

# EULER CHARACTERISTICS OF CENTRALIZER SUBCATEGORIES

JESPER M. MØLLER

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ABSTRACT. Let  $G$  be a finite group and  $p$  a prime number. By a theorem of K.S. Brown, the reduced Euler characteristic of  $\mathcal{S}_G^{p+*}$ , the poset of nonidentity  $p$ -subgroups of  $G$ , is a multiple of  $|G|_p$ , the  $p$ -part of the group order. We prove here an equivariant version of Brown’s theorem for the case where  $G$  is equipped with an action of a finite group  $A$ . The equivariant version asserts that the reduced Euler characteristic of  $\mathcal{C}_{\mathcal{S}_G^{p+*}}(A)$ , the poset of nonidentity  $A$ -normalized  $p$ -subgroups of  $G$ , is a multiple of  $|C_G(A)|_p$ , the  $p$ -part of the centralizer order.

We also examine the higher equivariant Euler characteristics  $\chi_r(\mathcal{S}_G^{p+*}, G)$ ,  $r \geq 0$ , introduced by Atiyah and Segal, for the conjugation action of  $G$  on  $\mathcal{S}_G^{p+*}$ . We observe that the  $r$ th equivariant Euler characteristics can be computed in the class function space from a virtual rational representation of  $G$ , the  $r$ th Euler class function. The second equivariant Euler characteristic is especially interesting in connection with the Knörr–Robinson conjecture.

## 1. INTRODUCTION

Let  $A$  be a finite group and  $\mathcal{C}$  a finite  $A$ -category. For any subset  $X$  of the action group  $A$ , the  $X$ -centralized subcategory,  $\mathcal{C}_{\mathcal{C}}(X)$ , is the subcategory of  $\mathcal{C}$  consisting of all  $\mathcal{C}$ -morphisms  $c \xrightarrow{\varphi} d$  such that  $c^x = c$ ,  $d^x = d$  and  $\varphi^x = \varphi$  for all endofunctors  $x$  in  $X$ . Atiyah and Segal [4] define the  $r$ th,  $r \geq 0$ , equivariant Euler characteristic to be the normalized sum

$$\chi_r(\mathcal{C}, A) = \frac{1}{|A|} \sum_{X \in \mathcal{C}_r(A)} \chi(\mathcal{C}_{\mathcal{C}}(X))$$

of the Euler characteristics of the  $X$ -centralized subcategories as  $X$  ranges over the set  $\mathcal{C}_r(A)$  of commuting  $r$ -tuples in  $A$ . We are implicitly assuming for instance that  $\mathcal{C}$  is an EI-category so that  $X$ -centralized subcategories have Euler characteristics in the sense of Leinster [23].

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`/home/moller/projects/euler/orbit/alperin03.tex`

Alternatively, the equivariant Euler characteristics may be computed in the virtual rational representation ring  $\mathbf{Q} \otimes R_{\mathbf{Q}}(A)$  of  $A$ : The  $r$ th equivariant Euler characteristic is the character inner product

$$\chi_r(\mathcal{C}, A) = \langle \alpha_r(\mathcal{C}, A), |C_A| \rangle_A$$

of the  $r$ th Euler class function of  $(\mathcal{C}, A)$ ,

$$\alpha_r(\mathcal{C}, A): A \rightarrow \mathbf{Q}, x \rightarrow \chi_{r-1}(C_{\mathcal{C}}(x), C_A(x)), \quad r \geq 1,$$

and the permutation character  $|C_A|$  for the action of  $A$  on itself by conjugation (Definition 2.10).

