PLURIASSOCIATIVE AND POLYDENDRIFORM ALGEBRAS

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Abstract. We introduce, by adopting the point of view and the tools offered by the theory of operads, a generalization on a nonnegative integer parameter γ of diassociative algebras of Loday, called γ-pluriassociative algebras. By Koszul duality of operads, we obtain a generalization of dendriform algebras, called γ-polydendriform algebras. In the same manner as dendriform algebras are suitable devices to split associative operations into two parts, γ-polydendriform algebras seem adapted structures to split associative operations into 2γ operations so that some partial sums of these operations are associative. We provide a complete study of the operads governing our generalizations of the diassociative and dendriform operads. Among other, we exhibit several presentations by generators and relations, compute their Hilbert series, show that they are Koszul, and construct free objects in the corresponding categories. We also provide consistent generalizations on a nonnegative integer of the duplicial, triassociative and tridendriform operads, and of some operads of the operadic butterfly.

Contents

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Introduction

Associative algebras play an obvious and primary role in algebraic combinatorics. In recent years, the study of natural operations on certain sets of combinatorial objects has given rise to more or less complicated algebraic structures on the vector spaces spanned by these sets. A primordial point to observe is that these structures maintain furthermore many links with combinatorics, combinatorial Hopf algebra theory, representation theory, and theoretical physics. Let us cite for instance the algebra of symmetric functions [Mac95] involving integer partitions, the algebra of noncommutative symmetric functions [GKL+95] involving integer compositions, the Malvenuto-Reutenauer algebra of free quasi-symmetric functions [MR95] (see also [DHT02]) involving permutations, the Loday-Ronco Hopf algebra of binary trees [LR98] (see also [HNT05]), and the Connes-Kreimer Hopf algebra of forests of rooted trees [CK98].

There are several ways to understand and to gather information about such structures. A very fruitful strategy consists in splitting their associative products $\star$ into two separate operations $\prec$ and $\succ$ in such a way that $\star$ turns to be the sum of $\prec$ and $\succ$. To be more precise, if $\mathcal{V}$ is a vector space endowed with an associative product $\star$, splitting $\star$ consists in providing two operations $\prec$ and $\succ$ defined on $\mathcal{V}$ and such that for all elements x and y of $\mathcal{V}$,

$$x \star y = x \prec y + x \succ y. \tag{0.0.1}$$

This splitting property is more concisely denoted by

$$\star = \prec + \succ . \tag{0.0.2}$$

One of the most obvious example occurs by considering the shuffle product on words. Indeed, this product can be separated into two operations according to the origin (first or second operand) of the last letter of the words appearing in the result [Ree58]. Other main examples include the split of the shifted shuffle product of permutations of the Malvenuto-Reutenauer Hopf algebra and of the product of binary trees of the Loday-Ronco Hopf algebra [Foi07]. The original formalization and the germs of generalization of these notions, due to Loday [Lod01], lead to the introduction of dendriform algebras. Dendriform algebras are vector spaces endowed with two operations $\prec$ and $\succ$ so that $\prec + \succ$ is associative and satisfy few other relations. Since any dendriform algebra is a quotient of a certain free dendriform algebra, the study of free dendriform algebras is worthwhile. Besides, the description of free dendriform algebras has a nice combinatorial interpretation involving binary trees and shuffle of binary trees.

In recent years, several generalizations of dendriform algebras were introduced and studied. Among these, one can cite dendriform trialgebras [LR04], quadri-algebras [AL04], ennea-algebras [Ler04], m-dendriform algebras of Leroux [Ler07], and m-dendriform algebras of Novelli [Nov14], all providing new ways to split associative products into more than two pieces. Besides, free objects in the corresponding categories of these algebras can be described by relatively complex combinatorial objects and more or less tricky operations on these. For instance, free dendriform trialgebras involve Schröder trees, free quadri-algebras involve noncrossing connected graphs on a circle, and free m-dendriform algebras of Leroux and free m-dendriform algebras of Novelli involves planar rooted trees where internal nodes have a constant number of children.

The theory of operads (see [LV12] for a complete exposition and also [Cha08]) seems to be one of the best tools to put all these algebraic structures under a same roof. Informally, an operad is a space of abstract operators that can be composed. The main interest of this theory is that any operad encodes a category of algebras and working with an operad amounts to work with the algebras all together of this category. Moreover, this theory gives a nice translation of connections that may exist between *a priori* two very different sorts of algebras. Indeed, any morphism between operads gives rise to a functor between the both encoded categories. We have to point out that operads were first introduced in the context of algebraic topology [May72, BV73] but they are more and more present in combinatorics [Cha08].

The first goal of this work is to define and justify a new generalization of dendriform algebras. Our long term primary objective is to develop new implements to split associative products in smaller pieces. Our main tool is the Koszul duality of operads, an important part of the theory introduced by Ginzburg and Kapranov [GK94]. We use the approach consisting in considering the diassociative operad Dias [Lod01], the Koszul dual of the dendriform operad Dendr, rather that focusing on Dendr. Since Dias admits a description far simpler than Dendr, starting by constructing a generalization of Dias to obtain a generalization of Dendr by Koszul duality is a convenient path to explore.

To obtain a generalization of the diassociative operad, we exploit a general functorial construction T introduced by the author [Gir12, Gir15] producing an operad from any monoid. We showed in these papers that this functor T provides an original construction for the diassociative operad. In the present paper, we rely on T to construct the operads Dias_γ, where γ is a nonnegative integer, in such a way that $\mathsf{Dias}_1 = \mathsf{Dias}$. The operads Dias_γ, called γ-pluriassociative operads, are set-operads involving words on the alphabet $\{0, 1, \dots, \gamma\}$ with exactly one occurrence of 0. Then, by computing the Koszul dual of Dias_γ, we obtain the operads Dendr_γ, satisfying $\mathsf{Dendr}_1 = \mathsf{Dendr}$. The operads Dendr_γ govern the category of the so-called γ-polydendriform algebras, that are algebras with 2γ operations $\leftharpoonup_a$, $\rightharpoonup_a$, $a \in \{1, \dots, \gamma\}$, satisfying some relations. Free objects in these categories involve binary trees where all edges connecting two internal nodes are labeled on $\{1, \dots, \gamma\}$. Moreover, the introduction of γ-polydendriform algebras offers to split an associative product $\star$ by

$$\star = \leftharpoonup_1 + \rightharpoonup_1 + \dots + \leftharpoonup_\gamma + \rightharpoonup_\gamma, \tag{0.0.3}$$

with, among others, the stiffening conditions that all partial sums

$$\leftharpoonup_1 + \rightharpoonup_1 + \dots + \leftharpoonup_a + \rightharpoonup_a \tag{0.0.4}$$

are associative for all $a \in \{1, \dots, \gamma\}$.

This work naturally leads to the consideration and the definition of numerous operads. Table 1 summarizes some information about these.

This work is organized as follows. Section 1 contains a conspectus of the tools used in this paper. We recall here the definition of the construction T [Gir12, Gir15] and provide a reformulation of results of Hoffbeck [Hof10] and Dotsenko and Khoroshkin [DK10] to prove that an operad is Koszul by using convergent rewrite rules. Besides, this part provides self-contained definitions about nonsymmetric operads, algebras over operads, free operads, rewrite

Operad	Objects	Dimensions	Symm.
Dias_γ	Some words on $\{0, 1, \dots, \gamma\}$	$n\gamma^{n-1}$	No
Dendr_γ	γ-edge valued binary trees	$\gamma^{n-1}\frac{1}{n+1}\binom{2n}{n}$	No
As_γ	γ-corollas	γ	No
DAs_γ	γ-alternating Schröder trees	$\sum_{k=0}^{n-2} \gamma^{k+1}(\gamma-1)^{n-k-2}\frac{1}{k+1}\binom{n-2}{k}\binom{n-1}{k}$	No
Dup_γ	γ-edge valued binary trees	$\gamma^{n-1}\frac{1}{n+1}\binom{2n}{n}$	No
Trias_γ	Some words on $\{0, 1, \dots, \gamma\}$	$(\gamma+1)^n - \gamma^n$	No
TDendr_γ	γ-edge valued Schröder trees	$\sum_{k=0}^{n-1}(\gamma+1)^k\gamma^{n-k-1}\frac{1}{k+1}\binom{n-1}{k}\binom{n}{k}$	No
Com_γ	—	—	Yes
Zin_γ	—	—	Yes

TABLE 1. The main operads defined in this paper. All these operads depend on a nonnegative integer parameter γ. The shown dimensions are the ones of the homogeneous components of arities $n \geqslant 2$ of the operads.

rules on trees, and Koszul duality. This section ends by some recalls about the diassociative and dendriform operads.

Section 2 is devoted to the introduction and the study of the operad Dias_γ. We begin by detailing the construction of Dias_γ as a suboperad of the operad obtained by the construction T applied on the monoid $\mathcal{M}_\gamma$ on $\{0, 1, \dots, \gamma\}$ with the operation max as product. More precisely, Dias_γ is defined as the suboperad of $\mathsf{T}\mathcal{M}_\gamma$ generated by the words $0a$ and $a0$ for all $a \in \{1, \dots, \gamma\}$. We then provide a presentation by generators and relations of Dias_γ (Theorem 2.2.6), and show that it is a Koszul operad (Theorem 2.3.1). We also establish some more properties of this operad: we compute its group of symmetries (Proposition 2.3.2), show that it is a basic operad in the sense of [Val07] (Proposition 2.3.3), and show that it is a rooted operad in the sense of [Cha14] (Proposition 2.3.3). We end this section by introducing an alternating basis of Dias_γ, the K-basis, defined through a partial ordering relation over the words indexing the bases of Dias_γ. After describing how the partial composition of Dias_γ expresses over the K-basis (Theorem 2.3.7), we provide a presentation of Dias_γ over this basis (Proposition 2.3.8). Despite the fact that this alternative presentation is more complex than the original one of Dias_γ provided by Theorem 2.2.6, the computation of the Koszul dual Dendr_γ of Dias_γ from this second presentation leads to a surprisingly plain presentation of Dendr_γ considered later in Section 4.

In Section 3, algebras over Dias_γ, called γ-pluriassociative algebras, are studied. The free γ-pluriassociative algebra over one generator is described as a vector space of words on the alphabet $\{0, 1, \dots, \gamma\}$ with exactly one occurrence of 0, endowed with 2γ binary operations (Proposition 3.1.1). We next study two different notions of units in γ-pluriassociative algebras, the bar-units and the wire-units, that are generalizations of definitions of Loday introduced into the context of diassociative algebras [Lod01]. We show that the presence of a wire-unit in a γ-pluriassociative algebra leads to many consequences on its structure (Proposition 3.2.1). Besides, we describe a general construction M to obtain γ-pluriassociative algebras by starting from γ-multiprojection algebras, that are algebraic structures with γ associative products and endowed with γ endomorphisms with extra relations (Theorem 3.3.2). The main interest of the construction M is that γ-multiprojection algebras are simpler algebraic structures than γ-pluriassociative algebras. The bar-units and wire-units of the γ-pluriassociative algebras obtained by this construction are then studied (Proposition 3.3.3). We end this section by listing five examples of γ-pluriassociative algebras constructed from γ-multiprojection algebras, including the free γ-pluriassociative algebra over one generator considered in Section 3.1.3.

Then, the operad Dendr_γ is introduced in Section 4 as the Koszul dual of Dias_γ (Theorem 4.1.1). Since Dias_γ is a Koszul operad, Dendr_γ also is, and then, by using results of Ginzburg and Kapranov [GK94], the alternating versions of the Hilbert series of Dias_γ and Dendr_γ are the inverses for each other for series composition. This leads to an expression for the Hilbert series of Dendr_γ (Proposition 4.1.2). Motivated by the knowledge of the dimensions of Dendr_γ, we consider binary trees where internal edges are labelled on $\{1, \dots, \gamma\}$, called γ-edge valued binary trees. These trees form a generalization of the common binary trees indexing the bases of Dendr, and index the bases of Dendr_γ. We continue the study of this operad by providing a new presentation obtained by considering the Koszul dual of Dias_γ over its K-basis (Theorem 4.1.4). This presentation of Dendr_γ is very compact since its space of relations can be expressed only by three sorts of relations ((4.1.17a), (4.1.17b), and (4.1.17c)), each one involving two or three terms. We also describe all the associative elements of Dendr_γ over its two bases (Propositions 4.1.3, 4.1.5, and 4.1.6). We end this section by constructing the free γ-polydendriform algebra over one generator (Theorem 4.2.3). Its underlying vector space is the vector space of the γ-edge valued binary trees and is endowed with 2γ products described by induction. These products are kinds of shuffle of trees, generalizing the shuffle of trees introduced by Loday [Lod01] intervening in the construction of free dendriform algebras.

Section 5 extends a part of the operadic butterfly [Lod01,Lod06], a diagram of operads gathering the most classical ones together, including the diassociative, dendriform, and associative operads. To extends this diagram into our context, we introduce a one-parameter nonnegative integer generalization As_γ of the associative operad. This operad, called γ-multiassociative operad, has γ associative generating operations, subjected to precise relations. We prove that this operad can be seen as a vector space of corollas labeled on $\{1, \dots, \gamma\}$ and that is Koszul (Proposition 5.1.1). Unlike the associative operad which is self-dual for Koszul duality, As_γ is not when $\gamma \geqslant 2$. The Koszul dual of As_γ, denoted by DAs_γ, is described by its presentation (Proposition 5.1.2) and is realized by means of γ-alternating Schröder trees, that are

Schröder trees where internal nodes are labeled on $\{1, \dots, \gamma\}$ with an alternating condition (Proposition 5.1.5). In passing, we provide an alternative and simpler basis for the space of relations of DAs_γ than the one obtained directly by considering the Koszul dual of As_γ (Proposition 5.1.3). We end this section by establishing a new version of the diagram gathering the diassociative, dendriform, and associative operads for the operads Dias_γ, As_γ, DAs_γ, and Dendr_γ (Theorem 5.2.3) by defining appropriate morphisms between these.

Finally, in Section 6, we sustain our previous ideas to propose one-parameter nonnegative integer generalizations of some more operads. We start by proposing a new operad Dup_γ generalizing the duplicial operad [Lod08], called γ-multiplicial operad. We prove that Dup_γ is Koszul and, like the bases of Dendr_γ, that the bases of Dup_γ are indexed by γ-edge valued binary trees (Proposition 6.1.2). The operads Dendr_γ and Dup_γ are nevertheless not isomorphic because there are 2γ associative elements in Dup_γ (Proposition 6.1.3) against only γ in Dendr_γ. Then, the free γ-multiplicial algebra over one generator is constructed (Theorem 6.1.6). Its underlying vector space is the vector space of the γ-edge valued binary trees and is endowed with 2γ products, similar to the over and under products on binary trees of Loday and Ronco [LR02]. Next, by using almost the same tools as the ones used in Sections 2 and 4, we propose a one-parameter nonnegative integer generalization Trias_γ of the triassociative operad Trias [LR04] and of its Koszul dual, the tridendriform operad TDendr. This follows a very simple idea: like Dias_γ, Trias_γ is defined as a suboperad of $\mathsf{T}\mathcal{M}_\gamma$ generated by the same generators as those of Dias_γ, plus the word 00. In a previous work [Gir12, Gir15], we showed that Trias_1 is the triassociative operad. We provide here a presentation (Theorem 6.2.2) of Trias_γ and deduce a presentation for its Koszul dual, denoted by TDendr_γ (Theorem 6.2.4). Since TDendr is the Koszul dual of Trias, the operads TDendr_γ are generalizations of TDendr. The knowledge of the Hilbert series of TDendr_γ (Proposition 6.2.5) leads to establish the fact that the bases of TDendr_γ are indexed by γ-edge valued Schröder trees, that are Schröder trees where internal edges are labelled on $\{1, \dots, \gamma\}$. We end this work by providing a one-parameter nonnegative integer generalization of all the operads intervening in the operadic butterfly. We then define the operads Com_γ, Lie_γ, Zin_γ, and Leib_γ, that are respective generalizations of the commutative operad, the Lie operad, the Zinbiel operad [Lod95] and the Leibniz operad [Lod93]. We provide analogous versions for our context of the arrows between the commutative operad and the Zinbiel operad (Proposition 6.3.1), and between the dendriform operad and the Zinbiel operad (Proposition 6.3.2).

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Notations and general conventions. All the algebraic structures of this article have a field of characteristic zero $\mathbb{K}$ as ground field. If S is a set, $\mathrm{Vect}(S)$ denotes the linear span of the elements of S. For any integers a and c, $[a, c]$ denotes the set $\{b \in \mathbb{N} : a \leqslant b \leqslant c\}$ and $[n]$, the set $[1, n]$. The cardinality of a finite set S is denoted by $\#S$. If u is a word, its letters are indexed from left to right from 1 to its length $|u|$. For any $i \in [|u|]$, u_i is the letter of u at position i. If a is a letter and n is a nonnegative integer, a^n denotes the word consisting in n occurrences of a. Notice that a^0 is the empty word ϵ.

1. Algebraic structures and main tools

This preliminary section sets our conventions and notations about operads and algebras over an operad, and describes the main tools we will use. The definitions of the diassociative and the dendriform operads are also recalled. This section does not contains new results but it is a self-contained set of definitions about operads intended to readers familiar with algebra or combinatorics but not necessarily with operadic theory.

1.1. Operads and algebras over an operad.

We list here several staple definitions about operads and algebras over an operad. We present also an important tool for this work: the construction T producing operads from monoids.

1.1.1. *Operads.* A *nonsymmetric operad in the category of vector spaces*, or a *nonsymmetric operad* for short, is a graded vector space $\mathcal{O} := \bigoplus_{n \geqslant 1} \mathcal{O}(n)$ together with linear maps

$$\circ_i : \mathcal{O}(n) \otimes \mathcal{O}(m) \to \mathcal{O}(n + m - 1), \qquad n, m \geqslant 1, i \in [n], \tag{1.1.1}$$

called *partial compositions*, and a distinguished element $\mathbb{1} \in \mathcal{O}(1)$, the *unit* of $\mathcal{O}$. This data has to satisfy the three relations

$$(x \circ_i y) \circ_{i+j-1} z = x \circ_i (y \circ_j z), \qquad x \in \mathcal{O}(n), y \in \mathcal{O}(m), z \in \mathcal{O}(k), i \in [n], j \in [m], \tag{1.1.2a}$$

$$(x \circ_i y) \circ_{j+m-1} z = (x \circ_j z) \circ_i y, \qquad x \in \mathcal{O}(n), y \in \mathcal{O}(m), z \in \mathcal{O}(k), i < j \in [n], \tag{1.1.2b}$$

$$\mathbb{1} \circ_1 x = x = x \circ_i \mathbb{1}, \qquad x \in \mathcal{O}(n), i \in [n]. \tag{1.1.2c}$$

Since we shall consider in this paper mainly nonsymmetric operads, we shall call these simply *operads*. Moreover, all considered operads are such that $\mathcal{O}(1)$ has dimension 1.

If x is an element of $\mathcal{O}$ such that $x \in \mathcal{O}(n)$ for a $n \geqslant 1$, we say that n is the *arity* of x and we denote it by $|x|$. An element x of $\mathcal{O}$ of arity 2 is *associative* if $x \circ_1 x = x \circ_2 x$. If $\mathcal{O}_1$ and $\mathcal{O}_2$ are operads, a linear map $\phi : \mathcal{O}_1 \to \mathcal{O}_2$ is an *operad morphism* if it respects arities, sends the unit of $\mathcal{O}_1$ to the unit of $\mathcal{O}_2$, and commutes with partial composition maps. We say that $\mathcal{O}_2$ is a *suboperad* of $\mathcal{O}_1$ if $\mathcal{O}_2$ is a graded subspace of $\mathcal{O}_1$, and $\mathcal{O}_1$ and $\mathcal{O}_2$ have the same unit and the same partial compositions. For any set $G \subseteq \mathcal{O}$, the *operad generated by* G is the smallest suboperad of $\mathcal{O}$ containing G. When the operad generated by G is $\mathcal{O}$ itself and G is minimal with respect to inclusion among the subsets of $\mathcal{O}$ satisfying this property, G is a *generating set* of $\mathcal{O}$ and its elements are *generators* of $\mathcal{O}$. An *operad ideal* of $\mathcal{O}$ is a graded subspace I of $\mathcal{O}$ such that, for any $x \in \mathcal{O}$ and $y \in I$, $x \circ_i y$ and $y \circ_j x$ are in I for all valid integers i and j. Given an operad ideal I of $\mathcal{O}$, one can define the *quotient operad* $\mathcal{O}/_I$ of $\mathcal{O}$ by I in the usual way. When $\mathcal{O}$ is such that all $\mathcal{O}(n)$ are finite for all $n \geqslant 1$, the *Hilbert series* of $\mathcal{O}$ is the series $\mathcal{H}_{\mathcal{O}}(t)$ defined by

$$\mathcal{H}_{\mathcal{O}}(t) := \sum_{n \geqslant 1} \dim \mathcal{O}(n) t^n. \tag{1.1.3}$$

Instead of working with the partial composition maps of $\mathcal{O}$, it is something useful to work with the maps

$$\circ : \mathcal{O}(n) \otimes \mathcal{O}(m_1) \otimes \cdots \otimes \mathcal{O}(m_n) \to \mathcal{O}(m_1 + \cdots + m_n), \qquad n, m_1, \dots, m_n \geqslant 1, \tag{1.1.4}$$

linearly defined for any $x \in \mathcal{O}$ of arity n and $y_1, \dots, y_{n-1}, y_n \in \mathcal{O}$ by

$$x \circ (y_1, \dots, y_{n-1}, y_n) := (\dots ((x \circ_n y_n) \circ_{n-1} y_{n-1}) \dots) \circ_1 y_1. \tag{1.1.5}$$

These maps are called *composition maps* of $\mathcal{O}$.

1.1.2. *Set-operads.* Instead of being a direct sum of vector spaces $\mathcal{O}(n)$, $n \geqslant 1$, $\mathcal{O}$ can be a graded disjoint union of sets. In this context, $\mathcal{O}$ is a *set-operad*. All previous definitions remain valid by replacing direct sums $\oplus$ by disjoint unions $\sqcup$, tensor products $\otimes$ by Cartesian products $\times$, and vector space dimensions dim by set cardinalities $\#$. Moreover, in the context of set-operads, we work with *operad congruences* instead of operad ideals. An operad congruence on a set-operad $\mathcal{O}$ is an equivalence relation $\equiv$ on $\mathcal{O}$ such that all elements of a same $\equiv$-equivalence class have the same arity and for all elements x, x', y, and y' of $\mathcal{O}$, $x \equiv x'$ and $y \equiv y'$ imply $x \circ_i y \equiv x' \circ_i y'$ for all valid integers i. The *quotient operad* $\mathcal{O}/_{\equiv}$ of $\mathcal{O}$ by $\equiv$ is the set-operad defined in the usual way.

Any set-operad $\mathcal{O}$ gives naturally rise to an operad on $\mathrm{Vect}(\mathcal{O})$ by extending the partial compositions of $\mathcal{O}$ by linearity. Besides this, any equivalence relation $\leftrightarrow$ of $\mathcal{O}$ such that all elements of a same $\leftrightarrow$-equivalence class have the same arity induces a subspace of $\mathrm{Vect}(\mathcal{O})$ generated by all $x - x'$ such that $x \leftrightarrow x'$, called *space induced* by $\leftrightarrow$. In particular, any operad congruence $\equiv$ on $\mathcal{O}$ induces an operad ideal of $\mathrm{Vect}(\mathcal{O})$.

1.1.3. *From monoids to operads.* In a previous work [Gir12, Gir15], the author introduced a construction which, from any monoid, produces an operad. This construction is described as follows. Let $\mathcal{M}$ be a monoid with an associative product $\bullet$ admitting a unit 1. We denote by $\mathsf{T}\mathcal{M}$ the operad $\mathsf{T}\mathcal{M} := \bigoplus_{n \geqslant 1} \mathsf{T}\mathcal{M}(n)$ where for all $n \geqslant 1$,

$$\mathsf{T}\mathcal{M}(n) := \mathrm{Vect}\left(\{u_1 \dots u_n : u_i \in \mathcal{M} \text{ for all } i \in [n]\}\right). \tag{1.1.6}$$

The partial composition of two words $u \in \mathsf{T}\mathcal{M}(n)$ and $v \in \mathsf{T}\mathcal{M}(m)$ is linearly defined by

$$u \circ_i v := u_1 \dots u_{i-1} \, (u_i \bullet v_1) \, \dots \, (u_i \bullet v_m) \, u_{i+1} \dots u_n, \qquad i \in [n]. \tag{1.1.7}$$

The unit of $\mathsf{T}\mathcal{M}$ is $\mathbb{1} := 1$. In other words, $\mathsf{T}\mathcal{M}$ is the vector space of words on $\mathcal{M}$ seen as an alphabet and the partial composition returns to insert a word v onto the ith letter u_i of a word u together with a left multiplication by u_i.

1.1.4. *Algebras over an operad.* Any operad $\mathcal{O}$ encodes a category of algebras whose objects are called *$\mathcal{O}$-algebras*. An $\mathcal{O}$-algebra $\mathcal{A}_\mathcal{O}$ is a vector space endowed with a right action

$$\cdot : \mathcal{A}_\mathcal{O}^{\otimes n} \otimes \mathcal{O}(n) \to \mathcal{A}_\mathcal{O}, \qquad n \geqslant 1, \tag{1.1.8}$$

satisfying the relations imposed by the structure of $\mathcal{O}$, that are

$$(e_1 \otimes \cdots \otimes e_{n+m-1}) \cdot (x \circ_i y) = \\ (e_1 \otimes \cdots \otimes e_{i-1} \otimes (e_i \otimes \cdots \otimes e_{i+m-1}) \cdot y \otimes e_{i+m} \otimes \cdots \otimes e_{n+m-1}) \cdot x, \tag{1.1.9}$$

for all $e_1 \otimes \cdots \otimes e_{n+m-1} \in \mathcal{A}_\mathcal{O}^{\otimes n+m-1}$, $x \in \mathcal{O}(n)$, $y \in \mathcal{O}(m)$, and $i \in [n]$. Notice that, by (1.1.9), if G is a generating set of $\mathcal{O}$, it is enough to define the action of each $x \in G$ on $\mathcal{A}_\mathcal{O}^{\otimes |x|}$ to wholly define $\cdot$.

In other words, any element x of $\mathcal{O}$ of arity n plays the role of a linear operation

$$x : \mathcal{A}_\mathcal{O}^{\otimes n} \to \mathcal{A}_\mathcal{O}, \tag{1.1.10}$$

taking n elements of $\mathcal{A}_\mathcal{O}$ as inputs and computing an element of $\mathcal{A}_\mathcal{O}$. By a slight but convenient abuse of notation, for any $x \in \mathcal{O}(n)$, we shall denote by $x(e_1, \dots, e_n)$, or by $e_1 \, x \, e_2$ if x has arity 2, the element $(e_1 \otimes \cdots \otimes e_n) \cdot x$ of $\mathcal{A}_\mathcal{O}$, for any $e_1 \otimes \cdots \otimes e_n \in \mathcal{A}_\mathcal{O}^{\otimes n}$. Observe that by (1.1.9), any associative element of $\mathcal{O}$ gives rise to an associative operation on $\mathcal{A}_\mathcal{O}$.

Arrows in the category of $\mathcal{O}$-algebras are *$\mathcal{O}$-algebra morphisms*, that are linear maps $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ between two $\mathcal{O}$-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ such that

$$\phi(x(e_1, \dots, e_n)) = x(\phi(e_1), \dots, \phi(e_n)), \tag{1.1.11}$$

for all $e_1, \dots, e_n \in \mathcal{A}_1$ and $x \in \mathcal{O}(n)$. We say that $\mathcal{A}_2$ is an *$\mathcal{O}$-subalgebra* of $\mathcal{A}_1$ if $\mathcal{A}_2$ is a subspace of $\mathcal{A}_1$ and $\mathcal{A}_1$ and $\mathcal{A}_2$ are endowed with the same right action of $\mathcal{O}$. If G is a set of elements of an $\mathcal{O}$-algebra $\mathcal{A}$, the *$\mathcal{O}$-algebra generated by G* is the smallest $\mathcal{O}$-subalgebra of $\mathcal{A}$ containing G. When the $\mathcal{O}$-algebra generated by G is $\mathcal{A}$ itself and G is minimal with respect to inclusion among the subsets of $\mathcal{A}$ satisfying this property, G is a *generating set* of $\mathcal{A}$ and its elements are *generators* of $\mathcal{A}$. An *$\mathcal{O}$-algebra ideal* of $\mathcal{A}$ is a subspace I of $\mathcal{A}$ such that for all operation x of $\mathcal{O}$ of arity n and elements $e_1, \dots, e_n$ of $\mathcal{O}$, $x(e_1, \dots, e_n)$ is in I whenever there is a $i \in [n]$ such that e_i is in I.

The *free $\mathcal{O}$-algebra over one generator* is the $\mathcal{O}$-algebra $\mathcal{F}_\mathcal{O}$ defined in the following way. We set $\mathcal{F}_\mathcal{O} := \oplus_{n \geqslant 1} \mathcal{F}_\mathcal{O}(n) := \oplus_{n \geqslant 1} \mathcal{O}(n)$, and for any $e_1, \dots, e_n \in \mathcal{F}_\mathcal{O}$ and $x \in \mathcal{O}(n)$, the right action of x on $e_1 \otimes \cdots \otimes e_n$ is defined by

$$x(e_1, \dots, e_n) := x \circ (e_1, \dots, e_n). \tag{1.1.12}$$

Then, any element x of $\mathcal{O}(n)$ endows $\mathcal{F}_\mathcal{O}$ with an operation

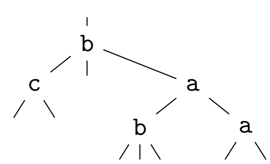
$$x : \mathcal{F}_\mathcal{O}(m_1) \otimes \cdots \otimes \mathcal{F}_\mathcal{O}(m_n) \to \mathcal{F}_\mathcal{O}(m_1 + \cdots + m_n) \tag{1.1.13}$$

respecting the graduation of $\mathcal{F}_\mathcal{O}$.

1.2. **Free operads, rewrite rules, and Koszul duality.** We recall here a description of free operads through syntax trees and presentations of operads by generators and relations. The Koszul duality and the Koszul property for operads are very important tools and notions in this paper. We recall these and describe an already known criterion to prove that a set-operad is Koszul by passing by rewrite rules on syntax trees.

1.2.1. *Syntax trees.* Unless otherwise specified, we use in the sequel the standard terminology (*i.e.*, *node*, *edge*, *root*, *parent*, *child*, *path*, *ancestor*, *etc.*) about planar rooted trees [Knu97]. Let $\mathfrak{t}$ be a planar rooted tree. The *arity* of a node of $\mathfrak{t}$ is its number of children. An *internal node* (resp. a *leaf*) of $\mathfrak{t}$ is a node with a nonzero (resp. null) arity. Given an internal node x of $\mathfrak{t}$, due to the planarity of $\mathfrak{t}$, the children of x are totally ordered from left to right and are thus indexed from 1 to the arity of x. If y is a child of x, y defines a *subtree* of $\mathfrak{t}$, that is the planar rooted tree with root y and consisting in the nodes of $\mathfrak{t}$ that have y as ancestor. We shall call *ith subtree* of x the subtree of $\mathfrak{t}$ rooted at the ith child of x. A *partial subtree* of $\mathfrak{t}$ is a subtree of $\mathfrak{t}$ in which some internal nodes have been replaced by leaves and its descendants has been forgotten. Besides, due to the planarity of $\mathfrak{t}$, its leaves are totally ordered from left to right and thus are indexed from 1 to the arity of $\mathfrak{t}$. In our graphical representations, each tree is depicted so that its root is the uppermost node.

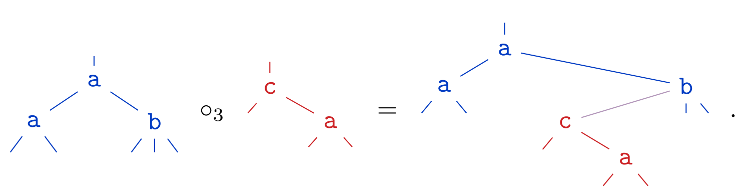
Let $S := \sqcup_{n \geqslant 1} S(n)$ be a graded set. By extension, we say that the *arity* of an element x of S is n provided that $x \in S(n)$. A *syntax tree on S* is a planar rooted tree such that its internal nodes of arity n are labeled on elements of arity n of S. The *degree* (resp. *arity*) of a syntax tree is its number of internal nodes (resp. leaves). For instance, if $S := S(2) \sqcup S(3)$ with $S(2) := \{\mathtt{a}, \mathtt{c}\}$ and $S(3) := \{\mathtt{b}\}$,



(1.2.1)

is a syntax tree on S of degree 5 and arity 8. Its root is labeled by $\mathtt{b}$ and has arity 3.

1.2.2. *Free operads.* Let S be a graded set. The *free operad* $\mathbf{Free}(S)$ *over* S is the operad wherein for any $n \geqslant 1$, $\mathbf{Free}(S)(n)$ is the vector space of syntax trees on S of arity n, the partial composition $\mathfrak{s} \circ_i \mathfrak{t}$ of two syntax trees $\mathfrak{s}$ and $\mathfrak{t}$ on S consists in grafting the root of $\mathfrak{t}$ on the ith leaf of $\mathfrak{s}$, and its unit is the tree consisting in one leaf. For instance, if $S := S(2) \sqcup S(3)$ with $S(2) := \{\mathtt{a}, \mathtt{c}\}$ and $S(3) := \{\mathtt{b}\}$, one has in $\mathbf{Free}(S)$,



(1.2.2)

We denote by $\mathrm{cor} : S \to \mathbf{Free}(S)$ the inclusion map, sending any x of S to the *corolla* labeled by x, that is the syntax tree consisting in one internal node labeled by x attached to a required number of leaves. In the sequel, if required by the context, we shall implicitly see any element x of S as the corolla $\mathrm{cor}(x)$ of $\mathbf{Free}(S)$. For instance, when x and y are two elements of S, we shall simply denote by $x \circ_i y$ the syntax tree $\mathrm{cor}(x) \circ_i \mathrm{cor}(y)$ for all valid integers i.

