\mathrm{ACYB}}_n(\beta))^{ab}$, see Definition 5.1 and Section 5.3.1, *successively*, starting from $x_{12}^{m_1} x_{23}$.

Proposition 5.80. *The function*

$$\mathbb{Z}_{\geq 0}^{n-1} \longrightarrow \mathbb{Z}_{\geq 0}^{n-1}, \qquad \boldsymbol{m} \longrightarrow P_{\boldsymbol{m}}(t=1; \beta=1)$$

can be extended to a piece-wise polynomial function on the space $\mathbb{R}_{\geq 0}^{n-1}$.

We start with the study of powers of Coxeter elements. Namely, for powers of *Coxeter* elements, one has[59]

$$\begin{aligned}
&P_{(x_{12}x_{23})^2}(\beta) = (6,6,1), \qquad P_{(x_{12}x_{23}x_{34})^2}(\beta) = (71,142,91,20,1) = (1,16,37,16,1)_{\beta+1},\\
&P_{(x_{12}x_{23}x_{34})^3}(\beta) = (1301,3903,4407,2309,555,51,1) = (1,45,315,579,315,45,1)_{\beta+1},\\
&P_{(x_{12}x_{23}x_{34}x_{45})^2}(\beta) = (1266,3798,4289,2248,541,50,1) = (1,44,306,564,306,44,1)_{\beta+1},\\
&P_{(x_{12}x_{23}x_{34})^3}(\beta=1) = 12527, \qquad P_{(x_{12}x_{23}x_{34})^4}(\beta=0) = 26599,\\
&P_{(x_{12}x_{23}x_{34})^4}(\beta=1) = 539601, \qquad P_{(x_{12}x_{23}x_{34}x_{45})^2}(\beta=1) = 12193,\\
&P_{(x_{12}x_{23}x_{34}x_{45})^3}(\beta=0) = 50000, \qquad P_{(x_{12}x_{23}x_{34}x_{45})^3}(\beta=1) = 1090199.
\end{aligned}$$

Lemma 5.81. *One has*

$$P_{x_{12}^n x_{23}^m}(\beta) = \sum_{k=0}^{\min(n,m)} \binom{n+m-k}{m}\binom{m}{k}\beta^k = \sum_{k=0}^{\min(n,m)} \binom{n}{k}\binom{m}{k}(1+\beta)^k.$$

Moreover,

- *polynomial* $P_{(x_{12}x_{23}\cdots x_{n-1,n})^m}(\beta-1)$ *is a symmetric polynomial in* β *with non-negative coefficients.*
- *polynomial* $P_{x_{12}^n x_{23}^m}(\beta)$ *counts the number of* (n,m)*-Delannoy paths according to the number of* NE *steps*[60].

Proposition 5.82. *Let* n *and* k, $0 \leq k \leq n$, *be integers. The number*

$$P_{(x_{12}x_{23})^n (x_{34})^k}(\beta=0)$$

is equal to the number of n *up,* n *down permutations in the symmetric group* $\mathbb{S}_{2n+k+1}$, *see* [131, *A*229892] *and Exercises* 5.30(2).

Conjecture 5.83. *Let* n, m, k *be nonnegative integers. Then the number*

$$P_{x_{12}^n x_{23}^m x_{34}^k}(\beta=0)$$

is equal to the number of n *up,* m *down and* k *up permutations in the symmetric group* $\mathbb{S}_{n+m+k+1}$.

[59] To simplify notation we set $P_w(\beta) := P_w(x_{ij}=1;\beta)$.

[60] Recall that a (n,m)-Delannoy path is a lattice paths from $(0,0)$ to (n,m) with steps $E=(1,0)$, $N=(0,1)$ and $NE=(1,1)$ only. For the definition and examples of the Delannoy paths and numbers, see [131, *A*001850, *A*008288] and http://mathworld.wolfram.com/DelannoyNumber.html.

For example,

- Take $n=2$, $k=0$, the six permutations in $\mathbb{S}_5$ with 2 up, 2 down are 12543, 13542, 14532, 23541, 24531, 34521.
- Take $n=3$, $k=1$, the twenty permutations in $\mathbb{S}_7$ with 3 up, 3 down are 1237654, 1247653, 1257643, 1267543, 1347652, 1357642, 1367542, 1457632, 1467532, 1567432, 2347651, 2357641, 2367541, 2457631, 2467531, 2567431, 3457621, 3467521, 3567421, 4567321, see [131, *A*229892].
- Take $n=3$, $m=2$, $k=1$, the number of 3 up, 2 down and 1 up permutations in $\mathbb{S}_7$ is equal to $50 = P_{321}(0)$: 1237645, 1237546, ..., 4567312.
- Take $n=1$, $m=3$, $k=2$, the number of 1 up, 3 down and 2 up permutations in $\mathbb{S}_7$ is equal to $55 = P_{132}(0)$, as it can be easily checked.

On the other hand, $P_{x_{12}^4 x_{23}^3 x_{34}^2 x_{45}}(\beta=0) = 7203 < 7910$, where 7910 is the number of 4 up, 3 down, 2 up and 1 down permutations in the symmetric group $\mathbb{S}_{11}$.

Conjecture 5.84. *Let $k_1,\dots,k_{n-1}$ be a sequence of non-negative integer numbers, consider monomial $M := x_{12}^{k_1} x_{23}^{k_2} \cdots x_{n-1,n}^{k_{n-1}}$. Then reduced polynomial $P_M(\beta-1)$ is a unimodal polynomial in β with non-negative coefficients.*

Example 5.85.

$$
\begin{aligned}
&P_{3,2,1}(\beta) = (1,14,27,8)_{\beta+1} = P_{1,2,3}(\beta), \qquad P_{2,3,1}(\beta) = (1,15,30,9)_{\beta+1} = P_{1,3,2}(\beta),\\
&P_{3,1,2}(\beta) = (1,11,18,4)_{\beta+1} = P_{2,1,3}(\beta),\\
&P_{4,3,2,1}(\beta) = (1,74,837,2630,2708,885,68)_{\beta+1}, \qquad P_{4,3,2,1}(0) = 7203 = 3\cdot 7^4,\\
&P_{5,4,3,2,1}(\beta) = (1,394,19177,270210,1485163,3638790,4198361,2282942,\\
&\qquad\qquad 553828,51945,1300)_{\beta+1},\\
&P_{5,4,3,2,1}(0) = 12502111 = 1019\times 12269.
\end{aligned}
$$

Exercises 5.86.

(1) Show that if $n \ge m$, then

$$
x_{ij}^n x_{jk}^m \Big|_{x_{ij}=1=x_{jk}} = \sum_{a=0}^{n} \binom{m+a-1}{a} \left(\sum_{p=0}^{n-a} \binom{m}{p} \beta^p \right) x_{ik}^{m+a}.
$$

(2) Show that if $n \ge m \ge k$, then

$$
\begin{aligned}
P_{x_{12}^n x_{23}^m x_{34}^k}(\beta) &= P_{x_{12}^n x_{23}^m}(\beta)\\
&\quad + \sum_{\substack{a\ge 1\\ b,p\ge 0}} \binom{m}{p}\binom{k}{a}\binom{a-1}{b}\binom{n+1}{p+a-b}\binom{m+a-1-b}{a}(\beta+1)^{p+a}.
\end{aligned}
$$

In particular, if $n \ge m \ge k$, then

$$
P_{x_{12}^n x_{23}^m x_{34}^k}(0) = \binom{m+n}{n} + \sum_{a\ge 1}\binom{k}{a}\left(\sum_{b=1}^{a}\binom{m+n+1}{m+b}\binom{a-1}{b-1}\binom{m+b-1}{a}\right).
$$

Note that the set of relations from the item (1) allows to give an explicit formula for the polynomial $P_M(\beta)$ for any *dominant* sequence $M = (m_1 \ge m_2 \ge \cdots \ge m_k) \in (\mathbb{Z}_{>0})^k$. Namely,

$$
P_M(\beta+1) = \sum_{\boldsymbol{a}} \prod_{j=2}^{k} \binom{m_j + a_{j-1} - 1}{a_{j-1}} \left(\sum_{\boldsymbol{b}} \prod_{j=1}^{k-1} \binom{m_{j+1}}{b_j} \beta^{b_j} \right),
$$

where the first sum runs over the following set $\mathcal{A}(M)$ of integer sequences $\boldsymbol{a}=(a_1,\ldots,a_{k-1})$

$$\mathcal{A}(M):=\{0\le a_j\le m_j+a_{j-1},\ j=1,\ldots,k-1\},\qquad a_0=0,$$

and the second sum runs over the set $\mathcal{B}(M)$ of all integer sequences $\boldsymbol{b}=(b_1,\ldots,b_{k-1})$

$$\mathcal{B}(M):=\bigcup_{\boldsymbol{a}\in\mathcal{A}(M)}\{0\le b_j\le\min(m_{j+1},m_j-a_j+a_{j-1})\},\qquad j=1,\ldots,k-1.$$

(3) Show that

$$\#|\mathcal{A}(n,1^{k-1})|=\frac{n+1}{k}\binom{2k+n}{k-1}=f^{(n+k,k)},$$

where $f^{(n+k,k)}$ denotes the number of standard Young tableaux of shape $(n+k,k)$. In particular, $\#|\mathcal{A}(1^k)|=C_{k+1}$.

(4) Let $n\ge m\ge 1$ be integers and set $M=(n,m,1^k)$. Show that

$$P_M(x_{ij}=1;\beta=0)=\sum_{p=0}^{n}\frac{m+p+1}{k}\binom{m+p-1}{p}\binom{m+2k+p}{k-1}:=P_k(n,m).$$

In particular, $P_1(n,m)=\binom{n+m}{n}+m\binom{n+m+1}{n}$,

$$P_k(n,1)=\frac{n+1}{k+1}\binom{2k+2+n}{k},\qquad P_k(2,2)=\left(79k^2+341k+360\right)\frac{(2k+2)!}{k!(k+5)!}.$$

Let us remark that

$$P_k(n,1)=\frac{n+1}{n+k+2}\binom{2(k+1)+n}{k+1}=F_{k+1}^{(2)}(n)=D(k,1,n,2),$$

where the $D(k,1,n,2)$ and $F_{k+1}^{(2)}(n)$ are defined in Section 5.2.4.

(5) Let $T\in\mathrm{STY}((n+k,k))$ be a standard Young tableau of shape $(n+k,k)$. Denote by $r(T)$ the number of integers $j\in[1,n+k]$ such that the integer j belongs to the second row of tableau T, whereas the number $j+1$ belongs to the first row of T.

Show that

$$P_{x_{12}^n x_{23}\cdots x_{k+1,k+2}}(\beta-1)=\sum_{T\in\mathrm{STY}((n+k,k))}\beta^{r(T)}.$$

(6) Let $M=(m_1,m_2,\ldots,m_{k-1})\in\mathbb{Z}_{>0}^{k-1}$ be a composition. Denote by $\overleftarrow{M}$ the composition $(m_{k-1},m_{k-2},\ldots,m_2,m_1)$, and set for short

$$P_M(\beta):=P_{\prod_{i=1}^{k-1}x_{i,i+1}^{m_i}}(x_{ij}=1;\beta).$$

Show that $P_M(\beta)=P_{\overleftarrow{M}}(\beta)$. Note that in general,

$$P_{\prod\limits_{i=1}^{k-1}x_{i,i+1}^{m_i}}(x_{ij};\beta)\ne P_{\prod\limits_{i=1}^{k-1}x_{i,i+1}^{m_{k-i}}}(x_{ij};\beta).$$

(7) Define polynomial $P_M(t,\beta)$ to be the following specialization

$$x_{ij}\longrightarrow 1,\quad i<j<n,\qquad x_{in}\longrightarrow t,\quad i=1,\ldots,n-1$$

of a polynomial $P_{\prod\limits_{i=1}^{k-1} x_{i,i+1}^{m_i}}(x_{ij};\beta)$.

Show that if $n \geq m$, then

$$P_{x_{12}^n x_{23}^m}(t,\beta) = \sum_{j=0}^{m} \binom{m}{j} \left(\sum_{k=m-1}^{n+m-j-1} \binom{k}{m-1} t^{k-m+1} \right) \beta^j.$$

See Lemma 5.31 for the case $t=1$.

(8) Define polynomials $\widetilde{R}_n(t)$ as follows

$$\widetilde{R}_n(t) := P_{(x_{12}x_{23}x_{34})^n}\big(-t^{-1}, \beta=-1\big)(-t)^{3n}.$$

Show that polynomials $\widetilde{R}_n(t)$ have non-negative coefficients, and

$$\widetilde{R}_n(0) = \frac{(3n)!}{6(n!)^3}.$$

(9) Consider reduced polynomial $P_{n,2,2}(\beta)$ corresponding to monomial $x_{12}^n(x_{23}x_{34})^2$ and set $\tilde{P}_{n,2,2}(\beta) := P_{n,2,2}(\beta-1)$. Show that

$$\tilde{P}_{n,2,2}(\beta) \in \mathbb{N}[\beta] \qquad \text{and} \qquad \tilde{P}_{n,2,2}(1) = T(n+5,3),$$

where the numbers $T(n,k)$ are defined in [131, A110952, A001701].