An especially interesting case of this general set-up arises when  $G$  a finite group,  $p$  is a prime, and  $\mathcal{C} = \mathcal{S}_G^{p+*}$  is the Brown poset of nontrivial  $p$ -subgroups of  $G$ , and  $A = G$  acts by conjugation. The  $G$ -poset  $\mathcal{S}_G^{p+*}$  has equivariant Euler characteristics  $\chi_r(\mathcal{S}_G^{p+*}, G)$  defined for every  $r \geq 0$ . We shall mainly focus on the cases  $r = 0, 1, 2$  as the significance of the higher equivariant Euler characteristics for  $r \geq 3$  remains unclear (Remark 2.17). It seems that these three first equivariant Euler characteristics carry some interesting information. For  $r = 0$  it is immediate from the definition that the equivariant Euler characteristic of the Brown poset

$$\chi_0(\mathcal{S}_G^{p+*}, G) = \chi(\mathcal{S}_G^{p+*})/|G|$$

is simply the usual Euler characteristic divided by the order of  $G$ . The strong form of a conjecture by Quillen asserts that

$$\chi_0(\mathcal{S}_G^{p+*}, G) = 0 \iff O_p(G) \neq 1$$

For  $r = 1$ , Webb [38] proved that

$$\chi_1(\mathcal{S}_G^{p+*}, G) = 1$$

when  $\mathcal{S}_G^{p+*}$  is nonempty. For  $r = 2$ , Knörr and Robinson [20, 37] conjecture that

$$\chi_2(\mathcal{S}_G^{p+*}, G) = k(G) - z_p(G)$$

The Knörr–Robinson Conjecture is known to be equivalent to (the non-blockwise form of) Alperin’s Weight Conjecture. (See Notation 1.4 for explanations of the symbols  $k(G)$  and  $z_p(G)$ .)

A little more generally, suppose  $G$  is an  $A$ -group. The poset  $\mathcal{S}_G^p$  of  $p$ -subgroups is then an  $A$ -poset. The objects of the centralized subposet  $C_{\mathcal{S}_G^p}(A)$  are the  $A$ -normalized  $p$ -subgroups of  $G$ . For any such  $A$ -normalized  $p$ -subgroup  $H$  of  $G$ , the sub-quotient  $N_G(H)/H$  is an  $A$ -group with associated centralized Brown poset,  $C_{\mathcal{S}_{N_G(H)/H}^{p+*}}(A)$ . The main result of this paper explores the local Euler characteristics given by the function  $H \rightarrow -\tilde{\chi}(C_{\mathcal{S}_{N_G(H)/H}^{p+*}}(A))$  defined for  $H \in C_{\mathcal{S}_G^p}(A)$ . It is easy to see that this function vanishes off the poset  $C_{\mathcal{S}_G^{p+\text{rad}}}(A)$  of  $A$ -normalized  $G$ -radical  $p$ -subgroups of  $G$  (Lemma 3.10).

**Theorem 1.1.** *Let  $G$  be a finite  $A$ -group and  $p$  a prime.*

(1) *If  $|C_G(A)_p|$  denotes the number of  $p$ -singular elements of  $G$  centralized by  $A$  then*

$$\sum_{H \in C_{\mathcal{S}_G^{p+\text{rad}}}(A)} -\tilde{\chi}(C_{\mathcal{S}_{N_G(H)/H}^{p+*}}(A)) |C_H(A)| = |C_G(A)_p|$$

(2) *For any  $A$ -normalized  $G$ -radical  $p$ -subgroup  $H$  of  $G$*

$$\sum_{H \leq K \in C_{\mathcal{S}_G^{p+\text{rad}}}(A)} -\tilde{\chi}(C_{\mathcal{S}_{N_G(K)/K}^{p+*}}(A)) = 1$$

*where the sum runs over the set of  $A$ -normalized  $G$ -radical  $p$ -subgroups  $K$  containing  $H$ .*

(3) *If  $|C_G(A)|_p$  denotes the  $p$ -part of the order of the centralizer subgroup then*

$$|C_G(A)|_p | \tilde{\chi}(C_{\mathcal{S}_G^{p+*}}(A))$$

The proof of Theorem 1.1, to be found in §3.2 and §4.2, is obtained by analyzing the Euler characteristics of the orbit category  $\mathcal{O}_G^p$  of  $p$ -subgroups of  $G$  and (a category related to) its  $A$ -centralized subcategory  $C_{\mathcal{O}_G^p}(A)$  (Propositions 4.6, 4.6, 4.25, 4.27). Theorem 1.1.(1) expresses the number of  $A$ -centralized  $p$ -singular elements in  $G$  as an affine combination of the number of  $A$ -centralized elements in the  $G$ -radical  $p$ -subgroups of  $G$ . The divisibility statement of Theorem 1.1.(3) follows from this affine relation by induction over group order.