For any operad $\mathcal{O}$, by seeing $\mathcal{O}$ as a graded set, $\mathbf{Free}(\mathcal{O})$ is the free operad of the syntax trees linearly labeled by elements of $\mathcal{O}$. The *evaluation map* of $\mathcal{O}$ is the map

$$\mathrm{eval}_{\mathcal{O}} : \mathbf{Free}(\mathcal{O}) \to \mathcal{O}, \tag{1.2.3}$$

recursively defined by

$$\mathrm{eval}_{\mathcal{O}}(\mathfrak{t}) := \begin{cases} \mathbb{1} & \text{if } \mathfrak{t} \text{ is the leaf,} \\ x \circ (\mathrm{eval}_{\mathcal{O}}(\mathfrak{s}_1), \dots, \mathrm{eval}_{\mathcal{O}}(\mathfrak{s}_n)) & \text{otherwise,} \end{cases} \tag{1.2.4}$$

where $\mathbb{1}$ is the unit of $\mathcal{O}$, x is the label of the root of $\mathfrak{t}$, and $\mathfrak{s}_1, \dots, \mathfrak{s}_n$ are, from left to right, the subtrees of the root of $\mathfrak{t}$. In other words, any tree $\mathfrak{t}$ of $\mathbf{Free}(\mathcal{O})$ can be seen as a tree-like expression for an element $\mathrm{eval}_{\mathcal{O}}(\mathfrak{t})$ of $\mathcal{O}$. Moreover, by induction on the degree of $\mathfrak{t}$, it appears that $\mathrm{eval}_{\mathcal{O}}$ is a well-defined surjective operad morphism.

1.2.3. *Presentations by generators and relations.* A *presentation* of an operad $\mathcal{O}$ consists in a pair $(\mathfrak{G}, \mathfrak{R})$ such that $\mathfrak{G} := \sqcup_{n \geqslant 1} \mathfrak{G}(n)$ is a graded set, $\mathfrak{R}$ is a subspace of $\mathbf{Free}(\mathfrak{G})$, and $\mathcal{O}$ is isomorphic to $\mathbf{Free}(\mathfrak{G})/_{\langle \mathfrak{R} \rangle}$, where $\langle \mathfrak{R} \rangle$ is the operad ideal of $\mathbf{Free}(\mathfrak{G})$ generated by $\mathfrak{R}$. We call $\mathfrak{G}$ the *set of generators* and $\mathfrak{R}$ the *space of relations* of $\mathcal{O}$. We say that $\mathcal{O}$ is *quadratic* if one can exhibit a presentation $(\mathfrak{G}, \mathfrak{R})$ of $\mathcal{O}$ such that $\mathfrak{R}$ is a homogeneous subspace of $\mathbf{Free}(\mathfrak{G})$ consisting in syntax trees of degree 2. Besides, we say that $\mathcal{O}$ is *binary* if one can exhibit a presentation $(\mathfrak{G}, \mathfrak{R})$ of $\mathcal{O}$ such that $\mathfrak{G}$ is concentrated in arity 2.

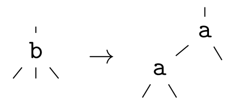
With knowledge of a presentation $(\mathfrak{G}, \mathfrak{R})$ of $\mathcal{O}$, it is easy to describe the category of the $\mathcal{O}$-algebras. Indeed, by denoting by $\pi : \mathbf{Free}(\mathfrak{G}) \to \mathbf{Free}(\mathfrak{G})/_{\langle \mathfrak{R} \rangle}$ the canonical surjection map, the category of $\mathcal{O}$-algebras is the category of vector spaces $\mathcal{A}_{\mathcal{O}}$ endowed with maps $\pi(g)$, $g \in \mathfrak{G}$, satisfying for all $r \in \mathfrak{R}$ the relations

$$r(e_1, \dots, e_n) = 0, \tag{1.2.5}$$

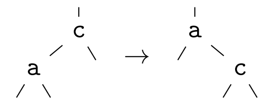
for all $e_1, \dots, e_n \in \mathcal{A}_{\mathcal{O}}$, where n is the arity of r.

1.2.4. *Rewrite rules.* Let S be a graded set. A *rewrite rule* on syntax trees on S is a binary relation $\to$ on $\mathbf{Free}(S)$ whenever for all trees $\mathfrak{s}$ and $\mathfrak{t}$ of $\mathbf{Free}(S)$, $\mathfrak{s} \to \mathfrak{t}$ only if $\mathfrak{s}$ and $\mathfrak{t}$ have the same arity. When $\to$ involves only syntax trees of degree two, $\to$ is *quadratic*. We say that a syntax tree $\mathfrak{s}'$ can be *rewritten* by $\to$ into $\mathfrak{t}'$ if there exist two syntax trees $\mathfrak{s}$ and $\mathfrak{t}$ satisfying $\mathfrak{s} \to \mathfrak{t}$ and $\mathfrak{s}'$ has a partial subtree equal to $\mathfrak{s}$ such that, by replacing it by $\mathfrak{t}$ in $\mathfrak{s}'$, we obtain $\mathfrak{t}'$. By a slight but convenient abuse of notation, we denote by $\mathfrak{s}' \to \mathfrak{t}'$ this property. When a syntax tree $\mathfrak{t}$ can be obtained by performing a sequence of $\to$-rewritings from a syntax tree $\mathfrak{s}$, we say that $\mathfrak{s}$ is *rewritable* by $\to$ into $\mathfrak{t}$ and we denote this property by $\mathfrak{s} \overset{*}{\to} \mathfrak{t}$. For instance, for

$S := S(2) \sqcup S(3)$ with $S(2) := \{\mathtt{a}, \mathtt{c}\}$ and $S(3) := \{\mathtt{b}\}$, consider the rewrite rule $\to$ on $\mathbf{Free}(S)$ satisfying

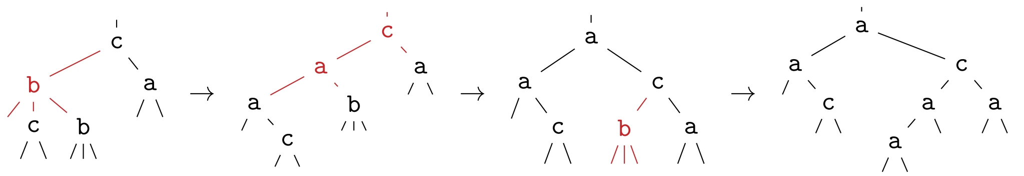


and



(1.2.6)

We then have the following sequence of rewritings



(1.2.7)

We shall use the standard terminology (*confluent*, *terminating*, *convergent*, *normal form*, *critical pair*, *etc.*) about rewrite rules (see [BN98]).

Any rewrite rule $\to$ on $\mathbf{Free}(S)$ defines an operad congruence $\equiv_\to$ on $\mathbf{Free}(S)$ seen as a set-operad, the *operad congruence induced* by $\to$, as the finest operad congruence on $\mathbf{Free}(S)$ containing the reflexive, symmetric, and transitive closure of $\to$.

1.2.5. *Koszul duality and Koszulity.* In [GK94], Ginzburg and Kapranov extended the notion of Koszul duality of quadratic associative algebras to quadratic operads. Starting with a binary and quadratic operad $\mathcal{O}$ admitting a presentation $(\mathfrak{G}, \mathfrak{R})$, the *Koszul dual* of $\mathcal{O}$ is the operad $\mathcal{O}^!$, isomorphic to the operad admitting the presentation $\left(\mathfrak{G}, \mathfrak{R}^\perp\right)$ where $\mathfrak{R}^\perp$ is the annihilator of $\mathfrak{R}$ in $\mathbf{Free}(\mathfrak{G})$ with respect to the scalar product

$$\langle -, - \rangle : \mathbf{Free}(\mathfrak{G})(3) \otimes \mathbf{Free}(\mathfrak{G})(3) \to \mathbb{K} \tag{1.2.8}$$

linearly defined, for all $x, x', y, y' \in \mathfrak{G}(2)$, by

$$\langle x \circ_i y, x' \circ_{i'} y' \rangle := \begin{cases} 1 & \text{if } x = x', y = y', \text{ and } i = i' = 1, \\ -1 & \text{if } x = x', y = y', \text{ and } i = i' = 2, \\ 0 & \text{otherwise.} \end{cases} \tag{1.2.9}$$

Then, knowing a presentation of $\mathcal{O}$, one can compute a presentation of $\mathcal{O}^!$.

Besides, we say a quadratic operad $\mathcal{O}$ is *Koszul* if its Koszul complex is acyclic [GK94,LV12]. In this work, to prove the Koszulity of an operad $\mathcal{O}$, we shall make use of a combinatorial tool introduced by Hoffbeck [Hof10] (see also [LV12]) consisting in exhibiting a particular basis of $\mathcal{O}$, a so-called *Poincaré-Birkhoff-Witt basis*.

In this paper, we shall use this tool only in the context of set-operads, which reformulates, thanks to the work of Dotsenko and Khoroshkin [DK10], as follows. A set-operad $\mathcal{O}$ is Kosuzl if there is a graded set S and a rewrite rule $\to$ on $\mathbf{Free}(S)$ such that $\mathcal{O}$ is isomorphic to $\mathbf{Free}(S)/_{\equiv_\to}$ and $\to$ is a convergent quadratic rewrite rule. Moreover, the set of normal forms of $\to$ forms a Poincaré-Birkhoff-Witt basis of $\mathcal{O}$.

Furthermore, when $\mathcal{O}$ and $\mathcal{O}^!$ are two operads Koszul dual one of the other, and moreover, when they are Koszul operads and admit Hilbert series, their Hilbert series satisfy [GK94]

$$\mathcal{H}_{\mathcal{O}}\left(-\mathcal{H}_{\mathcal{O}^!}(-t)\right) = t. \tag{1.2.10}$$

We shall make use of (1.2.10) to compute the dimensions of Koszul operads defined as Koszul duals of known ones.

1.3. **Diassociative and dendriform operads.** We recall here, by using the notions presented during the previous sections, the definitions and some properties of the diassociative and dendriform operads.

1.3.1. *Diassociative operad and diassociative algebras.* The *diassociative operad* Dias was introduced by Loday [Lod01] as the operad admitting the presentation $(\mathfrak{G}_{\mathsf{Dias}}, \mathfrak{R}_{\mathsf{Dias}})$ where $\mathfrak{G}_{\mathsf{Dias}} := \mathfrak{G}_{\mathsf{Dias}}(2) := \{\dashv, \vdash\}$ and $\mathfrak{R}_{\mathsf{Dias}}$ is the space induced by the equivalence relation $\equiv$ satisfying

$$\dashv \circ_1 \vdash \equiv \vdash \circ_2 \dashv, \tag{1.3.1a}$$

$$\dashv \circ_1 \dashv \equiv \dashv \circ_2 \dashv \equiv \dashv \circ_2 \vdash, \tag{1.3.1b}$$

$$\vdash \circ_1 \dashv \equiv \vdash \circ_1 \vdash \equiv \vdash \circ_2 \vdash . \tag{1.3.1c}$$

Note that Dias is a binary and quadratic operad.

This operad admits the following realization [Cha05]. For any $n \geqslant 1$, $\mathsf{Dias}(n)$ is the linear span of the $\mathfrak{e}_{n,k}$, $k \in [n]$, and the partial compositions linearly satisfy, for all $n, m \geqslant 1$, $k \in [n]$, $\ell \in [m]$, and $i \in [n]$,

$$\mathfrak{e}_{n,k} \circ_i \mathfrak{e}_{m,\ell} = \begin{cases} \mathfrak{e}_{n+m-1,k+m-1} & \text{if } i < k, \\ \mathfrak{e}_{n+m-1,k+\ell-1} & \text{if } i = k, \\ \mathfrak{e}_{n+m-1,k} & \text{otherwise } (i > k). \end{cases} \tag{1.3.2}$$

Since the partial composition of two basis elements of Dias produces exactly one basis element, Dias is well-defined as a set-operad. Moreover, this realization shows that $\dim \mathsf{Dias}(n) = n$ and hence, the Hilbert series of Dias satisfies

$$\mathcal{H}_{\mathsf{Dias}}(t) = \frac{t}{(1-t)^2}. \tag{1.3.3}$$

From the presentation of Dias, we deduce that any Dias-algebra, also called *diassociative algebra*, is a vector space $\mathcal{A}_{\mathsf{Dias}}$ endowed with linear operations $\dashv$ and $\vdash$ satisfying the relations encoded by (1.3.1a)—(1.3.1c).

From the realization of Dias, we deduce that the free diassociative algebra $\mathcal{F}_{\mathsf{Dias}}$ over one generator is the vector space Dias endowed with the linear operations

$$\dashv, \vdash \colon \mathcal{F}_{\mathsf{Dias}} \otimes \mathcal{F}_{\mathsf{Dias}} \to \mathcal{F}_{\mathsf{Dias}}, \tag{1.3.4}$$

satisfying, for all $n, m \geqslant 1$, $k \in [n]$, $\ell \in [m]$,

$$\mathfrak{e}_{n,k} \dashv \mathfrak{e}_{m,\ell} = (\mathfrak{e}_{n,k} \otimes \mathfrak{e}_{m,\ell}) \cdot \mathfrak{e}_{2,1} = (\mathfrak{e}_{2,1} \circ_2 \mathfrak{e}_{m,\ell}) \circ_1 \mathfrak{e}_{n,k} = \mathfrak{e}_{n+m,k}, \tag{1.3.5}$$

and

$$\mathfrak{e}_{n,k} \vdash \mathfrak{e}_{m,\ell} = (\mathfrak{e}_{n,k} \otimes \mathfrak{e}_{m,\ell}) \cdot \mathfrak{e}_{2,2} = (\mathfrak{e}_{2,2} \circ_2 \mathfrak{e}_{m,\ell}) \circ_1 \mathfrak{e}_{n,k} = \mathfrak{e}_{n+m,n+\ell}. \tag{1.3.6}$$

As shown in [Gir12, Gir15], the diassociative operad is isomorphic to the suboperad of $\mathsf{T}\mathcal{M}$ generated by 01 and 10 where $\mathcal{M}$ is the multiplicative monoid on $\{0, 1\}$. The concerned isomorphism sends any $\mathfrak{e}_{n,k}$ of Dias to the word $0^{k-1}\,1\,0^{n-k}$ of $\mathsf{T}\mathcal{M}$.

1.3.2. *Dendriform operad and dendriform algebras.* The *dendriform operad* Dendr was also introduced by Loday [Lod01]. It is the operad admitting the presentation $(\mathfrak{G}_{\mathsf{Dendr}}, \mathfrak{R}_{\mathsf{Dendr}})$ where $\mathfrak{G}_{\mathsf{Dendr}} := \mathfrak{G}_{\mathsf{Dendr}}(2) := \{\prec, \succ\}$ and $\mathfrak{R}_{\mathsf{Dendr}}$ is the vector space generated by

$$\prec \circ_1 \succ - \succ \circ_2 \prec, \tag{1.3.7a}$$

$$\prec \circ_1 \prec - \prec \circ_2 \prec - \prec \circ_2 \succ, \tag{1.3.7b}$$

$$\succ \circ_1 \prec + \succ \circ_1 \succ - \succ \circ_2 \succ . \tag{1.3.7c}$$

Note that Dendr is a binary and quadratic operad.

This operad admits a quite complicated realization [Lod01]. For all $n \geqslant 1$, the $\mathsf{Dendr}(n)$ are vector spaces of binary trees with n internal nodes. The partial composition of two binary trees can be described by means of intervals of the Tamari order [HT72], a partial order relation involving binary trees. This realization shows that $\dim \mathsf{Dendr}(n) = \mathrm{cat}(n)$ where

$$\mathrm{cat}(n) := \frac{1}{n+1}\binom{2n}{n} \tag{1.3.8}$$

is the nth *Catalan number*, counting the binary trees with respect to their number of internal nodes. Therefore, the Hilbert series of Dendr satisfies

$$\mathcal{H}_{\mathsf{Dendr}}(t) = \frac{1 - \sqrt{1-4t} - 2t}{2t}. \tag{1.3.9}$$

Throughout this article, we shall graphically represent binary trees in a slightly different manner than syntax trees. We represent the leaves of binary trees by squares $\square$, internal nodes by circles $\bigcirc$, and edges by thick segments $|$.

From the presentation of Dendr, we deduce that any Dendr-algebra, also called *dendriform algebra*, is a vector space $\mathcal{A}_{\mathsf{Dendr}}$ endowed with linear operations $\prec$ and $\succ$ satisfying the relations encoded by (1.3.7a)—(1.3.7c). Classical examples of dendriform algebras include Rota-Baxter algebras [Agu00] and shuffle algebras [Lod01].

The operation obtained by summing $\prec$ and $\succ$ is associative. Therefore, we can see a dendriform algebra as an associative algebra in which its associative product has been split into two parts satisfying Relations (1.3.7a), (1.3.7b), and (1.3.7c). More precisely, we say that an associative algebra $\mathcal{A}$ admits a *dendriform structure* if there exist two nonzero binary operations $\prec$ and $\succ$ such that the associative operation $\star$ of $\mathcal{A}$ satisfies $\star = \prec + \succ$, and $\mathcal{A}$ endowed with the operations $\prec$ and $\succ$, is a dendriform algebra

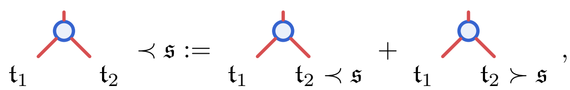
The free dendriform algebra $\mathcal{F}_{\mathsf{Dendr}}$ over one generator is the vector space Dendr of binary trees with at least one internal node endowed with the linear operations

$$\prec, \succ : \mathcal{F}_{\mathsf{Dendr}} \otimes \mathcal{F}_{\mathsf{Dendr}} \to \mathcal{F}_{\mathsf{Dendr}}, \tag{1.3.10}$$

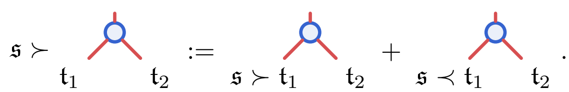
defined recursively, for any binary tree $\mathfrak{s}$ with at least one internal node, and binary trees $\mathfrak{t}_1$ and $\mathfrak{t}_2$ by

$$\mathfrak{s} \prec \boxdot := \mathfrak{s} =: \boxdot \succ \mathfrak{s}, \tag{1.3.11}$$

$$\boxdot \prec \mathfrak{s} := 0 =: \mathfrak{s} \succ \boxdot, \tag{1.3.12}$$



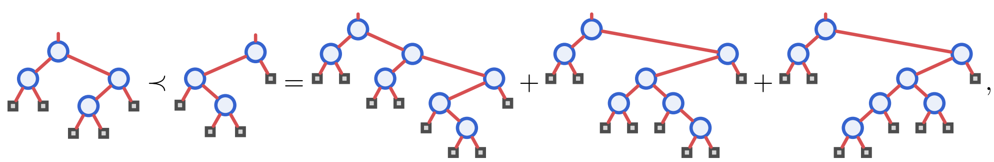
(1.3.13)



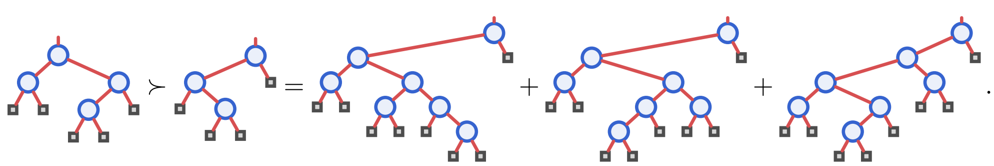
(1.3.14)

Note that neither $\boxdot \prec \boxdot$ nor $\boxdot \succ \boxdot$ are defined.

We have for instance,


(1.3.15)

and


(1.3.16)

As shown in [Lod01], the dendriform operad is the Koszul dual of the diassociative operad. This can be checked by a simple computation following what is explained in Section 1.2.5. Besides that, since theses two operads are Koszul operads, the alternating versions of their Hilbert series are the inverses for each other for series composition.

We invite the reader to take a look at [LR98, Agu00, Lod02, Foi07, EFMP08, EFM09, LV12] for a supplementary review of properties of dendriform algebras and of the dendriform operad.

2. Pluriassociative operads

In this section, we define the main object of this work: a generalization on a nonnegative integer parameter γ of the diassociative operad. We provide a complete study of this new operad.

2.1. Construction and first properties.

We define here our generalization of the diassociative operad using the functor T (whose definition is recalled in Section 1.1.3). We then describe the elements and establish the Hilbert series of our generalization.

2.1.1. *Construction.* For any integer $\gamma \geqslant 0$, let $\mathcal{M}_\gamma$ be the monoid $\{0\} \cup [\gamma]$ with the binary operation max as product, denoted by $\uparrow$. We define Dias_γ as the suboperad of $\mathsf{T}\mathcal{M}_\gamma$ generated by

$$\{0a, a0 : a \in [\gamma]\}. \tag{2.1.1}$$

By definition, Dias_γ is the vector space of words that can be obtained by partial compositions of words of (2.1.1). We have, for instance,

$$\mathsf{Dias}_2(1) = \mathrm{Vect}(\{0\}), \tag{2.1.2}$$

$$\mathsf{Dias}_2(2) = \mathrm{Vect}(\{01, 02, 10, 20\}), \tag{2.1.3}$$

$$\mathsf{Dias}_2(3) = \mathrm{Vect}(\{011, 012, 021, 022, 101, 102, 201, 202, 110, 120, 210, 220\}), \tag{2.1.4}$$

and

$$211\mathbf{2}01 \circ_4 31103 = 2113222301, \tag{2.1.5}$$

$$11\mathbf{1}101 \circ_3 20 = 1121101, \tag{2.1.6}$$

$$1\mathbf{0}13 \circ_2 210 = 121013. \tag{2.1.7}$$

It follows immediately from the definition of Dias_γ as a suboperad of $\mathsf{T}\mathcal{M}_\gamma$ that Dias_γ is a set-operad. Indeed, any partial composition of two basis elements of Dias_γ gives rises to exactly one basis element. We then shall see Dias_γ as a set-operad over all Section 2.

Notice that $\mathsf{Dias}_\gamma(2)$ is the set (2.1.1) of generators of Dias_γ. Besides, observe that Dias_0 is the trivial operad and that Dias_γ is a suboperad of $\mathsf{Dias}_{\gamma+1}$. We call Dias_γ the *γ-pluriassociative operad*.

2.1.2. *Elements and dimensions.*

Proposition 2.1.1. *For any integer $\gamma \geqslant 0$, as a set-operad, the underlying set of Dias_γ is the set of the words on the alphabet $\{0\} \cup [\gamma]$ containing exactly one occurrence of 0.*

Proof. Let us show that any word x of Dias_γ satisfies the statement of the proposition by induction on the length n of x. This is true when $n = 1$ because we necessarily have $x = 0$. Otherwise, when $n \geqslant 2$, there is a word y of Dias_γ of length $n-1$ and a generator g of Dias_γ such that $x = y \circ_i g$ for a $i \in [n-1]$. Then, x is obtained by replacing the ith letter a of y by the factor $u := u_1 u_2$ where $u_1 := a \uparrow g_1$ and $u_2 := a \uparrow g_2$. Since g contains exactly one 0, this operation consists in inserting a nonzero letter of $[\gamma]$ into y. Since by induction hypothesis y contains exactly one 0, it follows that x satisfies the statement of the proposition.

Conversely, let us show that any word x satisfying the statement of the proposition belongs to Dias_γ by induction on the length n of x. This is true when $n = 1$ because we necessarily have $x = 0$ and 0 belongs to Dias_γ since it is its unit. Otherwise, when $n \geqslant 2$, there is an integer $i \in [n-1]$ such that $x_i x_{i+1} \in \{0a, a0\}$ for an $a \in [\gamma]$. Let us suppose without loss of generality that $x_i x_{i+1} = a0$. By setting y as the word obtained by erasing the ith letter of x, we have $x = y \circ_i a0$. Thus, since by induction hypothesis y is an element of Dias_γ, it follows that x also is. □

We deduce from Proposition 2.1.1 that the Hilbert series of Dias_γ satisfies

$$\mathcal{H}_{\mathsf{Dias}_\gamma}(t) = \frac{t}{(1-\gamma t)^2} \tag{2.1.8}$$

and that for all $n \geqslant 1$, $\dim \mathsf{Dias}_\gamma(n) = n\gamma^{n-1}$. For instance, the first dimensions of Dias_1, Dias_2, Dias_3, and Dias_4 are respectively

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, \tag{2.1.9}$$

$$1, 4, 12, 32, 80, 192, 448, 1024, 2304, 5120, 11264, \tag{2.1.10}$$

$$1, 6, 27, 108, 405, 1458, 5103, 17496, 59049, 196830, 649539, \tag{2.1.11}$$

$$1, 8, 48, 256, 1280, 6144, 28672, 131072, 589824, 2621440, 11534336. \tag{2.1.12}$$

The second one is Sequence **A001787**, the third one is Sequence **A027471**, and the last one is Sequence **A002697** of [Slo].

2.2. **Presentation by generators and relations.** To establish a presentation of Dias_γ, we shall start by defining a morphism word_γ from a free operad to Dias_γ. Then, after showing that word_γ is a surjection, we will show that word_γ induces an operad isomorphism between a quotient of a free operad by a certain operad congruence $\equiv_\gamma$ and Dias_γ. The space of relations of Dias_γ of its presentation will be induced by $\equiv_\gamma$.

2.2.1. *From syntax trees to words.* For any integer $\gamma \geqslant 0$, let $\mathfrak{G}_{\mathsf{Dias}_\gamma} := \mathfrak{G}_{\mathsf{Dias}_\gamma}(2)$ be the graded set where

$$\mathfrak{G}_{\mathsf{Dias}_\gamma}(2) := \{\dashv_a, \vdash_a : a \in [\gamma]\}. \tag{2.2.1}$$

Let $\mathfrak{t}$ be a syntax tree of $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dias}_\gamma}\right)$ and x be a leaf of $\mathfrak{t}$. We say that an integer $a \in \{0\} \cup [\gamma]$ is *eligible* for x if $a = 0$ or there is an ancestor y of x labeled by $\dashv_a$ (resp. $\vdash_a$) and x is in the right (resp. left) subtree of y. The *image* of x is its greatest eligible integer. Moreover, let

$$\mathrm{word}_\gamma : \mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dias}_\gamma}\right)(n) \to \mathsf{Dias}_\gamma(n), \qquad n \geqslant 1, \tag{2.2.2}$$

the map where $\mathrm{word}_\gamma(\mathfrak{t})$ is the word obtained by considering, from left to right, the images of the leaves of $\mathfrak{t}$ (see Figure 1).

Lemma 2.2.1. *For any integer $\gamma \geqslant 0$, the map word_γ is an operad morphism from $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dias}_\gamma}\right)$ to Dias_γ.*

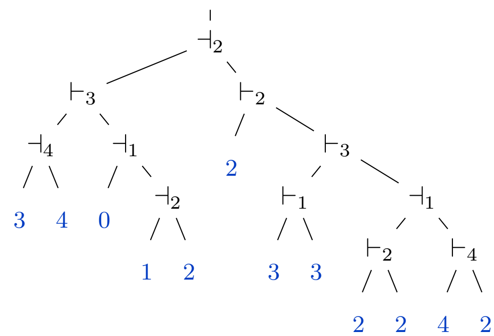


FIGURE 1. A syntax tree $\mathfrak{t}$ of $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dias}_\gamma}\right)$ where images of its leaves are shown. This tree satisfies $\mathrm{word}_\gamma(\mathfrak{t}) = 340122332242$.

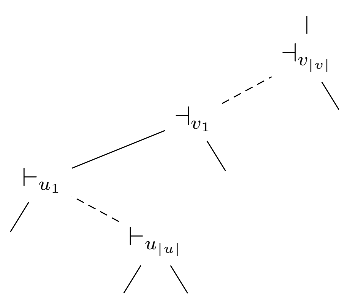
Proof. Let us first show that word_γ is a well-defined map. Let $\mathfrak{t}$ be a syntax tree of $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dias}_\gamma}\right)$ of arity n. Observe that by starting from the root of $\mathfrak{t}$, there is a unique maximal path obtained by following the directions specified by its internal nodes (a $\dashv_a$ means to go the left child while a $\vdash_a$ means to go to the right child). Then, the leaf at the end of this path is the only leaf with 0 as image. Others $n-1$ leaves have integers of $[\gamma]$ as images. By Proposition 2.1.1, this implies that $\mathrm{word}_\gamma(\mathfrak{t})$ is an element of $\mathsf{Dias}_\gamma(n)$.

To prove that word_γ is an operad morphism, we consider its following alternative description. If $\mathfrak{t}$ is a syntax tree of $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dias}_\gamma}\right)$, we can consider the tree $\mathfrak{t}'$ obtained by replacing in $\mathfrak{t}$ each label $\dashv_a$ (resp. $\vdash_a$) by the word $0a$ (resp. $a0$), where $a \in [\gamma]$. Then, by a straightforward induction on the number of internal nodes of $\mathfrak{t}$, we obtain that $\mathrm{eval}_{\mathsf{Dias}_\gamma}(\mathfrak{t}')$, where $\mathfrak{t}'$ is seen as a syntax tree of $\mathbf{Free}(\mathsf{Dias}_\gamma(2))$, is $\mathrm{word}_\gamma(\mathfrak{t})$. It then follows that word_γ is an operad morphism. $\square$

2.2.2. *Hook syntax trees.* Let us now consider the map

$$\mathrm{hook}_\gamma : \mathsf{Dias}_\gamma(n) \to \mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dias}_\gamma}\right)(n), \qquad n \geqslant 1, \tag{2.2.3}$$

defined for any word x of Dias_γ by

$\mathrm{hook}_\gamma(x) :=$  , (2.2.4)

where x decomposes, by Proposition 2.1.1, uniquely in $x = u0v$ where u and v are words on the alphabet $[\gamma]$. The dashed edges denote, depending on their orientation, a right comb (wherein internal nodes are labeled, from top to bottom by $\vdash_{u_1}, \ldots, \vdash_{u_{|u|}}$) or a left comb (wherein internal nodes are labeled, from bottom to top, by $\dashv_{v_1}, \dots, \dashv_{v_{|v|}}$). We shall call any syntax tree of the form (2.2.4) a *hook syntax tree.*

Lemma 2.2.2. *For any integer $\gamma \geqslant 0$, the map* word_γ *is a surjective operad morphism from* $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dias}_\gamma}\right)$ *onto* Dias_γ*. Moreover, for any element x of* Dias_γ*,* $\mathrm{hook}_\gamma(x)$ *belongs to the fiber of x under* word_γ*.*

Proof. The fact that x belongs to the fiber of x under word_γ is an immediate consequence of the definitions of word_γ and hook_γ, and the fact that by Proposition 2.1.1, any word x of Dias_γ decomposes uniquely in $x = u0v$ where u and v are words on the alphabet $[\gamma]$. Then, word_γ is surjective as a map. Moreover, since by Lemma 2.2.1, word_γ is an operad morphism, it is a surjective operad morphism. □

2.2.3. *A rewrite rule on syntax trees.* Let $\to_\gamma$ be the quadratic rewrite rule on $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dias}_\gamma}\right)$ satisfying

$$\vdash_{a'} \circ_2 \dashv_a \;\to_\gamma\; \dashv_a \circ_1 \vdash_{a'}, \qquad a, a' \in [\gamma], \tag{2.2.5a}$$

$$\dashv_a \circ_2 \vdash_b \;\to_\gamma\; \dashv_a \circ_1 \dashv_b, \qquad a < b \in [\gamma], \tag{2.2.5b}$$

$$\vdash_a \circ_1 \dashv_b \;\to_\gamma\; \vdash_a \circ_2 \vdash_b, \qquad a < b \in [\gamma], \tag{2.2.5c}$$

$$\dashv_a \circ_2 \dashv_b \;\to_\gamma\; \dashv_b \circ_1 \dashv_a, \qquad a < b \in [\gamma], \tag{2.2.5d}$$

$$\vdash_a \circ_1 \vdash_b \;\to_\gamma\; \vdash_b \circ_2 \vdash_a, \qquad a < b \in [\gamma], \tag{2.2.5e}$$

$$\dashv_d \circ_2 \dashv_c \;\to_\gamma\; \dashv_d \circ_1 \dashv_d, \qquad c \leqslant d \in [\gamma], \tag{2.2.5f}$$

$$\dashv_d \circ_2 \vdash_c \;\to_\gamma\; \dashv_d \circ_1 \dashv_d, \qquad c \leqslant d \in [\gamma], \tag{2.2.5g}$$

$$\vdash_d \circ_1 \dashv_c \;\to_\gamma\; \vdash_d \circ_2 \vdash_d, \qquad c \leqslant d \in [\gamma], \tag{2.2.5h}$$

$$\vdash_d \circ_1 \vdash_c \;\to_\gamma\; \vdash_d \circ_2 \vdash_d, \qquad c \leqslant d \in [\gamma], \tag{2.2.5i}$$

and denote by $\equiv_\gamma$ the operadic congruence on $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dias}_\gamma}\right)$ induced by $\to_\gamma$.

Lemma 2.2.3. *For any integer $\gamma \geqslant 0$ and any syntax trees $\mathfrak{t}_1$ and $\mathfrak{t}_2$ of* $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dias}_\gamma}\right)$*, $\mathfrak{t}_1 \equiv_\gamma \mathfrak{t}_2$ implies* $\mathrm{word}_\gamma(\mathfrak{t}_1) = \mathrm{word}_\gamma(\mathfrak{t}_2)$*.*

Proof. Let us denote by $\leftrightarrow_\gamma$ the symmetric closure of $\to_\gamma$. In the first place, observe that for any relation $\mathfrak{s}_1 \leftrightarrow_\gamma \mathfrak{s}_2$ where $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are syntax trees of $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dias}_\gamma}\right)(3)$, for any $i \in [3]$, the eligible integers for the ith leaves of $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are the same. Besides, by definition of $\equiv_\gamma$, since $\mathfrak{t}_1 \equiv_\gamma \mathfrak{t}_2$, one can obtain $\mathfrak{t}_2$ from $\mathfrak{t}_1$ by performing a sequence of $\leftrightarrow_\gamma$-rewritings. According to the previous observation, a $\leftrightarrow_\gamma$-rewriting preserve the eligible integers of all leaves of the tree on which they are performed. Therefore, the images of the leaves of $\mathfrak{t}_2$ are, from left to right, the same as the images of the leaves of $\mathfrak{t}_1$ and hence, $\mathrm{word}_\gamma(\mathfrak{t}_1) = \mathrm{word}_\gamma(\mathfrak{t}_2)$. □

Lemma 2.2.3 implies that the map

$$\bar{\mathrm{word}}_\gamma : \mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dias}_\gamma}\right)(n)/_{\equiv_\gamma} \to \mathsf{Dias}_\gamma(n), \qquad n \geqslant 1, \tag{2.2.6}$$

satisfying, for any $\equiv_\gamma$-equivalence class $[\mathfrak{t}]_{\equiv_\gamma}$,

$$\bar{\mathrm{word}}_\gamma\left([\mathfrak{t}]_\gamma\right) = \mathrm{word}_\gamma(\mathfrak{t}), \tag{2.2.7}$$

where $\mathfrak{t}$ is any tree of $[\mathfrak{t}]_{\equiv_\gamma}$ is well-defined.