Conjecture 5.87. *Let λ be a partition. The element $s_\lambda(\theta_1^{(n)},\dots,\theta_m^{(n)})$ of the algebra $3T_n^{(0)}$ can be written in this algebra as a sum of*

$$\left(\prod_{x\in\lambda} h(x)\right) \times \dim V_{\lambda'}{}^{(\mathfrak{gl}(n-m))} \times \dim V_{\lambda}{}^{(\mathfrak{gl}(m))}$$

monomials with all coefficients are equal to 1.

Here $s_\lambda(x_1,\dots,x_m)$ denotes the Schur function corresponding to the partition λ and the set of variables $\{x_1,\dots,x_m\}$; for $x\in\lambda$, $h(x)$ denotes the hook length corresponding to a box x; $V_\lambda^{(\mathfrak{gl}(n))}$ denotes the highest weight λ irreducible representation of the Lie algebra $\mathfrak{gl}(n)$.

Problems 5.88.

(1) *Define a bijection between monomials of the form $\prod\limits_{a=1}^{s} x_{i_a,j_a}$ involved in the polynomial $P(x_{ij};\beta)$, and dissections of a convex $(n+2)$-gon by s diagonals, such that no two diagonals intersect their interior.*

(2) *Describe permutations $w\in\mathbb{S}_n$ such that the Grothendieck polynomial $\mathfrak{G}_w(t_1,\dots,t_n)$ is equal to the "reduced polynomial" for a some monomial in the associative quasi-classical Yang–Baxter algebra $\widehat{\mathrm{ACYB}}_n(\beta)$.*

(3) *Study "reduced polynomials" corresponding to the monomials*

- *transposition: $s_{1n} := (x_{12}x_{23}\cdots x_{n-2,n-1})^2 x_{n-1,n}$,*
- *powers of the Coxeter element: $(x_{12}x_{23}\cdots x_{n-1,n})^k$,*

in the algebra $\widehat{\mathrm{ACYB}}_n(\alpha,\beta)^{ab}$.

(4) *Construct a bijection between the set of k-dissections of a convex $(n+k+1)$-gon and "pipe dreams" corresponding to the Grothendieck polynomial $\mathfrak{G}^{(\beta)}_{\pi_k^{(n)}}(x_1,\dots,x_n)$. As for a definition of "pipe dreams" for Grothendieck polynomials, see [78] and [42].*

Comments 5.89. We don't know any "good" combinatorial interpretation of polynomials which appear in Problem 5.88(3) for general n and k. For example,

$$P_{s_{13}}(x_{ij}=1;\beta)=(3,2)_{\beta}, \qquad P_{s_{14}}(x_{ij}=1;\beta)=(26,42,19,2)_{\beta},$$
$$P_{s_{15}}(x_{ij}=1;\beta)=(381,988,917,362,55,2)_{\beta}, \qquad P_{s_{15}}(x_{ij}=1;1)=2705.$$

On the other hand,

$$P_{(x_{12}x_{23})^2x_{34}(x_{45})^2}(x_{ij}=1;\beta)=(252,633,565,212,30,1),$$

that is in deciding on different reduced decompositions of the transposition s_{1n}. one obtains in general different reduced polynomials.

One can compare these formulas for polynomials $P_{s_{ab}}(x_{ij}=1;\beta)$ with those for the β-Grothendieck polynomials corresponding to transpositions (a,b), see Comments 5.37.

5.4.1 Reduced polynomials, Motzkin and Riordan numbers

In this subsection we investigate reduced polynomials associated with Coxeter element $C_n = u_{12}u_{23}\cdots u_{n-1,n}$ in commutative algebra $\widehat{\mathrm{ACYB}}_n(\alpha,\beta)$ in more detail. Recall that this algebra is generated over the ring $\mathbb{Z}[z,\alpha,\beta]$ by the set of elements $\{u_{i,j},\ 1\le i<j\le n\}$ subject to the following relations

$$u_{ij}u_{jk}=u_{ik}u_{ij}+u_{jk}u_{ik}+\beta u_{ik}+\alpha, \qquad i<j<k.$$

Show that

$$P_n(1,1,\beta=-1)=M_n,$$

where M_n denotes the n-th *Motzkin* number that is the number of *Motzkin n-paths*: paths from $(0,0)$ to $(n,0)$ in an $n\times n$ grid using only steps $U=(1,1)$, $(1,0)$ and $(1,-1)$. It is also the number of Dyck $(n+1)$-paths with no steps UUU, see [131, $A001006$] for a wide variety of combinatorial interpretations, and vast literature concerning the Motzkin numbers. For example,

$$P_7(0,1,\beta=-1)=36+37+24+18+5+6+0+1=127=M_7.$$

Therefore we treat the polynomials $P_n(t,\alpha,\beta=-1)$ as the (t,α)-Motzkin numbers. For example,

$$P_7(t,\alpha,\beta=-1)=t^7+6\alpha t^5+5\alpha t^4+(0,4,14)_{\alpha}t^3+(0,3,21)_{\alpha}t^2+(0,2,21,14)_{\alpha}t$$
$$+(0,1,14,21)_{\alpha}=t^7+\alpha(1,2,3,4,5,6)_t+\alpha^2(14,21,21,14)_t+\alpha^3(21,14)_t.$$

Therefore

$$P_7(t,1,\beta=-1)=1+21\alpha+70\alpha^2+35\alpha^3, \qquad P_7(1,1,\beta=-1)=127=M_7.$$

Show that

$$P_n(0,1,\beta=-1)=A005043(n),$$

known as the *Riordan number*, or *Motzkin sum* [131]. This number, denoted by MS_n, counts the number of Motzkin paths of length n with no horizontal steps at level zero; it is also equal to the number of Dyck paths of semilenght n with no *peaks* at odd level, see [131, $A005043$] for a bit more combinatorial interpretations, and literature concerning the Motzkin sum or Riordan numbers. For example,

$$P_7(t,1,-1) = (\mathbf{36}, 37, 24, 18, 5, 6, 0, 1), \qquad 36 = MS_7.$$

Show that the Riordan number MS_n is equal to the number of underdiagonal paths from $(0,0)$ to the line $x = n-2$, using only steps $(1,0)$, $(0,1)$ and $NE = (2,1)$ and beginning with the step $NE = (2,1)$. Note that the number of such paths with no steps NE is equal to the Catalan number Cat_{n-1}.

Let $\mathcal{MS} = \{n \in \mathbb{N} \,|\, n = 2^{2k}(2r+1) - 1,\ k \geq 1,\ r \geq 0\}$ be a subset of the set of all odd integers [31]. Show that

(a) $MS_n \equiv 1 \pmod 2$, if either $n \equiv 0 \pmod 2$ or $n \in \mathcal{MS}_n$,

(b) $MS_n \equiv 0 \pmod 2$, if n is an odd integer and $n \notin \mathcal{MS}$.

Show that

$$\frac{P_n(0,\alpha,\beta)}{\alpha}\Big|_{\alpha=0} = N_{n-1}(\beta+1),$$

where as before, $N_n(t)$ denotes the Narayana polynomial.

Let us set

$$P_n(0,\alpha,\beta) = \sum_{k\geq 0} c_k(\beta+1)\alpha^k.$$

Show that polynomials $c_k(\beta+1)$, $k \geq 0$ are symmetric (unimodal?) polynomials of the variable $\beta+1$.

Show that [131]

$$P_n(1,1,0) = A052709(n+1).$$

Show that [131]

$$P_n(0,1,0) = A052705(n)$$

that is the number of underdiagonal paths from $(0,0)$ to the line $x = n-2$, using only steps $R = (1,0)$, $V = (0,1)$ and $NE = (2,1)$.

For example,

$$P_7(0,10) = \mathbf{36} + 106 + 120 + 64 + 15 + 1 = 342 = A052705(7).$$

Show that [131]

$$\frac{\partial}{\partial\alpha} P_n(t,\alpha,\beta)\Big|_{\substack{\alpha=0,\\ \beta=0,\\ t=1}} = A05775(n-1),$$

that is the number of paths in the half-plane $x \geq 0$ from $(0,0)$ to $(n-1,2)$ or $(n-1,-3)$, and consisting of steps $U = (1,1)$, $D = (1,-1)$ and $H = (1,0)$. For example,

$$\text{l.h.s.} = 106 + 130 + 99 + 48 + 5 + 6 = 427 = A05775(6).$$

Let us set

$$P_n(t,\alpha,\beta=1) := \sum_{k,l\geq 0} c_{k,l}^{(n)} t^k \alpha^l.$$

Show that

$$(a) \quad \sum_{k=1}^{n} c_{k,n-k}^{(n)} t^k \alpha^{n-k} = (t+\alpha)^{n-1},$$

$$(b) \quad c_{k,n-k-1}^{(n)} = (k+1)\binom{n-1}{k+2}, \qquad 0 \leq k \leq n-3,$$

$$(c) \quad c_{1,0}^{(n)} = c_{0,0}^{(n)} + (-1)^{n-1}, \qquad n \geq 3.$$

5.4.2 Reduced polynomials, dissections and Lagrange inversion formula

Let $\{a_i, b_i, \beta_i, \alpha_i,\ 1 \le i \le n-1\}$ be a set of parameters, consider non commutative algebra generated over the ring $Z[\{a_i, b_i, \beta_i, \alpha_i\}_{1 \le i \le n-1}]$ by the set of generators $\{u_{ij},\ 1 \le i < j \le n\}$ subject to the set of relations

$$u_{ij}u_{jk} = a_i u_{ik}u_{ij} + b_i u_{jk}u_{ik} + \beta u_{ik} + \alpha_i, \qquad 1 \le i < j < k \le n.$$

Consider reduced expression $R_n(\{u_{ij}\}_{1 \le i < j \le n})$ in the above algebra which corresponds to the "Coxeter element"

$$C_n := u_{12}u_{23} \cdots u_{n-1,n}.$$

Note that the reduced expression $R_n(\{u_{ij}\})$ is a linear combination of noncommutative monomials in the generators $\{u_{ij},\ 1 \le i < j \le n\}$ with coefficients from the ring

$$K_n := \mathbb{Z}[\{a_i, b_i, \beta_i, \alpha_i\}_{1 \le i < n}].$$

Now to each monomial U which appears in the reduced expression $R_n(\{u_{ij}\})$ we associate a *dissection* $\mathcal{D} := \mathcal{D}_U$ of a convex $(n+1)$-gon as follows. First of all let us label the vertices of a convex $(n+1)$-gon selected, by the numbers $n+1, n, \ldots, 1$, written consequently and clockwise, starting from a fixed vertex, from here on named by $(n+1)$-vertex.

Next, let us take a monomial $U = u_{i_1,j_1} \cdots u_{i_p,j_p}$ which appears in the reduced expression $R_n(\{u_{ij}\})$ with coefficient $c(U) \in K_n$. We draw diagonals in a convex $(n+1)$-gon chosen which connect vertices labeled correspondingly by numbers i_s and j_s+1, $s = 1, \ldots, p$. It is clearly seen from the defining relations in the algebra in question when being applied to the Coxeter element above, that in fact, the diagonals we have drawn in a convex $(n+1)$-gon selected, do not meet at interior points of our convex $(n+1)$-gon. Therefore, to each monomial U which appears in the reduced polynomial associated with the Coxeter element C_n above, one can associate a *dissecion* $\mathcal{D} := \mathcal{D}_U$ of a convex $(n+1)$-gon selected. Moreover, it is not difficult to see (e.g., cf. [58]) that there exists a natural bijection $U \Longleftrightarrow \mathcal{D}_U$ between monomials which appear in the reduced expression $R_n(\{u_{ij}\})$ and the set of dissections of a convex $(n+1)$-gon. As a corollary, to each dissection $\mathcal{D} := \mathcal{D}_U$ of a conves $(n+1)$-gon one can attache the element $c(\mathcal{D}) := c(U) \in K_n$ which is equal to the coefficient in front of monomial U in the reduced expression corresponding to the Coxeter element C_n.

To continue, let $\boldsymbol{x} = (x_1, \ldots, x_{n-1})$, $\boldsymbol{y} = (y_1, \ldots, y_{n-1})$ and $\boldsymbol{z} = (z_1, \ldots, z_{n-1})$ be three sets of variables, and $\mathcal{D}$ be a dissection of a convex $(n+1)$-gon. We associate with dissection $\mathcal{D}$ a monomial $m(\mathcal{D}) \in K_n$ as follows

$$m(\mathcal{D}) := \prod_{k=1}^{n-1} x_k^{n(k)} y_k^{m(k)} z_{r(k)},$$

where $m(k) := m_k(\mathcal{D})$ (resp. $r(k) := r_k(\mathcal{D})$ and $n(k) := n_k(\mathcal{D})$) denotes the number of (convex) (m_k+2)-gons constituent a dissection $\mathcal{D}$ taken (resp. the number of diagonals issue out of the vertex labeled by $(n+1)$; $n_k(\mathcal{D})$) stands for the number of (oriented) diagonals and edges which issue out of the vertex labeled by k, $k = 1, \ldots, n$). Therefore we associate with the reduced polynomial corresponding to the Coxeter element $u_{12}, \ldots, u_{n-1,n}$ the following polynomial

$$\mathrm{PL}_n(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{\beta}, \boldsymbol{\alpha}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = \sum_{\mathcal{D}} m(\mathcal{D})c(\mathcal{D}),$$

where the sum runs over all dissections $\mathcal{D}$ of a convex $(n+1)$-gon.