By the theorem of K.S. Brown [8], reproved by Quillen [27] and Webb [38] among others, the  $p$ -part of the group order divides the reduced Euler characteristic of the Brown poset. Theorem 1.1.(3) is an equivariant generalization of Brown’s theorem.

In the special case where  $A$  is trivial, item (3) of Theorem 1.1 reduces to Brown's theorem while item (1),

$$(1.2) \quad \sum_{H \in \mathcal{S}_G^{p+\text{rad}}} -\tilde{\chi}(\mathcal{S}_{N_G(H)/H}^{p+*})|H| = |G|_p$$

expresses the number of  $p$ -singular elements in  $G$  as an affine combination of the orders of the  $G$ -radical  $p$ -subgroups. Frobenius proved in 1907 (or even earlier) that  $|G|_p$  divides  $|G_p|$  [11, 17]. Brown's theorem,  $|G|_p$  divides  $\tilde{\chi}(\mathcal{S}_G^{p+*})$ , came much later in 1975. The affine relation (1.2) provides a link between these two theorems showing that they actually are equivalent (Theorem 4.10). Thus the theorems of Sylow, Frobenius, and Brown are equivalent.

We note that already in [16, Theorem 6.3] Hawkes, Isaacs, and Özaydin prove a more general version of Equation (1.2) and also observe the connection in one direction between the theorems of Brown and Frobenius.

In Section 5 we compute the second equivariant Euler characteristic  $\chi_2(\mathcal{S}_G^{p+*}, G)$  for the Mathieu group  $G = M_{11}$  and check the validity of the Knörr–Robinson conjecture for this group (Example 5.6). We also compute (Table 2) the uniquely determined Artin decomposition of the second equivariant Euler class function

$$\tilde{a}_2(\mathcal{S}_G^{p+*}, G) = \sum_{[C] \in [\mathcal{S}_G^{p'+\text{cyc}}]} \tilde{a}_2(\mathcal{S}_G^{p+*}, G)(C) \frac{1_C^G}{|N_G(C) : C|}$$

into  $\mathbf{Z}$ -linear combinations of the class functions  $|N_G(C) : C|^{-1}1_C^G$  where  $C$  runs through the set of  $p$ -regular cyclic subgroups of  $G$  (Corollary 3.15).

**1.1. Notation.** The following definitions and notation will be used throughout this paper:

**Definition 1.3.** Let  $G$  be a finite group and  $p$  a prime number.

- (1) An element  $g \in G$  of order  $|g|$  is  $p$ -regular if  $p \nmid |g|$ ,  $p$ -irregular if  $p \mid |g|$ , and  $p$ -singular if  $|g|$  is a power of  $p$  [10, Definition 40.2, §82.1]
- (2) A finite group  $A$  is  $p$ -regular if  $p \nmid |A|$
- (3) An irreducible  $\mathbf{C}$ -character on  $G$  has  $p$ -defect 0 if  $p$  does not divide  $|G|/\chi(1)$  [18, p 134]
- (4)  $O_p(G)$  is the largest normal  $p$ -subgroup of  $G$
- (5) A  $p$ -subgroup  $H$  of  $G$  is  $G$ -radical if  $H = O_p(N_G(H))$  [3]