Lemma 2.2.4. *For any integer $\gamma \geqslant 0$, any syntax tree $\mathfrak{t}$ of $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dias}_\gamma}\right)$ can be rewritten, by a sequence of $\to_\gamma$-rewritings, into a hook syntax tree. Moreover, this hook syntax tree is* $\mathrm{hook}_\gamma(\mathrm{word}_\gamma(\mathfrak{t}))$.

Proof. In the following, to gain readability, we shall denote by $\dashv_*$ (resp. $\vdash_*$) any element $\dashv_a$ (resp. $\vdash_a$) of $\mathfrak{G}_{\mathsf{Dias}_\gamma}$ when taking into account the value of $a \in [\gamma]$ is not necessary. Using this notation, from (2.2.5a)—(2.2.5i), we observe that $\to_\gamma$ expresses as

$$\vdash_* \circ_2 \dashv_* \;\to_\gamma\; \dashv_* \circ_1 \vdash_*, \tag{2.2.8a}$$

$$\dashv_* \circ_2 \vdash_* \;\to_\gamma\; \dashv_* \circ_1 \dashv_*, \tag{2.2.8b}$$

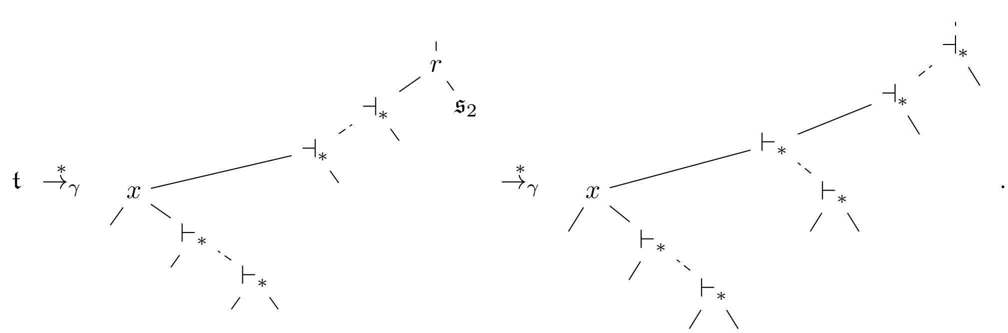
$$\vdash_* \circ_1 \dashv_* \;\to_\gamma\; \vdash_* \circ_2 \vdash_*, \tag{2.2.8c}$$

$$\dashv_* \circ_2 \dashv_* \;\to_\gamma\; \dashv_* \circ_1 \dashv_*, \tag{2.2.8d}$$

$$\vdash_* \circ_1 \vdash_* \;\to_\gamma\; \vdash_* \circ_2 \vdash_* \,. \tag{2.2.8e}$$

Let us first focus on the first part of the statement of the lemma to show that $\mathfrak{t}$ is rewritable by $\to_\gamma$ into a hook syntax tree. We reason by induction on the arity n of $\mathfrak{t}$. When $n \leqslant 2$, $\mathfrak{t}$ is immediately a hook syntax tree. Otherwise, $\mathfrak{t}$ has at least two internal nodes. Then, $\mathfrak{t}$ is made of a root connected to a first subtree $\mathfrak{t}_1$ and a second subtree $\mathfrak{t}_2$. By induction hypothesis, $\mathfrak{t}$ is rewritable by $\to_\gamma$ into a tree made of a root r of the same label as the one of the root of $\mathfrak{t}$, connected to a first subtree $\mathfrak{s}_1$ such that $\mathfrak{t}_1 \overset{*}{\to}_\gamma \mathfrak{s}_1$ and a second subtree $\mathfrak{s}_2$ such that $\mathfrak{t}_2 \overset{*}{\to}_\gamma \mathfrak{s}_2$, both being hook syntax trees. We have to deal two cases following the number of internal nodes of $\mathfrak{t}_1$.

Case 1. If $\mathfrak{t}_1$ has at least one internal node, we have the two $\overset{*}{\to}_\gamma$-relations



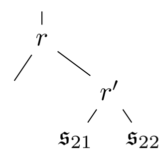
(2.2.9)

The first $\overset{*}{\to}_\gamma$-relation of (2.2.9) has just been explained. The second one comes from the application of the induction hypothesis on the upper part of the tree of the middle of (2.2.9) obtained by cutting the edge connecting the node x to its father. When the rightmost tree of (2.2.9) is not already a hook syntax tree, one has two cases following the label of x.

Case 1.1. If x is labeled by $\vdash_*$, by (2.2.8e), the bottom part of the rightmost tree of (2.2.9) consisting in internal nodes labeled by $\vdash_*$ is rewritable by $\to_\gamma$ into a right comb tree wherein internal nodes are labeled by $\vdash_*$. Then, the rightmost tree of (2.2.9) is rewritable by $\to_\gamma$ into a hook syntax tree, and then $\mathfrak{t}$ also is.

Case 1.2. Otherwise, x is labeled by $\dashv_*$. By definition of hook_γ, the second subtree of x is a leaf. By (2.2.8c), the bottom part of the rightmost tree of (2.2.9) consisting in x and internal nodes labeled by $\vdash_*$ can be rewritten by $\to_\gamma$ into a right comb tree wherein internal nodes are labeled by $\vdash_*$. Then, the rightmost tree of (2.2.9) is rewritable by $\to_\gamma$ into a hook syntax tree, and then $\mathfrak{t}$ also is.

Case 2. Otherwise, $\mathfrak{t}_1$ is the leaf. We then have the $\overset{*}{\to}_\gamma$-relation

$$\mathfrak{t} \overset{*}{\to}_\gamma$$

, (2.2.10)

where $\mathfrak{s}_{21}$ is the first subtree of the root of $\mathfrak{s}_2$, $\mathfrak{s}_{22}$ is the second subtree of the root of $\mathfrak{s}_2$, and r' is a node with the same label as the root of $\mathfrak{s}_2$.

Case 2.1. If $r \circ_2 r'$ is equal to $\vdash_* \circ_2 \dashv_*$, $\dashv_* \circ_2 \vdash_*$, or $\dashv_* \circ_2 \dashv_*$, respectively by (2.2.8a), (2.2.8b), and (2.2.8d), the rightmost tree of (2.2.10) can be rewritten by $\to_\gamma$ into a tree $\mathfrak{r}$ having a first subtree with at least one internal node. Hence, $\mathfrak{r}$ is of the form required to be treated by *Case 1.*, implying that $\mathfrak{t}$ is rewritable by $\to_\gamma$ into a hook syntax tree.

Case 2.2. Otherwise, $r \circ_2 r'$ is equal to $\vdash_* \circ_2 \vdash_*$. Since $\mathfrak{s}_2$ is by hypothesis a hook syntax tree, it is necessarily a right comb tree whose internal nodes are labeled by $\vdash_*$. Hence, the rightmost tree of (2.2.10) is already a hook syntax tree, showing that $\mathfrak{t}$ is rewritable by $\to_\gamma$ into a hook syntax tree.

Let us finally show the last part of the statement of the lemma. Observe that, by definition of hook_γ and word_γ, if $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are two different hook syntax trees, $\mathrm{word}_\gamma(\mathfrak{s}_1) \neq \mathrm{word}_\gamma(\mathfrak{s}_2)$. We have just shown that $\mathfrak{t}$ is rewritable by $\to_\gamma$ into a hook syntax tree $\mathfrak{s}$. Besides, by Lemma 2.2.3, one has $\mathrm{word}_\gamma(\mathfrak{t}) = \mathrm{word}_\gamma(\mathfrak{s})$. Then, $\mathfrak{s}$ is necessarily the hook syntax tree $\mathrm{hook}_\gamma(\mathrm{word}_\gamma(\mathfrak{t}))$. $\square$

2.2.4. *Presentation by generators and relations.*

Lemma 2.2.5. *For any integers $\gamma \geqslant 0$ and $n \geqslant 1$, the map $\bar{\mathrm{word}}_\gamma$ defines a bijection between* $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dias}_\gamma}\right)(n)/_{\equiv_\gamma}$ *and* $\mathsf{Dias}_\gamma(n)$.

Proof. Let us show that $\bar{\mathrm{word}}_\gamma$ is injective. Let $\mathfrak{t}_1$ and $\mathfrak{t}_2$ be two syntax trees of $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dias}_\gamma}\right)$ such that $\mathrm{word}_\gamma(\mathfrak{t}_1) = \mathrm{word}_\gamma(\mathfrak{t}_2)$ and let $\mathfrak{s} := \mathrm{hook}_\gamma(\mathrm{word}_\gamma(\mathfrak{t}_1)) = \mathrm{hook}_\gamma(\mathrm{word}_\gamma(\mathfrak{t}_2))$. By Lemma 2.2.4, one has $\mathfrak{t}_1 \overset{*}{\to}_\gamma \mathfrak{s}$ and $\mathfrak{t}_2 \overset{*}{\to}_\gamma \mathfrak{s}$, and hence, $\mathfrak{t}_1 \equiv_\gamma \mathfrak{t}_2$. By the definition of the map

$\bar{\mathrm{word}}_\gamma$ from the map word_γ, this show that $\bar{\mathrm{word}}_\gamma$ is injective. Besides, by Lemma 2.2.2, $\bar{\mathrm{word}}_\gamma$ is surjective, whence the statement of the lemma. □

Theorem 2.2.6. *For any integer $\gamma \geqslant 0$, the operad Dias_γ admits the following presentation. It is generated by $\mathfrak{G}_{\mathsf{Dias}_\gamma}$ and its space of relations $\mathfrak{R}_{\mathsf{Dias}_\gamma}$ is the space induced by the equivalence relation $\leftrightarrow_\gamma$ satisfying*

$$\dashv_a \circ_1 \vdash_{a'} \leftrightarrow_\gamma \vdash_{a'} \circ_2 \dashv_a, \qquad a, a' \in [\gamma], \tag{2.2.11a}$$

$$\dashv_a \circ_1 \dashv_b \leftrightarrow_\gamma \dashv_a \circ_2 \vdash_b, \qquad a < b \in [\gamma], \tag{2.2.11b}$$

$$\vdash_a \circ_1 \dashv_b \leftrightarrow_\gamma \vdash_a \circ_2 \vdash_b, \qquad a < b \in [\gamma], \tag{2.2.11c}$$

$$\dashv_b \circ_1 \dashv_a \leftrightarrow_\gamma \dashv_a \circ_2 \dashv_b, \qquad a < b \in [\gamma], \tag{2.2.11d}$$

$$\vdash_a \circ_1 \vdash_b \leftrightarrow_\gamma \vdash_b \circ_2 \vdash_a, \qquad a < b \in [\gamma], \tag{2.2.11e}$$

$$\dashv_d \circ_1 \dashv_d \leftrightarrow_\gamma \dashv_d \circ_2 \dashv_c \leftrightarrow_\gamma \dashv_d \circ_2 \vdash_c, \qquad c \leqslant d \in [\gamma], \tag{2.2.11f}$$

$$\vdash_d \circ_1 \dashv_c \leftrightarrow_\gamma \vdash_d \circ_1 \vdash_c \leftrightarrow_\gamma \vdash_d \circ_2 \vdash_d, \qquad c \leqslant d \in [\gamma]. \tag{2.2.11g}$$

Proof. By Lemma 2.2.5, the map $\bar{\mathrm{word}}_\gamma$ is, for any $n \geqslant 1$, a bijection between the sets $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dias}_\gamma}\right)(n)/_{\equiv_\gamma}$ and $\mathsf{Dias}_\gamma(n)$. Moreover, by Lemma 2.2.1, word_γ is an operad morphism, and then $\bar{\mathrm{word}}_\gamma$ also is. Hence, $\bar{\mathrm{word}}_\gamma$ is an operad isomorphism between $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dias}_\gamma}\right)/_{\equiv_\gamma}$ and Dias_γ. Therefore, since $\mathfrak{R}_{\mathsf{Dias}_\gamma}$ is the space induced by $\equiv_\gamma$, Dias_γ admits the stated presentation. □

The space of relations $\mathfrak{R}_{\mathsf{Dias}_\gamma}$ of Dias_γ exhibited by Theorem 2.2.6 can be rephrased in a more compact way as the space generated by

$$\dashv_a \circ_1 \vdash_{a'} - \vdash_{a'} \circ_2 \dashv_a, \qquad a, a' \in [\gamma], \tag{2.2.12a}$$

$$\dashv_a \circ_1 \dashv_{a \uparrow a'} - \dashv_a \circ_2 \vdash_{a'}, \qquad a, a' \in [\gamma], \tag{2.2.12b}$$

$$\vdash_a \circ_1 \dashv_{a'} - \vdash_a \circ_2 \vdash_{a \uparrow a'}, \qquad a, a' \in [\gamma], \tag{2.2.12c}$$

$$\dashv_{a \uparrow a'} \circ_1 \dashv_a - \dashv_a \circ_2 \dashv_{a'}, \qquad a, a' \in [\gamma], \tag{2.2.12d}$$

$$\vdash_a \circ_1 \vdash_{a'} - \vdash_{a \uparrow a'} \circ_2 \vdash_a, \qquad a, a' \in [\gamma]. \tag{2.2.12e}$$

Observe that, by Theorem 2.2.6, Dias_1 and the diassociative operad (see [Lod01] or Section 1.3.1) admit the same presentation. Then, for all integers $\gamma \geqslant 0$, the operads Dias_γ are generalizations of the diassociative operad.

2.3. **Miscellaneous properties.** From the description of the elements of Dias_γ and its structure revealed by its presentation, we develop here some of its properties. Unless otherwise specified, Dias_γ is still considered in this section as a set-operad.

2.3.1. *Koszulity.*

Theorem 2.3.1. *For any integer $\gamma \geqslant 0$, Dias_γ is a Koszul operad. Moreover, the set of hook syntax trees of* $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dias}_\gamma}\right)$ *forms a Poincaré-Birkhoff-Witt basis of* Dias_γ.

Proof. From the definition of hook syntax trees, it appears that no hook syntax tree can be rewritten by $\to_\gamma$ into another syntax tree. Hence, and by Lemma 2.2.4, $\to_\gamma$ is a terminating rewrite rule and its normal forms are hook syntax trees. Moreover, again by Lemma 2.2.4, since any syntax tree is rewritable by $\to_\gamma$ into a unique hook syntax tree, $\to_\gamma$ is a confluent rewrite rule, and hence, $\to_\gamma$ is convergent. Now, since by Theorem 2.2.6, the space of relations of Dias_γ is the space induced by the operad congruence induced by $\to_\gamma$, by the Koszulity criterion [Hof10, DK10, LV12] we have reformulated in Section 1.2.5, Dias_γ is a Koszul operad and the set of of hook syntax trees of $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dias}_\gamma}\right)$ forms a Poincaré-Birkhoff-Witt basis of Dias_γ. □

2.3.2. *Symmetries.* If $\mathcal{O}_1$ and $\mathcal{O}_2$ are two operads, a linear map $\phi : \mathcal{O}_1 \to \mathcal{O}_2$ is an *operad antimorphism* if it respects arities and anticommutes with partial composition maps, that is,

$$\phi(x \circ_i y) = \phi(x) \circ_{n-i+1} \phi(y), \qquad x \in \mathcal{O}(n), y \in \mathcal{O}, i \in [n]. \tag{2.3.1}$$

A *symmetry* of an operad $\mathcal{O}$ is either an automorphism or an antiautomorphism. The set of all symmetries of $\mathcal{O}$ form a group for the composition, called the *group of symmetries* of $\mathcal{O}$.

Proposition 2.3.2. *For any integer $\gamma \geqslant 0$, the group of symmetries of Dias_γ as a set-operad contains two elements: the identity map and the linear map sending any word of Dias_γ to its mirror image.*

Proof. Let us denote by $\mathbb{G}_\gamma$ the set $\{0a, a0 : a \in [\gamma]\}$. Since Dias_γ is generated by $\mathbb{G}_\gamma$, any automorphism or antiautomorphism ϕ of Dias_γ is wholly determined by the images of the elements of $\mathbb{G}_\gamma$. Besides let us observe that ϕ is in particular a permutation of $\mathbb{G}_\gamma$.

By contradiction, assume that ϕ is an automorphism of Dias_γ different from the identity map. We have two cases to explore.

Case 1. If there are $a, a' \in [\gamma]$ satisfying $\phi(0a) = a'0$, since ϕ is a permutation of $\mathbb{G}_\gamma$, there are $b, b' \in [\gamma]$ satisfying $\phi(b0) = 0b'$. Then, we have at the same time $b0 \circ_2 0a = b0a = 0a \circ_1 b0$,

$$\phi(b0 \circ_2 0a) = \phi(b0) \circ_2 \phi(0a) = 0b' \circ_2 a'0 = 0\,(b' \uparrow a')\,b', \tag{2.3.2}$$

and

$$\phi(0a \circ_1 b0) = \phi(0a) \circ_1 \phi(b0) = a'0 \circ_1 0b' = a'\,(a' \uparrow b')\,0. \tag{2.3.3}$$

This shows that $\phi(b0 \circ_2 0a) \neq \phi(0a \circ_1 b0)$ and hence, ϕ is not an operad morphism. By a similar argument, one can show that there are no $a, a' \in [\gamma]$ such that $\phi(a0) = 0a'$.

Case 2. Otherwise, for all $a \in [\gamma]$, we have $\phi(0a) = 0a'$ and $\phi(a0) = a''0$ for some $a', a'' \in [\gamma]$. Since, by hypothesis, ϕ is not the identity map, there exist $a \neq a' \in [\gamma]$ such that $\phi(0a) = 0a'$ or $\phi(a0) = a'0$. Let us assume, without loss of generality, that $\phi(0a) = 0a'$. Since ϕ is a permutation of $\mathbb{G}_\gamma$, there exist $b \neq b' \in [\gamma]$ such that $\phi(0b) = 0b'$. One can assume, without loss of generality, that $a < b$ and $b' < a'$. Then, we have at the same time $0a \circ_2 0b = 0ab = 0b \circ_1 0a$,

$$\phi(0a \circ_2 0b) = \phi(0a) \circ_2 \phi(0b) = 0a' \circ_2 0b' = 0a'a', \tag{2.3.4}$$

and

$$\phi(0b \circ_1 0a) = \phi(0b) \circ_1 \phi(0a) = 0b' \circ_1 0a' = 0a'b'. \tag{2.3.5}$$

This shows that $\phi(0a \circ_2 0b) \neq \phi(0b \circ_1 0a)$ and hence, that ϕ is not an operad morphism. By a similar argument, one can show that there are no $a \neq a' \in [\gamma]$ such that $\phi(a0) = \phi(a'0)$.

We then have shown that if ϕ is an automorphism of Dias_γ, it is necessarily the identity map.

Finally, by Proposition 2.1.1, if x is an element of Dias_γ, its mirror image also is in Dias_γ. Moreover, it is immediate to see that the map sending a word to its mirror image is an antiautomorphism of Dias_γ. Similar arguments as the ones developed previously show that it is the only. □

2.3.3. *Basic operad.* A set-operad $\mathcal{O}$ is *basic* if for all $y_1, \dots, y_n \in \mathcal{O}$, all the maps

$$\circ^{y_1,\dots,y_n} : \mathcal{O}(n) \to \mathcal{O}(|y_1| + \dots + |y_n|) \tag{2.3.6}$$

defined by

$$\circ^{y_1,\dots,y_n}(x) := x \circ (y_1, \dots, y_n), \qquad x \in \mathcal{O}(n), \tag{2.3.7}$$

are injective. This property for set-operads introduced by Vallette [Val07] is a very relevant one since there is a general construction producing a family of posets (see [MY91] and [CL07]) from a basic set-operad. This family of posets leads to the definition of an incidence Hopf algebra by a construction of Schmitt [Sch94].

Proposition 2.3.3. *For any integer $\gamma \geqslant 0$, Dias_γ is a basic operad.*

Proof. Let $n \geqslant 1$, $y_1, \dots, y_n$ be words of Dias_γ, and x and x' be two words of $\mathsf{Dias}_\gamma(n)$ such that $\circ^{y_1,\dots,y_n}(x) = \circ^{y_1,\dots,y_n}(x')$. Then, for all $i \in [n]$ and $j \in [|y_i|]$, we have $x_i \uparrow y_{i,j} = x'_i \uparrow y_{i,j}$ where $y_{i,j}$ is the jth letter of y_i. Since by Proposition 2.1.1, any word y_i contains a 0, we have in particular $x_i \uparrow 0 = x'_i \uparrow 0$ for all $i \in [n]$. This implies $x = x'$ and thus, that $\circ^{y_1,\dots,y_n}$ is injective. □

2.3.4. *Rooted operad.* We restate here a property on operads introduced by Chapoton [Cha14]. An operad $\mathcal{O}$ is *rooted* if there is a map

$$\mathrm{root} : \mathcal{O}(n) \to [n], \qquad n \geqslant 1, \tag{2.3.8}$$

satisfying, for all $x \in \mathcal{O}(n)$, $y \in \mathcal{O}(m)$, and $i \in [n]$,

$$\mathrm{root}(x \circ_i y) = \begin{cases} \mathrm{root}(x) + m - 1 & \text{if } i \leqslant \mathrm{root}(x) - 1, \\ \mathrm{root}(x) + \mathrm{root}(y) - 1 & \text{if } i = \mathrm{root}(x), \\ \mathrm{root}(x) & \text{otherwise } (i \geqslant \mathrm{root}(x) + 1). \end{cases} \tag{2.3.9}$$

We call such a map a *root map*. More intuitively, the root map of a rooted operad associates a particular input with any of its elements and this input is preserved by partial compositions.

It is immediate that any operad $\mathcal{O}$ is a rooted operad for the root maps $\mathrm{root}_{\mathrm{L}}$ and $\mathrm{root}_{\mathrm{R}}$, which send respectively all elements x of arity n to 1 or to n. For this reason, we say that an operad $\mathcal{O}$ is *nontrivially rooted* if it can be endowed with a root map different from $\mathrm{root}_{\mathrm{L}}$ and $\mathrm{root}_{\mathrm{R}}$.

Proposition 2.3.4. *For any integer $\gamma \geqslant 0$, Dias_γ is a nontrivially rooted operad for the root map sending any word of Dias_γ to the position of its 0.*

Proof. Thanks to Proposition 2.1.1, the map of the statement of the proposition is well-defined. The fact that 0 is the neutral element for the $\uparrow$ operation and the fact that any word of Dias_γ contains exactly one 0 imply that this map satisfies (2.3.9). Finally, this map is obviously different from $\mathrm{root}_{\mathrm{L}}$ and $\mathrm{root}_{\mathrm{R}}$, whence the statement of the proposition. □

2.3.5. *Alternative basis.* In this section, Dias_γ is considered as an operad in the category of vector spaces.

Let $\preccurlyeq_\gamma$ be the order relation on the underlying set of $\mathsf{Dias}_\gamma(n)$, $n \geqslant 1$, where for all words x and y of Dias_γ of a same arity n, we have

$$x \preccurlyeq_\gamma y \qquad \text{if } x_i \leqslant y_i \text{ for all } i \in [n]. \tag{2.3.10}$$

This order relation allows to define for all word x of Dias_γ the elements

$$\mathsf{K}_x^{(\gamma)} := \sum_{x \preccurlyeq_\gamma x'} \mu_\gamma(x, x')\, x', \tag{2.3.11}$$

where μ_γ is the Möbius function of the poset defined by $\preccurlyeq_\gamma$. For instance,

$$\mathsf{K}_{102}^{(2)} = 102 - 202, \tag{2.3.12}$$

$$\mathsf{K}_{102}^{(3)} = \mathsf{K}_{102}^{(4)} = 102 - 103 - 202 + 203, \tag{2.3.13}$$

$$\mathsf{K}_{23102}^{(3)} = 23102 - 23103 - 23202 + 23203 - 33102 + 33103 + 33202 - 33203. \tag{2.3.14}$$

Since, by Möbius inversion, for any word x of Dias_γ one has

$$x = \sum_{x \preccurlyeq_\gamma x'} \mathsf{K}_{x'}^{(\gamma)}, \tag{2.3.15}$$

the family of all $\mathsf{K}_x^{(\gamma)}$, where the x are words of Dias_γ, forms by triangularity a basis of Dias_γ, called the K-*basis*.

If u and v are two words of a same length n, we denote by $\mathrm{ham}(u, v)$ the *Hamming distance* between u and v that is the number of positions $i \in [n]$ such that $u_i \neq v_i$. Moreover, for any word x of Dias_γ of length n and any subset J of $[n]$, we denote by $\mathrm{Incr}_\gamma(x, J)$ the set of words obtained by incrementing by one some letters of x smaller than γ and greater than 0 whose positions are in J. We shall simply denote by $\mathrm{Incr}_\gamma(x)$ the set $\mathrm{Incr}_\gamma(x, [n])$. Proposition 2.1.1 ensures that all $\mathrm{Incr}_\gamma(x, J)$ are sets of words of Dias_γ.

Lemma 2.3.5. *For any integer $\gamma \geqslant 0$ and any word x of Dias_γ,*

$$\mathsf{K}^{(\gamma)}_x = \sum_{x' \in \mathrm{Incr}_\gamma(x)} (-1)^{\mathrm{ham}(x,x')}\, x'. \tag{2.3.16}$$

Proof. Let n be the arity of x. To compute $\mathsf{K}^{(\gamma)}_x$ from its definition (2.3.11), it is enough to know the Möbius function μ_γ of the poset $\mathbb{P}^{(\gamma)}_x$ consisting in the words x' of Dias_γ satisfying $x \preccurlyeq_\gamma x'$. Immediately from the definition of $\preccurlyeq_\gamma$, it appears that $\mathbb{P}^{(\gamma)}_x$ is isomorphic to the Cartesian product poset

$$\mathbb{T}^{(\gamma)}_x := \mathbb{T}\left(\gamma - x_1\right) \times \cdots \times \mathbb{T}\left(\gamma - x_{r-1}\right) \times \mathbb{T}(0) \times \mathbb{T}\left(\gamma - x_{r+1}\right) \times \cdots \times \mathbb{T}\left(\gamma - x_n\right), \tag{2.3.17}$$

where for any nonnegative integer k, $\mathbb{T}(k)$ denotes the poset over $\{0\} \cup [k]$ with the natural total order relation, and r is the position of, by Proposition 2.1.1, the only 0 of x. The map $\phi^{(\gamma)}_x : \mathbb{P}^{(\gamma)}_x \to \mathbb{T}^{(\gamma)}_x$ defined for all words x' of $\mathbb{P}^{(\gamma)}_x$ by

$$\phi^{(\gamma)}_x(x') := \left(x'_1 - x_1, \dots, x'_{r-1} - x_{r-1}, 0, x'_{r+1} - x_{r+1}, \dots, x'_n - x_n\right) \tag{2.3.18}$$

is an isomorphism of posets.

Recall that the Möbius function μ of $\mathbb{T}(k)$ satisfies, for all $a, a' \in \mathbb{T}(k)$,

$$\mu(a, a') = \begin{cases} 1 & \text{if } a' = a, \\ -1 & \text{if } a' = a + 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2.3.19}$$

Moreover, since by [Sta11], the Möbius function of a Cartesian product poset is the product of the Möbius functions of the posets involved in the product, through the isomorphism $\phi^{(\gamma)}_x$, we obtain that when x' is in $\mathrm{Incr}_\gamma(x)$, $\mu_\gamma(x, x') = (-1)^{\mathrm{ham}(x,x')}$ and that when x' is not in Incr_γ, $\mu_\gamma(x, x') = 0$. Therefore, (2.3.16) is established. $\square$

Lemma 2.3.6. *For any integer $\gamma \geqslant 0$, any word x of Dias_γ, and any nonempty set J of positions of letters of x that are greater than 0 and smaller than γ,*

$$\sum_{x' \in \mathrm{Incr}_\gamma(x,J)} (-1)^{\mathrm{ham}(x,x')} = 0. \tag{2.3.20}$$

Proof. The statement of the lemma follows by induction on the nonzero cardinality of J. $\square$

To compute a direct expression for the partial composition of Dias_γ over the K-basis, we have to introduce two notations. If x is a word of Dias_γ of length nonsmaller than 2, we denote by $\min(x)$ the smallest letter of x among its letters different from 0. Proposition 2.1.1 ensures that $\min(x)$ is well-defined. Moreover, for all words x and y of Dias_γ, a position i such that $x_i \neq 0$, and $a \in [\gamma]$, we denote by $x \circ_{a,i} y$ the word $x \circ_i y$ in which the 0 coming from y is replaced by a instead of x_i.

Theorem 2.3.7. *For any integer $\gamma \geqslant 0$, the partial composition of Dias_γ over the K-basis satisfies, for all words x and y of Dias_γ of arities nonsmaller than 2,*

$$\mathsf{K}^{(\gamma)}_x \circ_i \mathsf{K}^{(\gamma)}_y = \begin{cases} \mathsf{K}^{(\gamma)}_{x \circ_i y} & \text{if } \min(y) > x_i, \\ \sum_{a \in [x_i, \gamma]} \mathsf{K}^{(\gamma)}_{x \circ_{a,i} y} & \text{if } \min(y) = x_i, \\ 0 & \text{otherwise } (\min(y) < x_i). \end{cases} \tag{2.3.21}$$

Proof. First of all, by Lemma 2.3.5 together with (2.3.15), we obtain

$$\begin{aligned} \mathsf{K}^{(\gamma)}_x \circ_i \mathsf{K}^{(\gamma)}_y &= \sum_{\substack{x' \in \mathrm{Incr}_\gamma(x) \\ y' \in \mathrm{Incr}_\gamma(y)}} (-1)^{\mathrm{ham}(x,x') + \mathrm{ham}(y,y')} \left(\sum_{x' \circ_i y' \preccurlyeq_\gamma z} \mathsf{K}^{(\gamma)}_z \right) \\ &= \sum_{x \circ_i y \preccurlyeq_\gamma z} \sum_{\substack{x' \in \mathrm{Incr}_\gamma(x) \\ y' \in \mathrm{Incr}_\gamma(y) \\ x' \circ_i y' \preccurlyeq_\gamma z}} (-1)^{\mathrm{ham}(x,x') + \mathrm{ham}(y,y')} \mathsf{K}^{(\gamma)}_z. \end{aligned} \tag{2.3.22}$$

Let us denote by n (resp. m) the arity of x (resp. y) and let z be a word of Dias_γ such that $x \circ_i y \preccurlyeq_\gamma z$. Let $x' \in \mathrm{Incr}_\gamma(x)$ and $y' \in \mathrm{Incr}_\gamma(y)$. We have, by definition of the partial composition of Dias_γ,

$$x \circ_i y = x_1 \dots x_{i-1}\, t_1 \dots t_{r-1}\, x_i\, t_{r+1} \dots t_m\, x_{i+1} \dots x_n, \tag{2.3.23}$$

and

$$x' \circ_i y' = x'_1 \dots x'_{i-1}\, t'_1 \dots t'_{r-1}\, x'_i\, t'_{r+1} \dots t'_m\, x'_{i+1} \dots x'_n, \tag{2.3.24}$$

where r denotes the position of the only, by Proposition 2.1.1, 0 of y and for all $j \in [m] \setminus \{r\}$, $t_j := x_i \uparrow y_j$ and $t'_j := x'_i \uparrow y'_j$. By (2.3.22), the pair (x', y') contributes to the coefficient of $\mathsf{K}^{(\gamma)}_z$ in (2.3.22) if and only if $x \circ_i y \preccurlyeq_\gamma x' \circ_i y' \preccurlyeq z$. To compute this coefficient, we have three cases to consider following the value of $\min(y)$ compared to the value of x_i.

Case 1. Assume first that $\min(y) < x_i$. Then, there is at least a $s \in [m] \setminus \{r\}$ such that $y_s < x_i$. This implies that $t_s = x_i$ and that y'_s has no influence on t'_s and then, on $x' \circ_i y'$. Thus, the word $y'' := y'_1 \dots y'_{s-1} a y'_{s+1} \dots y'_m$ where a is the only possible letter such that $y'' \in \mathrm{Incr}_\gamma(y)$ and $a \neq y'_s$ satisfies $x' \circ_i y'' = x' \circ_i y'$. Therefore, since $\mathrm{ham}(y', y'') = 1$, the contribution of the pair (x', y') for the coefficient of $\mathsf{K}^{(\gamma)}_z$ in (2.3.22) is compensated by the contribution of the pair (x', y''). This shows that this coefficient is 0 and hence, $\mathsf{K}^{(\gamma)}_x \circ_i \mathsf{K}^{(\gamma)}_y = 0$.

Case 2. Assume now that $\min(y) > x_i$. Then, for all $j \in [m] \setminus \{r\}$, we have $y_j > x_i$ and thus, $t_j = y_j$. When $z = x \circ_i y$, we necessarily have $x' = x$ and $y' = y$. Hence, the coefficient of $\mathsf{K}^{(\gamma)}_{x \circ_i y}$ in (2.3.22) is 1. Else, when $z \neq x \circ_i y$, we have $x' \circ_i y' \in \mathrm{Incr}_\gamma(x \circ_i y, J)$, where J is the nonempty set of the positions of letters of z different from letters of $x \circ_i y$. Now, from (2.3.22), the coefficient of $\mathsf{K}^{(\gamma)}_z$ in (2.3.22) is

$$\sum_{x' \circ_i y' \in \mathrm{Incr}_\gamma(x \circ_i y, J)} (-1)^{\mathrm{ham}(x,x') + \mathrm{ham}(y,y')}. \tag{2.3.25}$$

Lemma 2.3.6 implies that this coefficient is 0. This shows that $\mathsf{K}^{(\gamma)}_x \circ_i \mathsf{K}^{(\gamma)}_y = \mathsf{K}^{(\gamma)}_{x \circ_i y}$.