To begin with we set $\boldsymbol{x} = \boldsymbol{1}$ and consider the following specializations

$$B_n(\boldsymbol{a}, \boldsymbol{y}) = \mathrm{PL}_n(\boldsymbol{a}, \boldsymbol{b} = \boldsymbol{1}, \boldsymbol{\beta} = \boldsymbol{1}, \boldsymbol{\alpha} = \boldsymbol{0}, \boldsymbol{y}, \boldsymbol{z} = \boldsymbol{1}),$$
$$P_n(\boldsymbol{z}, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{\beta}) = \mathrm{PL}_n(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{\beta}, \boldsymbol{\alpha} = \boldsymbol{0}, \boldsymbol{y} = \boldsymbol{1}, \boldsymbol{z}),$$

Show that

$$B_{n-1}(\boldsymbol{a}, \boldsymbol{y})) = \mathrm{Coeff}_{t^n}\big(z - f(ty_1, \ldots, ty_n)\big)^{[-1]},$$

where $f(y_1, \ldots, y_n) = \sum_{k=1}^{n-1} y_k u^{k+1}$, and for any formal power series $g(u)$, $\frac{d}{du} g(u)|_{u=0} = 1$, we denote by $g(u)^{[-1]}$ the Lagrange Inverse formal power series associated with that $g(u)$ that is a unique formal power series such that $g(g^{[-1]}(u)) = u = g^{[-1]}(g(u))$.

Now let us recall the statement of Lagrange's inversion theorem. Namely, let

$$f(x) = x - \sum_{k \ge 1} y_k x^{k+1}$$

be a formal power series. Then the inverse power series $f^{[-1]}(u)$ is given by the following formula

$$f^{[-1]}(y) = \sum_{n \ge 1} w_n u^n,$$

where

$$w_n := w_n(p_1, \ldots, p_n) = \frac{1}{n+1} \sum_{\substack{p_1, \ldots, p_n \ge 0 \\ \sum j p_j = n}} \binom{n + \sum p_j}{n, p_1, \ldots, p_n} y_1^{p_1} y_2^{p_2} \cdots y_n^{p_n},$$

where if $N = m_1 + \cdots + m_n$, then

$$\binom{N}{m_1, \ldots, m_n} = \frac{N!}{m_1! m_2! \cdots m_n!}$$

denotes the multinomial coefficient.

Therefore, the coefficient

$$b_n(p_1, \ldots, p_n) := \frac{1}{n+1} \binom{n + \sum p_j}{n, p_1, \ldots, p_n}, \qquad \sum_j j p_j = n$$

is equal to the number of dissections of a convex $(n+2)$-gon which contain exactly p_j convex $(j+2)$-gons, see, e.g., [38]. Equivalently, the number $b_n(p_1, \ldots, p_n)$ is equal to the number of cells of the associahedron $\mathcal{K}^{n-1}$ which are isomorphic to the cartesian product $(\mathcal{K}^0)^{p_1} \times \cdots \times (\mathcal{K}^{n-1})^{p_n}$ [90, 91]. Based on a natural and well-known bijection between the set of dissections of a convex $(n+2)$-gon and the set of plane trees with $(n+1)$ ends and such that the all other vertices have degree at least 2, see, e.g., [134], one can readily seen that the number $w_n(p_1, \ldots, p_n)$ defined above under constraint $\sum_j j p_j = n$, is equal to the number of plane trees with $n+1$ ends and having p_j vertices of degree $j+1$.

Example 5.90. For short we set $B_n = \mathrm{PL}_n(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{\beta}, \boldsymbol{\alpha}, \boldsymbol{x}, \boldsymbol{y})$.

(1) Quadrangular:

$$B_2 = y_1^2(a_1 z_1 + b_1 z_1 z_2) + y_2(\beta_1 z_1 + \alpha_1).$$

(2) Pentagon:

$$\begin{aligned}B_3 &= y_1^3\big(a_1^2z_1 + a_1b_1z_1 + a_2b_1^2z_1z_2 + a_1b_1z_1z_3 + b_1^2b_2z_1z_2z_3\big)\\ &\quad + y_1y_2\big(2a_1\beta_1z_1 + b_1\beta_1z_1 + b_1^2\beta_2z_1z_2 + b_1\beta_1z_1z_3 + a_1\alpha_1b_1\alpha_1 + \alpha_1z_3\big)\\ &\quad + y_3\big(\beta_1\alpha_1 + \beta_1^2z_1 + b_1^2\alpha_2z_1\big).\end{aligned}$$

(3) Hexagon:

$$\begin{aligned}B_4 &= y_1^4\big(\big(a_1^3 + 2a_1^2b_1 + a_1a_2b_1^2 + a_1b_1^2b_2\big)z_1\\ &\quad + a_1^2b_1b_2z_1z_2 + a_2b_1^3b_2z_1z_2 + a_1a_3b_1^2z_1z_3 + a_1^2b_1z_1z_4 + a_1b_1^2z_1z_4 + a_3b_1^3b_2^2z_1z_2z_3\\ &\quad + a_2b_1^2b_2z_1z_2z_4 + a_1b_1^2b_3z_1z_3z_4 + b_1^3b_2^2b_3z_1z_2z_3z_4\big) + y_1^2y_2\big(a_1^2\alpha_1 + 2a_1b_1\alpha_1 + a_2b_1^2\alpha_1\\ &\quad + b_1^2b_2\alpha_1 + (3a_1^2b\beta_1 + 4a_1b_1\beta_1 + a_2b_1^2\beta_1 + b_1^2 + b_2\beta_1 + a_1b_1^2\beta_2)z_1 + a_2b_1^2\beta_2z_1z_2\\ &\quad + b_1^3b_2\beta_2z_1z_2 + a_2b_1^3\beta_2z_1z_2 + a_1b_1^2b_3z_1z_3 + a_1b_1\beta_1z_1z_3 + a_3b_1^2z_1z_3 + b_1^2\beta_1z_1z_4\\ &\quad + a_1b_1\beta_1z_1z_4 + b_1^2b_2\beta_3z_1z_2z_3 + b_1^3b_2\beta_2z_1z_2z_4 + b_1^2b_3\beta_1z_1z_3z_4\big) + y_1y_3\big(a_1\beta_1\alpha_1\\ &\quad + 2b_1\beta_1\alpha_1 + \big(2a_1\beta_1^2 + 2b_1\beta_1^2a_1b_1^2\alpha_3 + a_2b_1^2\alpha_2 + b_1^3b_2\alpha_2\big)z_1 + b_1^3b_2\alpha_3z_1z_2 + b_1^3\beta_2^2z_1z_2\\ &\quad + b_1^3\alpha_3z_1z_4 + b_3\alpha_1z_3z_3z_4 + a_3\alpha_1z_3 + a_1\alpha_1z_4 + b_1\alpha_1z_4 + \beta_1\alpha_1z_4\big) + y_2^2\big(a_1\beta_1\alpha_1\\ &\quad + b_1^2\beta_2\alpha_2 + \big(b_1^2\beta_1\beta_2 + a_1\beta_1^2 + a_1\beta_1^2\alpha_2\big)z_1 + \beta_3\alpha_1z_3 + b_1\beta_1\beta_3z_1z_3\big) + y_4(\alpha_1\alpha_3 + \beta_1^2\alpha_1\\ &\quad + b_1^2\alpha_1\alpha_2\big(b_1^2\beta_1\alpha_2 + b_1^3\beta_2\alpha_2 + \beta_1^3 + b_1^2\beta_1\alpha_3\big)z_1\big).\end{aligned}$$

Special cases. Generalized *Schröder* or *Lagrange* polynomials:

$$P_n(\boldsymbol{a},\boldsymbol{b},\boldsymbol{\beta},\boldsymbol{y},\boldsymbol{z}) = B_n\big|_{\boldsymbol{\alpha}=\boldsymbol{0}}.$$

For example,

$$\begin{aligned}P_4(\boldsymbol{a},\boldsymbol{b},\boldsymbol{y}) &= y_1^4\big(\big(a_1^3 + 2a_1^2b_1 + a_1a_2b_1^2 + a_1b_1^2b_2\big)z_1 + a_1^2b_1b_2z_1z_2 + a_2b_1^3b_2z_1z_2 + a_1a_3b_1^2z_1z_3\\ &\quad + a_1^2b_1z_1z_4 + a_1b_1^2z_1z_4 + a_3b_1^3b_2^2z_1z_2z_3 + a_2b_1^2b_2z_1z_2z_4 + a_1b_1^2b_3z_1z_3z_4\\ &\quad + b_1^3b_2^2b_3z_1z_2z_3z_4\big) + y_1^2y_2\big(\big(3a_1^2b\beta_1 + 4a_1b_1\beta_1 + a_2b_1^2\beta_1 + b_1^2 + b_2\beta_1\\ &\quad + a_1b_1^2\beta_2\big)z_1 + a_2b_1^2\beta_2z_1z_2 + b_1^3b_2\beta_2z_1z_2 + a_2b_1^3\beta_2z_1z_2 + a_1b_1^2b_3z_1z_3\\ &\quad + a_1b_1\beta_1z_1z_3 + a_3b_1^2z_1z_3 + b_1^2\beta_1z_1z_4 + a_1b_1\beta_1z_1z_4 + b_1^2b_2\beta_3z_1z_2z_3\\ &\quad + b_1^3b_2\beta_2z_1z_2z_4 + b_1^2b_3\beta_1z_1z_3z_4\big) + y_1y_3\big(2a_1\beta_1^2 + 2b_1\beta_1^2 + b_1^3\beta_2^2z_1z_2\\ &\quad + b_1\beta_1^2z_1z_4\big) + y_2^2\big(\big(b_1^2\beta_1\beta_2 + a_1\beta_1^2\big)z_1 + b_1\beta_1\beta_3z_1z_3\big) + y_4\beta_1^3z_1.\end{aligned}$$

After the specialization $a_i = b_i = \beta_i = z_i = 1$, $i = 1,2,3,4$, one will obtain

$$P_4(\boldsymbol{a}=\boldsymbol{1},\boldsymbol{b}=\boldsymbol{1},\boldsymbol{\beta}=\boldsymbol{1},\boldsymbol{y},\boldsymbol{z}=\boldsymbol{1}) = 14y_1^4 + 21y_1^2y_2 + 6y_1y_3 + 3y_2^2 + y_4.$$

Generalized *Narayana* polynomials:

$$\begin{aligned}P_n(\boldsymbol{a},\boldsymbol{b},\boldsymbol{y},\boldsymbol{z}) &= B_n\big|_{\substack{\boldsymbol{\alpha}=\boldsymbol{0}\\ \boldsymbol{\beta}=\boldsymbol{0}}},\\ P_n(\boldsymbol{a},\boldsymbol{b},\boldsymbol{y},\boldsymbol{z}) &= y_1^4\big(\big(a_1^3 + 2a_1^2b_1 + a_1a_2b_1^2 + a_1b_1^2b_2\big)z_1 + a_1^2b_1b_2z_1z_2 + a_2b_1^3b_2z_1z_2\\ &\quad + a_1a_3b_1^2z_1z_3 + a_1^2b_1z_1z_4 + a_1b_1^2z_1z_4 + a_3b_1^3b_2^2z_1z_2z_3 + a_2b_1^2b_2z_1z_2z_4\\ &\quad + a_1b_1^2b_3z_1z_3z_4 + b_1^3b_2^2b_3z_1z_2z_3z_4\big).\end{aligned}$$

Generalized *Motzkin–Schröder* polynomials:

$$\mathrm{MS}_n(\boldsymbol{a},\boldsymbol{b},\boldsymbol{y},\boldsymbol{z}) = B_n\big|_{\boldsymbol{a}=\boldsymbol{0}}.$$

For example,

$$\mathrm{MS}_4(\boldsymbol{a},\boldsymbol{b},\boldsymbol{y},\boldsymbol{z}) = y_1^1 y_2\big(a_1^2\alpha_1 + 2a_1b_1\alpha_1 + a_2b_1^2\alpha_1 + b_1^2b_2\alpha_1\big) + y_1y_3(a_1\beta_1\alpha_1 + 2b_1\beta_1\alpha_1)$$
$$+ y_2^2\big(a_1\beta_1\alpha_1 + b_1^2\beta_2\alpha_2\big) + y_4\big(\alpha_1\alpha_3 + b_1^2\alpha_1\alpha_2 + \beta_1^2\alpha_1\big).$$

Generalized *Motzkin* polynomials:

$$M_n(\boldsymbol{b},\boldsymbol{y},\boldsymbol{z}) = B_n\big|_{\substack{\boldsymbol{a}=\boldsymbol{0}\\ \boldsymbol{\beta}=\boldsymbol{0}}}.$$

For example,

$$M_4(\boldsymbol{b},\boldsymbol{y},\boldsymbol{z}) = y_1^4 b_1^3 b_2^2 b_3 z_1z_2z_3z_4 + y_1^2y_2b_1^2b_2\alpha_1 + y_1y_3\big(b_1^3b_2\alpha_2 + b_1\alpha_1z_4 + b_1^3b_2z_1z_3$$
$$+ b_1^3\alpha_2z_1z_4 + b_3\alpha_1z_3z_4\big) + y_4\big(\alpha_2\alpha_3 + b_1^2\alpha_1\alpha_2\big).$$

Generalized *Motzkin–Riordan* polynomials:

$$\mathrm{MR}_n(\boldsymbol{a},\boldsymbol{b},\boldsymbol{\beta},\boldsymbol{\alpha},\boldsymbol{y}) = B_n\big|_{\boldsymbol{z}=\boldsymbol{0}}.$$

Generalized *Riordan* polynomials:

$$\mathrm{RI}_n(\boldsymbol{b},\boldsymbol{\alpha},\boldsymbol{y}) = B_n\big|_{\substack{\boldsymbol{z}=\boldsymbol{0}\top\boldsymbol{a}=\boldsymbol{0}\\ \boldsymbol{\beta}=\boldsymbol{0}}}.$$

For example,

$$\mathrm{RI}_4(\boldsymbol{b},\boldsymbol{\alpha},\boldsymbol{y}) = y_1^2y_2b_1^2b_2\alpha_1 + y_4\big(\alpha_2\alpha_3 + b_1^2\alpha_1\alpha_2\big).$$

Let us set $B_n(y_1,\dots,y_n) = B_n(\boldsymbol{a}=\boldsymbol{1},\boldsymbol{b}=\boldsymbol{1},\boldsymbol{\beta}=\boldsymbol{1},\boldsymbol{y})$. Let β be a new parameter. Show that

$$B(1,\beta,\dots,\beta^{n-1}) = \mathfrak{G}^{(\beta)}_{1\times w_0^{(n-1)}}(\underbrace{1,\dots,1}_{n}),$$

where $\mathfrak{G}^{(\beta)}_w(X)$ denotes the β-Grothendieck polynomial corresponding to a permutation $w\in\mathbb{S}_n$. In particular,

$$B_n(\underbrace{1,\dots,1}_{n}) = \mathrm{Sch}_n,$$

where Sch_n denotes the n-th Schröder number, that is the numbers of paths from $(0,0)$ to $(2n,0)$, using only steps northeast $U=(1,1)$ or or $D=(1,-1))$ or double $H=(2,0)$, that never fall below the x-axis.

Assume that n is devisible by an integer $d\geq 1$. Show that if $\boldsymbol{y}=(y_j=\delta_{j+1,d})$, then

$$B_n(0,\dots,0,\underbrace{1}_{d-1},0,\dots,0) = \mathrm{FC}^{(d+1)}_{n/d},$$

where FC^p_m denotes the Fuss–Catalan number, see, e.g., [134], and [131, *A*001764] for a variety of combinatorial interpretations the Fuss–Catalan numbers $\mathrm{FC}^{(3)}_n$.

More generally, let $2<d_1<\dots<d_k$ be a sequence of integers, and set

$$\boldsymbol{y} = (\delta_{i+1,d_j},\ 1\leq j\leq k).$$

Show that the specialization $B_n(\boldsymbol{y})$ counts the number of dissections of a convex $(n+2)$-gon on parts which are convex $(d+2)$-gons, where each d belongs to the set $\{d_1,\dots,d_k\}$. We would like to point out that the polynomials

$$\mathrm{FS}^{(d)}_n := \mathrm{Coeff}_{y_d^n}\big(P_{nd}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{\beta},\boldsymbol{y}=(\delta_{i+1,d}),\boldsymbol{z})\big).$$

can be treated as a multi-parameter analogue of the Fuss–Catalan numbers $\mathrm{FC}^{(d+1)}_n$.

Colored dissections [127]. A colored dissection of a convex polygon is a dissection where each $(d+1)$-gon appearing in the dissection can be colored by one of b_d possible colors[61], $d\geq 2$ [127].

[61] We assume that if $b_d=0$, then the dissection in question doesn't contain parts which are $(d+1)$-gons.

Show [127] that if $b_2,\ldots,b_n$ be a sequence of non-negative integers, $B_n(b_2,\ldots,\ldots,b_n)$ is equal to the number of colored dissections of a convex $(n+2)$-gon.

Consider the specialization $y_i=i-1$, $i=1,\ldots,n$. Show that

$$B_n(y):=\mathrm{SL}(0,1,\ldots,n-1)=\mathrm{Fine}(n+1),$$

where $\mathrm{Fine}(m)$ denotes the m-th *Fine number*, that is the number of ordered rooted trees with m edges having root of even degree [131, $A000957$]. Therefore, the Fine number $Fine(n+1)$ counts the number of dissections of a convex $(n+2)$-gon such that each $(d+3)$-gon appearing in the dissection can be colored by d possible colors, $d\geq 1$.

Consider the specialization $y_{3k+1}=1$, $y_{3k+2}=0$, $y_{3k+3}=-1$, $k\geq 0$. Show that

$$B_n(y_1,\ldots,y_n)=M_n,$$

where M_n denotes the n-th *Motzkin* number [131, $A001006$].

Recall that it is the number of ways to draw any number of nonintersecting chord joining n labeled points on a circle. The number M_n is also equals to the number of *Motzkin* paths, that is paths from $(0,0)$ to (n,n) in the $n\times n$ grid using only steps $U=(1,1)$, $H=(1,0)$ and $D=(1,-1)$, see [131, $A001006$] for references and a wide variety of combinatorial interpretations of Motzkin's numbers.

Consider the specialization $y_{3k+1}=0$, $y_{3k+2}=(-1)^k$, $y_{3k+3}=(-1)^k$, $k\geq 0$. Show that

$$B_n(y_1,\ldots,y_n)=\mathrm{MS}_n,$$

where MS_n denotes the *Motzkin sum* or *Riordan* number [131, $A005043$].

Recall that it is the number of Motzkin paths of length n with no horizontal steps $H=(1,0)$ at level zero, see [131, $A005043$] for references and a wide variety of combinatorial interpretations of Riordan's numbers.

Consider the specialization $y_{2k+1}=(-1)^k$, $y_{2k}=(-1)^{k+1}$, $k\geq 0$. Show that [131]

$$B_n(y_1,\ldots,y_n)=A052709(n),$$

that is the number of underdiagonal lattice paths from $(0,0)$ to $(n-1,n-1)$ and such that each step is either $H=(1,0)$, $V=(0,1)$, or $D=(2,1)$.

Consider specialization $y_k=(-1)^k\frac{n!}{k!}$, $k\geq 1$. Show that

$$B_n(y_1,\ldots,y_n)=n^{n-2},$$

that is the number of *parking functions*, see, e.g., [55, 134] and the literature quoted therein.

Consider the specialization $y_k=\frac{n!}{k!}$. Show that [131]

$$B_n(y_1,\ldots,y_n)=A052894(n),$$

where $A052894(n)$ denotes the number of *Schröder trees*[62].

A Appendixes

A.1 Grothendieck polynomials

Definition A.1. Let β be a parameter. The Id-Coxeter algebra $\mathrm{IdC}_n(\beta)$ is an associative algebra over the ring of polynomials $\mathbb{Z}[\beta]$ generated by elements $\langle e_1,\ldots,e_{n-1}\rangle$ subject to the set of relations

[62] Schröder trees have been introduced in a paper by W.Y.C. Chen [23]. Namely, these are trees for which the set of subtrees at any vertex is endowed with the structure of ordered partition. Recall that *an ordered partition* of a set in which the blocks are linearly ordered [23].

- $e_i e_j = e_j e_i$ if $|i-j| \geq 2$,
- $e_i e_j e_i = e_j e_i e_j$ if $|i-j| = 1$,
- $e_i^2 = \beta e_i$ $1 \leq i \leq n-1$.

It is well-known that the elements $\{e_w,\, w \in \mathbb{S}_n\}$ form a $\mathbb{Z}[\beta]$-linear *basis* of the algebra $\mathrm{IdC}_n(\beta)$. Here for a permutation $w \in \mathbb{S}_n$ we denoted by e_w the product $e_{i_1} e_{i_2} \cdots e_{i_\ell} \in \mathrm{IdC}_n(\beta)$, where $(i_1, i_2, \ldots, i_\ell)$ is any *reduced* word for a permutation w, i.e., $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ and $\ell = \ell(w)$ is the length of w.

Let $x_1, x_2, \ldots, x_{n-1}, x_n = y, x_{n+1} = z, \ldots$ be a set of mutually commuting variables. We assume that x_i and e_j commute for all values of i and j. Let us define

$$h_i(x) = 1 + x e_i, \qquad A_i(x) = \prod_{a=n-1}^{i} h_a(x), \qquad i = 1, \ldots, n-1.$$

Lemma A.2. *One has*

(1) *addition formula:*

$$h_i(x) h_i(y) = h_i(x \oplus y),$$

where we set $(x \oplus y) := x + y + \beta xy$*;*

(2) *Yang–Baxter relation:*

$$h_i(x) h_{i+1}(x \oplus y) h_i(y) = h_{i+1}(y) h_i(x \oplus y) h_{i+1}(x).$$

Corollary A.3.

(1) $[h_{i+1}(x) h_i(x), h_{i+1}(y) h_i(y)] = 0$.

(2) $[A_i(x), A_i(y)] = 0$, $i = 1, 2, \ldots, n-1$.

The second equality follows from the first one by induction using the addition formula, whereas the fist equality follows directly from the Yang–Baxter relation.

Definition A.4 (Grothendieck expression)**.**

$$\mathfrak{G}_n(x_1, \ldots, x_{n-1}) := A_1(x_1) A_2(x_2) \cdots A_{n-1}(x_{n-1}).$$

Theorem A.5 ([42])**.** *The following identity*

$$\mathfrak{G}_n(x_1, \ldots, x_{n-1}) = \sum_{w \in \mathbb{S}_n} \mathfrak{G}_w^{(\beta)}(X_{n-1}) e_w$$

holds in the algebra $\mathrm{IdC}_n \otimes \mathbb{Z}[x_1, \ldots, x_{n-1}]$.

Definition A.6. We will call polynomial $\mathfrak{G}_w^{(\beta)}(X_{n-1})$ as the β-*Grothendieck* polynomial corresponding to a permutation w.

Corollary A.7.

(1) *If* $\beta = -1$*, the polynomials* $\mathfrak{G}_w^{(-1)}(X_{n-1})$ *coincide with the Grothendieck polynomials introduced by Lascoux and M.-P. Schützenberger* [86].

(2) *The* β*-Grothendieck polynomial* $\mathfrak{G}_w^{(\beta)}(X_{n-1})$ *is divisible by* $x_1^{w(1)-1}$.

(3) *For any integer* $k \in [1, n-1]$ *the polynomial* $\mathfrak{G}_w^{(\beta-1)}(x_k = q,\, x_a = 1,\, \forall\, a \neq k)$ *is a polynomial in the variables* q *and* β *with non-negative integer coefficients.*

Sketch of proof. It is enough to show that the specialized Grothendieck expression $\mathfrak{G}_n(x_k = q, x_a = 1, \forall\, a \neq k)$ can be written in the algebra $IdC_n(\beta-1) \otimes \mathbb{Z}[q, \beta]$ as a linear combination of elements $\{e_w\}_{w \in \mathbb{S}_n}$ with coefficients which are polynomials in the variables q and β with non-negative coefficients. Observe that one can rewrite the relation $e_k^2 = (\beta-1)e_k$ in the following form $e_k(e_k+1) = \beta e_k$. Now, all possible negative contributions to the expression $\mathfrak{G}_n(x_k = q, x_a = 1, \forall\, a \neq k)$ can appear only from products of a form $c_a(q) := (1+qe_k)(1+e_k)^a$. But using the Addition formula one can see that $(1+qe_k)(1+e_k) = 1+(1+q\beta)e_k$. It follows by induction on a that $c_a(q)$ is a polynomial in the variables q and β with *non-negative* coefficients. ■

Definition A.8.

- The double β-Grothendieck expression $\mathfrak{G}_n(X_n, Y_n)$ is defined as follows

$$\mathfrak{G}_n(X_n, Y_n) = \mathfrak{G}_n(X_n)\mathfrak{G}_n(-Y_n)^{-1} \in \mathrm{IdC}_n(\beta) \otimes \mathbb{Z}[X_n, Y_n].$$

- The double β-Grothendieck polynomials $\{\mathfrak{G}_w(X_n, Y_n)\}_{w \in \mathbb{S}_n}$ are defined from the decomposition

$$\mathfrak{G}_n(X_n, Y_n) = \sum_{w \in \mathbb{S}_n} \mathfrak{G}_w(X_n, Y_n) e_w$$

of the double β-Grothendieck expression in the algebra $\mathrm{IdC}_n(\beta)$.

More details about β-Grothendieck and related polynomials can be found in [71, 84].

A.2 Cohomology of partial flag varieties

Let $n = n_1 + \cdots + n_k$, $n_i \in \mathbb{Z}_{\geq 1} \forall i$, be a composition of n, $k \geq 2$. For each $j = 1, \ldots, k$ define the numbers $N_j = n_1 + \cdots + n_j$, $N_0 = 0$, and $M_j = n_j + \cdots + n_k$. Denote by $\boldsymbol{X} := \boldsymbol{X}_{n_1,\ldots,n_k} = \{x_a^{(i)} \,|\, i = 1, \ldots, k,\ 1 \leq a \leq n_i\}$ (resp. $\boldsymbol{Y}, \ \ldots$) a set of variables of the cardinality n. We set $\deg(x_a^{(i)}) = a$, $i = 1, \ldots, k$. For each $i = 1, \ldots, k$ define quasihomogeneous polynomial of degree n_i in variables $\boldsymbol{X}^{(i)} = \{x_a^{(i)} \,|\, 1 \leq a \leq n_i\}$

$$p_{n_i}\big(\boldsymbol{X}^{(i)}, t\big) = t^{n_i} + \sum_{a=1}^{n_i} x_a^{(i)} t^{n_i - a},$$

and put

$$p_{n_1,\ldots,n_k}(\boldsymbol{X}, t) = \prod_{i=1}^{k} p_{n_i}(\boldsymbol{X}^{(i)}, t).$$

We summarize in the theorem below some well-known results about the classical and quantum cohomology and K-theory rings of type A_{n-1} partial flag varieties $\mathcal{F}l_{n_1,\ldots,n_k}$. Let $q_1, \ldots, q_{k-1}$, $\deg(q_i) = n_i + n_{i+1}$, $i = 1, \ldots, k-1$, be a set of "quantum parameters".