Case 3. The last case occurs when $\min(y) = x_i$. Then, for all $j \in [m] \setminus \{r\}$, we have $y_j \geqslant x_i$ and thus, $t_j = y_j$. Moreover, there is at least a $s \in [m] \setminus \{r\}$ such that $y_s = x_i$. When $z = x \circ_{a,i} y$ with $a \in [x_i, \gamma]$, we necessarily have $x' = x$ and $y' = y$. Therefore, for all $a \in [x_i, \gamma]$, the $\mathsf{K}^{(\gamma)}_{x \circ_{a,i}}$ have coefficient 1 in (2.3.22). The same argument as the one exposed for *Case 2.* shows that when $z \neq x \circ_{a,i} y$ for all $a \in [x_i, \gamma]$, the coefficient of $\mathsf{K}^{(\gamma)}_z$ is zero. Hence, $\mathsf{K}^{(\gamma)}_x \circ_i \mathsf{K}^{(\gamma)}_y = \sum_{a \in [x_i, \gamma]} \mathsf{K}^{(\gamma)}_{x \circ_{a,i} y}$.

□

We have for instance

$$\mathsf{K}^{(5)}_{20413} \circ_1 \mathsf{K}^{(5)}_{304} = \mathsf{K}^{(5)}_{3240413}, \tag{2.3.26}$$

$$\mathsf{K}^{(5)}_{20413} \circ_2 \mathsf{K}^{(5)}_{304} = \mathsf{K}^{(5)}_{2304413}, \tag{2.3.27}$$

$$\mathsf{K}^{(5)}_{20413} \circ_3 \mathsf{K}^{(5)}_{304} = 0, \tag{2.3.28}$$

$$\mathsf{K}^{(5)}_{20413} \circ_4 \mathsf{K}^{(5)}_{304} = \mathsf{K}^{(5)}_{2043143}, \tag{2.3.29}$$

$$\mathsf{K}^{(5)}_{20413} \circ_5 \mathsf{K}^{(5)}_{304} = \mathsf{K}^{(5)}_{2041334} + \mathsf{K}^{(5)}_{2041344} + \mathsf{K}^{(5)}_{2041354}. \tag{2.3.30}$$

Theorem 2.3.7 implies in particular that the structure coefficients of the partial composition of Dias_γ over the K-basis are 0 or 1. It is possible to define another bases of Dias_γ by reversing in (2.3.11) the relation $\preccurlyeq_\gamma$ and by suppressing or keeping the Möbius function μ_γ. This gives obviously rise to three other bases. It worth to note that, as small computations reveal, over all these additional bases, the structure coefficients of the partial composition of Dias_γ can be negative or different from 1. This observation makes the K-basis even more particular and interesting. It has some other properties, as next section will show.

2.3.6. *Alternative presentation.* The K-basis introduced in the previous section leads to state a new presentation for Dias_γ in the following way.

For any integer $\gamma \geqslant 0$, let $\dashv\!\!|_a$ and $|\!\!\vdash_a$, $a \in [\gamma]$, be the elements of $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dias}_\gamma}\right)(2)$ defined by

$$\dashv\!\!|_a := \begin{cases} \dashv_\gamma & \text{if } a = \gamma, \\ \dashv_a - \dashv_{a+1} & \text{otherwise,} \end{cases} \tag{2.3.31a}$$

and

$$|\!\!\vdash_a := \begin{cases} \vdash_\gamma & \text{if } a = \gamma, \\ \vdash_a - \vdash_{a+1} & \text{otherwise.} \end{cases} \tag{2.3.31b}$$

Then, since for all $a \in [\gamma]$ we have

$$\dashv_a = \sum_{a \leqslant b \in [\gamma]} \dashv\!\!|_b \tag{2.3.32a}$$

and

$$\vdash_a = \sum_{a \leqslant b \in [\gamma]} |\!\!\vdash_b, \tag{2.3.32b}$$

by triangularity, the family $\mathfrak{G}'_{\mathsf{Dias}_\gamma} := \{\dashv\!\!|_a, |\!\!\vdash_a : a \in [\gamma]\}$ forms a basis of $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dias}_\gamma}\right)(2)$ and then, generates $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dias}_\gamma}\right)$ as an operad. This change of basis from $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dias}_\gamma}\right)$ to

$\mathbf{Free}(\mathfrak{G}'_{\mathsf{Dias}_\gamma})$ comes from the change of basis from the usual basis of Dias_γ to the K-basis. Let us now express a presentation of Dias_γ through the family $\mathfrak{G}'_{\mathsf{Dias}_\gamma}$.

Proposition 2.3.8. *For any integer $\gamma \geqslant 0$, the operad Dias_γ admits the following presentation. It is generated by $\mathfrak{G}'_{\mathsf{Dias}_\gamma}$ and its space of relations is $\mathfrak{R}'_{\mathsf{Dias}_\gamma}$ is generated by*

$$\dashv\!\!|_a \circ_1 |\!\!\vdash_{a'} - |\!\!\vdash_{a'} \circ_2 \dashv\!\!|_a, \qquad a, a' \in [\gamma], \tag{2.3.33a}$$

$$|\!\!\vdash_b \circ_1 |\!\!\vdash_a, \qquad a < b \in [\gamma], \tag{2.3.33b}$$

$$\dashv\!\!|_b \circ_2 \dashv\!\!|_a, \qquad a < b \in [\gamma], \tag{2.3.33c}$$

$$|\!\!\vdash_b \circ_1 \dashv\!\!|_a, \qquad a < b \in [\gamma], \tag{2.3.33d}$$

$$\dashv\!\!|_b \circ_2 |\!\!\vdash_a, \qquad a < b \in [\gamma], \tag{2.3.33e}$$

$$|\!\!\vdash_a \circ_1 |\!\!\vdash_b - |\!\!\vdash_b \circ_2 |\!\!\vdash_a, \qquad a < b \in [\gamma], \tag{2.3.33f}$$

$$\dashv\!\!|_b \circ_1 \dashv\!\!|_a - \dashv\!\!|_a \circ_2 \dashv\!\!|_b, \qquad a < b \in [\gamma], \tag{2.3.33g}$$

$$|\!\!\vdash_a \circ_1 \dashv\!\!|_b - |\!\!\vdash_a \circ_2 |\!\!\vdash_b, \qquad a < b \in [\gamma], \tag{2.3.33h}$$

$$\dashv\!\!|_a \circ_1 \dashv\!\!|_b - \dashv\!\!|_a \circ_2 |\!\!\vdash_b, \qquad a < b \in [\gamma], \tag{2.3.33i}$$

$$|\!\!\vdash_a \circ_1 |\!\!\vdash_a - \left(\sum_{a \leqslant b \in [\gamma]} |\!\!\vdash_a \circ_2 |\!\!\vdash_b\right), \qquad a \in [\gamma], \tag{2.3.33j}$$

$$\left(\sum_{a \leqslant b \in [\gamma]} \dashv\!\!|_a \circ_1 \dashv\!\!|_b\right) - \dashv\!\!|_a \circ_2 \dashv\!\!|_a, \qquad a \in [\gamma], \tag{2.3.33k}$$

$$|\!\!\vdash_a \circ_1 \dashv\!\!|_a - \left(\sum_{a \leqslant b \in [\gamma]} |\!\!\vdash_b \circ_2 |\!\!\vdash_a\right), \qquad a \in [\gamma], \tag{2.3.33l}$$

$$\left(\sum_{a \leqslant b \in [\gamma]} \dashv\!\!|_b \circ_1 \dashv\!\!|_a\right) - \dashv\!\!|_a \circ_2 |\!\!\vdash_a, \qquad a \in [\gamma]. \tag{2.3.33m}$$

Proof. Let us show that $\mathfrak{R}'_{\mathsf{Dias}_\gamma}$ is equal to the space of relations $\mathfrak{R}_{\mathsf{Dias}_\gamma}$ of Dias_γ defined in the statement of Theorem 2.2.6. First of all, recall that the map $\mathrm{word}_\gamma : \mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dias}_\gamma}\right) \to \mathsf{Dias}_\gamma$ defined in Section 2.2.1 satisfies $\mathrm{word}_\gamma(\dashv_a) = 0a$ and $\mathrm{word}_\gamma(\vdash_a) = a0$ for all $a \in [\gamma]$. By Theorem 2.2.6, for any $x \in \mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dias}_\gamma}\right)(3)$, x is in $\mathfrak{R}_{\mathsf{Dias}_\gamma}$ if and only if $\mathrm{word}_\gamma(x) = 0$.

Besides, by definition of $\dashv\!\!|_a$, $|\!\!\vdash_a$, $a \in [\gamma]$, and by making use of the K-basis of Dias_γ, we have $\mathrm{word}_\gamma(\dashv\!\!|_a) = \mathsf{K}^{(\gamma)}_{0a}$ and $\mathrm{word}_\gamma(|\!\!\vdash_a) = \mathsf{K}^{(\gamma)}_{a0}$. By using the partial composition rules for Dias_γ over the K-basis of Theorem 2.3.7, straightforward computations show that $\mathrm{word}_\gamma(x) = 0$ for all elements x among (2.3.33a)—(2.3.33m). This implies that $\mathfrak{R}'_{\mathsf{Dias}_\gamma}$ is a subspace of $\mathfrak{R}_{\mathsf{Dias}_\gamma}$.

Now, one can observe that elements (2.3.33a)—(2.3.33m) are linearly independent. Then, $\mathfrak{R}'_{\mathsf{Dias}_\gamma}$ has dimension $5\gamma^2$ which is also, by Theorem 2.2.6, the dimension of $\mathfrak{R}_{\mathsf{Dias}_\gamma}$. Hence, $\mathfrak{R}'_{\mathsf{Dias}_\gamma}$ and $\mathfrak{R}_{\mathsf{Dias}_\gamma}$ are equal. The statement of the proposition follows. □

Despite the apparent complexity of the presentation of Dias_γ exhibited by Proposition 2.3.8, as we will see in Section 4, the Koszul dual of Dias_γ computed from this presentation has a very simple and manageable expression.

3. Pluriassociative algebras

We now focus on algebras over γ-pluriassociative operads. For this purpose, we construct free Dias_γ-algebras over one generator, and define and study two notions of units for Dias_γ-algebras. We end this section by introducing a convenient way to define Dias_γ-algebras and give several examples of such algebras.

3.1. **Category of pluriassociative algebras and free objects.**
Let us study the category of Dias_γ-algebras and the units for algebras in this category.

3.1.1. *Pluriassociative algebras.* We call *γ-pluriassociative algebra* any Dias_γ-algebra. From the presentation of Dias_γ provided by Theorem 2.2.6, any γ-pluriassociative algebra is a vector space endowed with linear operations $\dashv_a, \vdash_a$, $a \in [\gamma]$, satisfying the relations encoded by (2.2.12a)—(2.2.12e).

3.1.2. *General definitions.* Let $\mathcal{P}$ be a γ-pluriassociative algebra. We say that $\mathcal{P}$ is *commutative* if for all $x, y \in \mathcal{P}$ and $a \in [\gamma]$, $x \dashv_a y = y \vdash_a x$. Besides, $\mathcal{P}$ is *pure* for all $a, a' \in [\gamma]$, $a \neq a'$ implies $\dashv_a \neq \dashv_{a'}$ and $\vdash_a \neq \vdash_{a'}$.

Given a subset C of $[\gamma]$, one can keep on the vector space $\mathcal{P}$ only the operations $\dashv_a$ and $\vdash_a$ such that $a \in C$. By renumbering the indexes of these operations from 1 to $\#C$ by respecting their former relative numbering, we obtain a $\#C$-pluriassociative algebra. We call it the *$\#C$-pluriassociative subalgebra induced by C* of $\mathcal{P}$.

3.1.3. *Free pluriassociative algebras.* Recall that $\mathcal{F}_{\mathsf{Dias}_\gamma}$ denotes the free Dias_γ-algebra over one generator. By definition, $\mathcal{F}_{\mathsf{Dias}_\gamma}$ is the linear span of the set of the words on $\{0\} \cup [\gamma]$ with exactly one occurrence of 0. Let us endow this space with the linear operations

$$\dashv_a, \vdash_a \colon \mathcal{F}_{\mathsf{Dias}_\gamma} \otimes \mathcal{F}_{\mathsf{Dias}_\gamma} \to \mathcal{F}_{\mathsf{Dias}_\gamma}, \qquad a \in [\gamma], \tag{3.1.1}$$

satisfying, for any such words u and v,

$$u \dashv_a v := u \, \mathrm{h}_a(v) \tag{3.1.2a}$$

and

$$u \vdash_a v := \mathrm{h}_a(u) \, v, \tag{3.1.2b}$$

where $\mathrm{h}_a(u)$ (resp. $\mathrm{h}_a(v)$) is the word obtained by replacing in u (resp. v) any occurrence of a letter smaller than a by a.

Proposition 3.1.1. *For any integer $\gamma \geqslant 0$, the vector space $\mathcal{F}_{\mathsf{Dias}_\gamma}$ of nonempty words on $\{0\} \cup [\gamma]$ containing exactly one occurrence of 0 endowed with the operations $\dashv_a$, $\vdash_a$, $a \in [\gamma]$, is the free γ-pluriassociative algebra over one generator.*

Proof. The fact that $\mathcal{F}_{\mathsf{Dias}_\gamma}$ is the stated vector space is a consequence of the description of the elements of Dias_γ provided by Proposition 2.1.1. Since Dias_γ is by definition the suboperad of $\mathsf{T}\mathcal{M}_\gamma$ generated by $\{0a, a0 : a \in [\gamma]\}$, $\mathcal{F}_{\mathsf{Dias}_\gamma}$ is endowed with 2γ binary operations where any generator $0a$ (resp. $a0$) gives rise to the operation $\dashv_a$ (resp. $\vdash_a$) of $\mathcal{F}_{\mathsf{Dias}_\gamma}$. Moreover, by making use of the realization of Dias_γ, we have for all $u, v \in \mathcal{F}_{\mathsf{Dias}_\gamma}$ and $a \in [\gamma]$,

$$u \dashv_a v = (u \otimes v) \cdot 0a = (0a \circ_2 v) \circ_1 u = u\, \mathrm{h}_a(v) \tag{3.1.3a}$$

and

$$u \vdash_a v = (u \otimes v) \cdot a0 = (a0 \circ_2 v) \circ_1 u = \mathrm{h}_a(u)\, v. \tag{3.1.3b}$$

□

One has for instance in $\mathcal{F}_{\mathsf{Dias}_4}$,

$$101241 \dashv_2 203 = 10124122\mathbf{2}3 \tag{3.1.4}$$

and

$$101241 \vdash_3 203 = \mathbf{3333}4\mathbf{3}203. \tag{3.1.5}$$

3.2. **Bar and wire-units.** Loday has defined in [Lod01] some notions of units in diassociative algebras. We generalize here these definitions to the context of γ-pluriassociative algebras.

3.2.1. *Bar-units.* Let $\mathcal{P}$ be a γ-pluriassociative algebra and $a \in [\gamma]$. We say that an element e of $\mathcal{P}$ is an *a-bar-unit*, or simply a *bar-unit* when taking into account the value of a is not necessary, of $\mathcal{P}$ if for all $x \in \mathcal{P}$,

$$x \dashv_a e = x = e \vdash_a x. \tag{3.2.1}$$

As we shall see below, a γ-pluriassociative algebra can have, for a given $a \in [\gamma]$, several a-bar-units. The *a-halo* of $\mathcal{P}$, denoted by $\mathrm{Halo}_a(\mathcal{P})$, is the set of the a-bar-units of $\mathcal{P}$.

3.2.2. *Wire-units.* Let $\mathcal{P}$ be a γ-pluriassociative algebra and $a \in [\gamma]$. We say that an element e of $\mathcal{P}$ is an *a-wire-unit*, or simply a *wire-unit* when taking into account the value of a is not necessary, of $\mathcal{P}$ if for all $x \in \mathcal{P}$,

$$e \dashv_a x = x = x \vdash_a e. \tag{3.2.2}$$

As shows the following proposition, the presence of a wire-unit in $\mathcal{P}$ has some implications.

Proposition 3.2.1. *Let $\gamma \geqslant 0$ be an integer and $\mathcal{P}$ be a γ-pluriassociative algebra admitting a b-wire-unit e for a $b \in [\gamma]$. Then*

- *(i) for all $a \in [b]$, the operations $\dashv_a$, $\dashv_b$, $\vdash_a$, and $\vdash_b$ of $\mathcal{P}$ are equal;*
- *(ii) e is also an a-wire-unit for all $a \in [b]$;*
- *(iii) e is the only wire-unit of $\mathcal{P}$;*
- *(iv) if e' is an a-bar unit for a $a \in [b]$, then $e' = e$.*

Proof. Let us show part *(i)*. By Relation (2.2.12d) of γ-pluriassociative algebras and by the fact that e is a b-wire-unit of $\mathcal{P}$, we have for all elements y and z of $\mathcal{P}$ and all $a \in [b]$,

$$y \dashv_a z = e \dashv_b (y \dashv_a z) = e \dashv_b (y \vdash_a z) = y \vdash_a z. \tag{3.2.3}$$

Thus, the operations $\dashv_a$ and $\vdash_a$ of $\mathcal{P}$ are equal. Moreover, for the same reasons, we have

$$y \dashv_a z = e \dashv_b (y \dashv_a z) = (e \dashv_b y) \dashv_b z = y \dashv_b z. \tag{3.2.4}$$

Then, the operations $\dashv_a$ and $\dashv_b$ of $\mathcal{P}$ are equal, whence *(i)*.

Now, by *(i)* and by the fact that e is a b-wire-unit, we have for all elements x of $\mathcal{P}$ and all $a \in [b]$,

$$e \dashv_a x = e \dashv_b x = x = x \vdash_b e = x \vdash_a e, \tag{3.2.5}$$

showing *(ii)*.

To prove *(iii)*, assume that e' is a b'-wire-unit of $\mathcal{P}$ for a $b' \in [\gamma]$. By *(i)* and by the fact that e is a b-wire-unit, one has

$$e = e \vdash_{b'} e' = e \dashv_b e' = e', \tag{3.2.6}$$

showing *(iii)*.

To establish *(iv)*, let us first prove that e is a b-bar-unit. By *(i)* and by the fact that e is a b-wire-unit, we have for all elements x of $\mathcal{P}$,

$$e \vdash_b x = e \dashv_b x = x = x \vdash_b e = x \dashv_b e. \tag{3.2.7}$$

Now, since e' is an a-bar-unit for an $a \in [b]$, by *(i)* and by the fact that e is a b-wire-unit,

$$e = e' \vdash_a e = e' \vdash_b e = e'. \tag{3.2.8}$$

This shows *(iv)*. □

Relying on Proposition 3.2.1, we define the *height* of a γ-pluriassociative algebra $\mathcal{P}$ as zero if $\mathcal{P}$ has no wire-unit, otherwise as the greatest integer $h \in [\gamma]$ such that the unique wire-unit e of $\mathcal{P}$ is a h-wire-unit. Observe that any pure γ-pluriassociative algebra has height 0 or 1.

3.3. **Construction of pluriassociative algebras.** We now present a general way to construct γ-pluriassociative algebras. Our construction is a natural generalization of some constructions introduced by Loday [Lod01] in the context of diassociative algebras. We introduce in this section new algebraic structures, the so-called γ-multiprojection algebras, which are inputs of our construction.

3.3.1. *Multiassociative algebras.* For any integer $\gamma \geqslant 0$, a *γ-multiassociative algebra* is a vector space $\mathcal{M}$ endowed with linear operations

$$\star_a : \mathcal{M} \otimes \mathcal{M} \to \mathcal{M}, \qquad a \in [\gamma], \tag{3.3.1}$$

satisfying, for all $x, y, z \in \mathcal{M}$, the relations

$$(x \star_a y) \star_b z = (x \star_b y) \star_{a'} z = x \star_{a''} (y \star_b z) = x \star_b (y \star_{a'''} z), \qquad a, a', a'', a''' \leqslant b \in [\gamma]. \tag{3.3.2}$$

These algebras are obvious generalizations of associative algebras since all of its operations are associative. Observe that by (3.3.2), all bracketings of an expression involving elements of a γ-multiassociative algebra and some of its operations are equal. Then, since the bracketings of such expressions are not significant, we shall denote these without parenthesis. In Section 5 we will study the operads governing these for a very specific purpose.

If $\mathcal{M}_1$ and $\mathcal{M}_2$ are two γ-multiassociative algebras, a linear map $\phi : \mathcal{M}_1 \to \mathcal{M}_2$ is a *γ-multiassociative algebra morphism* if it commutes with the operations of $\mathcal{M}_1$ and $\mathcal{M}_2$. We say that $\mathcal{M}$ is *commutative* when all operations of $\mathcal{M}$ are commutative. Besides, for an $a \in [\gamma]$, an element $\mathbb{1}$ of $\mathcal{M}$ is an *a-unit*, or simply a *unit* when taking into account the value of a is not necessary, of $\mathcal{M}$ if for all $x \in \mathcal{M}$, $\mathbb{1} \star_a x = x = x \star_a \mathbb{1}$. When $\mathcal{M}$ admits a unit, we say that $\mathcal{M}$ is *unital*. As shows the following proposition, the presence of a unit in $\mathcal{M}$ has some implications.

Proposition 3.3.1. *Let $\gamma \geqslant 0$ be an integer and $\mathcal{M}$ be a γ-multiassociative algebra admitting a b-unit $\mathbb{1}$ for a $b \in [\gamma]$. Then*

(i) for all $a \in [b]$, the operations $\star_a$ and $\star_b$ of $\mathcal{M}$ are equal;
(ii) $\mathbb{1}$ is also an a-unit for all $a \in [b]$;
(iii) $\mathbb{1}$ is the only unit of $\mathcal{M}$.

Proof. By Relation (3.3.2) of γ-multiassociative algebras and by the fact that $\mathbb{1}$ is a b-unit of $\mathcal{M}$, we have for all elements y and z of $\mathcal{M}$ and all $a \in [b]$,

$$y \star_a z = y \star_a z \star_b \mathbb{1} = y \star_b z \star_b \mathbb{1} = y \star_b z. \tag{3.3.3}$$

Therefore, $\star_a = \star_b$, showing *(i)*.

Now, by *(i)* and by the fact that $\mathbb{1}$ is a b-unit, we have for all elements x of $\mathcal{M}$ and all $a \in [b]$,

$$\mathbb{1} \star_a x = \mathbb{1} \star_b x = x = x \star_b \mathbb{1} = x \star_a \mathbb{1}, \tag{3.3.4}$$

showing *(ii)*.

To prove *(iii)*, assume that $\mathbb{1}'$ is a b'-unit of $\mathcal{M}$ for a $b' \in [\gamma]$. By *(i)* and by the fact that $\mathbb{1}$ is a b-unit, one has

$$\mathbb{1} = \mathbb{1} \star_{b'} \mathbb{1}' = \mathbb{1} \star_b \mathbb{1}' = \mathbb{1}', \tag{3.3.5}$$

establishing *(iii)*. ☐

Relying on Proposition 3.3.1, similarly to the case of γ-pluriassociative algebras, we define the *height* of a γ-multiassociative algebra $\mathcal{M}$ as zero if $\mathcal{M}$ has no unit, otherwise as the greatest integer $h \in [\gamma]$ such that the unit $\mathbb{1}$ of $\mathcal{M}$ is an h-unit.

3.3.2. *Multiprojection algebras.* We call *γ-multiprojection algebra* any γ-multiassociative algebra $\mathcal{M}$ endowed with endomorphisms

$$\pi_a : \mathcal{M} \to \mathcal{M}, \qquad a \in [\gamma], \tag{3.3.6}$$

satisfying

$$\pi_a \circ \pi_{a'} = \pi_{a \uparrow a'}, \qquad a, a' \in [\gamma]. \tag{3.3.7}$$

By extension, the *height* of $\mathcal{M}$ is its height as a γ-multiassociative algebra. We say that $\mathcal{M}$ is *unital* as a γ-multiprojection algebra if $\mathcal{M}$ is unital as a γ-multiassociative algebra and its only, by Proposition 3.3.1, unit $\mathbb{1}$ satisfies $\pi_a(\mathbb{1}) = \mathbb{1}$ for all $a \in [h]$ where h is the height of $\mathcal{M}$.

3.3.3. *From multiprojection algebras to pluriassociative algebras.* Next result describes how to construct γ-pluriassociative algebras from γ-multiprojection algebras.

Theorem 3.3.2. *For any integer $\gamma \geqslant 0$ and any γ-multiprojection algebra $\mathcal{M}$, the vector space $\mathcal{M}$ endowed with binary linear operations $\dashv_a$, $\vdash_a$, $a \in [\gamma]$, defined for all $x, y \in \mathcal{M}$ by*

$$x \dashv_a y := x \star_a \pi_a(y) \tag{3.3.8a}$$

and

$$x \vdash_a y := \pi_a(x) \star_a y, \tag{3.3.8b}$$

where the $\star_a$, $a \in [\gamma]$, are the operations of $\mathcal{M}$ and the π_a, $a \in [\gamma]$, are its endomorphisms, is a γ-pluriassociative algebra, denoted by $\mathrm{M}(\mathcal{M})$*.*

Proof. This is a verification of the relations of γ-pluriassociative algebras in $\mathrm{M}(\mathcal{M})$. Let x, y, and z be three elements of $\mathrm{M}(\mathcal{M})$ and $a, a' \in [\gamma]$.

By (3.3.2), we have

$$(x \vdash_{a'} y) \dashv_a z = \pi_{a'}(x) \star_{a'} y \star_a \pi_a(z) = x \vdash_{a'} (y \dashv_a z), \tag{3.3.9}$$

showing that (2.2.12a) is satisfied in $\mathrm{M}(\mathcal{M})$.

Moreover, by (3.3.2) and (3.3.7), we have

$$\begin{aligned} x \dashv_a (y \vdash_{a'} z) &= x \star_a \pi_a(\pi_{a'}(y) \star_{a'} z) \\ &= x \star_a \pi_{a \uparrow a'}(y) \star_{a'} \pi_a(z) \\ &= x \star_{a \uparrow a'} \pi_{a \uparrow a'}(y) \star_a \pi_a(z) \\ &= (x \dashv_{a \uparrow a'} y) \dashv_a z, \end{aligned} \tag{3.3.10}$$

so that (2.2.12b), and for the same reasons (2.2.12c), check out in $\mathrm{M}(\mathcal{M})$.

Finally, again by (3.3.2) and (3.3.7), we have

$$\begin{aligned} x \dashv_a (y \dashv_{a'} z) &= x \star_a \pi_a(y \star_{a'} \pi_{a'}(z)) \\ &= x \star_a \pi_a(y) \star_{a'} \pi_{a \uparrow a'}(z) \\ &= x \star_a \pi_a(y) \star_{a \uparrow a'} \pi_{a \uparrow a'}(z) \\ &= (x \dashv_a y) \dashv_{a \uparrow a'} z, \end{aligned} \tag{3.3.11}$$

showing that (2.2.12d), and for the same reasons (2.2.12e), are satisfied in $\mathrm{M}(\mathcal{M})$. □

When $\mathcal{M}$ is commutative, since for all $x, y \in \mathrm{M}(\mathcal{M})$ and $a \in [\gamma]$,

$$x \dashv_a y = x \star_a \pi_a(y) = \pi_a(y) \star_a x = y \vdash_a x, \tag{3.3.12}$$

it appears that $\mathrm{M}(\mathcal{M})$ is a commutative γ-pluriassociative algebra.

When $\mathcal{M}$ is unital, $\mathrm{M}(\mathcal{M})$ has several properties, summarized in the next proposition.

Proposition 3.3.3. *Let $\gamma \geqslant 0$ be an integer, $\mathcal{M}$ be a unital γ-multiprojection algebra of height h. Then, by denoting by $\mathbb{1}$ the unit of $\mathcal{M}$ and by π_a, $a \in [\gamma]$, its endomorphisms,*

(i) for any $a \in [h]$, $\mathbb{1}$ is an a-bar-unit of $\mathrm{M}(\mathcal{M})$*;*
(ii) for any $a \leqslant b \in [h]$, $\mathrm{Halo}_a(\mathrm{M}(\mathcal{M}))$ *is a subset of* $\mathrm{Halo}_b(\mathrm{M}(\mathcal{M}))$*;*

(iii) *for any $a \in [h]$, the linear span of* $\mathrm{Halo}_a(\mathrm{M}(\mathcal{M}))$ *forms an $h-a+1$-pluriassociative subalgebra of the $h-a+1$-pluriassociative subalgebra of* $\mathrm{M}(\mathcal{M})$ *induced by $[a, h]$;*

(iv) *for any $a \in [h]$, π_a is the identity map if and only if $\mathbb{1}$ is an a-wire-unit of* $\mathrm{M}(\mathcal{M})$.

Proof. Let us denote by $\star_a$, $a \in [\gamma]$, the operations of $\mathcal{M}$.

Since $\mathbb{1}$ is an h-unit of $\mathcal{M}$, for all elements x of $\mathrm{M}(\mathcal{M})$ and all $a \in [h]$,

$$x \dashv_a \mathbb{1} = x \star_a \pi_a(\mathbb{1}) = x \star_a \mathbb{1} = x = \mathbb{1} \star_a x = \pi_a(\mathbb{1}) \star_a x = \mathbb{1} \vdash_a x, \tag{3.3.13}$$

showing *(i)*.

Assume that e is an element of $\mathrm{Halo}_a(\mathrm{M}(\mathcal{M}))$ for an $a \in [h]$, that is, e is an a-bar-unit of $\mathrm{M}(\mathcal{M})$. Then, for all elements x of $\mathrm{M}(\mathcal{M})$,

$$x \dashv_a e = x \star_a \pi_a(e) = x = \pi_a(e) \star_a x = e \vdash_a x, \tag{3.3.14}$$

showing that $\pi_a(e)$ is the unit for the operation $\star_a$ on $\mathrm{M}(\mathcal{M})$ and therefore, $\pi_a(e) = \mathbb{1}$. Since $\mathcal{M}$ is unital, we have $\pi_b(\mathbb{1}) = \mathbb{1}$ for all $b \in [h]$. Hence, and by (3.3.7), for all $a \leqslant b \in [h]$,

$$\pi_b(e) = \pi_b(\pi_a(e)) = \pi_b(\mathbb{1}) = \mathbb{1}. \tag{3.3.15}$$

Then, for all elements x of $\mathrm{M}(\mathcal{M})$ and all $a \leqslant b \in [h]$,

$$x \dashv_b e = x \star_b \pi_b(e) = x \star_b \mathbb{1} = x = \mathbb{1} \star_b x = \pi_b(e) \star_b x = e \vdash_b x, \tag{3.3.16}$$

showing that e is also a b-bar-unit of $\mathrm{M}(\mathcal{M})$, whence *(ii)*.

Let $a \in [\gamma]$ and e and e' be elements of $\mathrm{Halo}_a(\mathrm{M}(\mathcal{M}))$. By *(ii)*, e and e' are b-bar-units of $\mathrm{M}(\mathcal{M})$ for all $a \leqslant b \in [h]$ and hence,

$$e \dashv_b e' = e = e' \vdash_b e. \tag{3.3.17}$$

Therefore, the linear span of $\mathrm{Halo}_a(\mathrm{M}(\mathcal{M}))$ is stable for the operations $\dashv_b$ and $\vdash_b$. This implies *(iii)*.

Finally, assume that π_a is the identity map for an $a \in [h]$. Then, for all elements x of $\mathrm{M}(\mathcal{M})$,

$$\mathbb{1} \dashv_a x = \mathbb{1} \star_a \pi_a(x) = \mathbb{1} \star_a x = x = x \star_a \mathbb{1} = \pi_a(x) \star_a \mathbb{1} = x \vdash_a \mathbb{1}, \tag{3.3.18}$$

showing that $\mathbb{1}$ is an a-wire unit of $\mathrm{M}(\mathcal{M})$. Conversely, if $\mathbb{1}$ is an a-wire unit of $\mathrm{M}(\mathcal{M})$, for all elements x of $\mathrm{M}(\mathcal{M})$, the relations $\mathbb{1} \dashv_a x = x = x \vdash_a \mathbb{1}$ imply $\mathbb{1} \star_a \pi_a(x) = x = \pi_a(x) \star_a \mathbb{1}$ and hence, $\pi_a(x) = x$. This shows *(iv)*. □

3.3.4. *Examples of constructions of pluriassociative algebras.* The construction M of Theorem 3.3.2 allows to build several γ-pluriassociative algebras. Here follows few examples.

The γ-pluriassociative algebra of positive integers. Let $\gamma \geqslant 1$ be an integer and consider the vector space Pos of positive integers, endowed with the operations $\star_a$, $a \in [\gamma]$, all equal to the operation $\uparrow$ extended by linearity and with the endomorphisms π_a, $a \in [\gamma]$, linearly defined for any positive integer x by $\pi_a(x) := a \uparrow x$. Then, Pos is a non-unital γ-multiprojection algebra. By Theorem 3.3.2, $\mathrm{M}(\mathsf{Pos})$ is a γ-pluriassociative algebra. We have for instance

$$2 \dashv_3 5 = 5, \tag{3.3.19}$$

and

$$1 \vdash_3 2 = 3. \tag{3.3.20}$$

We can observe that $\mathrm{M}(\mathsf{Pos})$ is commutative, pure, and its 1-halo is $\{1\}$. Moreover, when $\gamma \geqslant 2$, $\mathrm{M}(\mathsf{Pos})$ has no wire-unit and no a-bar-unit for $a \geqslant 2 \in [\gamma]$. This example is important because it provides a counterexample for *(ii)* of Proposition 3.3.3 in the case when the construction M is applied to a non-unital γ-multiprojection algebra.

The γ-pluriassociative algebra of finite sets. Let $\gamma \geqslant 1$ be an integer and consider the vector space Sets of finite sets of positive integers, endowed with the operations $\star_a$, $a \in [\gamma]$, all equal to the union operation $\cup$ extended by linearity and with the endomorphisms π_a, $a \in [\gamma]$, linearly defined for any finite set of positive integers x by $\pi_a(x) := x \cap [a, \gamma]$. Then, Sets is a γ-multiprojection algebra. By Theorem 3.3.2, $\mathrm{M}(\mathsf{Sets})$ is a γ-pluriassociative algebra. We have for instance

$$\{2, 4\} \dashv_3 \{1, 3, 5\} = \{2, 3, 4, 5\}, \tag{3.3.21}$$

and

$$\{1, 2, 4\} \vdash_3 \{1, 3, 5\} = \{1, 3, 4, 5\}. \tag{3.3.22}$$

We can observe that $\mathrm{M}(\mathsf{Sets})$ is commutative and pure. Moreover, $\emptyset$ is a 1-wire-unit of $\mathrm{M}(\mathsf{Sets})$ and, by Proposition 3.2.1, it is its only wire-unit. Therefore, $\mathrm{M}(\mathsf{Sets})$ has height 1. Observe that for any $a \in [\gamma]$, the a-halo of $\mathrm{M}(\mathsf{Sets})$ consists in the subsets of $[a-1]$. Besides, since Sets is a unital γ-multiprojection algebra, $\mathrm{M}(\mathsf{Sets})$ satisfies all properties exhibited by Proposition 3.3.3.