Theorem A.9. *There are canonical isomorphisms*

$$H^*(\mathcal{F}l_{n_1,\ldots,n_k}, \mathbb{Z}) \cong \mathbb{Z}[\boldsymbol{X}_{n_1,\ldots,n_k}]\big/\big\langle p_{n_1,\ldots,n_k}(\boldsymbol{X}, t) - t^n \big\rangle,$$
$$K^\bullet(\mathcal{F}l_{n_1,\ldots,n_k}, \mathbb{Z}) \cong \mathbb{Z}[\boldsymbol{Y}^{\pm 1}]\big/\big\langle p_{n_1,\ldots,n_k}(\boldsymbol{Y}, t) - (1+t)^n \big\rangle,$$

$$H_T^*(\mathcal{F}l_{n_1,\dots,n_k},\mathbb{Z}) \cong \mathbb{Z}[\boldsymbol{X},\boldsymbol{Y}]\Big/\left\langle \prod_{i=1}^{k}\prod_{a=1}^{n_i}(x_a^{(i)}+t) - p_{n_1,\dots,n_k}(\boldsymbol{Y},t)\right\rangle,$$
$$QH^*(\mathcal{F}l_{n_1,\dots,n_k}) \cong \mathbb{Z}[\boldsymbol{X}_{n_1,\dots,n_k},q_1,\dots,q_{k-1}]/\langle \Delta_{n_1,\dots,n_k}(\boldsymbol{X},t)-t^n\rangle \quad \text{(cf. [4])},$$
$$QH_T^*(\mathcal{F}l_{n_1,\dots,n_k}) \cong \mathbb{Z}[\boldsymbol{X},\boldsymbol{Y},q_1,\dots,q_{k-1}]/\langle \Delta_{n_1,\dots,n_k}(\boldsymbol{X},t)-p_{n_1,\dots,n_k}(\boldsymbol{Y},t)\rangle \quad \text{(cf. [4])},$$

where[63]

$$\Delta_{n_1,\dots,n_k}(\boldsymbol{X},t) = \\ \det\begin{vmatrix} p_{n_1}(\boldsymbol{X}^{(1)},t) & q_1 & 0 & \cdots & \cdots & \cdots & 0 \\ -1 & p_{n_2}(\boldsymbol{X}^{(2)},t) & q_2 & 0 & \cdots & \cdots & 0 \\ 0 & -1 & p_{n_3}(\boldsymbol{X}^{(3)},t) & q_3 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & -1 & p_{n_{k-1}}(\boldsymbol{X}^{(k-1)},t) & q_{k-1} \\ 0 & \cdots & \cdots & \cdots & 0 & -1 & p_{n_k}(\boldsymbol{X}^{(k)},t) \end{vmatrix}.$$

Here for any polynomial $P(\boldsymbol{x},t)=\sum_{j=0}^{r} b_j(\boldsymbol{x})t^{r-j}$ in variables $\boldsymbol{x}=(x_1,x_2,\dots)$, we denote by $\langle P(\boldsymbol{x},t)\rangle$ the ideal in the ring $\mathbb{Z}[\boldsymbol{x}]$ generated by the coefficients $b_0(\boldsymbol{x}),\dots,b_r(\boldsymbol{x})$. A similar meaning have the symbols

$$\left\langle \prod_{i=1}^{k}\prod_{a=1}^{n_i}(x_a^{(i)}+t) - p_{n_1,\dots,n_k}(\boldsymbol{y},t)\right\rangle, \qquad \langle \Delta_{n_1,\dots,n_k}(\boldsymbol{x},t)-t^n\rangle$$

and so on.

Note that $\dim(\mathcal{F}_{n_1,\dots,n_k})=\sum_{i<j} n_in_j$ and the Hilbert polynomial $\mathrm{Hilb}(\mathcal{F}_{n_1,\dots,n_k},q)$ of the partial flag variety $\mathcal{F}_{n_1,\dots,n_k}$ is equal to the q-multinomial coefficient $\left[{n \atop n_1,\dots,n_k}\right]_q$, and also is equal to the q-dimension of the weight $(n_1,\dots,n_k)$ subspace of the n-th tensor power $(\mathbb{C}^n)^{\otimes n}$ of the fundamental representation of the Lie algebra $\mathfrak{gl}(n)$.

Comments A.10. The cohomology and (small) quantum cohomology rings $H^*(\mathcal{F}_{n_1,\dots,n_k},\mathbb{Z})$ and $QH^*(\mathcal{F}_{n_1,\dots,n_k},\mathbb{Z})$, of the partial flag variety $\mathcal{F}_{n_1,\dots,n_k}$ admit yet another representations we are going to present. To start with, let as before $n=n_1+\cdots+n_k$, $n_i\in\mathbb{Z}_{\geq 1}$, $\forall\, i$, be a composition. Consider the set of variables $\widehat{\boldsymbol{X}}=X_{n_1,\dots,n_{k-1}}:=\{x_a^{(i)}\,|\,1\leq i\leq n_a,\ a=1,\dots,k-1\}$, and set as before $\deg x_a^{(i)}=a$. Note that the number of variables $\widehat{\boldsymbol{X}}$ is equal to $n-n_k$. To continue, let's define *elementary quasihomogeneous polynomials of degree* r

$$e_r(\widehat{\boldsymbol{X}})=\sum_{I,A} x_{a_1}^{(i_1)}\cdots x_{a_s}^{(i_s)}, \qquad e_0(\widehat{\boldsymbol{X}})=1, \qquad e_{-r}(\widehat{\boldsymbol{X}})=0, \qquad r>0,$$

where the sum runs over sequences of integers $I=(i_1,\dots,i_s)$ and $A=(a_1,\dots,a_s)$ such that

- $1\leq i_1<\cdots<i_s\leq k-1$,
- $1\leq a_j\leq n_{i_j}$, $j=1,\dots,s$, and $r=a_1+\cdots+a_s$,

and *complete homogeneous polynomials* of degree p

$$h_p(\widehat{\boldsymbol{X}})=\det\left|e_{j-i+1}(\widehat{\boldsymbol{X}})\right|_{1\leq i,j\leq p}.$$

[63] We prefer to use quantum parameters $\{q_i\,|\,1\leq i\leq k-1\}$ instead of the parameters $\{(-1)^{n_i}q_i\,|\,1\leq i\leq k-1\}$ have been used in [4].

Finally, let's define the ideal $J_{n_1,\dots,n_k}$ in the ring of polynomials $\mathbb{Z}[X_{n_1,\dots,n_{k-1}}]$ generated by polynomials

$$h_{n_k+1}\big(\widehat{\boldsymbol{X}}\big),\ \dots,\ h_n\big(\widehat{\boldsymbol{X}}\big).$$

Note that the ideal $J_{n_1,\dots,n_k}$ is generated by $n-n_k=\#(X_{n_1,\dots,n_{k-1}})$ elements.

Proposition A.11. *There exists an isomorphism of rings*

$$H^*(\mathcal{F}_{n_1,\dots,n_k},\mathbb{Z})\cong\mathbb{Z}[X_{n_1,\dots,n_{k-1}}]/J_{n_1,\dots,n_k}.$$

In a similar way one can describe relations in the (small) quantum cohomology ring of the partial flag variety $\mathcal{F}_{n_1,\dots,n_k}$. To accomplish this let's introduce *quantum quasihomogeneous elementary polynomials* of degree j $e_j^{(\boldsymbol{q})}(\boldsymbol{X}_{n_1,\dots,n_r})$ through the decomposition

$$\Delta_{n_1,\dots,n_r}(\boldsymbol{X}_{n_1,\dots,n_r})=\sum_{j=0}^{N_r}e_j^{(\boldsymbol{q})}(\boldsymbol{X}_{n_1,\dots,n_r})t^{N_r-j},\qquad e_0^{(\boldsymbol{q})}(\boldsymbol{x})=1,\qquad e_{-p}^{(\boldsymbol{q})}(\boldsymbol{x})=0,\qquad p>0.$$

To exclude redundant variables $\big\{x_a^{(k)},\ 1\le a\le n_k\big\}$, let us define *quantum quasihomogeneous Schur polynomials* $s_\alpha^{(\boldsymbol{q})}(\boldsymbol{X}_{n_1,\dots,n_r})$ corresponding to a composition $\alpha=(\alpha_1\le\alpha_2\le\dots\le\alpha_p)$ as follows

$$s_\alpha^{(\boldsymbol{q})}(\boldsymbol{X}_{n_1,\dots,n_r})=\det\big|e_{j-i+\alpha_i}^{(\boldsymbol{q})}(\boldsymbol{X}_{n_1,\dots,n_r})\big|_{1\le i,j\le p}.$$

Proposition A.12. *The (small) quantum cohomology ring $QH^*(\mathcal{F}_{n_1,\dots,n_k},\mathbb{Z})$ is isomorphic to the quotient of the ring of polynomials $\mathbb{Z}[q_1,\dots,q_{k-1}]$ $[\mathbf{X}_{n_1,\dots,n_{k-1}}]$ by the ideal $I_{n_1,\dots,n_{k-1}}$ generated by the elements*

$$g_r(\boldsymbol{X}_{n_1,\dots,n_{k-1}}):=s_{(1^{n_k},r)}^{(q_1,\dots,q_{k-1})}(\boldsymbol{X}_{n_1,\dots,n_{k-1}})-q_{k-1}e_{r-n_{k-1}}^{(q_1,\dots,q_{k-2})}(\boldsymbol{X}_{n_1,\dots,n_{k-2}}),$$

where $n_k+1\le r\le n$.

It is easy to see that the Jacobi matrix

$$\left(\frac{\partial}{\partial x_a^{(i)}}g_r(\boldsymbol{X}_{n_1,\dots,n_{k-1}})\right)_{\substack{\{a=1,\dots,k-1,\,1\le i\le n_a\\ n_k+1\le r\le n\}}}$$

corresponding to the set of polynomials $g_r(\boldsymbol{X}_{n_1,\dots,n_{k-1}})$, $n_k\le r\le n$, has nonzero determinant, and the component of maximal degree $n_{\max}:=\sum\limits_{l<j}n_in_j$ in the ring $QH^*(\mathcal{F}_{n_1,\dots,n_k},\mathbb{Z})$ is a $\mathbb{Z}[q_1,\dots,q_{k-1}]$-module of rank one with generator

$$\Lambda=\prod_{i=1}^{k-1}\prod_{a=1}^{n_a}\big(x_a^{(i)}\big)^{M_i}.$$

Therefore, one can define a scalar product (the Grothendieck residue)

$$\langle\bullet,\bullet\rangle\colon\ HQ^*(\mathcal{F}_{n_1,\dots,n_k},\mathbb{Z})\times HQ^*(\mathcal{F}_{n_1,\dots,n_k},\mathbb{Z})\longrightarrow\mathbb{Z}[q_1,\dots,q_{k-1}]$$

setting for elements f and g of degrees a and b, $\langle f,h\rangle=0$, if $a+b\ne n_{\max}$, and $\langle f,h\rangle=\lambda(q)$, if $a+b=n_{\max}$ and $fh=\lambda(q)\Lambda$. It is well known that the Grothendieck pairing $\langle\bullet,\bullet\rangle$ is nondegenerate (for any choice of parameters $q_1,\dots,q_{k-1}$).

Finally we state "a mirror presentation" of the small quantum cohomology ring of partial flag varieties. To start with, let $n = n_1 + \cdots + n_k$, $k \in \mathbb{Z}_{ge2}$ be a composition of size n, and consider the set

$$\Sigma(\boldsymbol{n}) = \{(i,j) \in \mathbb{Z} \times \mathbb{Z} \,|\, 1 \le i \le N_a,\ M_{a+1} + 1 \le j \le M_a,\ a = 1, \dots, k-1\},$$

where $N_a = n_1 + \cdots + n_a$, $N_0 = 0, N_k = n$ $M_a = n_{a+1} + \cdots + n_k$, $M_0 = n, M_k = 0$.

With these data given, let us introduce the set of variables

$$Z_{\boldsymbol{n}} = \{z_{i,j} \,|\, (i,j) \in \Sigma(\boldsymbol{n})\},$$

and define "boundary conditions" as follows

- $z_{i,M_a+1} = 0$, if $N_{a-1} + 2 \le i \le N_a$, $a = 1, \dots, k-1$,
- $z_{N_a+1,j} = \infty$, if $M_{a+1} + 2 \le j \le M_a$, $a = 1, \dots, k-1$,
- $z_{N_{a-1}+1,M_a+1} = q_a$, $a = 1, \dots, k$, where $q_1, \dots, q_k$ are "quantum parameters.

Now we are ready, follow [53], to define *superpotential*

$$W_{q,\boldsymbol{n}} = \sum_{(p,j)\in\Sigma(\boldsymbol{n})} \left(\frac{z_{i,j+1}}{z_{i,j}} + \frac{z_{i,j}}{z_{i+1,j}} \right).$$

Conjecture A.13 (cf. [53])**.** *There exists an isomorphism of rings*

$$QH^*_{[2]}(\mathcal{F}l_{n_1,\dots,n_k}, \mathbb{Z}) \cong \mathbb{Z}\big[q_1^{\pm 1}, \dots, q_k^{\pm 1}\big]\big[Z_{\boldsymbol{n}}^{\pm 1}\big] / J(W_{q,\boldsymbol{n}}),$$

where $QH^*_{[2]}(\mathcal{F}l_{n_1,\dots,n_k}, \mathbb{Z})$ denotes the subring of the ring $QH^*(\mathcal{F}l_{n_1,\dots,n_k}, \mathbb{Z})$ generated by the elements from $H^2(\mathcal{F}l_{n_1,\dots,n_k}, \mathbb{Z})$.

$J(W_{q,\boldsymbol{n}})$ stands for the ideal generated by the partial derivatives of the superpotential $W_{q,\boldsymbol{n}}$:

$$J(W_{q,\boldsymbol{n}}) = \left\langle \frac{\partial W_q}{\partial z_{i,j}} \right\rangle, \qquad (i,j) \in \Sigma(\boldsymbol{n}).$$

Note that variables $\{z_{i,j} \in \Sigma(\boldsymbol{n}),\ i \ne N_a + 1,\ a = 0, \dots, k-2\}$ are redundant, whereas the variables $\{z_{a,j} := z_{N_a+1,j}^{-1},\ j = 1, \dots, n_a,\ a = 0, \dots, k-2\}$ satisfy the system of algebraic equations.

In the case of complete flag variety $\mathcal{F}l_n$ corresponds to partition $\boldsymbol{n} = (1^n)$ and the superpotential $W_{q,1^n}$ is equal to

$$W_{q,1^n} = \sum_{1 \le i < j \le n-1} \left(\frac{z_{i,j+1}}{z_{i,j}} + \frac{z_{i,j}}{z_{i-1,j+1}} \right),$$

where we set $z_{i,n} := q_i$, $i = 1, \dots, n$. The ideal $J(W_{q,1^n})$ is generated by elements

$$\frac{\partial W_{q,1^n}}{z_{i,j}} = \frac{1}{z_{i,j-1}} + \frac{1}{z_{i-1,j+1}} - \frac{z_{i,j+1} + z_{i-1,j-1}}{z_{i,j}^2}.$$

One can check that the ideal $J(W_{q,1^n})$ can be also generated by elements of the form

$$\sum_{j=0}^{i} A_j^{(i)}(q_1, \dots, q_{n-i+1}, z_{n-1}, \dots, z_{n-i+1}) z_{n-i}^{j-i-1} = 1, \qquad A_0^{(i)} = q_1 \cdots q_{n-i+1},$$

where $z_i := z_{1,i}^{-1}$, $i = 1, \dots n-1$. For example,

$$z_1^n q_1 \cdots q_n = 1, \qquad q_1 q_2 z_{n-1}^2 - q_2 z_{n-2} = 1,$$
$$q_1 q_2 q_3 z_{n-2}^3 - 2 q_1 q_2 q_3 z_{n-1} z_{n-2} z_{n-3} + q_2 q_3 z_{n-3}^2 + q_3 z_{n-4} = 1.$$

Therefore the number of critical points of the superpotential W_q is equal to $n! = \dim H^*(\mathcal{F}l_n, \mathbb{Z})$, as it should be. Note also that $QH^*(\mathcal{F}l_n, \mathbb{Z}) = QH^*_{[2]}(\mathcal{F}l_n, \mathbb{Z})$.

A.3 Multiparamater 3-term relations algebras

A.3.1 Equivariant multiparameter 3-term relations algebras

Let $\boldsymbol{q} = \{q_{ij}\}_{1\le i\neq j\le n}$, $q_{ij} = q_{ji}$, be a collection of mutually commuting parameters and $\boldsymbol{\beta} = \{\beta_{ij}\}_{1\le i\neq j\le n}$, $\beta_{ij} = \beta_{ji}$ and $\boldsymbol{\ell} = \{\ell_{ij}\}_{1\le i\neq j\le n}$, $\ell_{ij} = \ell_{ji}$, be two sets of mutually commuting variables each.

Definition A.14. Denote by $3QT_n(\boldsymbol{\beta},\boldsymbol{\ell},\boldsymbol{q})$ an associative algebra generated over the ring $\mathbb{Z}[\boldsymbol{\beta},\boldsymbol{\ell},\boldsymbol{q}]$ by the set of generators $\{x_1,\dots,x_n\}$ and that $\{u_{ij}\}_{1\le i\neq j\le n}$ subject to the set of relations

(1) locality conditions: $[x_i,x_j]=0$, $[u_{ij},u_{kl}]=0$, $[x_k,u_{ij}]=0$ if i, j, k, l are pairwise distinct,

(2) generalized unitarity conditions: $u_{ij}+u_{ji}=\beta_{ij}$,

(3) Hecke type conditions: $u_{ij}u_{ji}=-q_{ij}$ if $i\neq j$,

(4) twisted 3-term relations: $u_{ij}u_{jk}=u_{jk}u_{ik}-u_{ik}u_{ji}, u_{jk}u_{ij}=u_{ik}u_{jk}-u_{ji}u_{ik}$ if i, j, k are distinct,

(5) crossing relations: $x_iu_{ji}=-u_{ij}x_j-\ell_{ij}$ if $i\neq j$.

As before we define the (additive) Dunkl elements to be

$$\theta_i = x_i+\sum_{j\neq i}u_{ij}, \qquad i=1,\dots,n. \tag{A.1}$$

It should be pointed out that the Dunkl elements do not commute with variables $\{x_i\}$, $\{\beta_{ij}\}$ and $\{\ell_{ij}\}$.

It is clearly seen from the defining relations listed in Definition A.14 that for any triple of distinct indices (i,j,k) the elements $\{x_i,x_j,x_k,u_{ji},u_{ik},u_{jk}\}$ satisfy the twisted dynamical Yang–Baxter relations, and thus the Dunkl elements $\{\theta_i\}_{1\le i\le n}$ generate a commutative subalgebra in the algebra $3QT_n(\boldsymbol{\beta},\boldsymbol{\ell})$.

On the other hand, one can show that the set of defining relations involve in the definition of algebra $3QT_n(\boldsymbol{\beta},\boldsymbol{\ell})$ implies the following set of compatibility relations among the set of generators $\{u_{ij}\}$ and the set of variables $\{\beta_{ij}\}$ and $\{\ell_{ij}\}$

$$\ell_{ij}u_{jk}+u_{ij}\ell_{jk}+\beta_{ij}u_{jk}x_i+u_{ij}\beta_{jk}x_j = u_{jk}\ell_{ik}+\ell_{ik}u_{ij}+u_{jk}\beta_{ik}x_i+\beta_{ij}x_iu_{ij},$$

if i, j, k are distinct.

These relations are satisfied, for example, if either $\beta_{ij}=\beta$, and $\ell_{ij}=h$, $\forall\, i,j$ for some parameters (i.e., a central elements) β and h, or variables $\{\beta_{ij}\}$ and $\{\ell_{ij}\}$ satisfy the exchange relations with generators $\{u_{ij}\}$, namely, the commutativity relations

$$[\beta_{ij},u_{km}]=0, \qquad [\ell_{ij},u_{km}]=0 \qquad \text{if} \quad \{i,j\}\cap\{k,m\}=\varnothing$$

and the exchange relations

$$\beta_{ij}u_{jk}=u_{jk}\beta_{ik}, \qquad \ell_{ij}u_{jk}=u_{jk}\ell_{ik} \qquad \text{if} \quad k\neq i,j.$$

It happens that in the first case, if $\beta=0$, then the (commutative) algebra generated by additive Dunkl's elements and elementary symmetric polynomials $\{e_k(X_n)\}_{1\le k\le n}$ (resp. multiplicative Dunkl's elements) is isomorphic to the equivariant quantum cohomology ring (resp. to the equivariant quantum K-theory ring) of the type A_{n-1} complete flag variety. In the second case a geometric interpretation of the algebra generated by Dunkl's elements is missing.

Our main objective in this section is to to describe (part of) relations among Dunkl's element using defining relations involve in the Definition A.14 of the algebra $3QT_n(\boldsymbol{\beta},\boldsymbol{\ell},\boldsymbol{q})$, under the following constraints

$$\ell_{ij} = h_{\max(i,j)}, \qquad h_2,\dots,h_n \quad \text{are all central.}$$

Note, that except the case $\beta_j = \beta$ and $h_i = h_j$, $\forall\, i,j$, our assumption violates the crossing relations between the elements β_{ij}, ℓ_{ij} and $u_{j,k}$, but nevertheless allows to compute explicitly (part of) relations among the Dunkl's elements. We expect that an abstract algebra generated over $\mathbb{Q}[\boldsymbol{\beta},\boldsymbol{h}]$ by a set of mutually commuting elements $\theta_1,\dots,\theta_n$ and elementary symmetric polynomials $\{e_k(X_n)\}_{1\le k\le n}$ subject to the set of relations descending from those for Dunkl's elements which were mentioned above, has some interesting combinatorial/geometric interpretations. Below we state some results concerning relations among Dunkl elements in the algebra $3QT_n(\boldsymbol{\beta},\boldsymbol{\ell},\boldsymbol{q})$.

Theorem A.15 (cf. Theorem 3.17, Section 3)**.** *Let $k\ge 1$ be an integer. There exist polynomials*

$$R_k(\boldsymbol{q},\boldsymbol{h},z_1,\dots,z_n) \in \mathbb{Z}[\beta,\boldsymbol{q},\{h_j-h_i\}_{1\le i<j\le n}][Z_n],$$
$$T_k(\beta,\boldsymbol{h},z_1,\dots,z_n) \in \mathbb{Z}[\beta,\boldsymbol{h}][Z_n]^{\mathbb{S}_n}$$

such that

$$R_k(\boldsymbol{q},\boldsymbol{h},z_1,\dots,z_n) = e_k^{(\boldsymbol{q}+\boldsymbol{h})}(z_1,\dots,z_n) + \textit{monomials of total degree}$$
$$\le k-2 \qquad \textit{w.r.t. variables } \{z_i\}_{1\le i\le n},$$
$$T_k(\beta,\boldsymbol{h},z_1,\dots,z_n) = e_k(z_1,\dots,z_n) + \sum_{j<k} c_{j,k} e_j(X_n),\, c_{j,k}\in\mathbb{Z}[\beta,\boldsymbol{h}],$$
$$R_k(\theta_1,\dots,\theta_n) = T_k(x_1,\dots,x_n),$$

where $e_k^{(\boldsymbol{q}+\boldsymbol{h})}(z_1,\dots,z_n)$ denotes the multiparameter quantum elementary polynomial corresponding to the set of parameters $\{(\boldsymbol{q}+\boldsymbol{h})\} = \{q_{ij}+h_j\}_{1\le i<j\le n}$.

It is not difficult to see that the unitarity and crossing conditions imply the following relations

$$[x_i+x_j,u_{kl}] = 0 = [x_ix_j,u_{kl}], \qquad [x_i^2,u_{kl}] = 0$$

are valid for all indices $i\ne j$, $k\ne l$. As a consequence of these relations one can deduce that the all symmetric polynomials $e_k(X_n) := e_k(x_1,\dots,x_n)$, $k=1,\dots,n$, belong to the *center* of the algebra $3QT_n(\boldsymbol{q},\boldsymbol{h})$, and therefore one has $[\theta_i,e_k(X_n)]=0$ for all i and k. Let us denote by $QH(\beta,\boldsymbol{h})$ a commutative subalgebra in the algebra $3QT_n(\beta,\boldsymbol{h})$ generated by the elementary symmetric polynomials $\{e_k(X_n)\}_{1\le k\le n}$ and the Dunkl elements $\{\theta_i\}_{1\le i\le n}$. It is an interesting *problem* to give a geometric/cohomological interpretation of the commutative algebra $QH(\beta,\boldsymbol{h})$. We don't know any geometric interpretation of that commutative algebra, except the special case [75]

$$\beta = 0, \qquad h_j = 1, \quad \forall\, j, \qquad q_{ij} := q_i\delta_{i+1,j}. \tag{A.2}$$

Proposition A.16 ([75])**.** *Under assumptions* (A.2)*, the algebra $QH(0,\mathbf{0})$ isomorphic to the equivariant quantum cohomology $QH_T^*(\mathcal{F}l_n)$ of the complete flag variety $\mathcal{F}l_n$.*

Examples A.17. Let us list the relations among the Dunkl elements in the algebra $3QT_n(\beta,\boldsymbol{h})$ for $n=3,4$, and $\beta_j=\beta$, $\forall j$.

$$(1)\ e_1(\theta_1,\dots,\theta_n) = e_1(X_n) + \binom{n}{2}\beta,$$

(2) $e_2^{(\boldsymbol{q}+\boldsymbol{h})}(\theta_1,\dots,\theta_n) = e_2(X_n) + (n-1)\beta e_1(X_n) + \dfrac{n(n-1)(n-2)(3n-1)}{24}\beta^2, \quad n \geq 3,$

(3) $e_3^{(\boldsymbol{q}+\boldsymbol{h})}(\theta_1,\theta_2,\theta_3) = e_3(X_3) + h_3\beta,$

$e_3^{(\boldsymbol{q}+\boldsymbol{h})}(\theta_1,\theta_2,\theta_3,\theta_4) = e_3(X_4) + \beta e_2(X_4) + 2\beta^2 e_1(X_4) + 6\beta^3 + \beta(h_3+3h_4),$

(4) $e_4^{(\boldsymbol{q}+\boldsymbol{h})}(\theta_1,\theta_2,\theta_3,\theta_4) + \beta(h_4-h_3)\theta_4 = e_4(X_4) + \beta h_4 e_1(X_4) + 5\beta^2 h_4.$

Note that $\frac{n(n-1)(n-2)(3n-1)}{24} = s(n-2,2) = e_2(1,2,\dots,n-1)$ is equal to the Stirling number of the first kind.