The γ-pluriassociative algebra of words. Let $\gamma \geqslant 1$ be an integer and consider the vector space Words of the words of positive integers. Let us endow Words with the operations $\star_a$, $a \in [\gamma]$, all equal to the concatenation operation extended by linearity and with the endomorphisms π_a, $a \in [\gamma]$, where for any word x of positive integers, $\pi_a(x)$ is the longest subword of x consisting in letters greater than or equal to a. Then, Words is a γ-multiprojection algebra. By Theorem 3.3.2, $\mathrm{M}(\mathsf{Words})$ is a γ-pluriassociative algebra. We have for instance

$$412 \dashv_3 14231 = 41243, \tag{3.3.23}$$

and

$$11 \vdash_2 323 = 323. \tag{3.3.24}$$

We can observe that $\mathrm{M}(\mathsf{Words})$ is not commutative and is pure. Moreover, ϵ is a 1-wire-unit of $\mathrm{M}(\mathsf{Words})$ and by Proposition 3.2.1, it is its only wire-unit. Therefore, $\mathrm{M}(\mathsf{Words})$ has height 1. Observe that for any $a \in [\gamma]$, the a-halo of $\mathrm{M}(\mathsf{Words})$ consists in the words on the alphabet $[a-1]$. Besides, since Words is a unital γ-multiprojection algebra, $\mathrm{M}(\mathsf{Words})$ satisfies all properties exhibited by Proposition 3.3.3.

The γ-pluriassociative algebras $\mathrm{M}(\mathsf{Sets})$ and $\mathrm{M}(\mathsf{Words})$ are related in the following way. Let I_{com} be the subspace of $\mathrm{M}(\mathsf{Words})$ generated by the $x - x'$ where x and x' are words of positive integers and have the same commutative image. Since I_{com} is a γ-pluriassociative algebra ideal of $\mathrm{M}(\mathsf{Words})$, one can consider the quotient γ-pluriassociative algebra $\mathsf{CWords} := \mathrm{M}(\mathsf{Words})/_{I_{\mathrm{com}}}$. Its elements can be seen as commutative words of positive integers.

Moreover, let I_{occ} be the subspace of $\mathrm{M}(\mathsf{CWords})$ generated by the $x - x'$ where x and x' are commutative words of positive integers and for any letter $a \in [\gamma]$, a appears in x if and only if a appears in x'. Since I_{occ} is a γ-pluriassociative algebra ideal of $\mathrm{M}(\mathsf{CWords})$, one can consider the quotient γ-pluriassociative algebra $\mathrm{M}(\mathsf{CWords})/_{I_{\mathrm{occ}}}$. Its elements can be seen as finite subsets of positive integers and we observe that $\mathrm{M}(\mathsf{CWords})/_{I_{\mathrm{occ}}} = \mathrm{M}(\mathsf{Sets})$.

The γ-pluriassociative algebra of marked words. Let $\gamma \geqslant 1$ be an integer and consider the vector space MWords of the words of positive integers where letters can be marked or not, with at least one occurrence of a marked letter. We denote by $\bar{a}$ any *marked letter* a and we say that the *value* of $\bar{a}$ is a. Let us endow MWords with the linear operations $\star_a$, $a \in [\gamma]$, where for all words u and v of MWords, $u \star_a v$ is obtained by concatenating u and v, and by replacing therein all marked letters by $\bar{c}$ where $c := \max(u) \uparrow a \uparrow \max(v)$ where $\max(u)$ (resp. $\max(v)$) denotes the greatest value among the marked letters of u (resp. v). For instance,

$$2\bar{1}31\bar{3} \star_2 3\bar{4}\bar{1}21 = 2\bar{\mathbf{4}}31\bar{\mathbf{4}}3\bar{4}\bar{\mathbf{4}}21, \tag{3.3.25}$$

and

$$\bar{2}11\bar{1} \star_3 34\bar{2} = \bar{\mathbf{3}}11\bar{\mathbf{3}}34\bar{\mathbf{3}}. \tag{3.3.26}$$

We also endow MWords with the endomorphisms π_a, $a \in [\gamma]$, where for any word u of MWords, $\pi_a(u)$ is obtained by replacing in u any occurrence of a nonmarked letter smaller than a by a. For instance,

$$\pi_3\left(2\bar{2}14\bar{4}3\bar{5}\right) = \mathbf{3}\bar{2}\mathbf{3}4\bar{4}\mathbf{3}\bar{5}. \tag{3.3.27}$$

One can show without difficulty that MWords is a γ-multiprojection algebra. By Theorem 3.3.2, $\mathrm{M}(\mathsf{MWords})$ is a γ-pluriassociative algebra. We have for instance

$$3\bar{2}5 \dashv_3 4\bar{4}1 = 3\bar{\mathbf{4}}54\bar{4}\mathbf{3}, \tag{3.3.28}$$

and

$$1\bar{3}4\bar{1}3 \vdash_2 31\bar{2}3\bar{1}1 = \mathbf{2}\bar{3}4\bar{\mathbf{3}}331\bar{\mathbf{3}}3\bar{\mathbf{3}}1. \tag{3.3.29}$$

We can observe that $\mathrm{M}(\mathsf{MWords})$ is not commutative, pure, and has no wire-units neither bar-units.

The free γ-pluriassociative algebra over one generator. Let $\gamma \geqslant 0$ be an integer. We give here a construction of the free γ-pluriassociative algebra $\mathcal{F}_{\mathsf{Dias}_\gamma}$ over one generator described in Section 3.1.3 passing through the following γ-multiprojection algebra and the construction M. Consider the vector space of nonempty words on the alphabet $\{0\} \cup [\gamma]$ with exactly one occurrence of 0, endowed with the operations $\star_a$, $a \in [\gamma]$, all equal to the concatenation operation extended by linearity and with the endomorphisms h_a, $a \in [\gamma]$, defined in Section 3.1.3. This vector space is a γ-multiprojection algebra. Therefore, by Theorem 3.3.2, it gives rise by the construction M to a γ-pluriassociative algebra and it appears that it is $\mathcal{F}_{\mathsf{Dias}_\gamma}$. Besides, we can now observe that $\mathcal{F}_{\mathsf{Dias}_\gamma}$ is not commutative, pure, and has no wire-units neither bar-units.

4. Polydendriform operads

At this point, the situation is ripe enough to introduce our generalization on a nonnegative integer γ of the dendriform operad and dendriform algebras. We first construct this operad, compute its dimensions, and give then two presentations by generators and relations. This section ends by a description of free algebras over one generator in the category encoded by our generalization.

4.1. Construction and properties.

Theorem 2.2.6, by exhibiting a presentation of Dias_γ, shows that this operad is binary and quadratic. It then admits a Koszul dual, denoted by Dendr_γ and called *γ-polydendriform operad*.

4.1.1. *Definition and presentation.* A description of Dendr_γ is provided by the following presentation by generators and relations.

Theorem 4.1.1. *For any integer $\gamma \geqslant 0$, the operad Dendr_γ admits the following presentation. It is generated by $\mathfrak{G}_{\mathsf{Dendr}_\gamma} := \mathfrak{G}_{\mathsf{Dendr}_\gamma}(2) := \{\leftharpoonup_a, \rightharpoonup_a : a \in [\gamma]\}$ and its space of relations $\mathfrak{R}_{\mathsf{Dendr}_\gamma}$ is generated by*

$$\leftharpoonup_a \circ_1 \rightharpoonup_{a'} - \rightharpoonup_{a'} \circ_2 \leftharpoonup_a, \qquad a, a' \in [\gamma], \tag{4.1.1a}$$

$$\leftharpoonup_a \circ_1 \leftharpoonup_b - \leftharpoonup_a \circ_2 \rightharpoonup_b, \qquad a < b \in [\gamma], \tag{4.1.1b}$$

$$\rightharpoonup_a \circ_1 \leftharpoonup_b - \rightharpoonup_a \circ_2 \rightharpoonup_b, \qquad a < b \in [\gamma], \tag{4.1.1c}$$

$$\leftharpoonup_a \circ_1 \leftharpoonup_b - \leftharpoonup_a \circ_2 \leftharpoonup_b, \qquad a < b \in [\gamma], \tag{4.1.1d}$$

$$\rightharpoonup_a \circ_1 \rightharpoonup_b - \rightharpoonup_a \circ_2 \rightharpoonup_b, \qquad a < b \in [\gamma], \tag{4.1.1e}$$

$$\leftharpoonup_d \circ_1 \leftharpoonup_d - \left(\sum_{c \in [d]} \leftharpoonup_d \circ_2 \leftharpoonup_c + \leftharpoonup_d \circ_2 \rightharpoonup_c\right), \qquad d \in [\gamma], \tag{4.1.1f}$$

$$\left(\sum_{c \in [d]} \rightharpoonup_d \circ_1 \rightharpoonup_c + \rightharpoonup_d \circ_1 \leftharpoonup_c\right) - \rightharpoonup_d \circ_2 \rightharpoonup_d, \qquad d \in [\gamma]. \tag{4.1.1g}$$

Proof. By Theorem 2.2.6, we know that Dias_γ is a binary and quadratic operad, and that its space of relations $\mathfrak{R}_{\mathsf{Dias}_\gamma}$ is the space induced by the equivalence relation $\leftrightarrow_\gamma$ defined by (2.2.11a)–(2.2.11g). Now, by a straightforward computation, and by identifying $\leftharpoonup_a$ (resp. $\rightharpoonup_a$) with $\dashv_a$ (resp. $\vdash_a$) for any $a \in [\gamma]$, we obtain that the space $\mathfrak{R}_{\mathsf{Dendr}_\gamma}$ of the statement of the theorem satisfies $\mathfrak{R}_{\mathsf{Dias}_\gamma}^{\perp} = \mathfrak{R}_{\mathsf{Dendr}_\gamma}$. Hence, Dendr_γ admits the claimed presentation. □

Theorem 4.1.1 provides a quite complicated presentation of Dendr_γ. We shall below define a more convenient basis for the space of relations of Dendr_γ.

4.1.2. *Elements and dimensions.*

Proposition 4.1.2. *For any integer $\gamma \geqslant 0$, the Hilbert series $\mathcal{H}_{\mathsf{Dendr}_\gamma}(t)$ of the operad* Dendr_γ *satisfies*

$$\mathcal{H}_{\mathsf{Dendr}_\gamma}(t) = t + 2\gamma t\, \mathcal{H}_{\mathsf{Dendr}_\gamma}(t) + \gamma^2 t\, \mathcal{H}_{\mathsf{Dendr}_\gamma}(t)^2. \tag{4.1.2}$$

Proof. By setting $\bar{\mathcal{H}}_{\mathsf{Dendr}_\gamma}(t) := \mathcal{H}_{\mathsf{Dendr}_\gamma}(-t)$, from (4.1.2), we obtain

$$t = \frac{-\bar{\mathcal{H}}_{\mathsf{Dendr}_\gamma}(t)}{\left(1 + \gamma\, \bar{\mathcal{H}}_{\mathsf{Dendr}_\gamma}(t)\right)^2}. \tag{4.1.3}$$

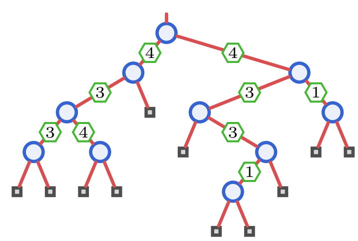
Moreover, by setting $\bar{\mathcal{H}}_{\mathsf{Dias}_\gamma}(t) := \mathcal{H}_{\mathsf{Dias}_\gamma}(-t)$, where $\mathcal{H}_{\mathsf{Dias}_\gamma}(t)$ is defined by (2.1.8), we have

$$\bar{\mathcal{H}}_{\mathsf{Dias}_\gamma}\left(\bar{\mathcal{H}}_{\mathsf{Dendr}_\gamma}(t)\right) = \frac{-\bar{\mathcal{H}}_{\mathsf{Dendr}_\gamma}(t)}{\left(1 + \gamma\, \bar{\mathcal{H}}_{\mathsf{Dendr}_\gamma}(t)\right)^2} = t, \tag{4.1.4}$$

showing that $\bar{\mathcal{H}}_{\mathsf{Dias}_\gamma}(t)$ and $\bar{\mathcal{H}}_{\mathsf{Dendr}_\gamma}(t)$ are the inverses for each other for series composition.

Now, since by Theorem 2.3.1 and Proposition 2.1.1, Dias_γ is a Koszul operad and its Hilbert series is $\mathcal{H}_{\mathsf{Dias}_\gamma}(t)$, and since Dendr_γ is by definition the Koszul dual of Dias_γ, the Hilbert series of these two operads satisfy (1.2.10). Therefore, (4.1.4) implies that the Hilbert series of Dendr_γ is $\mathcal{H}_{\mathsf{Dendr}_\gamma}(t)$. □

By examining the expression for $\mathcal{H}_{\mathsf{Dendr}_\gamma}(t)$ of the statement of Proposition 4.1.2, we observe that for any $n \geqslant 1$, $\mathsf{Dendr}_\gamma(n)$ can be seen as the vector space $\mathcal{F}_{\mathsf{Dendr}_\gamma}(n)$ of binary trees with n internal nodes wherein its $n-1$ edges connecting two internal nodes are labeled on $[\gamma]$. We call these trees *γ-edge valued binary trees*. In our graphical representations of γ-edge valued binary trees, any edge label is drawn into a hexagon located half the edge. For instance,



(4.1.5)

is a 4-edge valued binary tree and a basis element of $\mathsf{Dendr}_4(10)$.

We deduce from Proposition 4.1.2 that the Hilbert series of Dendr_γ satisfies

$$\mathcal{H}_{\mathsf{Dendr}_\gamma}(t) = \frac{1 - \sqrt{1 - 4\gamma t} - 2\gamma t}{2\gamma^2 t}, \tag{4.1.6}$$

and we also obtain that for all $n \geqslant 1$, $\dim \mathsf{Dendr}_\gamma(n) = \gamma^{n-1}\mathrm{cat}(n)$. For instance, the first dimensions of Dendr_1, Dendr_2, Dendr_3, and Dendr_4 are respectively

$$1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, \tag{4.1.7}$$

$$1, 4, 20, 112, 672, 4224, 27456, 183040, 1244672, 8599552, 60196864, \tag{4.1.8}$$

$$1, 6, 45, 378, 3402, 32076, 312741, 3127410, 31899582, 330595668, 3471254514, \tag{4.1.9}$$

$$1, 8, 80, 896, 10752, 135168, 1757184, 23429120, 318636032, 4402970624, 61641588736. \tag{4.1.10}$$

The first one is Sequence **A000108**, the second one is Sequence **A003645**, and the third one is Sequence **A101600** of [Slo]. Last sequence is not listed in [Slo] at this time.

4.1.3. *Associative operations.* In the same manner as in the dendriform operad the sum of its two operations produces an associative operation, in the γ-dendriform operad there is a way to build associative operations, as shows next statement.

Proposition 4.1.3. *For any integers $\gamma \geqslant 0$ and $b \in [\gamma]$, the element*

$$\bullet_b := \pi \left(\sum_{a \in [b]} \leftharpoonup_a + \rightharpoonup_a \right) \tag{4.1.11}$$

of Dendr_γ*, where π :* **Free** $\left(\mathfrak{G}_{\mathsf{Dendr}_\gamma}\right) \to \mathsf{Dendr}_\gamma$ *is the canonical surjection map, is associative.*

Proof. By setting

$$x := \sum_{a \in [b]} \leftharpoonup_a + \rightharpoonup_a, \tag{4.1.12}$$

we have

$$\begin{aligned} x \circ_1 x - x \circ_2 x = & \leftharpoonup_a \circ_1 \leftharpoonup_{a'} + \leftharpoonup_a \circ_1 \rightharpoonup_{a'} + \rightharpoonup_a \circ_1 \leftharpoonup_{a'} + \rightharpoonup_a \circ_1 \rightharpoonup_{a'} \\ & - \leftharpoonup_a \circ_2 \leftharpoonup_{a'} - \leftharpoonup_a \circ_2 \rightharpoonup_{a'} - \rightharpoonup_a \circ_2 \leftharpoonup_{a'} - \rightharpoonup_a \circ_2 \rightharpoonup_{a'} . \end{aligned} \tag{4.1.13}$$

We the observe that (4.1.13) is the sum of elements (4.1.1a)—(4.1.1g) which generate, by Theorem 4.1.1, the space of relations of Dendr_γ. Therefore, we have $\pi(x \circ_1 x - x \circ_2 x) = 0$, implying $\bullet_b \circ_1 \bullet_b - \bullet_b \circ_2 \bullet_b = 0$ and the associativity of $\bullet_b$. □

4.1.4. *Alternative presentation.* For any integer $\gamma \geqslant 0$, let $\prec_b$ and $\succ_b$, $b \in [\gamma]$, the elements of **Free** $\left(\mathfrak{G}_{\mathsf{Dendr}_\gamma}\right)$ defined by

$$\prec_b := \sum_{a \in [b]} \leftharpoonup_a, \tag{4.1.14a}$$

and

$$\succ_b := \sum_{a \in [b]} \rightharpoonup_a . \tag{4.1.14b}$$

Then, since for all $b \in [\gamma]$ we have

$$\leftharpoonup_b = \begin{cases} \prec_1 & \text{if } b = 1, \\ \prec_b - \prec_{b-1} & \text{otherwise,} \end{cases} \tag{4.1.15a}$$

and

$$\rightharpoonup_b = \begin{cases} \succ_1 & \text{if } b = 1, \\ \succ_b - \succ_{b-1} & \text{otherwise,} \end{cases} \tag{4.1.15b}$$

by triangularity, the family $\mathfrak{G}'_{\mathsf{Dendr}_\gamma} := \{\prec_b, \succ_b \colon b \in [\gamma]\}$ forms a basis of **Free** $\left(\mathfrak{G}_{\mathsf{Dendr}_\gamma}\right)(2)$ and then, generates **Free** $\left(\mathfrak{G}_{\mathsf{Dendr}_\gamma}\right)$ as an operad. This change of basis from **Free** $\left(\mathfrak{G}_{\mathsf{Dendr}_\gamma}\right)$ to **Free**$(\mathfrak{G}'_{\mathsf{Dendr}_\gamma})$ is similar to the change of basis from **Free**$(\mathfrak{G}'_{\mathsf{Dias}_\gamma})$ to **Free** $\left(\mathfrak{G}_{\mathsf{Dias}_\gamma}\right)$ introduced in Section 2.3.6. Let us now express a presentation of Dendr_γ through the family $\mathfrak{G}'_{\mathsf{Dendr}_\gamma}$.

Theorem 4.1.4. *For any integer $\gamma \geqslant 0$, the operad Dendr_γ admits the following presentation. It is generated by $\mathfrak{G}'_{\mathsf{Dendr}_\gamma}$ and its space of relations $\mathfrak{R}'_{\mathsf{Dendr}_\gamma}$ is generated by*

$$\prec_a \circ_1 \succ_{a'} - \succ_{a'} \circ_2 \prec_a, \qquad a, a' \in [\gamma], \tag{4.1.16a}$$

$$\prec_a \circ_1 \prec_b - \prec_a \circ_2 \succ_b - \prec_a \circ_2 \prec_a, \qquad a < b \in [\gamma], \tag{4.1.16b}$$

$$\succ_a \circ_1 \succ_a + \succ_a \circ_1 \prec_b - \succ_a \circ_2 \succ_b, \qquad a < b \in [\gamma], \tag{4.1.16c}$$

$$\prec_b \circ_1 \prec_a - \prec_a \circ_2 \prec_b - \prec_a \circ_2 \succ_a, \qquad a < b \in [\gamma], \tag{4.1.16d}$$

$$\succ_a \circ_1 \prec_a + \succ_a \circ_1 \succ_b - \succ_b \circ_2 \succ_a, \qquad a < b \in [\gamma], \tag{4.1.16e}$$

$$\prec_a \circ_1 \prec_a - \prec_a \circ_2 \succ_a - \prec_a \circ_2 \prec_a, \qquad a \in [\gamma], \tag{4.1.16f}$$

$$\succ_a \circ_1 \succ_a + \succ_a \circ_1 \prec_a - \succ_a \circ_2 \succ_a, \qquad a \in [\gamma]. \tag{4.1.16g}$$

Proof. Let us show that $\mathfrak{R}'_{\mathsf{Dendr}_\gamma}$ is equal to the space of relations $\mathfrak{R}_{\mathsf{Dendr}_\gamma}$ of Dendr_γ defined in the statement of Theorem 4.1.1. By this last theorem, for any $x \in \mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dendr}_\gamma}\right)(3)$, x is in $\mathfrak{R}_{\mathsf{Dendr}_\gamma}$ if and only if $\pi(x) = 0$ where $\pi : \mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dendr}_\gamma}\right) \to \mathsf{Dendr}_\gamma$ is the canonical surjection map. By straightforward computations, by expanding any element x of (4.1.16a)—(4.1.16g) over the elements $\leftharpoonup_a$, $\rightharpoonup_a$, $a \in [\gamma]$, by using (4.1.14a) and (4.1.14b) we obtain that x can be expressed as a sum of elements of $\mathfrak{R}_{\mathsf{Dendr}_\gamma}$. This implies that $\pi(x) = 0$ and hence that $\mathfrak{R}'_{\mathsf{Dendr}_\gamma}$ is a subspace of $\mathfrak{R}_{\mathsf{Dendr}_\gamma}$.

Now, one can observe that elements (4.1.16a)—(4.1.16f) are linearly independent. Then, $\mathfrak{R}'_{\mathsf{Dendr}_\gamma}$ has dimension $3\gamma^2$ which is also, by Theorem 4.1.1, the dimension of $\mathfrak{R}_{\mathsf{Dendr}_\gamma}$. The statement of the theorem follows. □

The presentation of Dendr_γ provided by Theorem 4.1.4 is easier to handle than the one provided by Theorem 4.1.1. The main reason is that Relations (4.1.1f) and (4.1.1g) of the first presentation involve a nonconstant number of terms, while all relations of this second presentation always involve only two or three terms. As a very remarkable fact, it is worthwhile to note that the presentation of Dendr_γ provided by Theorem 4.1.4 can be directly obtained by considering the Koszul dual of Dias_γ over the K-basis (see Sections 2.3.5 and 2.3.6). Therefore, an alternative way to establish this presentation consists in computing the Koszul dual of Dias_γ seen through the presentation having $\mathfrak{R}'_{\mathsf{Dendr}_\gamma}$ as space of relations, which is made of the relations of Dias_γ expressed over the K-basis (see Proposition 2.3.8).

From now on, $\downarrow$ denotes the operation min on integers. Using this notation, the space of relations $\mathfrak{R}'_{\mathsf{Dendr}_\gamma}$ of Dendr_γ exhibited by Theorem 4.1.4 can be rephrased in a more compact way as the space generated by

$$\prec_a \circ_1 \succ_{a'} - \succ_{a'} \circ_2 \prec_a, \qquad a, a' \in [\gamma], \tag{4.1.17a}$$

$$\prec_a \circ_1 \prec_{a'} - \prec_{a \downarrow a'} \circ_2 \prec_a - \prec_{a \downarrow a'} \circ_2 \succ_{a'}, \qquad a, a' \in [\gamma], \tag{4.1.17b}$$

$$\succ_{a \downarrow a'} \circ_1 \prec_{a'} + \succ_{a \downarrow a'} \circ_1 \succ_a - \succ_a \circ_2 \succ_{a'}, \qquad a, a' \in [\gamma]. \tag{4.1.17c}$$

Over the family $\mathfrak{G}'_{\mathsf{Dendr}_\gamma}$, one can build associative operations in Dendr_γ in the following way.

Proposition 4.1.5. *For any integers $\gamma \geqslant 0$ and $b \in [\gamma]$, the element*

$$\odot_b := \pi(\prec_b + \succ_b) \tag{4.1.18}$$

of Dendr_γ*, where* $\pi : \mathbf{Free}(\mathfrak{G}'_{\mathsf{Dendr}_\gamma}) \to \mathsf{Dendr}_\gamma$ *is the canonical surjection map, is associative.*

Proof. By definition of the $\prec_b$ and $\succ_b$, $b \in [\gamma]$, we have

$$\prec_b + \succ_b = \sum_{a \in [b]} \leftharpoonup_a + \rightharpoonup_a . \tag{4.1.19}$$

We hence observe that $\odot_b = \bullet_b$, where $\bullet_b$ is the element of Dendr_γ defined in the statement of Proposition 4.1.3. Hence, by this latter proposition, $\odot_b$ is associative. □

Proposition 4.1.6. *For any integer $\gamma \geqslant 0$, any associative element of* Dendr_γ *is proportional to $\odot_b$ for a $b \in [\gamma]$.*

Proof. Let $\pi : \mathbf{Free}(\mathfrak{G}'_{\mathsf{Dendr}_\gamma}) \to \mathsf{Dendr}_\gamma$ be the canonical surjection map. Consider the element

$$x := \sum_{a \in [\gamma]} \alpha_a \prec_a + \beta_a \succ_a \tag{4.1.20}$$

of $\mathbf{Free}(\mathfrak{G}'_{\mathsf{Dendr}_\gamma})$, where $\alpha_a, \beta_a \in \mathbb{K}$ for all $a \in [\gamma]$, such that $\pi(x)$ is associative in Dendr_γ. Since we have $\pi(r) = 0$ for all elements r of $\mathfrak{R}'_{\mathsf{Dendr}_\gamma}$ (see (4.1.17a), (4.1.17b), and (4.1.17c)), the fact that $\pi(x \circ_1 x - x \circ_2 x) = 0$ implies the constraints

$$\begin{aligned} \alpha_a \, \beta_{a'} &= \beta_{a'} \, \alpha_a, && a, a' \in [\gamma], \\ \alpha_a \, \alpha_{a'} &= \alpha_{a \downarrow a'} \, \alpha_a = \alpha_{a \downarrow a'} \, \beta_{a'}, && a, a' \in [\gamma], \\ \beta_{a \downarrow a'} \, \alpha_{a'} &= \beta_{a \downarrow a'} \, \beta_a = \beta_a \, \beta_{a'}, && a, a' \in [\gamma], \end{aligned} \tag{4.1.21}$$

on the coefficients intervening in x. Moreover, since the syntax trees $\succ_b \circ_1 \succ_a$, $\succ_b \circ_1 \prec_a$, $\prec_b \circ_2 \prec_a$, and $\prec_b \circ_2 \succ_a$ do not appear in $\mathfrak{R}'_{\mathsf{Dendr}_\gamma}$ for all $a < b \in [\gamma]$, we have the further constraints

$$\begin{aligned} \beta_b \, \beta_a &= 0, && a < b \in [\gamma], \\ \beta_b \, \alpha_a &= 0, && a < b \in [\gamma], \\ \alpha_b \, \alpha_a &= 0, && a < b \in [\gamma], \\ \alpha_b \, \beta_a &= 0, && a < b \in [\gamma]. \end{aligned} \tag{4.1.22}$$

These relations imply that there are at most one $c \in [\gamma]$ and one $d \in [\gamma]$ such that $\alpha_c \neq 0$ and $\beta_d \neq 0$. In this case, these relations imply also that $c = d$, and $\alpha_c = \beta_c$. Therefore, x is of the form $x = \alpha_a \prec_a + \alpha_a \succ_a$ for an $a \in [\gamma]$, whence the statement of the proposition. □

4.2. **Category of polydendriform algebras and free objects.** The aim of this section is to describe the category of Dendr_γ-algebras and more particularly the free Dendr_γ-algebra over one generator.

4.2.1. *Polydendriform algebras.* We call *γ-polydendriform algebra* any Dendr_γ-algebra. From the presentation of Dendr_γ provided by Theorem 4.1.1, any γ-polydendriform algebra is a vector space endowed with linear operations $\leftharpoonup_a, \rightharpoonup_a$, $a \in [\gamma]$, satisfying the relations encoded by (4.1.1a)—(4.1.1g). By considering the presentation of Dendr_γ exhibited by Theorem 4.1.4, any γ-polydendriform algebra is a vector space endowed with linear operations $\prec_a, \succ_a$, $a \in [\gamma]$, satisfying the relations encoded by (4.1.17a)—(4.1.17c).

4.2.2. *Two ways to split associativity.* Like dendriform algebras, which offer a way to split an associative operation into two parts, γ-polydendriform algebras propose two ways to split associativity depending on its chosen presentation.

On the one hand, in a γ-polydendriform algebra $\mathcal{D}$ over the operations $\leftharpoonup_a$, $\rightharpoonup_a$, $a \in [\gamma]$, by Proposition 4.1.3, an associative operation $\bullet$ is split into the 2γ operations $\leftharpoonup_a$, $\rightharpoonup_a$, $a \in [\gamma]$, so that for all $x, y \in \mathcal{D}$,

$$x \bullet y = \sum_{a \in [\gamma]} x \leftharpoonup_a y + x \rightharpoonup_a y, \tag{4.2.1}$$

and all partial sums operations $\bullet_b$, $b \in [\gamma]$, satisfying

$$x \bullet_b y = \sum_{a \in [b]} x \leftharpoonup_a y + x \rightharpoonup_a x, \tag{4.2.2}$$

also are associative.

On the other hand, in a γ-polydendriform algebra over the operations $\prec_a$, $\succ_a$, $a \in [\gamma]$, by Proposition 4.1.5, several associative operations $\odot_a$, $a \in [\gamma]$, are each split into two operations $\prec_a$, $\succ_a$, $a \in [\gamma]$, so that for all $x, y \in \mathcal{D}$,

$$x \odot_a y = x \prec_a y + x \succ_a y. \tag{4.2.3}$$

Therefore, we can observe that γ-polydendriform algebras over the operations $\leftharpoonup_a$, $\rightharpoonup_a$, $a \in [\gamma]$, are adapted to study associative algebras (by splitting its single product in the way we have described above) while γ-polydendriform algebras over the operations $\prec_a$, $\succ_a$, $a \in [\gamma]$, are adapted to study vectors spaces endowed with several associative products (by splitting each one in the way we have described above). Algebras with several associative products will be studied in Section 5.

4.2.3. *Free polydendriform algebras.* From now, in order to simplify and make uniform next definitions, we consider that in any γ-edge valued binary tree $\mathfrak{t}$, all edges connecting internal nodes of $\mathfrak{t}$ with leaves are labeled by ∞. By convention, for all $a \in [\gamma]$, we have $a \downarrow \infty = a = \infty \downarrow a$.

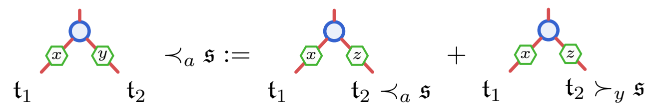
Let us endow the vector space $\mathcal{F}_{\mathsf{Dendr}_\gamma}$ of γ-edge valued binary trees with linear operations

$$\prec_a, \succ_a \colon \mathcal{F}_{\mathsf{Dendr}_\gamma} \otimes \mathcal{F}_{\mathsf{Dendr}_\gamma} \to \mathcal{F}_{\mathsf{Dendr}_\gamma}, \qquad a \in [\gamma], \tag{4.2.4}$$

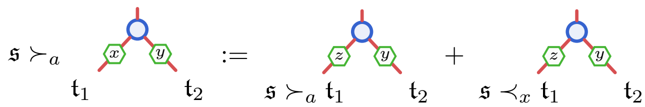
recursively defined, for any γ-edge valued binary tree $\mathfrak{s}$ and any γ-edge valued binary trees or leaves $\mathfrak{t}_1$ and $\mathfrak{t}_2$ by

$$\mathfrak{s} \prec_a \mid := \mathfrak{s} =: \mid \succ_a \mathfrak{s}, \tag{4.2.5}$$

$$\square \prec_a \mathfrak{s} := 0 =: \mathfrak{s} \succ_a \square, \tag{4.2.6}$$



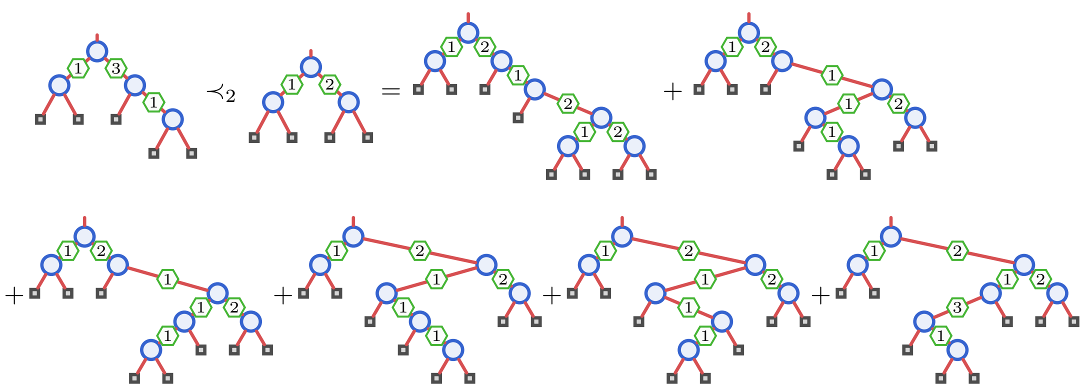
$$, \qquad z := a \downarrow y, \tag{4.2.7}$$



$$, \qquad z := a \downarrow x. \tag{4.2.8}$$

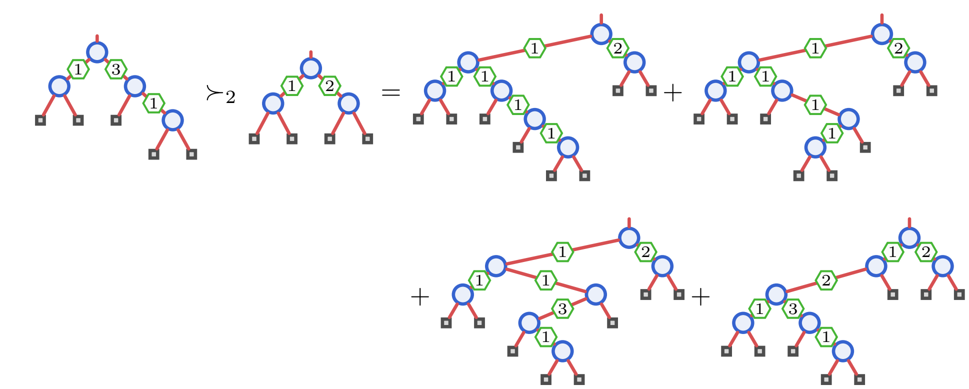
Note that neither $\square \prec_a \square$ nor $\square \succ_a \square$ are defined.