Conjecture A.18. *The polynomial $R_k(\boldsymbol{q},\boldsymbol{h},Z_n)$, see Theorem 2.29, can be written as a polynomial in the variables $\{h_{ij} := h_j - h_i,\ 1 \leq i < j \leq n,\ z_1,\dots,z_n,\ \beta,\ q_{ij},\ 1 \leq i < j \leq n\}$ with nonnegative coefficients.*

Exercises A.19 (Pieri formula in the algebra $3T_n(0,h)$, [75]). Assume that $\beta = 0$ and $h_2 = \cdots = h_n = h$, and denote by $\theta_i^{(n)}$, $i = 1,\dots,n$ the Dunkl elements (A.1) in the algebra $3T_n(0,h)$. Show that

$$e_k\big(\theta_1^{(n)},\dots,\theta_m^{(n)}\big) = \sum_{r\geq 0}(-h)^r N(m-k,2\,r)\left\{\sum_{\substack{S\subset[1,m]\\ I=\{i_a\},\, J=\{j_a\}}} X_S u_{i_1,j_1}\cdots u_{i_{|I|},j_{|J|}}\right\},$$

where

$$N(a,2b) = (2b-1)!!\binom{a+2b}{2b},$$

$X_S = \prod_{s\in S} x_s$, and the second summation runs over triples of sets $\{S,I,J\}$ such that $S \subset [1,m]$, $I \subset [1,m]\backslash S$, $|I|+|S|+2r = k$, $|I| = |J|$, $1 \leq i_a < m < j_a \leq n$ and $j_1 \leq \cdots \leq j_{|I|}$.

A.3.2 Algebra $3QT_n(\boldsymbol{\beta},\boldsymbol{h})$, generalized unitary case

Let $\boldsymbol{\beta} = (\beta_1,\dots,\beta_{n-1})$, $\boldsymbol{h} = (h_2,\dots,h_n)$ and $\{q_{ij}\}_{1\leq i<j\leq n}$ be collections of mutually commuting parameters as in the previous section. As before we define the Dunkl elements θ_i, $i = 1,\dots,n$, by the formula (A.1). It is necessary to *stress* that the Dunkl elements $\{\theta\}_{1\leq i\leq n}$ *do not commute* in the algebra $3QT_n(\boldsymbol{\beta},\boldsymbol{h})$ but satisfy a noncommutative analogue of the relations displayed in Theorem A.15. Namely, one needs to replace the both elementary polynomials $e_k(Z_n)$ and the quantum multiparameter elementary polynomials $e_k^{(\boldsymbol{q})}(Z_n)$ by its noncommutative versions. Recall that the noncommutative elementary polynomial $\underline{e}_k(Z_n)$ is equal to

$$\sum_{1\leq j_1<j_2<\cdots<j_k\leq n} z_{j_1}\ z_{j_2}\cdots z_{j_k}$$

and the noncommutative quantum multiparameters elementary polynomial $\underline{e}_k^{(\boldsymbol{q})}(Z_n)$ is equal to

$$\sum_{\ell}\ \sum_{\substack{1\leq i_1<\cdots<j_\ell\leq n\\ i_1<j_1,\dots,i_\ell<j_\ell}} \underline{e}_{k-2\ell}(Z_{\overline{I\cup J}})\prod_{a=1}^{\ell} u_{i_a,j_a},$$

where $I = (i_1,\dots,i_\ell)$, $J = (j_1,\dots,j_\ell)$ should be distinct elements of the set $\{1,\dots,n\}$, and $Z_{\overline{I\cup J}}$ denotes set of variables z_a for which the subscript a is neither one of i_m nor one of the j_m.

Example A.20.

$$\underline{e}_2^{(\boldsymbol{q}+\boldsymbol{h})}(\theta_1,\ldots,\theta_n) = e_2(X_n) + \left(\sum_{j=1}^{n-1}\beta_j\right) e_1(X_n) + \sum_{1\le a<b\le n-1} ab\beta_a\beta_b,$$

$$\begin{aligned}&\underline{e}_3^{(\boldsymbol{q}+\boldsymbol{h})}(\theta_1,\theta_2,\theta_3,\theta_4) + (\beta_3-\beta_1)(\theta_3\theta_4 + q_{34} + h_4 + \beta_2(\theta_1+\theta_2)) + (\beta_3-\beta_2)((\theta_1+\theta_2)\theta_4\\ &\quad + q_{14} + q_{24} + 2h_4 + \beta_1\theta_3) = e_3(X_4) + \beta_3 e_2(X_4) + (\beta_1\beta_3 + \beta_2\beta_3 + \beta_3^2 - \beta_1\beta_2)e_1(X_4)\\ &\quad + (3\beta_3^2 - \beta_1\beta_2)(\beta_1 + 2\beta_2) + \beta_1(h_3+h_4) + 2\beta_2 h_4,\end{aligned}$$

$$\begin{aligned}&\underline{e}_4^{(\boldsymbol{q}+\boldsymbol{h})}(\theta_1,\theta_2,\theta_3,\theta_4) + (\beta_2 h_4 - \beta_1 h_3)\theta_4 + h_4(\beta_2-\beta_1)\theta_3\\ &\quad = e_4(X_4) + \beta_2 h_4 e_1(X_4) + \beta_2 h_4(2\beta_2 + 3\beta_3).\end{aligned}$$

Project A.21 (noncommutative universal Schubert polynomials)**.** *Let $w \in \mathbb{S}_n$ be a permutation and $\mathfrak{S}_w(Z_n)$ be the corresponding Schubert polynomial.*

(1) *There exists a* (*noncommutative*) *polynomial $\mathfrak{Sh}_w(\{u_{ij}\}_{1\le i<j\le n})$ with non-negative integer coefficients such that the following identity*

$$\mathfrak{S}_w(\theta_1,\ldots,\theta_n) = \mathfrak{Sh}_w(\{u_{ij}\}_{1\le i<j\le n})$$

holds in the algebra $3T_n^{(0)}$, where $\{\theta_j\}_{1\le j\le n}$ are the Dunkl elements in the algebra $3T_n^{(0)}$.

(2) *There exist polynomials $R_w(\beta,\boldsymbol{q},\boldsymbol{h},Z_n) \in \mathbb{N}[\beta,\boldsymbol{q},h_j - h_{i_{1\le i<j\le n}}][Z_n]$ and $T_w(\beta,\boldsymbol{h},Z_n) \in \mathbb{Z}[\beta,\boldsymbol{h}][Z_n]$ such that the following identity*

$$R_w(\beta,\boldsymbol{q},\boldsymbol{h},\theta_1,\ldots,\theta_n) = T_w(\beta,\boldsymbol{h},X_n) + \mathfrak{Sh}_w(\{u_{ij}\}_{1\le i<j\le n})$$

holds in the algebra $3QT_n(\beta,\boldsymbol{h})$.

3) *Let $r \in \mathbb{Z}_{\ge 2}$ and $N = n_1 + \cdots + n_r$, $n_j \in \mathbb{Z}_{\ge 1}$, $\forall\, j$, be a composition of N, and set $N_j = n_1 + \cdots + n_j$, $j \ge 1$, $N_0 = 0$. Eliminate the Dunkl elements $\theta_{N_{r-1}+1}^{(N)},\ldots,\theta_N^{(N)}$ from the set of relations among the Dunkl elements $\theta_1^{(N)},\ldots,\theta_N^{(N)}$ in the algebra $3QT_n(\beta,\boldsymbol{h})$, by the use of the degree $1,\ldots,n_r$ relations among the former. As a result one obtains a set consisting of N_{r-1} relations among the N_{r-1} elements*

$$\theta_{j.k_j}^{(N)} := e_{k_j}^{(\boldsymbol{q})}\big(\theta_{N_{j-1}+1}^{(N)},\ldots,\theta_{N_j}^{(N)}\big), \qquad 1 \le k_j \le n_j, \qquad 1 \le j \le r-1.$$

Give a geometric interpretation of the commutative subalgebra $QH_{n_1,\ldots,n_r}(\beta,\boldsymbol{h}) \subset 3QT_n(\beta,\boldsymbol{h})$ generated by the set of elements $\theta_{j,k_j}^{(N)}$, $1 \le k_j \le n_j$, $j = 1,\ldots,r-1$.

A.4 Koszul dual of quadratic algebras and Betti numbers

Let k be a field of zero characteristic, $F^{(n)} := k\langle x_1,\ldots,x_n\rangle = \bigoplus_{j\ge 0} F_j^{(n)}$ be the free associative algebra generated by $\{x_i,\, 1 \le i \le n\}$. Let $A = F^{(n)}/I$ be a quadratic algebra, i.e., the ideal of relations I is generated by the elements of degree 2, $I \subset F_2^{(n)}$. Let $F^{(n)*} = \operatorname{Hom}(F_n,k) = \bigoplus_{j\ge 0} F_j^{(n)*}$ with a multiplication induced by the rule $fg(ab) = f(a)g(b)$, $f \in F_i^{(n)*}$, $g \in F_j^{(n)*}$, $a \in F_i^{(n)}$, $b \in F_j^{(n)}$. Let $I_2^{\perp} = \{f \in F_2^{(n)*}, f(I_2) = 0\}$, and denote by $I^{\perp}$ the two-sided ideal in $F^{(n)*}$ generated by the set $I_2^{\perp}$.

Definition A.22. *The Koszul* (*or quadratic*) *dual* $A^!$ of a quadratic algebra A is defined to be $A^! := F^{(n)*}/I^{\perp}$.

The Koszul dual of a quadratic algebra A is a quadratic algebra and $(A^!)^! = A$.

Examples A.23.

(1) Let $A = F^{(n)}$ be the free associative algebra, then the quadratic dual

$$A^! = k\langle y_1, \dots, y_n\rangle/(y_i y_j,\ 1 \le i, j \le n).$$

(2) If $A = k[x_1, \dots, x_n]$ is the ring of polynomials, then

$$A^! = k[y_1, \dots, y_n]/([y_i, y_j]_-,\ 1 \le i, j \le n),$$

where we put by definition $[y_i, y_j]_- = y_i y_j + y_j y_i$ if $i \neq j$, and $[y_i, y_i]_- = y_i^2$.

(3) Let $A = F^{(n)}/(f_1, \dots, f_r)$, where $f_i = \sum\limits_{1 \le j,k \le n} a_{ijk} x_j x_k$, $i = 1, \dots, r$ are linear independent elements of degree 2 in $F^{(n)}$. Then the quadratic dual of A is equal to the quotient algebra $A^! = k\langle y_1, \dots, y_n\rangle/J$, where the ideal $J = \langle g_1, \dots, g_s\rangle$, $s = n^2 - r$, is generated by elements $g_m = \sum\limits_{1 \le j,k \le n} b_{mjk} y_j y_k$. The coefficients b_{mjk}, $m = l, \dots, s$, $1 \le j, k \le n$, can be defined from the system of linear equations $\sum\limits_{1 \le j,k \le n} a_{ijk} b_{mjk} = 0$, $i = 1, \dots, r$, $m = 1, \dots, s$ (cf. [95, Chapter 5]).

Let $A = \bigoplus_{j \ge 0} A_j$ be a graded finitely generated algebra over field k.

Definition A.24. *The Hilbert series* of a graded algebra A is defined to be the generating function of dimensions of its homogeneous components: $\operatorname{Hilb}(A, t) = \sum\limits_{k \ge 0} \dim A_k t^k$.

The Betti–Poincaré numbers $B_A(n, m)$ of a graded algebra A are defined to be $B_A(i, j) := \dim \operatorname{Tor}_i^A(k, k)_j$. *The Poincaré* series of algebra A is defined to be the generating function for the Betti numbers: $P_A(s, t) := \sum\limits_{i \ge 0, j \ge 0} B_A(i, j) s^i t^j$.

Let B is a k-module and A is a B-module. The *Betti number* $\beta_{ij}^{B(A)}$ of A over B is the rank of the free module $B[-j]$ the ith module of a minimal resolution of A over B that is $\beta_{ij}^B(A) = \dim_k \operatorname{Ext}_i^B(A, k)_j$. The graded Betti series of A over B is the generating function

$$\operatorname{Betti}_B(A, x, y) := \sum_{i \in \mathbb{N},\, j \in \mathbb{Z}} \beta_{ij}^B(A) x^i y^{-j} \in \mathbb{Z}[y, y^{-1}][[x]].$$

Definition A.25. A quadratic algebra A is called *Koszul* iff the Betti numbers $B_A(i, j)$ are equal to zero unless $i = j$.

It is well-known that $\operatorname{Hilb}(A, t) P_A(-1, t) = 1$, and a quadratic algebra A is *Koszul*, if and only if $B_A(i, j) = 0$ for all $i \neq j$. In this case $\operatorname{Hilb}(A, t)\ \operatorname{Hilb}(A^!, -t) = 1$.

Example A.26. Let $F_n^{(0)}$ be a quotient of the free associative algebra F_n over field k with the set of generators $\{x_1, \dots, x_n\}$ by the two-sided ideal generated by the set of elements $\{x_1^2, \dots, x_n^2\}$. Then the algebra $F_f^{(0)}n$ is *Koszul*, and $\operatorname{Hilb}\big(F_n^{(0)}, t\big) = \frac{1+t}{1-(n-1)t}$.

We refer the reader to a nice written book by A. Polishchuk and L. Positselski [116] to read more widely in the theory of quadratic algebras, see also [94].

A.5 On relations in the algebra Z_n^0

Let us define algebra Z_n^0 to be the subalgebra in $3T_n^0$ generated by the elements $u_{i,n}$, $1 \le i \le n-1$. It is clear that Z_n^0 is a $\mathbb{S}_{n-1}$-module,and well-known [46] that if one sets $\operatorname{Hilb}(Z_k^0, t) := Z_k(t)$, then

$$\operatorname{Hilb}\big(3T_n^{(0)}, t\big) = \prod_{k=2}^{n} Z_k(t).$$

There exists a natural action of algebra $3T_{n-1}^0$ on that Z_n^0. To define it, it's convenient to put $x_i := u_{i,n}$, $1 \le i \le n-1$.

Definition A.27 (cf. [67] and Section 2.3.4)**.** Define operators $\nabla_{i,j}, 1 \le i < j \le n-1$, which act on Z_n^0, by the following rules

- $\nabla_{i,j}(x_k) = 0$ if $k \neq i, j$,
- $\nabla_{i,j}(x_i) = x_i x_j, \nabla_{i,j}(x_j) = -x_j x_i$,
- twisted Leibniz rule:

$$\nabla_{i,j}(x \cdot y) = \nabla_{i,j}(x) \cdot y + s_{i,j}(x) \cdot \nabla_{i,j}(y)$$

  for $x, y \in Z_n^0$ and all $1 \le i < j \le n-1$. Here $s_{i,j} \in \mathbb{S}_{n-1}$ denotes the transposition that interchanges i and j and fixes each $k \neq i, j$.

Proposition A.28. *The operators $\nabla_{i,j}$, $1 \le i < j \le n-1$, satisfy all defining relations of algebra $3T_{n-1}^0$.*

In particular, the operators $\nabla_{i,j}$, satisfy the Coxeter and Yang–Baxter relations:

- Yang–Baxter relations:

$$\nabla_{i,j}\nabla_{i,k}\nabla_{j,k} = \nabla_{j,k}\nabla_{i,k}\nabla_{i,j},$$

- Coxeter relations. Let $\nabla_j = \nabla_{j,j+1}$, $1 \le j \le n-2$, then

$$\nabla_j\nabla_{j+1}\nabla_j = \nabla_{j+1}\nabla_j\nabla_{j+1}, \qquad [\nabla_i, \nabla_j] = 0 \qquad \text{if} \quad |i-j| \ge 2.$$

Therefore, for each $w \in \mathbb{S}_{n-1}$ one can define the operator $\nabla_w = \nabla_{a_1} \cdots \nabla_{a_l}$, where the sequence $(a_1, \dots, a_l)$ is a reduce decomposition of the element w.

Denote by $\mathcal{R}_n$ the *kernel* of the epimorphism $\iota \colon Z_n \longrightarrow F_{n-1}$ given by $\iota(u_{k,n}) = x_k$, where $F_{n-1} := \mathbb{Q}\langle x_1, \dots, x_{n-1}\rangle$ denotes the free associative algebra generated by the elements $x_1, \dots, x_{n-1}$. There exists the decomposition $\mathcal{R}_n = \bigoplus_{k\ge 2} \mathcal{R}_{n,k}$, where $\mathcal{R}_{n,k}$ denotes the degree k part of $\mathcal{R}_n$. We denote by $r_{n,k}$ the dimension of the space $\mathcal{R}_{n,k} / \sum\limits_{j=1}^{n-1} (x_{j,n}\mathcal{R}_{n,k-1} + \mathcal{R}_{n,k-1}x_{j,n})$, and put $r_n := (r_{n,2}, r_{n,3}, \dots)$.

Example A.29.

$$r_3 = (2,1), \qquad r_4 = (3,3,2), \qquad r_5 = (4,6,8,6,3),$$
$$r_6 = (5,10,20,30,39,40,39,30,20,10,4).$$

Remark A.30. The same formulas for the action of $\nabla_{i,j}$ on Z_n^0 given in Definition A.27, define an action of operators $\nabla_{i,j}$ on the free algebra F_{n-1}. In this way we obtain a representation of the algebra $3T_{n-1}$ on that F_{n-1}, cf. Section 2.3.4.

Let us denote by $\widehat{F}_n$ the quotient of the free associative algebra $F_n = \langle x_1, \dots, x_n\rangle$ by the two-sided ideal generated by the elements $\{x_i^2 x_j - x_j x_i^2,\ 1 \le i, j \le n\}$. It is not difficult to see that the operators $\nabla_{i,j}$, $1 \le i < j \le n$, define a representation of the algebra $3T_n^0$ on that $\widehat{F}_n$. Note that

$$\widehat{F}_n \backsimeq F_{n-1} \otimes \mathbb{Z}[y_1, y_2, \dots, y_n],$$

where $\deg(y_1) = 1$, $\deg(y_j) = 2$, $j = 2, \dots, n$. Therefore,

$$\operatorname{Hilb}\bigl(\widehat{F}_n, t\bigr) = \frac{1}{(1-t)(1-(n-1)t)(1-t^2)^{n-1}}.$$

Conjecture A.31. *The kernel $\mathcal{R}_n$ coincides with the two-sided ideal in the free algebra F_{n-1} generated by elements of the form $\prod_{k=1}^{s} \nabla_{i_k,j_k}(x_a^2)$ for some positive integers s and $1 \leq a \leq n-1$.*

In other words, the *all* relations in the algebra Z_n^0 are consequence of the following relations $u\nabla_w(x_1^2) = 0$ for some $u, w \in \mathbb{S}_{n-1}$.

Challenge A.32.

(1) Compute the numbers $r_{n,k}$.

(2) Prove (or disprove) that there exists a positive integer $k_{\max} := k_{\max}^{(n)}$ such that $r_{n,k_{\max}} \neq 0$, but $r_{n,k} = 0$ for all integers $k > k_{\max}^{(n)}$.

(3) These examples suggest that there might be exist a certain symmetry $r_{n,k} = r_{n,k_{\max}-k+2}$, if $3 \leq k < k_{\max}$, between the numbers $r_{n,k}$, and moreover, $r_{n,k_{\max}} = r_{n,2} - 1$. If so, how to explain these properties of the numbers $r_{n,k}$?

We *expect* that if $n \geq 4$, then $k_{\max}^{(n)} = 2\binom{n-2}{[(n-2)/2]}$.

Example A.33 (cyclic relations in the algebra Z_n^0)**.** The following relation

$$\prod_{j=1}^{n-1} \nabla_{n-j,n-j+1}(x_1^2) = \sum_{i=1}^{n} x_i \left(\prod_{a=i+1}^{n} x_a \prod_{a=1}^{i-1} x_a \right) x_i$$

holds in the free algebra F_n. Therefore in the algebra Z_n^0 one has the following cyclic relation of the degree n and length $n-1$:

$$\sum_{i=1}^{n-1} x_i \left(\prod_{a=i+1}^{n-1} x_a \prod_{a=1}^{i-1} x_a \right) x_i = 0.$$

If $n \geq 5$, then by applying to monomials of the form $\prod_{j=2}^{n-1} \nabla_{n-j,n-j+1}(x_1^2)$ the action of either operators $\nabla_{a,n-1}$, $2 \leq a \leq n-3$, or those $\nabla_{a,b}$, $1 \leq a \leq b-2 \leq n-4$, new, more complicated relations in the algebra Z_n^0, i.e., non-cyclic relations, can *appear*. These are relations of the length $2n$ and degree $n+1$ in the algebra Z_n^0. Conjecturally all relations in the algebra Z_n^0 can be obtained by this method.

Proposition A.34.

$$r_{n,k} = (k-2)!\binom{n-1}{k-1}, \qquad 2 \leq k \leq 5,$$

$$r_{n,6} = 4!\binom{n-1}{5} + 3\binom{n-1}{4}, \qquad r_{n,7} = 5!\binom{n-1}{6} + 40\binom{n-1}{5},$$

$$r_{n,8} = 6!\binom{n-1}{7} + 430\binom{n-1}{6} + 39\binom{n-1}{5}.$$

A.5.1 Hilbert series $\mathrm{Hilb}(3T_n^0, t)$ and $\mathrm{Hilb}\big((3T_n^0)^!, t\big)$: Examples[64]

Examples A.35.

$$\mathrm{Hilb}(3T_3^0, t) = [2]^2[3], \qquad \mathrm{Hilb}(3T_4^0, t) = [2]^2[3]^2[4]^2, \qquad \mathrm{Hilb}(3T_5^0, t) = [4]^4[5]^2[6]^4,$$

[64] All computations in this section were performed by using the computer system `Bergman`, except computations of $\mathrm{Hilb}(3T_6^0, t)$ in degrees from twelfth till fifteenth. The last computations were made by J. Backelin, S. Lundqvist and J.-E. Roos from Stockholm University, using the computer algebra system `aalg` mainly developed by S. Lundqvist.

$$\begin{aligned}
\mathrm{Hilb}\big(3T_6^0,t\big) &= (1,15,125,765,3831,16605,64432,228855,755777,2347365,6916867,\\
&\qquad 19468980,52632322,137268120,346652740,850296030,\dots)\\
&= \mathrm{Hilb}\big(3T_5^0,t\big)(1,5,20,70,220,640,1751,4560,11386,27425,64015,\\
&\qquad 145330,321843,696960,1478887,3080190,\dots),\\
\mathrm{Hilb}\big(3T_7^0,t\big) &= \mathrm{Hilb}\big(3T_6^0,t\big)(1,6,30,135,560,2190,8181,29472,103032,\\
&\qquad 351192,1170377,\dots),\\
\mathrm{Hilb}\big(3T_8^0,t\big) &= \mathrm{Hilb}\big(3T_7^0,t\big)(1,7,42,231,1190,5845,27671,127239,571299,2514463,\\
\mathrm{Hilb}\big(\big(3T_3^0\big)^!,t\big)(1-t) &= (1,2,2,1), \qquad \mathrm{Hilb}\big(\big(3T_4^0\big)^!,t\big)(1-t)^2 = (1,4,6,2,-5,-4,-1),\\
\mathrm{Hilb}\big(\big(3T_5^0\big)^!,t\big)(1-t)^2 &= (1,8,26,40,19,-18,-22,-8,-1),\\
\mathrm{Hilb}\big(\big(3T_6^0\big)^!,t\big)(1-t)^3 &= (1,12,58,134,109,-112,-245,-73,68,50,12,1),\\
\mathrm{Hilb}\big(\big(3T_7^0\big)^!,t\big)(1-t)^3 &= (1,18,136,545,1169,1022,-624,-1838,-837,312,374,123,18,1).
\end{aligned}$$

We *expect* that $\mathrm{Hilb}((3T_n^0)^!,t)$ is a rational function with the only pole at $t=1$ of order $[n/2]$, and the polynomial $\mathrm{Hilb}((3T_n^0)^!,t)(1-t)^{[n/2]}$ has degree equals to $[5n/2]-4$, if $n \geq 2$.

A.6 Summation and Duality transformation formulas [63]

Summation formula. Let $a_1+\dots+a_m=b$. Then

$$\sum_{i=1}^m [a_i]\left(\prod_{j\neq i}\frac{[x_i-x_j+a_j]}{[x_i-x_j]}\right)\frac{[x_i+y-b]}{[x_i+y]} = [b]\prod_{1\leq i\leq m}\frac{[y+x_i-a_i]}{[y+x_i]}.$$

Duality transformation, case $N=1$. Let $a_1+\dots+a_m=b_1+\dots+b_n$. Then

$$\begin{aligned}
&\sum_{i=1}^m [a_i]\prod_{j\neq i}\frac{[x_i-x_j+a_j]}{[x_i-x_j]}\prod_{1\leq k\leq n}\frac{[x_i+y_k-b_k]}{[x_i+y_k]}\\
&\quad = \sum_{k=1}^n [b_k]\prod_{l\neq k}\frac{[y_k-y_l+b_l]}{[y_k-y_l]}\prod_{1\leq i\leq m}\frac{[y_k+x_i-a_i]}{[y_k+x_i]}.
\end{aligned}$$

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These notes represent an update version of Section 5 of my notes [72], which have been designed as an extended version of [66], and are based on my talks given at[65]

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