For example, we have



$$\tag{4.2.9}$$

and

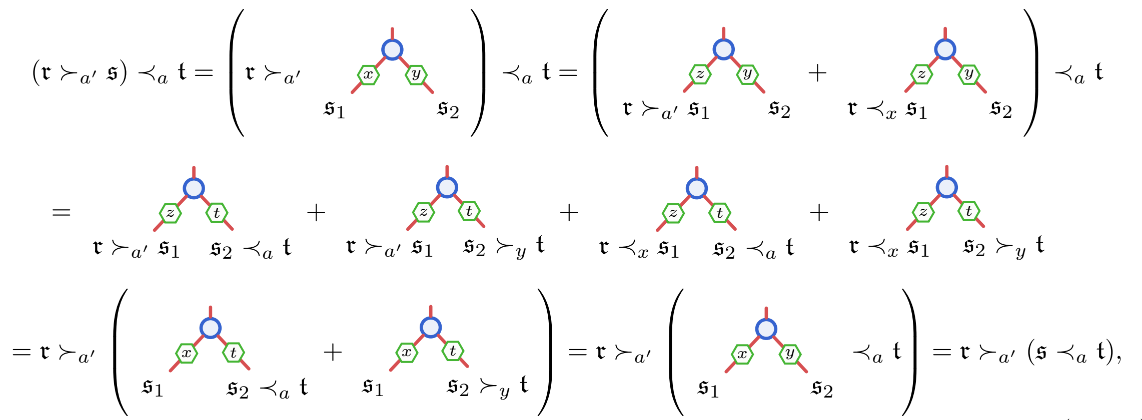


$$\tag{4.2.10}$$

Lemma 4.2.1. *For any integer $\gamma \geqslant 0$, the vector space $\mathcal{F}_{\mathsf{Dendr}_\gamma}$ of γ-edge valued binary trees endowed with the operations $\prec_a$, $\succ_a$, $a \in [\gamma]$, is a γ-polydendriform algebra.*

Proof. We have to check that the operations $\prec_a$, $\succ_a$, $a \in [\gamma]$, of $\mathcal{F}_{\mathsf{Dendr}_\gamma}$ satisfy Relations (4.1.17a), (4.1.17b), and (4.1.17c) of γ-polydendriform algebras. Let $\mathfrak{r}$, $\mathfrak{s}$, and $\mathfrak{t}$ be three γ-edge valued binary trees and $a, a' \in [\gamma]$.

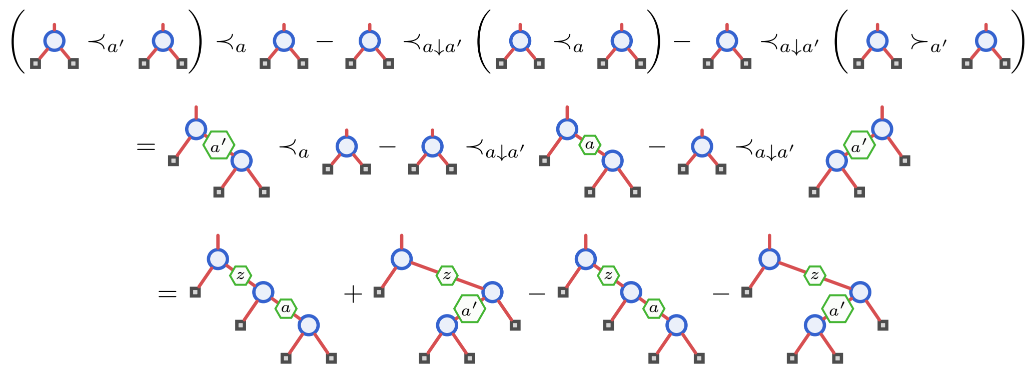
Denote by $\mathfrak{s}_1$ (resp. $\mathfrak{s}_2$) the left subtree (resp. right subtree) of $\mathfrak{s}$ and by x (resp. y) the label of the left (resp. right) edge incident to the root of $\mathfrak{s}$. We have



(4.2.11)

where $z := a' \downarrow x$ and $t := a \downarrow y$. This shows that (4.1.17a) is satisfied in $\mathcal{F}_{\mathsf{Dendr}_\gamma}$.

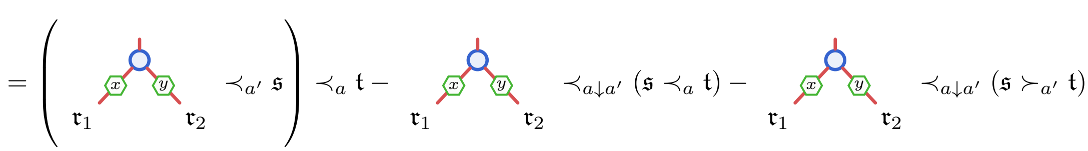
We now prove that Relations (4.1.17b) and (4.1.17c) hold by induction on the sum of the number of internal nodes of $\mathfrak{r}$, $\mathfrak{s}$, and $\mathfrak{t}$. Base case holds when all these trees have exactly one internal node, and since

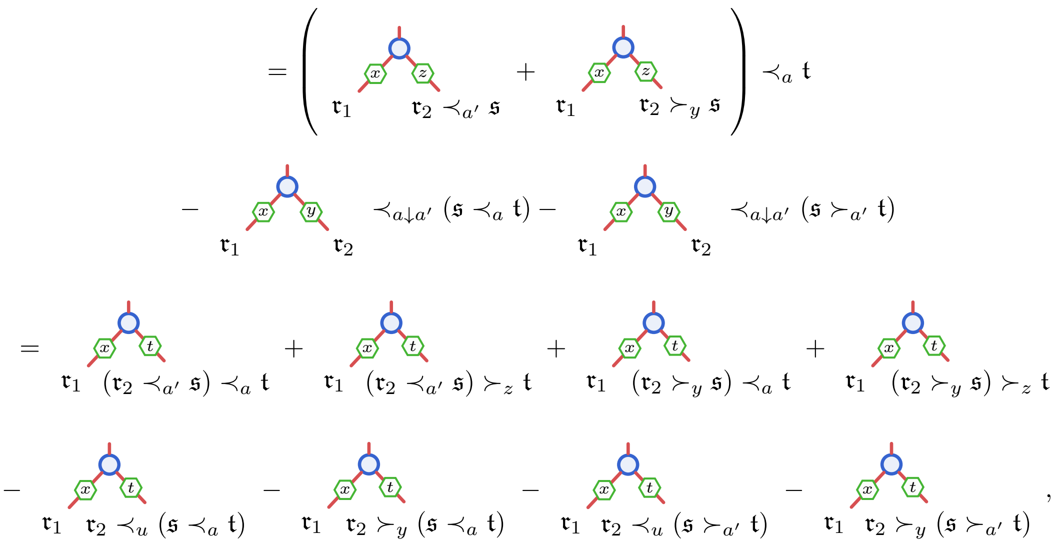


$$= 0, \quad (4.2.12)$$

where $z := a \downarrow a'$, (4.1.17b) holds on trees with exactly one internal node. For the same arguments, we can show that (4.1.17c) holds on trees with exactly one internal node. Denote now by $\mathfrak{r}_1$ (resp. $\mathfrak{r}_2$) the left subtree (resp. right subtree) of $\mathfrak{r}$ and by x (resp. y) the label of the left (resp. right) edge incident to the root of $\mathfrak{r}$. We have

$$(\mathfrak{r} \prec_{a'} \mathfrak{s}) \prec_a \mathfrak{t} - \mathfrak{r} \prec_{a \downarrow a'} (\mathfrak{s} \prec_a \mathfrak{t}) - \mathfrak{r} \prec_{a \downarrow a'} (\mathfrak{s} \succ_{a'} \mathfrak{t})$$



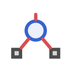


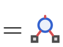
(4.2.13)

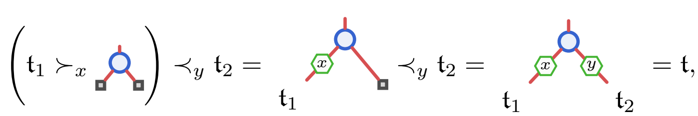
where $z := y \downarrow a'$, $t := z \downarrow a = y \downarrow a' \downarrow a$, and $u := a \downarrow a'$. Now, by induction hypothesis, Relation (4.1.17b) holds on $\mathfrak{r}_2$, $\mathfrak{s}$, and $\mathfrak{t}$. Hence, the sum of the first, fifth, and seventh terms of (4.2.13) is zero. Again by induction hypothesis, Relation (4.1.17c) holds on $\mathfrak{r}_2$, $\mathfrak{s}$, and $\mathfrak{t}$. Thus, the sum of the second, fourth, and last terms of (4.2.13) is zero. Finally, by what we just have proven in the first part of this proof, the sum of the third and sixth terms of (4.1.17c) is zero. Therefore, (4.2.13) is zero and (4.1.17b) is satisfied in $\mathcal{F}_{\mathsf{Dendr}_\gamma}$.

Finally, for the same arguments, we can show that (4.1.17c) is satisfied in $\mathcal{F}_{\mathsf{Dendr}_\gamma}$, implying the statement of the lemma. □

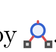
Lemma 4.2.2. *For any integer $\gamma \geqslant 0$, the γ-pluriassociative algebra $\mathcal{F}_{\mathsf{Dendr}_\gamma}$ of γ-edge valued binary trees endowed with the operations $\prec_a$, $\succ_a$, $a \in [\gamma]$, is generated by*

 . (4.2.14)

Proof. First, Lemma 4.2.1 shows that $\mathcal{F}_{\mathsf{Dendr}_\gamma}$ is a γ-polydendriform algebra. Let $\mathcal{D}$ be the γ-polydendriform subalgebra of $\mathcal{F}_{\mathsf{Dendr}_\gamma}$ generated by . Let us show that any γ-edge valued binary tree $\mathfrak{t}$ is in $\mathcal{D}$ by induction on the number n of its internal nodes. When $n = 1$, $\mathfrak{t} =$  and hence the property is satisfied. Otherwise, let $\mathfrak{t}_1$ (resp. $\mathfrak{t}_2$) be the left (resp. right) subtree of the root of $\mathfrak{t}$ and denote by x (resp. y) the label of the left (resp. right) edge incident to the root of $\mathfrak{t}$. Since $\mathfrak{t}_1$ and $\mathfrak{t}_2$ have less internal nodes than $\mathfrak{t}$, by induction hypothesis, $\mathfrak{t}_1$ and $\mathfrak{t}_2$ are in $\mathcal{D}$. Moreover, by definition of the operations $\prec_a$, $\succ_a$, $a \in [\gamma]$, of $\mathcal{F}_{\mathsf{Dendr}_\gamma}$, one has



(4.2.15)

showing that $\mathfrak{t}$ also is in $\mathcal{D}$. Therefore, $\mathcal{D}$ is $\mathcal{F}_{\mathsf{Dendr}_\gamma}$, showing that $\mathcal{F}_{\mathsf{Dendr}_\gamma}$ is generated by . □

Theorem 4.2.3. *For any integer $\gamma \geqslant 0$, the vector space $\mathcal{F}_{\mathsf{Dendr}_\gamma}$ of γ-edge valued binary trees endowed with the operations $\prec_a$, $\succ_a$, $a \in [\gamma]$, is the free γ-polydendriform algebra over one generator.*

Proof. By Lemmas 4.2.1 and 4.2.2, $\mathcal{F}_{\mathsf{Dendr}_\gamma}$ is a γ-polydendriform algebra over one generator.

Moreover, since by Proposition 4.1.2, for any $n \geqslant 1$, the dimension of $\mathcal{F}_{\mathsf{Dendr}_\gamma}(n)$ is the same as the dimension of $\mathsf{Dendr}_\gamma(n)$, there cannot be relations in $\mathcal{F}_{\mathsf{Dendr}_\gamma}(n)$ involving $\mathfrak{g}$ that are not γ-polydendriform relations (see (4.1.17a), (4.1.17b), and (4.1.17c)). Hence, $\mathcal{F}_{\mathsf{Dendr}_\gamma}$ is free as a γ-polydendriform algebra over one generator. □

5. Multiassociative operads

There is a well-known diagram, whose definition is recalled below, gathering the diassociative, associative, and dendriform operads. The main goal of this section is to define a one-parameter nonnegative integer generalization of the associative operad to obtain a new version of this diagram, suited to the context of pluriassociative and polydendriform operads.

5.1. **Two generalizations of the associative operad.**

The associative operad is generated by one binary element. This operad admits two different generalizations generated by γ binary elements with the particularity that one is the Koszul dual of the other. We introduce and study in this section these two operads.

5.1.1. *Nonsymmetric associative operad.* Recall that the *nonsymmetric associative operad*, or the *associative operad* for short, is the operad As admitting the presentation $(\mathfrak{G}_{\mathsf{As}}, \mathfrak{R}_{\mathsf{As}})$, where $\mathfrak{G}_{\mathsf{As}} := \mathfrak{G}_{\mathsf{As}}(2) := \{\star\}$ and $\mathfrak{R}_{\mathsf{As}}$ is generated by $\star \circ_1 \star - \star \circ_2 \star$. It admits the following realization. For any $n \geqslant 1$, $\mathsf{As}(n)$ is the vector space of dimension one generated by the corolla of arity n and the partial composition $\mathfrak{c}_1 \circ_i \mathfrak{c}_2$ where $\mathfrak{c}_1$ is the corolla of arity n and $\mathfrak{c}_2$ is the corolla of arity m is the corolla of arity $n + m - 1$ for all valid i.

5.1.2. *Multiassociative operads.* For any integer $\gamma \geqslant 0$, we define As_γ as the operad admitting the presentation $\left(\mathfrak{G}_{\mathsf{As}_\gamma}, \mathfrak{R}_{\mathsf{As}_\gamma}\right)$, where $\mathfrak{G}_{\mathsf{As}_\gamma} := \mathfrak{G}_{\mathsf{As}_\gamma}(2) := \{\star_a : a \in [\gamma]\}$ and $\mathfrak{R}_{\mathsf{As}_\gamma}$ is generated by

$$\star_a \circ_1 \star_b - \star_b \circ_2 \star_b, \qquad a \leqslant b \in [\gamma], \tag{5.1.1a}$$

$$\star_b \circ_1 \star_a - \star_b \circ_2 \star_b, \qquad a < b \in [\gamma], \tag{5.1.1b}$$

$$\star_a \circ_2 \star_b - \star_b \circ_2 \star_b, \qquad a < b \in [\gamma], \tag{5.1.1c}$$

$$\star_b \circ_2 \star_a - \star_b \circ_2 \star_b, \qquad a < b \in [\gamma]. \tag{5.1.1d}$$

This space of relations can be rephrased in a more compact way as the space generated by

$$\star_a \circ_1 \star_{a'} - \star_{a \uparrow a'} \circ_2 \star_{a \uparrow a'}, \qquad a, a' \in [\gamma], \tag{5.1.2a}$$

$$\star_a \circ_2 \star_{a'} - \star_{a \uparrow a'} \circ_2 \star_{a \uparrow a'}, \qquad a, a' \in [\gamma]. \tag{5.1.2b}$$

We call As_γ the *γ-multiassociative operad*.

It follows immediately that As_γ is a set-operad and that it provides a generalization of the associative operad. The algebras over As_γ are the γ-multiassociative algebras introduced in Section 3.3.1.

Let us now provide a realization of As_γ. A *γ-corolla* is a rooted tree with at most one internal node labeled on $[\gamma]$. Denote by $\mathcal{F}_{\mathsf{As}_\gamma}(n)$ the vector space of γ-corollas of arity $n \geqslant 1$, by $\mathcal{F}_{\mathsf{As}_\gamma}$ the graded vector space of all γ-corollas, and let

$$\star : \mathcal{F}_{\mathsf{As}_\gamma} \otimes \mathcal{F}_{\mathsf{As}_\gamma} \to \mathcal{F}_{\mathsf{As}_\gamma} \tag{5.1.3}$$

be the linear operation where, for any γ-corollas $\mathfrak{c}_1$ and $\mathfrak{c}_2$, $\mathfrak{c}_1 \star \mathfrak{c}_2$ is the γ-corolla with $n+m-1$ leaves and labeled by $a \uparrow a'$ where n (resp. m) is the number of leaves of $\mathfrak{c}_1$ (resp. $\mathfrak{c}_2$) and a (resp. a') is the label of $\mathfrak{c}_1$ (resp. $\mathfrak{c}_2$).

Proposition 5.1.1. *For any integer $\gamma \geqslant 0$, the operad As_γ is the vector space $\mathcal{F}_{\mathsf{As}_\gamma}$ of γ-corollas and its partial compositions satisfy, for any γ-corollas $\mathfrak{c}_1$ and $\mathfrak{c}_2$, $\mathfrak{c}_1 \circ_i \mathfrak{c}_2 = \mathfrak{c}_1 \star \mathfrak{c}_2$ for all valid integer i. Besides, As_γ is a Koszul operad and the set of right comb syntax trees of* $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{As}_\gamma}\right)$ *where all internal nodes have a same label forms a Poincaré-Birkhoff-Witt basis of As_γ.*

Proof. In this proof, we consider that $\mathfrak{G}_{\mathsf{As}_\gamma}$ is totally ordered by the relation $\leqslant$ satisfying $\star_a \leqslant \star_b$ whenever $a \leqslant b \in [\gamma]$. It is immediate that the vector space $\mathcal{F}_{\mathsf{As}_\gamma}$ endowed with the partial compositions described in the statement of the proposition is an operad. Let us prove that this operad admits the presentation $\left(\mathfrak{G}_{\mathsf{As}_\gamma}, \mathfrak{R}_{\mathsf{As}_\gamma}\right)$.

For this purpose, consider the quadratic rewrite rule $\to_\gamma$ on $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{As}_\gamma}\right)$ satisfying

$$\star_a \circ_1 \star_b \to_\gamma \star_b \circ_2 \star_b, \qquad a \leqslant b \in [\gamma], \tag{5.1.4a}$$

$$\star_b \circ_1 \star_a \to_\gamma \star_b \circ_2 \star_b, \qquad a < b \in [\gamma], \tag{5.1.4b}$$

$$\star_a \circ_2 \star_b \to_\gamma \star_b \circ_2 \star_b, \qquad a < b \in [\gamma], \tag{5.1.4c}$$

$$\star_b \circ_2 \star_a \to_\gamma \star_b \circ_2 \star_b, \qquad a < b \in [\gamma]. \tag{5.1.4d}$$

Observe first that the space induced by the operad congruence induced by $\to_\gamma$ is $\mathfrak{R}_{\mathsf{As}_\gamma}$ (see (5.1.1a)—(5.1.1d)). Moreover, $\to_\gamma$ is a terminating rewrite rule and its normal forms are right comb syntax trees of $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{As}_\gamma}\right)$ where all internal nodes have a same label. Besides, one can show that for any syntax tree $\mathfrak{t}$ of $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{As}_\gamma}\right)$, we have $\mathfrak{t} \overset{*}{\to}_\gamma \mathfrak{s}$ with $\mathfrak{s}$ is a right comb syntax tree where all internal nodes labeled by the greatest label of $\mathfrak{t}$. Therefore, $\to_\gamma$ is a convergent rewrite rule and the operad As, admitting by definition the presentation $\left(\mathfrak{G}_{\mathsf{As}_\gamma}, \mathfrak{R}_{\mathsf{As}_\gamma}\right)$, has bases indexed by such trees.

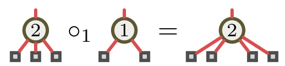
Now, let

$$\phi : \mathsf{As}_\gamma \simeq \mathbf{Free}\left(\mathfrak{G}_{\mathsf{As}_\gamma}\right) /_{\left\langle \mathfrak{R}_{\mathsf{As}_\gamma} \right\rangle} \to \mathcal{F}_{\mathsf{As}_\gamma} \tag{5.1.5}$$

be the map satisfying $\phi(\pi(\star_a)) = \mathfrak{c}_a$ where $\mathfrak{c}_a$ is the γ-corolla of arity 2 with internal node labeled by $a \in [\gamma]$ and $\pi : \mathbf{Free}\left(\mathfrak{G}_{\mathsf{As}_\gamma}\right) \to \mathsf{As}_\gamma$ is the canonical surjection map. Since we have $\phi(\pi(x)) \circ_i \phi(\pi(y)) = \phi(\pi(x')) \circ_{i'} \phi(\pi(y'))$ for all relations $x \circ_i y \to_\gamma x' \circ_{i'} y'$ of (5.1.4a)—(5.1.4d), ϕ extends in a unique way into an operad morphism. First, since the set G_γ of all γ-corollas of arity two is a generating set of $\mathcal{F}_{\mathsf{As}_\gamma}$ and the image of ϕ contains G_γ, ϕ is surjective. Second, since by definition of $\mathcal{F}_{\mathsf{As}_\gamma}$, the bases of $\mathcal{F}_{\mathsf{As}_\gamma}$ are indexed by γ-corollas, in accordance with what we have shown in the previous paragraph of this proof, $\mathcal{F}_{\mathsf{As}_\gamma}$ and As_γ are isomorphic as graded vector spaces. Hence, ϕ is an operad isomorphism, showing that As_γ admits the claimed realization.

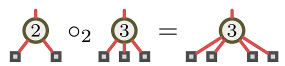
Finally, the existence of the convergent rewrite rule $\to_\gamma$ implies, by the Koszulity criterion [Hof10, DK10, LV12] we have reformulated in Section 1.2.5, that As_γ is Koszul and that its Poincaré-Birkhoff-Witt basis is the one described in the statement of the proposition. $\square$

We have for instance in As_3,



(5.1.6)

and



(5.1.7)

We deduce from Proposition 5.1.1 that the Hilbert series of As_γ satisfies

$$\mathcal{H}_{\mathsf{As}_\gamma}(t) = \frac{t + (\gamma - 1)t^2}{1 - t}. \tag{5.1.8}$$

and that for all $n \geqslant 2$, $\dim \mathsf{As}_\gamma(n) = \gamma$.

5.1.3. *Dual multiassociative operads.* Since As_γ is a binary and quadratic operad, its admits a Koszul dual, denoted by DAs_γ and called γ-*dual multiassociative operad.* The presentation of this operad is provided by next result.

Proposition 5.1.2. *For any integer $\gamma \geqslant 0$, the operad DAs_γ admits the following presentation. It is generated by $\mathfrak{G}_{\mathsf{DAs}_\gamma} := \mathfrak{G}_{\mathsf{DAs}_\gamma}(2) := \{\maltese_a : a \in [\gamma]\}$ and its space of relations $\mathfrak{R}_{\mathsf{DAs}_\gamma}$ is generated by*

$$\maltese_b \circ_1 \maltese_b - \maltese_b \circ_2 \maltese_b + \left(\sum_{a<b} \maltese_a \circ_1 \maltese_b + \maltese_b \circ_1 \maltese_a - \maltese_a \circ_2 \maltese_b - \maltese_b \circ_2 \maltese_a \right), \qquad b \in [\gamma]. \tag{5.1.9}$$

Proof. By a straightforward computation, and by identifying $\maltese_a$ with $\star_a$ for any $a \in [\gamma]$, we obtain that the space $\mathfrak{R}_{\mathsf{DAs}_\gamma}$ of the statement of the proposition satisfies $\mathfrak{R}^{\perp}_{\mathsf{DAs}_\gamma} = \mathfrak{R}_{\mathsf{As}_\gamma}$. Hence, DAs admits the claimed presentation. $\square$

For any integer $\gamma \geqslant 0$, let $\diamond_b$, $b \in [\gamma]$, the elements of $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{DAs}_\gamma}\right)$ defined by

$$\diamond_b := \sum_{a \in [b]} \maltese_a. \tag{5.1.10}$$

Then, since for all $b \in [\gamma]$ we have

$$\maltese_b = \begin{cases} \diamond_1 & \text{if } b = 1, \\ \diamond_b - \diamond_{b-1} & \text{otherwise,} \end{cases} \tag{5.1.11}$$

by triangularity, the family $\mathfrak{G}'_{\mathsf{DAs}_\gamma} := \{\diamond_b : b \in [\gamma]\}$ forms a basis of $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{DAs}_\gamma}\right)(2)$ and then, generates $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{DAs}_\gamma}\right)$ as an operad. Let us now express a presentation of DAs_γ through the family $\mathfrak{G}'_{\mathsf{DAs}_\gamma}$.

Proposition 5.1.3. *For any integer $\gamma \geqslant 0$, the operad DAs_γ admits the following presentation. It is generated by $\mathfrak{G}'_{\mathsf{DAs}_\gamma}$ and its space of relations $\mathfrak{R}'_{\mathsf{DAs}_\gamma}$ is generated by*

$$\diamond_a \circ_1 \diamond_a - \diamond_a \circ_2 \diamond_a, \qquad a \in [\gamma]. \tag{5.1.12}$$

Proof. Let us show that $\mathfrak{R}'_{\mathsf{DAs}_\gamma}$ is equal to the space of relations $\mathfrak{R}_{\mathsf{DAs}_\gamma}$ of DAs_γ defined in the statement of Proposition 5.1.2. By this last proposition, for any $x \in \mathbf{Free}\left(\mathfrak{G}_{\mathsf{DAs}_\gamma}\right)(3)$, x is in $\mathfrak{R}_{\mathsf{DAs}_\gamma}$ if and only if $\pi(x) = 0$ where $\pi : \mathbf{Free}\left(\mathfrak{G}_{\mathsf{DAs}_\gamma}\right) \to \mathsf{DAs}$ is the canonical surjection map. By a straightforward computation, by expanding (5.1.12) over the elements $\boxminus_a$, $a \in [\gamma]$, by using (5.1.10) we obtain that (5.1.12) can be expressed as a sum of elements of $\mathfrak{R}_{\mathsf{DAs}_\gamma}$. This implies that $\pi(x) = 0$ and hence that $\mathfrak{R}'_{\mathsf{DAs}_\gamma}$ is a subspace of $\mathfrak{R}_{\mathsf{DAs}_\gamma}$.

Now, one can observe that for all $a \in [\gamma]$, the elements (5.1.12) are linearly independent. Then, $\mathfrak{R}'_{\mathsf{DAs}_\gamma}$ has dimension γ which is also, by Proposition 5.1.2, the dimension of $\mathfrak{R}_{\mathsf{DAs}_\gamma}$. The statement of the proposition follows. □

Observe, from the presentation provided by Proposition 5.1.3 of DAs_γ, that DAs_2 is the operad denoted by *2as* in [LR06].

Notice that the Koszul dual of DAs_γ through its presentation $\left(\mathfrak{G}'_{\mathsf{DAs}_\gamma}, \mathfrak{R}'_{\mathsf{DAs}_\gamma}\right)$ of Proposition 5.1.3 gives rise to the following presentation for As_γ. This last operad admits the presentation $\left(\mathfrak{G}'_{\mathsf{As}_\gamma}, \mathfrak{R}'_{\mathsf{As}_\gamma}\right)$ where $\mathfrak{G}'_{\mathsf{As}_\gamma} := \mathfrak{G}'_{\mathsf{As}_\gamma}(2) := \{\triangle_a : a \in [\gamma]\}$ and $\mathfrak{R}'_{\mathsf{As}_\gamma}$ is generated by

$$\triangle_a \circ_1 \triangle_{a'}, \qquad a \neq a' \in [\gamma], \tag{5.1.13a}$$

$$\triangle_a \circ_2 \triangle_{a'}, \qquad a \neq a' \in [\gamma], \tag{5.1.13b}$$

$$\triangle_a \circ_1 \triangle_a - \triangle_a \circ_2 \triangle_a, \qquad a \in [\gamma]. \tag{5.1.13c}$$

Indeed, $\mathfrak{R}'_{\mathsf{As}_\gamma}$ is the space $\mathfrak{R}_{\mathsf{As}_\gamma}$ through the identification

$$\triangle_a = \begin{cases} \star_\gamma & \text{if } a = \gamma, \\ \star_a - \star_{a+1} & \text{otherwise.} \end{cases} \tag{5.1.14}$$

Proposition 5.1.4. *For any integer $\gamma \geqslant 0$, the Hilbert series $\mathcal{H}_{\mathsf{DAs}_\gamma}(t)$ of the operad DAs_γ satisfies*

$$\mathcal{H}_{\mathsf{DAs}_\gamma}(t) = t + t\,\mathcal{H}_{\mathsf{DAs}_\gamma}(t) + (\gamma - 1)\,\mathcal{H}_{\mathsf{DAs}_\gamma}(t)^2. \tag{5.1.15}$$

Proof. By setting $\bar{\mathcal{H}}_{\mathsf{DAs}_\gamma}(t) := \mathcal{H}_{\mathsf{DAs}_\gamma}(-t)$, from (5.1.15), we obtain

$$t = \frac{-\bar{\mathcal{H}}_{\mathsf{DAs}_\gamma}(t) + (\gamma - 1)\bar{\mathcal{H}}_{\mathsf{DAs}_\gamma}(t)^2}{1 + \bar{\mathcal{H}}_{\mathsf{DAs}_\gamma}(t)}. \tag{5.1.16}$$

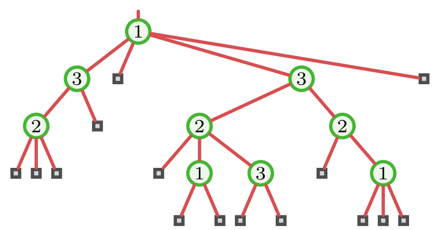
Moreover, by setting $\bar{\mathcal{H}}_{\mathsf{As}_\gamma}(t) := \mathcal{H}_{\mathsf{As}_\gamma}(-t)$, where $\mathcal{H}_{\mathsf{As}_\gamma}(t)$ is defined by (5.1.8), we have

$$\bar{\mathcal{H}}_{\mathsf{As}_\gamma}\left(\bar{\mathcal{H}}_{\mathsf{DAs}_\gamma}(t)\right) = \frac{-\bar{\mathcal{H}}_{\mathsf{DAs}_\gamma}(t) + (\gamma - 1)\bar{\mathcal{H}}_{\mathsf{DAs}_\gamma}(t)^2}{1 + \bar{\mathcal{H}}_{\mathsf{DAs}_\gamma}(t)} = t, \tag{5.1.17}$$

showing that $\bar{\mathcal{H}}_{\mathsf{As}_\gamma}(t)$ and $\bar{\mathcal{H}}_{\mathsf{DAs}_\gamma}(t)$ are the inverses for each other for series composition.

Now, since by Proposition 5.1.1, As_γ is a Koszul operad and its Hilbert series is $\mathcal{H}_{\mathsf{As}_\gamma}(t)$, and since DAs_γ is by definition the Koszul dual of As_γ, the Hilbert series of these two operads satisfy (1.2.10). Therefore, (5.1.17) implies that the Hilbert series of DAs_γ is $\mathcal{H}_{\mathsf{DAs}_\gamma}(t)$. □

A *Schröder tree* [Sta01, Sta11] is a planar rooted tree such that internal nodes have two of more children. By examining the expression for $\mathcal{H}_{\mathsf{DAs}_\gamma}(t)$ of the statement of Proposition 5.1.4, we observe that for any $n \geqslant 1$, $\mathsf{DAs}_\gamma(n)$ can be seen as the vector space $\mathcal{F}_{\mathsf{DAs}_\gamma}(n)$ of Schröder trees with n internal nodes, all labeled on $[\gamma]$ such that the label of an internal node is different from the labels of its children that are internal nodes. We call these trees *γ-alternating Schröder trees*. Let us also denote by $\mathcal{F}_{\mathsf{DAs}_\gamma}$ the graded vector space of all γ-alternating Schröder trees. For instance,



(5.1.18)

is a 3-alternating Schröder tree and a basis element of $\mathsf{DAs}_3(9)$.

We deduce also from Proposition 5.1.4 that

$$\mathcal{H}_{\mathsf{DAs}_\gamma}(t) = \frac{1 - \sqrt{1 - (4\gamma - 2)t + t^2} - t}{2(\gamma - 1)}. \tag{5.1.19}$$

By denoting by $\operatorname{nar}(n, k)$ the *Narayana number* [Nar55] defined by

$$\operatorname{nar}(n, k) := \frac{1}{k+1}\binom{n-1}{k}\binom{n}{k}, \tag{5.1.20}$$

we obtain that for all $n \geqslant 1$,

$$\dim \mathsf{DAs}_\gamma(n) = \sum_{k=0}^{n-2} \gamma^{k+1}(\gamma - 1)^{n-k-2} \operatorname{nar}(n-1, k). \tag{5.1.21}$$

This formula is a consequence of the fact that $\operatorname{nar}(n-1, k)$ is the number of binary trees with n leaves and with exactly k internal nodes having a internal node as a left child, the fact that the number $\operatorname{schr}(n)$ of Schröder trees with n leaves expresses as

$$\operatorname{schr}(n) = \sum_{k=0}^{n-2} 2^k \operatorname{nar}(n-1, k), \tag{5.1.22}$$

and the fact that any Schröder tree $\mathfrak{s}$ with n leaves can be encoded by a binary tree $\mathfrak{t}$ with n leaves where any left oriented edge connecting two internal nodes of $\mathfrak{t}$ is labeled on $[2]$ ($\mathfrak{s}$ is obtained from $\mathfrak{t}$ by contracting all edges labeled by 2).

For instance, the first dimensions of DAs_1, DAs_2, DAs_3, and DAs_4 are respectively

$$1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, \tag{5.1.23}$$

$$1, 2, 6, 22, 90, 394, 1806, 8558, 41586, 206098, 1037718, \tag{5.1.24}$$

$$1, 3, 15, 93, 645, 4791, 37275, 299865, 2474025, 20819307, 178003815, \tag{5.1.25}$$

$$1, 4, 28, 244, 2380, 24868, 272188, 3080596, 35758828, 423373636, 5092965724. \tag{5.1.26}$$

The second one is Sequence **A006318**, the third one is Sequence **A103210**, and the last one is Sequence **A103211** of [Slo].

Let us now establish a realization of DAs_γ.

Proposition 5.1.5. *For any nonnegative integer γ, the operad DAs_γ is the vector space $\mathcal{F}_{\mathsf{DAs}_\gamma}$ of γ-alternating Schröder trees. Moreover, for any γ-alternating Schröder trees $\mathfrak{s}$ and $\mathfrak{t}$, $\mathfrak{s} \circ_i \mathfrak{t}$ is the γ-alternating Schröder tree obtained by grafting the root of $\mathfrak{t}$ on the ith leaf x of $\mathfrak{s}$ and then, if the father y of x and the root z of $\mathfrak{t}$ have a same label, by contracting the edge connecting y and z.*

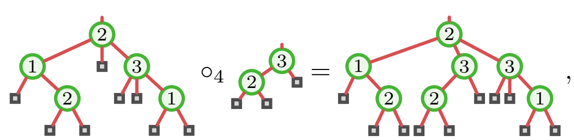
Proof. First, it is immediate that the vector space $\mathcal{F}_{\mathsf{DAs}_\gamma}$ endowed with the partial compositions described in the statement of the proposition is an operad.

Let

$$\phi : \mathsf{DAs}_\gamma \simeq \mathbf{Free}\left(\mathfrak{G}'_{\mathsf{DAs}_\gamma}\right) /_{\left\langle \mathfrak{R}'_{\mathsf{DAs}_\gamma} \right\rangle} \to \mathcal{F}_{\mathsf{DAs}_\gamma} \tag{5.1.27}$$

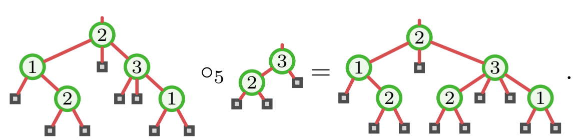
be the map satisfying $\phi(\pi(\diamond_a)) := \mathfrak{c}_a$ where $\mathfrak{c}_a$ is the γ-alternating Schröder with two leaves and one internal node labeled by $a \in [\gamma]$ and $\pi : \mathbf{Free}(\mathfrak{G}'_{\mathsf{DAs}_\gamma}) \to \mathsf{DAs}_\gamma$ is the canonical surjection map. Since we have $\phi(\pi(\diamond_a)) \circ_1 \phi(\pi(\diamond_a)) = \phi(\pi(\diamond_a)) \circ_2 \phi(\pi(\diamond_a))$ for all $a \in [\gamma]$, ϕ extends in a unique way into an operad morphism. First, since the set G_γ of all γ-alternating Schröder trees with two leaves and one internal node is a generating set of $\mathcal{F}_{\mathsf{DAs}_\gamma}$ and the image of ϕ contains G_γ, ϕ is surjective. Second, since by definition of $\mathcal{F}_{\mathsf{DAs}_\gamma}$, the bases of $\mathcal{F}_{\mathsf{DAs}_\gamma}$ are indexed by γ-alternating Schröder trees, by Proposition 5.1.4, $\mathcal{F}_{\mathsf{DAs}_\gamma}$ and DAs_γ are isomorphic as graded vector spaces. Hence, ϕ is an operad isomorphism, showing that DAs_γ admits the claimed realization. □

We have for instance in DAs_3,



, (5.1.28)

and



. (5.1.29)

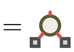
5.2. **A diagram of operads.** We now define morphisms between the operads Dias_γ, As_γ, DAs_γ, and Dendr_γ to obtain a generalization of a classical diagram involving the diassociative, associative, and dendriform operads.

5.2.1. *Relating the diassociative and dendriform operads.* The diagram

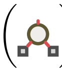
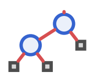
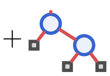
$$
\begin{tikzcd}
\mathsf{Dendr} & \mathsf{As} \arrow[l, "\zeta"'] \arrow[loop above, dotted, "!"] & \mathsf{Dias} \arrow[l, "\eta"'] \arrow[ll, bend right=60, dotted, leftrightarrow, "!"']
\end{tikzcd}
\tag{5.2.1}
$$

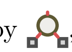
is a well-known diagram of operads, being a part of the so-called *operadic butterfly* [Lod01, Lod06] and summarizing in a nice way the links between the dendriform, associative, and diassociative operads. The operad As, being at the center of the diagram, is it own Koszul dual, while Dias and Dendr are Koszul dual one of the other.

The operad morphisms $\eta : \mathsf{Dias} \to \mathsf{As}$ and $\zeta : \mathsf{As} \to \mathsf{Dendr}$ are linearly defined through the realizations of Dias and Dendr recalled in Section 1.3 by

$\eta(\mathfrak{e}_{2,1}) :=$  $=: \eta(\mathfrak{e}_{2,2})\,,$ (5.2.2)

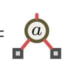
and

$\zeta\Big($  $\Big) :=$  $+$  $.$ (5.2.3)

Since Dias is generated by $\mathfrak{e}_{2,1}$ and $\mathfrak{e}_{2,2}$, and since As is generated by , η and ζ are wholly defined.

5.2.2. *Relating the pluriassociative and polydendriform operads.*

Proposition 5.2.1. *For any integer $\gamma \geqslant 0$, the map $\eta_\gamma : \mathsf{Dias}_\gamma \to \mathsf{As}_\gamma$ satisfying*

$\eta_\gamma(0a) =$  $= \eta_\gamma(a0), \qquad a \in [\gamma],$ (5.2.4)

extends in a unique way into an operad morphism. Moreover, this morphism is surjective.

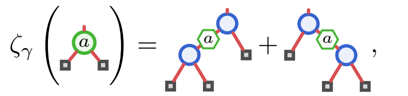
Proof. Theorem 2.2.6 and Proposition 5.1.5 allow to interpret the map η_γ over the presentations of Dias_γ and As_γ. Then, via this interpretation, one has

$$\eta_\gamma(\pi(\dashv_a)) = \pi'(\star_a) = \eta_\gamma(\pi(\vdash_a)), \qquad a \in [\gamma], \tag{5.2.5}$$

where $\pi : \mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dias}_\gamma}\right) \to \mathsf{Dias}_\gamma$ and $\pi' : \mathbf{Free}\left(\mathfrak{G}_{\mathsf{As}_\gamma}\right) \to \mathsf{As}_\gamma$ are canonical surjection maps. Now, for any element x of $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dias}_\gamma}\right)$ generating the space of relations $\mathfrak{R}_{\mathsf{Dias}_\gamma}$ of Dias_γ, we can check that $\eta_\gamma(\pi(x)) = 0$. This shows that η_γ extends in a unique way into an operad morphism. Finally, this morphism is a surjection since its image contains the set of all γ-corollas of arity 2, which is a generating set of As_γ. $\square$

By Proposition 5.2.1, the map η_γ, whose definition is only given in arity 2, defines an operad morphism. Nevertheless, by induction on the arity, one can prove that for any word x of Dias_γ, $\eta_\gamma(x)$ is the γ-corolla of arity $|x|$ labeled by the greatest letter of x.

Proposition 5.2.2. *For any integer $\gamma \geqslant 0$, the map $\zeta_\gamma : \mathsf{DAs}_\gamma \to \mathsf{Dendr}_\gamma$ satisfying*



$$a \in [\gamma], \tag{5.2.6}$$

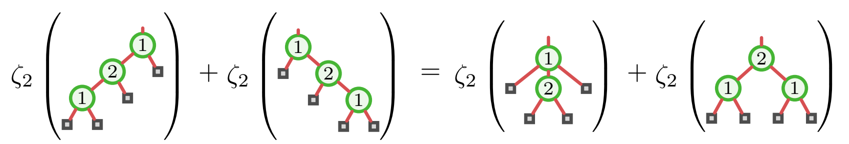
extends in a unique way into an operad morphism.

Proof. Propositions 5.1.3 and 5.1.5, and Theorem 4.1.4 allow to interpret the map ζ_γ over the presentations of DAs_γ and Dendr_γ. Then, via this interpretation, one has

$$\zeta_\gamma(\pi(\diamond_a)) = \pi'(\prec_a + \succ_a), \qquad a \in [\gamma], \tag{5.2.7}$$

where $\pi : \mathbf{Free}(\mathfrak{G}'_{\mathsf{DAs}_\gamma}) \to \mathsf{DAs}_\gamma$ and $\pi' : \mathbf{Free}\left(\mathfrak{G}'_{\mathsf{Dendr}_\gamma}\right) \to \mathsf{Dendr}_\gamma$ are canonical surjection maps. We now observe that the image of $\pi(\diamond_a)$ is $\odot_a$, where $\odot_a$ is the element of Dendr_γ defined in the statement of Proposition 4.1.5. Then, since by this last proposition this element is associative, for any element x of $\mathbf{Free}(\mathfrak{G}'_{\mathsf{DAs}_\gamma})$ generating the space of relations of $\mathfrak{R}'_{\mathsf{DAs}_\gamma}$ of DAs_γ, $\zeta_\gamma(\pi(x)) = 0$. This shows that ζ_γ extends in a unique way into an operad morphism. □

We have to observe that the morphism ζ_γ defined in the statement of Proposition 5.2.2 is injective only for $\gamma \leqslant 1$. Indeed, when $\gamma \geqslant 2$, we have the relation



$$\tag{5.2.8}$$

Theorem 5.2.3. *For any integer $\gamma \geqslant 0$, the operads Dias_γ, Dendr_γ, As_γ, and DAs_γ fit into the diagram*

$$\begin{tikzcd} \mathsf{Dendr}_\gamma & \mathsf{DAs}_\gamma \arrow[l, "\zeta_\gamma"'] \arrow[r, dotted, leftrightarrow, "!"] & \mathsf{As}_\gamma & \mathsf{Dias}_\gamma \arrow[l, "\eta_\gamma"'] \arrow[lll, dotted, leftrightarrow, bend right, "!"'] \end{tikzcd}, \tag{5.2.9}$$

where η_γ is the surjection defined in the statement of Proposition 5.2.1 and ζ_γ is the operad morphism defined in the statement of Proposition 5.2.2.

Proof. This is a direct consequence of Propositions 5.2.1 and 5.2.2. □

Diagram (5.2.9) is a generalization of (5.2.1) in which the associative operad split into operads As_γ and DAs_γ.

6. Further generalizations

In this last section, we propose some one-parameter nonnegative integer generalizations of well-known operads. For this, we use similar tools as the ones used in the first sections of this paper.

6.1. **Duplicial operad.** We construct here a one-parameter nonnegative integer generalization of the duplicial operad and describe the free algebras over one generator in the category encoded by this generalization.

6.1.1. *Multiplicial operads.* It is well-known [LV12] that the dendriform operad and the duplicial operad Dup [Lod08] are both specializations of a same operad D_q with one parameter $q \in \mathbb{K}$. This operad admits the presentation $\left(\mathfrak{G}_{\mathsf{D}_q}, \mathfrak{R}_{\mathsf{D}_q}\right)$, where $\mathfrak{G}_{\mathsf{D}_q} := \mathfrak{G}_{\mathsf{Dendr}}$ and $\mathfrak{R}_{\mathsf{D}_q}$ is the vector space generated by

$$\prec \circ_1 \succ - \succ \circ_2 \prec, \tag{6.1.1a}$$

$$\prec \circ_1 \prec - \prec \circ_2 \prec -q \prec \circ_2 \succ, \tag{6.1.1b}$$

$$q \succ \circ_1 \prec + \succ \circ_1 \succ - \succ \circ_2 \succ . \tag{6.1.1c}$$

One can observe that D_1 is the dendriform operad and that D_0 is the duplicial operad.

On the basis of this observation, from the presentation of Dendr_γ provided by Theorem 4.1.4 and its concise form provided by Relations (4.1.17a), (4.1.17b), and (4.1.17c) for its space of relations, we define the operad $\mathsf{D}_{q,\gamma}$ with two parameters, an integer $\gamma \geqslant 0$ and $q \in \mathbb{K}$, in the following way. We set $\mathsf{D}_{q,\gamma}$ as the operad admitting the presentation $\left(\mathfrak{G}_{\mathsf{D}_{q,\gamma}}, \mathfrak{R}_{\mathsf{D}_{q,\gamma}}\right)$, where $\mathfrak{G}_{\mathsf{D}_{q,\gamma}} := \mathfrak{G}'_{\mathsf{Dendr}_\gamma}$ and $\mathfrak{R}_{\mathsf{D}_{q,\gamma}}$ is the vector space generated by

$$\prec_a \circ_1 \succ_{a'} - \succ_{a'} \circ_2 \prec_a, \qquad a, a' \in [\gamma], \tag{6.1.2a}$$

$$\prec_a \circ_1 \prec_{a'} - \prec_{a \downarrow a'} \circ_2 \prec_a -q \prec_{a \downarrow a'} \circ_2 \succ_{a'}, \qquad a, a' \in [\gamma], \tag{6.1.2b}$$

$$q \succ_{a \downarrow a'} \circ_1 \prec_{a'} + \succ_{a \downarrow a'} \circ_1 \succ_a - \succ_a \circ_2 \succ_{a'}, \qquad a, a' \in [\gamma]. \tag{6.1.2c}$$

One can observe that $\mathsf{D}_{1,\gamma}$ is the operad Dendr_γ.

Let us define the operad Dup_γ, called *γ-multiplicial operad*, as the operad $\mathsf{D}_{0,\gamma}$. By using respectively the symbols $\hookleftarrow_a$ and $\hookrightarrow_a$ instead of $\prec_a$ and $\succ_a$ for all $a \in [\gamma]$, we obtain that the space of relations $\mathfrak{R}_{\mathsf{Dup}_\gamma}$ of Dup_γ is generated by

$$\hookleftarrow_a \circ_1 \hookrightarrow_{a'} - \hookrightarrow_{a'} \circ_2 \hookleftarrow_a, \qquad a, a' \in [\gamma], \tag{6.1.3a}$$

$$\hookleftarrow_a \circ_1 \hookleftarrow_{a'} - \hookleftarrow_{a \downarrow a'} \circ_2 \hookleftarrow_a, \qquad a, a' \in [\gamma], \tag{6.1.3b}$$

$$\hookrightarrow_{a \downarrow a'} \circ_1 \hookrightarrow_a - \hookrightarrow_a \circ_2 \hookrightarrow_{a'}, \qquad a, a' \in [\gamma]. \tag{6.1.3c}$$

We denote by $\mathfrak{G}_{\mathsf{Dup}_\gamma}$ the set of generators $\{\hookleftarrow_a, \hookrightarrow_a : a \in [\gamma]\}$ of Dup_γ.

In order to establish some properties of Dup_γ, let us consider the quadratic rewrite rule $\to_\gamma$ on $\mathbf{Free}(\mathfrak{G}_{\mathsf{Dup}_\gamma})$ satisfying

$$\hookleftarrow_a \circ_1 \hookrightarrow_{a'} \;\to_\gamma\; \hookrightarrow_{a'} \circ_2 \hookleftarrow_a, \qquad a, a' \in [\gamma], \tag{6.1.4a}$$

$$\hookleftarrow_a \circ_1 \hookleftarrow_{a'} \;\to_\gamma\; \hookleftarrow_{a \downarrow a'} \circ_2 \hookleftarrow_a, \qquad a, a' \in [\gamma], \tag{6.1.4b}$$

$$\hookrightarrow_a \circ_2 \hookrightarrow_{a'} \;\to_\gamma\; \hookrightarrow_{a \downarrow a'} \circ_1 \hookrightarrow_a, \qquad a, a' \in [\gamma]. \tag{6.1.4c}$$

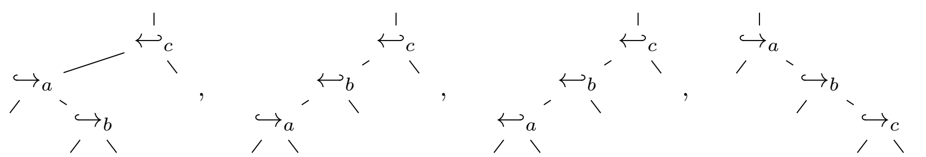
Observe that the space induced by the operad congruence induced by $\to_\gamma$ is $\mathfrak{R}_{\mathsf{Dup}_\gamma}$.

Lemma 6.1.1. *For any integer $\gamma \geqslant 0$, the rewrite rule $\to_\gamma$ is convergent and the generating series $\mathcal{G}_\gamma(t)$ of its normal forms counted by arity satisfies*

$$\mathcal{G}_\gamma(t) = t + 2\gamma t\, \mathcal{G}_\gamma(t) + \gamma^2 t\, \mathcal{G}_\gamma(t)^2. \tag{6.1.5}$$

Proof. Let us first prove that $\to_\gamma$ is terminating. Consider the map $\phi : \mathbf{Free}(\mathfrak{G}_{\mathsf{Dup}_\gamma}) \to \mathbb{N}^2$ defined, for any syntax tree $\mathfrak{t}$ by $\phi(\mathfrak{t}) := (\alpha + \alpha', \beta)$, where α (resp. α', β) is the sum, for all internal nodes of $\mathfrak{t}$ labeled by $\hookleftarrow_a$ (resp. $\hookrightarrow_a$, $\hookrightarrow_a$), $a \in [\gamma]$, of the number of internal nodes in its right (resp. left, right) subtree. For the lexicographical order $\leqslant$ on $\mathbb{N}^2$, we can check that for all $\to_\gamma$-rewritings $\mathfrak{s} \to_\gamma \mathfrak{t}$ where $\mathfrak{s}$ and $\mathfrak{t}$ are syntax trees with two internal nodes, we have $\phi(\mathfrak{s}) \neq \phi(\mathfrak{t})$ and $\phi(\mathfrak{s}) \leqslant \phi(\mathfrak{t})$. This implies that any syntax tree $\mathfrak{t}$ obtained by a sequence of $\to_\gamma$-rewritings from a syntax tree $\mathfrak{s}$ satisfies $\phi(\mathfrak{s}) \neq \phi(\mathfrak{t})$ and $\phi(\mathfrak{s}) \leqslant \phi(\mathfrak{t})$. Then, since the set of syntax trees of $\mathbf{Free}(\mathfrak{G}_{\mathsf{Dup}_\gamma})$ of a fixed arity is finite, this shows that $\to_\gamma$ is a terminating rewrite rule.

Let us now prove that $\to_\gamma$ is convergent. We call *critical tree* any syntax tree $\mathfrak{s}$ with three internal nodes that can be rewritten by $\to_\gamma$ into two different trees $\mathfrak{t}$ and $\mathfrak{t}'$. The pair $(\mathfrak{t}, \mathfrak{t}')$ is a *critical pair* for $\to_\gamma$. Critical trees for $\to_\gamma$ are, for all $a, b, c \in [\gamma]$,



(6.1.6)

Since $\to_\gamma$ is terminating, by the diamond lemma [New42] (see also [BN98]), to prove that $\to_\gamma$ is confluent, it is enough to check that for any critical tree $\mathfrak{s}$, there is a normal form $\mathfrak{r}$ of $\to_\gamma$ such that $\mathfrak{s} \to_\gamma \mathfrak{t} \overset{*}{\to}_\gamma \mathfrak{r}$ and $\mathfrak{s} \to_\gamma \mathfrak{t}' \overset{*}{\to}_\gamma \mathfrak{r}$, where $(\mathfrak{t}, \mathfrak{t}')$ is a critical pair. This can be done by hand for each of the critical trees depicted in (6.1.6).

Let us finally prove that the generating series of the normal forms of $\to_\gamma$ is (6.1.5). Since $\to_\gamma$ is terminating, its normal forms are the syntax trees that have no partial subtree equal to $\hookleftarrow_a \circ_1 \hookrightarrow_{a'}$, $\hookleftarrow_a \circ_1 \hookleftarrow_{a'}$, or $\hookrightarrow_a \circ_2 \hookrightarrow_{a'}$ for all $a, a' \in [\gamma]$. Then, the normal forms of $\to_\gamma$ are the syntax trees wherein any internal node labeled by $\hookleftarrow_a$, $a \in [\gamma]$, has a leaf as left child and any internal node labeled by $\hookrightarrow_a$, $a \in [\gamma]$, has a leaf or an internal node labeled by $\hookleftarrow_{a'}$, $a' \in [\gamma]$, as right child. Therefore, by denoting by $\mathcal{G}'_\gamma(t)$ the generating series of the normal forms of $\to_\gamma$ equal to the leaf or with a root labeled by $\hookleftarrow_a$, $a \in [\gamma]$, we obtain

$$\mathcal{G}'_\gamma(t) = t + \gamma t \, \mathcal{G}_\gamma(t) \tag{6.1.7}$$

and

$$\mathcal{G}_\gamma(t) = \mathcal{G}'_\gamma(t) + \gamma \, \mathcal{G}_\gamma(t)\mathcal{G}'_\gamma(t). \tag{6.1.8}$$

An elementary computation shows that $\mathcal{G}(t)$ satisfies (6.1.5). □

Proposition 6.1.2. *For any integer $\gamma \geqslant 0$, the operad Dup_γ is Koszul and for any integer $n \geqslant 1$, $\mathsf{Dup}_\gamma(n)$ is the vector space of γ-edge valued binary trees with n internal nodes.*

Proof. Since the space induced by the operad congruence induced by $\to_\gamma$ is $\mathfrak{R}_{\mathsf{Dup}_\gamma}$, and since by Lemma 6.1.1, $\to_\gamma$ is convergent, by the Koszulity criterion [Hof10, DK10, LV12] we have reformulated in Section 1.2.5, Dup_γ is a Koszul operad. Moreover, again because $\to_\gamma$ is convergent, as a vector space, $\mathsf{Dup}_\gamma(n)$ is isomorphic to the vector space of the normal forms of $\to_\gamma$ with $n \geqslant 1$ internal nodes. Since the generating series $\mathcal{G}_\gamma(t)$ of the normal forms of $\to_\gamma$

is also the generating series of γ-edge valued binary trees (see Proposition 4.1.2), the second part of the statement of the proposition follows. $\square$

Since Proposition 6.1.2 shows that the operads Dup_γ and Dendr_γ have the same underlying vector space, asking if these two operads are isomorphic is natural. Next result implies that this is not the case.

Proposition 6.1.3. *For any integer $\gamma \geqslant 0$, any associative element of Dup_γ is proportional to $\pi(\leftarrow_a)$ or $\pi(\hookrightarrow_a)$ for an $a \in [\gamma]$, where $\pi : \mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dup}_\gamma}\right) \to \mathsf{Dup}_\gamma$ is the canonical surjection map.*

Proof. Let $\pi : \mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dup}_\gamma}\right) \to \mathsf{Dup}_\gamma$ be the canonical surjection map. Consider the element

$$x := \sum_{a \in [\gamma]} \alpha_a \leftarrow_a + \beta_a \hookrightarrow_a \tag{6.1.9}$$

of $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dup}_\gamma}\right)$, where $\alpha_a, \beta_a \in \mathbb{K}$ for all $a \in [\gamma]$, such that $\pi(x)$ is associative in Dup_γ. Since we have $\pi(r) = 0$ for all elements r of $\mathfrak{R}_{\mathsf{Dup}_\gamma}$ (see (6.1.3a), (6.1.3b), and (6.1.3c)), the fact that $\pi(x \circ_1 x - x \circ_2 x) = 0$ implies the constraints

$$\begin{aligned} \alpha_a \, \beta_{a'} - \beta_{a'} \, \alpha_a &= 0, & a, a' &\in [\gamma], \\ \alpha_a \, \alpha_{a'} - \alpha_{a \downarrow a'} \, \alpha_a &= 0, & a, a' &\in [\gamma], \\ \beta_a \, \beta_{a'} - \beta_{a \downarrow a'} \, \beta_a &= 0, & a, a' &\in [\gamma], \end{aligned} \tag{6.1.10}$$

on the coefficients intervening in x. Moreover, since the syntax trees $\hookrightarrow_b \circ_1 \hookrightarrow_a$, $\hookrightarrow_a \circ_1 \leftarrow_{a'}$, $\leftarrow_b \circ_2 \leftarrow_a$, and $\leftarrow_a \circ_2 \hookrightarrow_{a'}$ do not appear in $\mathfrak{R}_{\mathsf{Dup}_\gamma}$ for all $a < b \in [\gamma]$ and $a, a' \in [\gamma]$, we have the further constraints

$$\begin{aligned} \beta_b \, \beta_a &= 0, & a < b &\in [\gamma], \\ \beta_a \, \alpha_{a'} &= 0, & a, a' &\in [\gamma], \\ \alpha_b \, \alpha_a &= 0, & a < b &\in [\gamma], \\ \alpha_a \, \beta_{a'} &= 0, & a, a' &\in [\gamma]. \end{aligned} \tag{6.1.11}$$

These relations imply that there are at most one $c \in [\gamma]$ and one $d \in [\gamma]$ such that $\alpha_c \neq 0$ and $\beta_d \neq 0$. In this case, the relations imply also that $\alpha_c = 0$ or $\beta_d = 0$, or both. Therefore, x is of the form $x = \alpha_a \leftarrow_a$ or $x = \beta_a \hookrightarrow_a$ for an $a \in [\gamma]$, whence the statement of the proposition. $\square$

By Proposition 6.1.3 there are exactly 2γ nonproportional associative operations in Dup_γ while, by Proposition 4.1.6 there are exactly γ such operations in Dendr_γ. Therefore, Dup_γ and Dendr_γ are not isomorphic.

6.1.2. *Free multiplicial algebras.* We call *γ-multiplicial algebra* any Dup_γ-algebra. From the definition of Dup_γ, any γ-multiplicial algebra is a vector space endowed with linear operations $\hookleftarrow_a, \hookrightarrow_a$, $a \in [\gamma]$, satisfying the relations encoded by (6.1.3a)—(6.1.3c).

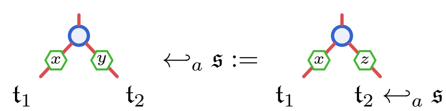
In order the simplify and make uniform next definitions, we consider that in any γ-edge valued binary tree $\mathfrak{t}$, all edges connecting internal nodes of $\mathfrak{t}$ with leaves are labeled by ∞. By convention, for all $a \in [\gamma]$, we have $a \downarrow \infty = a = \infty \downarrow a$. Let us endow the vector space $\mathcal{F}_{\mathsf{Dup}_\gamma}$ of γ-edge valued binary trees with linear operations

$$\hookleftarrow_a, \hookrightarrow_a : \mathcal{F}_{\mathsf{Dup}_\gamma} \otimes \mathcal{F}_{\mathsf{Dup}_\gamma} \to \mathcal{F}_{\mathsf{Dup}_\gamma}, \qquad a \in [\gamma], \tag{6.1.12}$$

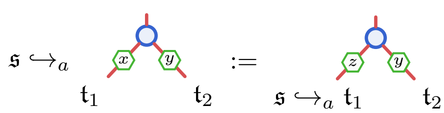
recursively defined, for any γ-edge valued binary tree $\mathfrak{s}$ and any γ-edge valued binary trees or leaves $\mathfrak{t}_1$ and $\mathfrak{t}_2$ by

$$\mathfrak{s} \hookleftarrow_a \square := \mathfrak{s} =: \square \hookrightarrow_a \mathfrak{s}, \tag{6.1.13}$$

$$\square \hookleftarrow_a \mathfrak{s} := 0 =: \mathfrak{s} \hookrightarrow_a \square, \tag{6.1.14}$$



$, \qquad z := a \downarrow y,$ (6.1.15)

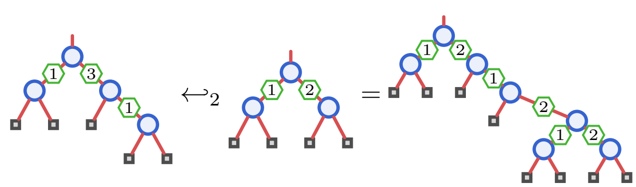


$, \qquad z := a \downarrow x.$ (6.1.16)

Note that neither $\square \prec_a \square$ nor $\square \hookrightarrow_a \square$ are defined.

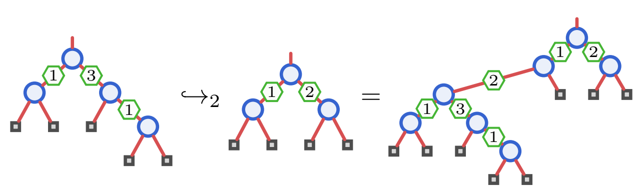
These recursive definitions for the operations $\hookleftarrow_a$, $\hookrightarrow_a$, $a \in [\gamma]$, lead to the following direct reformulations. If $\mathfrak{s}$ and $\mathfrak{t}$ are two γ-edge valued binary trees, $\mathfrak{t} \hookleftarrow_a \mathfrak{s}$ (resp. $\mathfrak{s} \hookrightarrow_a \mathfrak{t}$) is obtained by replacing each label y (resp. x) of any edge in the rightmost (resp. leftmost) path of $\mathfrak{t}$ by $a \downarrow y$ (resp. $a \downarrow x$) to obtain a tree $\mathfrak{t}'$, and by grafting the root of $\mathfrak{s}$ on the rightmost (resp. leftmost) leaf of $\mathfrak{t}'$. These two operations are respective generalizations of the operations *under* and *over* on binary trees introduced by Loday and Ronco [LR02].

For example, we have



, (6.1.17)

and

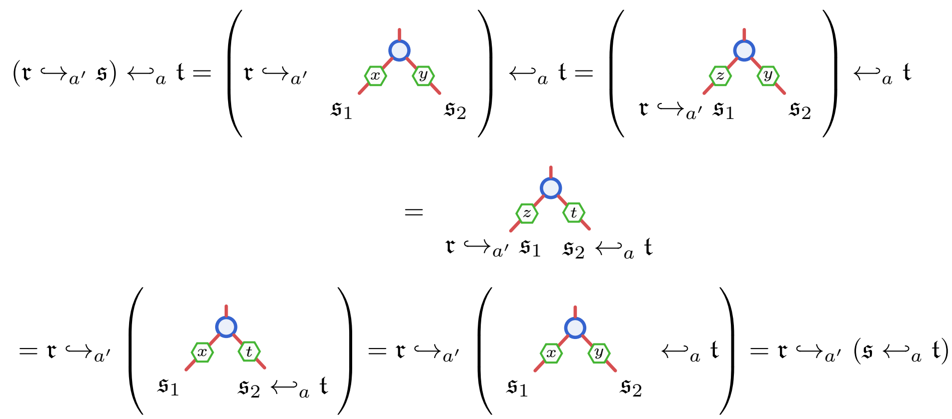


. (6.1.18)

Lemma 6.1.4. *For any integer $\gamma \geqslant 0$, the vector space $\mathcal{F}_{\mathsf{Dup}_\gamma}$ of γ-edge valued binary trees endowed with the operations $\hookleftarrow_a$, $\hookrightarrow_a$, $a \in [\gamma]$, is a γ-multiplicial algebra.*

Proof. We have to check that the operations $\hookleftarrow_a$, $\hookrightarrow_a$, $a \in [\gamma]$, of $\mathcal{F}_{\mathsf{Dup}_\gamma}$ satisfy Relations (6.1.3a), (6.1.3b), and (6.1.3c) of γ-multiplicial algebras. Let $\mathfrak{r}$, $\mathfrak{s}$, and $\mathfrak{t}$ be three γ-edge valued binary trees and $a, a' \in [\gamma]$.

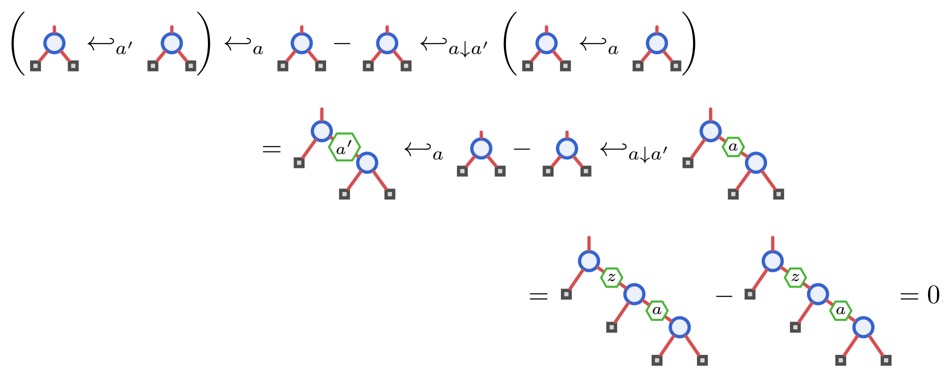
Denote by $\mathfrak{s}_1$ (resp. $\mathfrak{s}_2$) the left subtree (resp. right subtree) of $\mathfrak{s}$ and by x (resp. y) the label of the left (resp. right) edge incident to the root of $\mathfrak{s}$. We have



(6.1.19)

where $z := a' \downarrow x$ and $t := a \downarrow y$. This shows that (6.1.3a) is satisfied in $\mathcal{F}_{\mathsf{Dup}_\gamma}$.

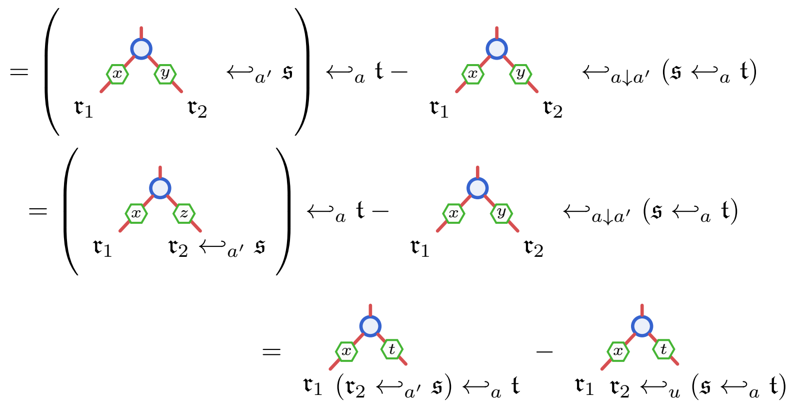
We now prove that Relations (6.1.3b) and (6.1.3c) hold by induction on the sum of the number of internal nodes of $\mathfrak{r}$, $\mathfrak{s}$, and $\mathfrak{t}$. Base case holds when all these trees have exactly one internal node, and since



(6.1.20)

where $z := a \downarrow a'$, (6.1.3b) holds on trees with one internal node. For the same arguments, we can show that (6.1.3c) holds on trees with exactly one internal node. Denote now by $\mathfrak{r}_1$ (resp. $\mathfrak{r}_2$) the left subtree (resp. right subtree) of $\mathfrak{r}$ and by x (resp. y) the label of the left (resp. right) edge incident to the root of $\mathfrak{r}$. We have

$$(\mathfrak{r} \hookleftarrow_{a'} \mathfrak{s}) \hookleftarrow_a \mathfrak{t} - \mathfrak{r} \hookleftarrow_{a \downarrow a'} (\mathfrak{s} \hookleftarrow_a \mathfrak{t})$$

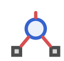


(6.1.21)

where $z := y \downarrow a'$, $t := z \downarrow a = y \downarrow a' \downarrow a$, and $u := a \downarrow a'$. Now, since by induction hypothesis Relation (6.1.3b) holds on $\mathfrak{r}_2$, $\mathfrak{s}$, and $\mathfrak{t}$, (6.1.21) is zero. Therefore, (6.1.3b) is satisfied in $\mathcal{F}_{\mathsf{Dup}_\gamma}$.

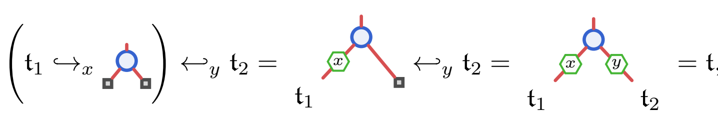
Finally, for the same arguments, we can show that (6.1.3c) is satisfied in $\mathcal{F}_{\mathsf{Dup}_\gamma}$, implying the statement of the lemma. □

Lemma 6.1.5. *For any integer $\gamma \geqslant 0$, the γ-multiplicial algebra $\mathcal{F}_{\mathsf{Dup}_\gamma}$ of γ-edge valued binary trees endowed with the operations $\leftarrow_a$, $\hookrightarrow_a$, $a \in [\gamma]$, is generated by*



. (6.1.22)

Proof. First, Lemma 6.1.4 shows that $\mathcal{F}_{\mathsf{Dup}_\gamma}$ is a γ-multiplicial algebra. Let $\mathcal{M}$ be the γ-multiplicial subalgebra of $\mathcal{F}_{\mathsf{Dup}_\gamma}$ generated by . Let us show that any γ-edge valued binary tree $\mathfrak{t}$ is in $\mathcal{M}$ by induction on the number n of its internal nodes. When $n = 1$, $\mathfrak{t} =$  and hence the property is satisfied. Otherwise, let $\mathfrak{t}_1$ (resp. $\mathfrak{t}_2$) be the left (resp. right) subtree of the root of $\mathfrak{t}$ and denote by x (resp. y) the label of the left (resp. right) edge incident to the root of $\mathfrak{t}$. Since $\mathfrak{t}_1$ and $\mathfrak{t}_2$ have less internal nodes than $\mathfrak{t}$, by induction hypothesis, $\mathfrak{t}_1$ and $\mathfrak{t}_2$ are in $\mathcal{M}$. Moreover, by definition of the operations $\leftarrow_a$, $\hookrightarrow_a$, $a \in [\gamma]$, of $\mathcal{F}_{\mathsf{Dup}_\gamma}$, one has



(6.1.23)

showing that $\mathfrak{t}$ also is in $\mathcal{M}$. Therefore, $\mathcal{M}$ is $\mathcal{F}_{\mathsf{Dup}_\gamma}$, showing that $\mathcal{F}_{\mathsf{Dup}_\gamma}$ is generated by . □

Theorem 6.1.6. *For any integer $\gamma \geqslant 0$, the vector space $\mathcal{F}_{\mathsf{Dup}_\gamma}$ of γ-valued binary trees endowed with the operations $\leftarrow_a$, $\hookrightarrow_a$, $a \in [\gamma]$, is the free γ-multiplicial algebra over one generator.*

Proof. By Lemmas 6.1.4 and 6.1.5, $\mathcal{F}_{\mathsf{Dup}_\gamma}$ is a γ-multiplicial algebra over one generator.

Moreover, since by Proposition 6.1.2, for any $n \geqslant 1$, the dimension of $\mathcal{F}_{\mathsf{Dup}_\gamma}(n)$ is the same as the dimension of $\mathsf{Dup}_\gamma(n)$, there cannot be relations in $\mathcal{F}_{\mathsf{Dup}_\gamma}(n)$ involving $\mathfrak{g}$ that are not γ-multiplicial relations (see (6.1.3a), (6.1.3b), and (6.1.3c)). Hence, $\mathcal{F}_{\mathsf{Dup}_\gamma}$ is free as a γ-multiplicial algebra over one generator. □

6.2. **Triassociative and tridendriform operads.** Our original idea of using the T construction (see Sections 1.1.3 and 2.1.1) to obtain a generalization of the diassociative operad admits an analogue in the context of the triassociative operad [LR04]. We describe in this section a one-parameter nonnegative integer generalization of the triassociative operad and of its Koszul dual, the tridendriform operad.

Since the proofs of the results contained in this section are very similar to the ones of Sections 2 and 4, we omit proofs here.

6.2.1. *Pluritriassociative operads.* For any integer $\gamma \geqslant 0$, we define Trias_γ as the suboperad of $\mathcal{M}_\gamma$ generated by

$$\{0a, 00, a0 : a \in [\gamma]\}. \tag{6.2.1}$$

By definition, Trias_γ is the vector space of words that can be obtained by partial compositions of words of (6.2.1). We have, for instance,

$$\mathsf{Trias}_2(1) = \mathrm{Vect}(\{0\}), \tag{6.2.2}$$

$$\mathsf{Trias}_2(2) = \mathrm{Vect}(\{00, 01, 02, 10, 20\}), \tag{6.2.3}$$

$$\begin{split}\mathsf{Trias}_2(3) = \mathrm{Vect}(\{&000, 001, 002, 010, 011, 012, 020, 021,\\ &022, 100, 101, 102, 110, 120, 200, 201, 202, 210, 220\}),\end{split} \tag{6.2.4}$$

It follows immediately from the definition of Trias_γ as a suboperad of $\mathsf{T}\mathcal{M}_\gamma$ that Trias_γ is a set-operad. Moreover, one can observe that Trias_γ is generated by the same generators as the ones of Dias_γ (see (2.1.1)), plus the word 00. Therefore, Dias_γ is a suboperad of Trias_γ. Besides, note that Trias_0 is the associative operad and that Trias_γ is a suboperad of $\mathsf{Trias}_{\gamma+1}$. We call Trias_γ the *γ-pluritriassociative operad*.

6.2.2. *Elements and dimensions.*

Proposition 6.2.1. *For any integer $\gamma \geqslant 0$, as a set-operad, the underlying set of Trias_γ is the set of the words on the alphabet $\{0\} \cup [\gamma]$ containing at least one occurrence of 0.*

We deduce from Proposition 6.2.1 that the Hilbert series of Trias_γ satisfies

$$\mathcal{H}_{\mathsf{Trias}_\gamma}(t) = \frac{t}{(1-\gamma t)(1-\gamma t - t)} \tag{6.2.5}$$

and that for all $n \geqslant 1$, $\dim \mathsf{Trias}_\gamma(n) = (\gamma+1)^n - \gamma^n$. For instance, the first dimensions of Trias_1, Trias_2, Trias_3, and Trias_4 are respectively

$$1, 3, 7, 15, 31, 63, 127, 255, 511, 1023, 2047, \tag{6.2.6}$$

$$1, 5, 19, 65, 211, 665, 2059, 6305, 19171, 58025, 175099, \tag{6.2.7}$$

$$1, 7, 37, 175, 781, 3367, 14197, 58975, 242461, 989527, 4017157, \tag{6.2.8}$$

$$1, 9, 61, 369, 2101, 11529, 61741, 325089, 1690981, 8717049, 44633821. \tag{6.2.9}$$

The first one is Sequence **A000225**, the second one is Sequence **A001047**, the third one is Sequence **A005061**, and the last one is Sequence **A005060** of [Slo].

6.2.3. *Presentation and Koszulity.* We follow the same strategy as the one used in Section 2.2 to establish a presentation by generators and relations of Trias_γ and prove that it is a Koszul operad. As announced above, we omit complete proofs here but we describe the analogue for Trias_γ of the maps word_γ and hook_γ defined in Section 2.2 for the operad Dias_γ.

For any integer $\gamma \geqslant 0$, let $\mathfrak{G}_{\mathsf{Trias}_\gamma} := \mathfrak{G}_{\mathsf{Trias}_\gamma}(2)$ be the graded set where

$$\mathfrak{G}_{\mathsf{Trias}_\gamma}(2) := \{\dashv_a, \perp, \vdash_a : a \in [\gamma]\}. \tag{6.2.10}$$

Let $\mathfrak{t}$ be a syntax tree of $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{Trias}_\gamma}\right)$ and x be a leaf of $\mathfrak{t}$. We say that an integer $a \in \{0\} \cup [\gamma]$ is *eligible* for x if $a = 0$ or there is an ancestor y of x labeled by $\dashv_a$ (resp. $\vdash_a$) and x is in the right (resp. left) subtree of y. The *image* of x is its greatest eligible integer. Moreover, let

$$\mathrm{wordt}_\gamma : \mathbf{Free}\left(\mathfrak{G}_{\mathsf{Trias}_\gamma}\right)(n) \to \mathsf{Trias}_\gamma(n), \qquad n \geqslant 1, \tag{6.2.11}$$

the map where $\mathrm{wordt}_\gamma(\mathfrak{t})$ is the word obtained by considering, from left to right, the images of the leaves of $\mathfrak{t}$ (see Figure 2). Observe that wordt_γ is an extension of word_γ (see (2.2.2)).

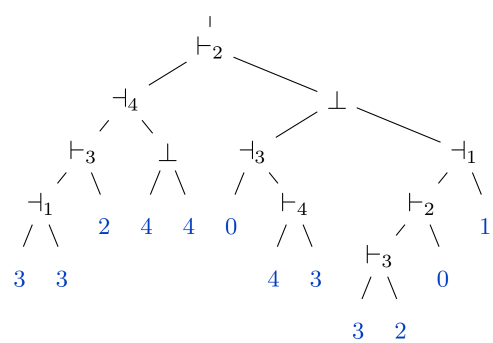


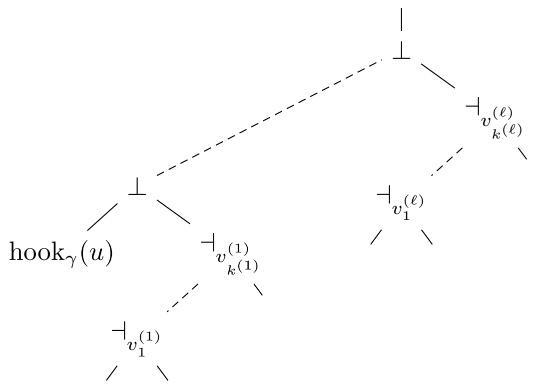
FIGURE 2. A syntax tree $\mathfrak{t}$ of $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{Trias}_\gamma}\right)$ where images of its leaves are shown. This tree satisfies $\mathrm{wordt}_\gamma(\mathfrak{t}) = 332440433201$.

Consider now the map

$$\mathrm{hookt}_\gamma : \mathsf{Trias}_\gamma(n) \to \mathbf{Free}\left(\mathfrak{G}_{\mathsf{Trias}_\gamma}\right)(n), \qquad n \geqslant 1, \tag{6.2.12}$$

defined for any word x of Trias_γ by

$\mathrm{hookt}_\gamma(x) :=$



, (6.2.13)

where x decomposes, by Proposition 6.2.1, uniquely in $x = u0v^{(1)} \dots 0v^{(\ell)}$ where u is a word of Dias_γ and for all $i \in [\ell]$, the $v^{(i)}$ are words on the alphabet $[\gamma]$. The length $|v^{(i)}|$ of any v_i is denoted by $k^{(i)}$. The dashed edges denote left comb trees wherein internal nodes are labeled as specified. Observe that hookt_γ is an extension of hook_γ (see (2.2.3)). We shall call any syntax tree of the form (6.2.13) an *extended hook syntax tree*.

Theorem 6.2.2. *For any integer $\gamma \geqslant 0$, the operad Trias_γ admits the following presentation. It is generated by $\mathfrak{G}_{\mathsf{Trias}_\gamma}$ and its space of relations $\mathfrak{R}_{\mathsf{Trias}_\gamma}$ is the space induced by the equivalence relation $\leftrightarrow_\gamma$ satisfying*

$$\perp \circ_1 \perp \;\leftrightarrow_\gamma\; \perp \circ_2 \perp, \tag{6.2.14a}$$

$$\dashv_a \circ_1 \perp \;\leftrightarrow_\gamma\; \perp \circ_2 \dashv_a, \qquad a \in [\gamma], \tag{6.2.14b}$$

$$\perp \circ_1 \vdash_a \;\leftrightarrow_\gamma\; \vdash_a \circ_2 \perp, \qquad a \in [\gamma], \tag{6.2.14c}$$

$$\perp \circ_1 \dashv_a \;\leftrightarrow_\gamma\; \perp \circ_2 \vdash_a, \qquad a \in [\gamma], \tag{6.2.14d}$$

$$\dashv_a \circ_1 \vdash_{a'} \;\leftrightarrow_\gamma\; \vdash_{a'} \circ_2 \dashv_a, \qquad a, a' \in [\gamma], \tag{6.2.14e}$$

$$\dashv_a \circ_1 \dashv_b \;\leftrightarrow_\gamma\; \dashv_a \circ_2 \vdash_b, \qquad a < b \in [\gamma], \tag{6.2.14f}$$

$$\vdash_a \circ_1 \dashv_b \;\leftrightarrow_\gamma\; \vdash_a \circ_2 \vdash_b, \qquad a < b \in [\gamma], \tag{6.2.14g}$$

$$\dashv_b \circ_1 \dashv_a \;\leftrightarrow_\gamma\; \dashv_a \circ_2 \dashv_b, \qquad a < b \in [\gamma], \tag{6.2.14h}$$

$$\vdash_a \circ_1 \vdash_b \;\leftrightarrow_\gamma\; \vdash_b \circ_2 \vdash_a, \qquad a < b \in [\gamma], \tag{6.2.14i}$$

$$\dashv_d \circ_1 \dashv_d \;\leftrightarrow_\gamma\; \dashv_d \circ_2 \perp \;\leftrightarrow_\gamma\; \dashv_d \circ_2 \dashv_c \;\leftrightarrow_\gamma\; \dashv_d \circ_2 \vdash_c, \qquad c \leqslant d \in [\gamma], \tag{6.2.14j}$$

$$\vdash_d \circ_1 \dashv_c \;\leftrightarrow_\gamma\; \vdash_d \circ_1 \vdash_c \;\leftrightarrow_\gamma\; \vdash_d \circ_1 \perp \;\leftrightarrow_\gamma\; \vdash_d \circ_2 \vdash_d, \qquad c \leqslant d \in [\gamma]. \tag{6.2.14k}$$

Observe that, by Theorem 6.2.2, Trias_1 and the triassociative operad [LR04] admit the same presentation. Then, for all integers $\gamma \geqslant 0$, the operads Trias_γ are generalizations of the triassociative operad.

Theorem 6.2.3. *For any integer $\gamma \geqslant 0$, Trias_γ is a Koszul operad. Moreover, the set of extended hook syntax trees of $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{Trias}_\gamma}\right)$ forms a Poincaré-Birkhoff-Witt basis of Trias_γ.*

6.2.4. *Polytridendriform operads.* Theorem 6.2.2, by exhibiting a presentation of Trias_γ, shows that this operad is binary and quadratic. It then admits a Koszul dual, denoted by TDendr_γ and called *γ-polytridendriform operad*.

Theorem 6.2.4. *For any integer $\gamma \geqslant 0$, the operad TDendr_γ admits the following presentation. It is generated by $\mathfrak{G}_{\mathsf{TDendr}_\gamma} := \mathfrak{G}_{\mathsf{TDendr}_\gamma}(2) := \{\leftharpoonup_a, \wedge, \rightharpoonup_a \colon a \in [\gamma]\}$ and its space of relations $\mathfrak{R}_{\mathsf{TDendr}_\gamma}$ is generated by*

$$\wedge \circ_1 \wedge - \wedge \circ_2 \wedge, \tag{6.2.15a}$$

$$\leftharpoonup_a \circ_1 \wedge - \wedge \circ_2 \leftharpoonup_a, \qquad a \in [\gamma], \tag{6.2.15b}$$

$$\wedge \circ_1 \rightharpoonup_a - \rightharpoonup_a \circ_2 \wedge, \qquad a \in [\gamma], \tag{6.2.15c}$$

$$\wedge \circ_1 \leftharpoonup_a - \wedge \circ_2 \rightharpoonup_a, \qquad a \in [\gamma], \tag{6.2.15d}$$

$$\leftharpoonup_a \circ_1 \rightharpoonup_{a'} - \rightharpoonup_{a'} \circ_2 \leftharpoonup_a, \qquad a, a' \in [\gamma], \tag{6.2.15e}$$

$$\leftharpoonup_a \circ_1 \leftharpoonup_b - \leftharpoonup_a \circ_2 \rightharpoonup_b, \qquad a < b \in [\gamma], \tag{6.2.15f}$$

$$\rightharpoonup_a \circ_1 \leftharpoonup_b - \rightharpoonup_a \circ_2 \rightharpoonup_b, \qquad a < b \in [\gamma], \tag{6.2.15g}$$

$$\leftharpoonup_b \circ_1 \leftharpoonup_a - \leftharpoonup_a \circ_2 \leftharpoonup_b, \qquad a < b \in [\gamma], \tag{6.2.15h}$$

$$\rightharpoonup_a \circ_1 \rightharpoonup_b - \rightharpoonup_b \circ_2 \rightharpoonup_a, \qquad a < b \in [\gamma], \tag{6.2.15i}$$

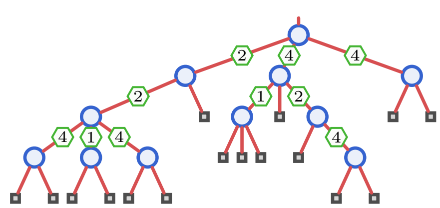
$$\leftharpoonup_d \circ_1 \leftharpoonup_d - \leftharpoonup_d \circ_2 \wedge - \left(\sum_{c \in [d]} \leftharpoonup_d \circ_2 \leftharpoonup_c + \leftharpoonup_d \circ_2 \rightharpoonup_c \right), \qquad d \in [\gamma], \tag{6.2.15j}$$

$$\left(\sum_{c \in [d]} \rightharpoonup_d \circ_1 \leftharpoonup_c + \rightharpoonup_d \circ_1 \rightharpoonup_c \right) + \rightharpoonup_d \circ_1 \wedge - \rightharpoonup_d \circ_2 \rightharpoonup_d, \qquad d \in [\gamma]. \tag{6.2.15k}$$

Proposition 6.2.5. *For any integer $\gamma \geqslant 0$, the Hilbert series $\mathcal{H}_{\mathsf{TDendr}_\gamma}(t)$ of the operad TDendr_γ satisfies*

$$\mathcal{H}_{\mathsf{TDendr}_\gamma}(t) = t + (2\gamma + 1)t\,\mathcal{H}_{\mathsf{TDendr}_\gamma}(t) + \gamma(\gamma + 1)t\,\mathcal{H}_{\mathsf{TDendr}_\gamma}(t)^2. \tag{6.2.16}$$

By examining the expression for $\mathcal{H}_{\mathsf{TDendr}_\gamma}(t)$ of the statement of Proposition 6.2.5, we observe that for any $n \geqslant 1$, $\mathsf{TDendr}(n)$ can be seen as the vector space $\mathcal{F}_{\mathsf{TDendr}_\gamma}(n)$ of Schröder trees with $n + 1$ leaves wherein its edges connecting two internal nodes are labeled on $[\gamma]$. We call these trees *γ-edge valued Schröder trees.* For instance,



(6.2.17)

is a 4-edge valued Schröder tree and a basis element of $\mathsf{TDendr}_4(16)$.

We deduce from Proposition 6.2.5 that

$$\mathcal{H}_{\mathsf{TDendr}_\gamma}(t) = \frac{1 - \sqrt{1 - (4\gamma + 2)t + t^2} - (2\gamma + 1)t}{2(\gamma + \gamma^2)t}. \tag{6.2.18}$$

Moreover, we obtain that for all $n \geqslant 1$,

$$\dim \mathsf{TDendr}_\gamma(n) = \sum_{k=0}^{n-1} (\gamma+1)^k \gamma^{n-k-1} \operatorname{nar}(n,k), \tag{6.2.19}$$

where $\operatorname{nar}(n,k)$ is defined in (5.1.20). For instance, the first dimensions of TDendr_1, TDendr_2, TDendr_3, and TDendr_4 are respectively

$$1, 3, 11, 45, 197, 903, 4279, 20793, 103049, 518859, 2646723, \tag{6.2.20}$$

$$1, 5, 31, 215, 1597, 12425, 99955, 824675, 6939769, 59334605, 513972967, \tag{6.2.21}$$

$$1, 7, 61, 595, 6217, 68047, 770149, 8939707, 105843409, 1273241431, 15517824973, \tag{6.2.22}$$

$$1, 9, 101, 1269, 17081, 240849, 3511741, 52515549, 801029681, 12414177369, 194922521301. \tag{6.2.23}$$

The first one is Sequence **A001003** of [Slo]. The others sequences are not listed in [Slo] at this time.

6.3. **Operads of the operadic butterfly.** The *operadic butterfly* [Lod01,Lod06] is a diagram gathering seven famous operads. We have seen in Section 5.2 that this diagram gathers the diassociative, associative, and dendriform operads. It involves also the *commutative operad* Com, the *Lie operad* Lie, the *Zinbiel operad* Zin [Lod95], and the *Leibniz operad* Leib [Lod93]. It is of the form

$$\begin{tikzcd}
 & \mathsf{Dendr} \arrow[dl] \arrow[rr, dotted, leftrightarrow, "!"] & & \mathsf{Dias} \arrow[dl] & \\
\mathsf{Zin} \arrow[rrrr, dotted, leftrightarrow, bend right, "!"] & & \mathsf{As} \arrow[ul] \arrow[dl] \arrow[loop above, dotted, leftrightarrow, "!"] & & \mathsf{Leib} \arrow[ul] \arrow[dl] \\
 & \mathsf{Com} \arrow[ul] \arrow[rr, dotted, leftrightarrow, "!"] & & \mathsf{Lie} \arrow[ul] &
\end{tikzcd} \tag{6.3.1}$$

and as it shows, some operads are Koszul dual of some others (in particular, $\mathsf{Com}^! = \mathsf{Lie}$ and $\mathsf{Zin}^! = \mathsf{Leib}$).

We have to emphasize the fact the operads Com, Lie, Zin, and Leib of the operadic butterfly are symmetric operads. The computation of the Koszul dual of a symmetric operad does not follows what we have presented in Section 1.2.5. We invite the reader to consult [GK94] or [LV12] for a complete description.

For simplicity, in what follows, we shall consider algebras over symmetric operads instead of symmetric operads.

6.3.1. *A generalization of the operadic butterfly.* A possible continuation to this work consists in constructing a diagram

$$\begin{tikzcd}
& \mathsf{Dendr}_\gamma \arrow[dl] \arrow[rrr, dotted, leftrightarrow, "!"] & & & \mathsf{Dias}_\gamma \arrow[dl] & \\
\mathsf{Zin}_\gamma \arrow[rrrrr, dotted, leftrightarrow, bend right, "!"] & & \mathsf{DAs}_\gamma \arrow[ul] \arrow[dl] \arrow[r, dotted, leftrightarrow, "!"] & \mathsf{As}_\gamma & & \mathsf{Leib}_\gamma \arrow[ul] \arrow[dl] \\
& \mathsf{Com}_\gamma \arrow[ul] \arrow[rrr, dotted, leftrightarrow, "!"] & & & \mathsf{Lie}_\gamma \arrow[ul] &
\end{tikzcd} \tag{6.3.2}$$

where DAs_γ is the γ-dual multiassociative operad defined in Section 5.1.3 and Com_γ, Lie_γ, Zin_γ, and Leib_γ, respectively are one-parameter nonnegative integer generalizations of the operads Com, Lie, Zin, and Leib. Let us now define these operads.

6.3.2. *Commutative and Lie operads.* The symmetric operad Com is the symmetric operad describing the category of algebras $\mathcal{C}$ with one binary operation $\diamond$, subjected for any elements x, y, and z of $\mathcal{C}$ to the two relations

$$x \diamond y = y \diamond x, \tag{6.3.3a}$$

$$(x \diamond y) \diamond z = x \diamond (y \diamond z). \tag{6.3.3b}$$

This operad has the property to be a commutative version of $\mathsf{As} = \mathsf{DAs}_1$.

We define the symmetric operad Com_γ by using the same idea of being a commutative version of DAs_γ. Therefore, Com_γ is the symmetric operad describing the category of algebras $\mathcal{C}$ with binary operations $\diamond_a$, $a \in [\gamma]$, subjected for any elements x, y, and z of $\mathcal{C}$ to the two sorts of relations

$$x \diamond_a y = y \diamond_a x, \qquad a \in [\gamma], \tag{6.3.4a}$$

$$(x \diamond_a y) \diamond_a z = x \diamond_a (y \diamond_a z), \qquad a \in [\gamma]. \tag{6.3.4b}$$

Moreover, we define the symmetric operad Lie_γ as the Koszul dual of Com_γ.

6.3.3. *Zinbiel and Leibniz operads.* The symmetric operad Zin is the symmetric operad describing the category of algebras $\mathcal{Z}$ with one generating binary operation $\sqcup\!\sqcup$, subjected for any elements x, y, and z of $\mathcal{Z}$ to the relation

$$(x \sqcup\!\sqcup y) \sqcup\!\sqcup z = x \sqcup\!\sqcup (y \sqcup\!\sqcup z) + x \sqcup\!\sqcup (z \sqcup\!\sqcup y). \tag{6.3.5}$$

This operad has the property to be a commutative version of $\mathsf{Dendr} = \mathsf{Dendr}_1$. Indeed, Relation (6.3.5) is obtained from Relations (1.3.7a), (1.3.7b), and (1.3.7c) of dendriform algebras with the condition that for any elements x and y, $x \prec y = y \succ x$, and by setting $x \sqcup\!\sqcup y := x \prec y$.

We define the symmetric operad Zin_γ by using the same idea of having the property to be a commutative version of Dendr_γ. Therefore, Zin_γ is the symmetric operad describing the category of algebras $\mathcal{Z}$ with binary operations $\sqcup\!\sqcup_a$, $a \in [\gamma]$, subjected for any elements x, y, and z of $\mathcal{Z}$ to the relation

$$(x \sqcup\!\sqcup_{a'} y) \sqcup\!\sqcup_a z = x \sqcup\!\sqcup_{a \downarrow a'} (y \sqcup\!\sqcup_a z) + x \sqcup\!\sqcup_{a \downarrow a'} (z \sqcup\!\sqcup_{a'} y), \qquad a, a' \in [\gamma]. \tag{6.3.6}$$

Relation (6.3.6) is obtained from Relations (4.1.17a), (4.1.17b), and (4.1.17c) of γ-polydendriform algebras with the condition that for any elements x and y and $a \in [\gamma]$, $x \prec_a y = y \succ_a x$, and by setting $x \sqcup\!\sqcup_a y := x \prec_a y$. Moreover, we define the symmetric operad Leib_γ as the Koszul dual of Zin_γ.

Proposition 6.3.1. *For any integer $\gamma \geqslant 0$ and any Zin_γ-algebra $\mathcal{Z}$, the binary operations $\diamond_a$, $a \in [\gamma]$, defined for all elements x and y of $\mathcal{Z}$ by*

$$x \diamond_a y := x \sqcup\!\sqcup_a y + y \sqcup\!\sqcup_a x, \qquad a \in [\gamma], \tag{6.3.7}$$

endow $\mathcal{Z}$ with a Com_γ-algebra structure.

Proof. Since for all $a \in [\gamma]$ and all elements x and y of $\mathcal{Z}$, by (6.3.6), we have

$$x \diamond_a y - y \diamond_a x = x \sqcup\!\sqcup_a y + y \sqcup\!\sqcup_a x - y \sqcup\!\sqcup_a x - x \sqcup\!\sqcup_a y = 0, \tag{6.3.8}$$

the operations $\diamond_a$ satisfy Relation (6.3.4a) of Com_γ-algebras. Moreover, since for all $a \in [\gamma]$ and all elements x, y, and z of $\mathcal{Z}$, by (6.3.6), we have

$$\begin{aligned}
(x \diamond_a y) \diamond_a z - x \diamond_a (y \diamond_a z)& \\
= (x \sqcup\!\sqcup_a y + y \sqcup\!\sqcup_a x) \sqcup\!\sqcup_a z + z \sqcup\!\sqcup_a (x \sqcup\!\sqcup_a y + y \sqcup\!\sqcup_a x)& \\
\quad - x \sqcup\!\sqcup_a (y \sqcup\!\sqcup_a z + z \sqcup\!\sqcup_a y) - (y \sqcup\!\sqcup_a z + z \sqcup\!\sqcup_a y) \sqcup\!\sqcup_a x& \\
= (x \sqcup\!\sqcup_a y) \sqcup\!\sqcup_a z + (y \sqcup\!\sqcup_a x) \sqcup\!\sqcup_a z + z \sqcup\!\sqcup_a (x \sqcup\!\sqcup_a y) + z \sqcup\!\sqcup_a (y \sqcup\!\sqcup_a x)& \\
\quad - x \sqcup\!\sqcup_a (y \sqcup\!\sqcup_a z) - x \sqcup\!\sqcup_a (z \sqcup\!\sqcup_a y) - (y \sqcup\!\sqcup_a z) \sqcup\!\sqcup_a x - (z \sqcup\!\sqcup_a y) \sqcup\!\sqcup_a x& \\
= (y \sqcup\!\sqcup_a x) \sqcup\!\sqcup_a z - (y \sqcup\!\sqcup_a z) \sqcup\!\sqcup_a x& \\
= y \sqcup\!\sqcup_a (x \sqcup\!\sqcup_a z) + y \sqcup\!\sqcup_a (z \sqcup\!\sqcup_a x) - y \sqcup\!\sqcup_a (z \sqcup\!\sqcup_a x) - y \sqcup\!\sqcup_a (x \sqcup\!\sqcup_a z)& \\
= 0,&
\end{aligned} \tag{6.3.9}$$

the operations $\diamond_a$ satisfy Relation (6.3.4b) of Com_γ-algebras. Hence, $\mathcal{Z}$ is a Com_γ-algebra. $\square$

Proposition 6.3.2. *For any integer $\gamma \geqslant 0$, and any Zin_γ-algebra $\mathcal{Z}$, the binary operations $\prec_a$, $\succ_a$, $a \in [\gamma]$ defined for all elements x and y of $\mathcal{Z}$ by*

$$x \prec_a y := x \sqcup\!\sqcup_a y, \qquad a \in [\gamma], \tag{6.3.10}$$

and

$$x \succ_a y := y \sqcup\!\sqcup_a x, \qquad a \in [\gamma], \tag{6.3.11}$$

endow $\mathcal{Z}$ with a γ-polydendriform algebra structure.

Proof. Since, for all $a, a' \in [\gamma]$ and all elements x, y, and z of $\mathcal{Z}$, by (6.3.6), we have

$$\begin{aligned}
(x \succ_{a'} y) \prec_a z - x \succ_{a'} (y \prec_a z)& \\
= (y \sqcup\!\sqcup_{a'} x) \sqcup\!\sqcup_a z - (y \sqcup\!\sqcup_a z) \sqcup\!\sqcup_{a'} x& \\
= y \sqcup\!\sqcup_{a \downarrow a'} (x \sqcup\!\sqcup_a z) + y \sqcup\!\sqcup_{a \downarrow a'} (z \sqcup\!\sqcup_{a'} x) - y \sqcup\!\sqcup_{a \downarrow a'} (z \sqcup\!\sqcup_{a'} x) - y \sqcup\!\sqcup_{a \downarrow a'} (x \sqcup\!\sqcup_a z)& \\
= 0,&
\end{aligned} \tag{6.3.12}$$

the operations $\prec_a$ and $\succ_a$ satisfy Relation (4.1.17a) of γ-polydendriform algebras. Moreover, since for all $a, a' \in [\gamma]$ and all elements x, y, and z of $\mathcal{Z}$, by (6.3.6), we have

$$\begin{aligned}
(x \prec_{a'} y) \prec_a z - x \prec_{a \downarrow a'} (y \prec_a z) - x \prec_{a \downarrow a'} (y \succ_{a'} z) \\
&= (x \sqcup\!\sqcup_{a'} y) \sqcup\!\sqcup_a z - x \sqcup\!\sqcup_{a \downarrow a'} (y \sqcup\!\sqcup_a z) - x \sqcup\!\sqcup_{a \downarrow a'} (z \sqcup\!\sqcup_{a'} y) \\
&= x \sqcup\!\sqcup_{a \downarrow a'} (y \sqcup\!\sqcup_a z) + x \sqcup\!\sqcup_{a \downarrow a'} (z \sqcup\!\sqcup_{a'} y) - x \sqcup\!\sqcup_{a \downarrow a'} (y \sqcup\!\sqcup_a z) - x \sqcup\!\sqcup_{a \downarrow a'} (z \sqcup\!\sqcup_{a'} y) \\
&= 0,
\end{aligned} \tag{6.3.13}$$

the operations $\prec_a$ and $\succ_a$ satisfy Relation (4.1.17b) of γ-polydendriform algebras. Finally, since for all $a, a' \in [\gamma]$ and all elements x, y, and z of $\mathcal{Z}$, we have

$$\begin{aligned}
(x \prec_{a'} y) \succ_{a \downarrow a'} z + (x \succ_a y) \succ_{a \downarrow a'} z - x \succ_a (y \succ_{a'} z) \\
&= z \sqcup\!\sqcup_{a \downarrow a'} (x \sqcup\!\sqcup_{a'} y) + z \sqcup\!\sqcup_{a \downarrow a'} (y \sqcup\!\sqcup_a x) - (z \sqcup\!\sqcup_{a'} y) \sqcup\!\sqcup_a x \\
&= z \sqcup\!\sqcup_{a \downarrow a'} (x \sqcup\!\sqcup_{a'} y) + z \sqcup\!\sqcup_{a \downarrow a'} (y \sqcup\!\sqcup_a x) - z \sqcup\!\sqcup_{a \downarrow a'} (y \sqcup\!\sqcup_a x) - z \sqcup\!\sqcup_{a \downarrow a'} (x \sqcup\!\sqcup_{a'} y) \\
&= 0,
\end{aligned} \tag{6.3.14}$$

the operations $\prec_a$ and $\succ_a$ satisfy Relation (4.1.17c) of γ-polydendriform algebras. Hence $\mathcal{Z}$ is a γ-polydendriform algebra. □

The constructions stated by Propositions 6.3.1 and 6.3.2 producing from a Zin_γ-algebra respectively a Com_γ-algebra and a γ-polydendriform algebra are functors from the category of Zin_γ-algebras respectively to the category of Com_γ-algebras and the category of γ-polydendriform algebras. These functors respectively translate into symmetric operad morphisms from Com_γ to Zin_γ and from Dendr_γ to Zin_γ. These morphisms are generalizations of known morphisms between Com, Dendr, and Zin of (6.3.1) (see [Lod01, Lod06, Zin12]).

A complete study of the operads Com_γ, Lie_γ, Zin_γ, and Leib_γ, and suitable definitions for all the morphisms intervening in (6.3.2) is worth to interest for future works.

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