**Notation 1.4.** Let  $G$  be a finite group and  $p$  a prime number.

- $G_p = \bigcup \text{Syl}_p(G)$  is the set of  $p$ -singular elements of  $G$ , the union of the Sylow  $p$ -subgroups
- $|G|_p$  is the  $p$ -part and  $|G|_{p'}$  the  $p'$ -part of the group order  $|G| = |G|_p |G|_{p'}$
- $[G]$  is the set of conjugacy classes of elements of  $G$
- $k(G) = |[G]| = |\text{Irr}(G)|$  is the number of irreducible  $\mathbf{C}$ -characters of  $G$
- $z_p(G) = |\{\chi \in \text{Irr}(G) \mid |G|_p \mid \chi(1)\}|$  is the number of irreducible  $\mathbf{C}$ -characters of  $p$ -defect 0
- $k_{p'}(G) = |\{[g] \in [G] \mid p \nmid |g|\}|$  is the number of  $p$ -regular conjugacy classes in  $G$
- $\mathcal{S}_G$  is the poset of subgroups of  $G$  and  $[\mathcal{S}_G]$  is the set of conjugacy classes of subgroups (Section 3)
- $N_G(H, K) = \{g \in G \mid H^g \leq K\}$  is the transporter set
- $\mathcal{O}_G$  is the orbit category of  $G$  with morphism sets  $\mathcal{O}_G(H, K) = N_G(H, K)/K$  (Section 4)
- $\mathcal{F}_G$  is the fusion category of  $G$  with morphism sets  $\mathcal{F}_G(H, K) = C_G(H) \backslash N_G(H, K)$
- $R_{\mathbf{C}}(G)$  is the ring of virtual  $\mathbf{C}$ -characters and  $R_{\mathbf{Q}}(G)$  the subring of virtual  $\mathbf{Q}$ -characters of  $G$  [29, 12.1]

If  $\mathcal{C}$  is a finite category of subgroups of  $G$  then  $\mathcal{C}^*, \mathcal{C}^p, \mathcal{C}^{p'}, \mathcal{C}^{\text{abe}}, \mathcal{C}^{\text{eab}}, \mathcal{C}^{\text{rad}}$  is the full subcategory of  $\mathcal{C}$  generated by all nonidentity subgroups,  $p$ -subgroups,  $p$ -regular subgroups, abelian subgroups, elementary abelian subgroups,  $G$ -radical  $p$ -subgroups, respectively. We shall also use various combinations of superscripts;  $\mathcal{C}^{p+*+\text{rad}}$ , for instance, denotes the full subcategory of  $\mathcal{C}$  generated by all nonidentity  $G$ -radical  $p$ -subgroups of  $G$ .  $\mathcal{C}(H, K)$  denotes the set of  $\mathcal{C}$ -morphisms from  $H$  to  $K$  and  $\mathcal{C}(H) = \mathcal{C}(H, H)$  for the  $\mathcal{C}$ -endomorphism monoid of  $H$ .  $H/\mathcal{C}$  is the coslice of  $\mathcal{C}$  under  $H$  and  $H//\mathcal{C}$  the strict coslice of nonisomorphisms under  $H$  [12, Definition 3.2].

## 2. $A$ -CATEGORIES AND $A$ -POSETS

Let  $A$  be a finite group,  $\mathcal{C}$  a small  $A$ -category, and  $X$  a subset of  $A$ .

**Definition 2.1.** The  $X$ -centralized subcategory of  $\mathcal{C}$  is the subcategory,  $C_{\mathcal{C}}(X)$ , of  $\mathcal{C}$  with objects  $\text{Ob}(C_{\mathcal{C}}(X)) = \text{C}_{\text{Ob}(\mathcal{C})}(X)$  and with morphism sets  $C_{\mathcal{C}}(X)(c, d) = C_{\mathcal{C}(c, d)}(X)$ ,  $c, d \in \text{Ob}(C_{\mathcal{C}}(X))$ .

In mathematical symbols, the  $X$ -centralized subcategory  $C_{\mathcal{C}}(X)$  is the subcategory of  $\mathcal{C}$  consisting of all morphisms  $c \xrightarrow{\varphi} d$  in  $\mathcal{C}$  such that  $(c \rightarrow d) = (c^x \xrightarrow{\varphi^x} d^x)$  for all  $x \in X$ . In words, the  $X$ -centralized subcategory  $C_{\mathcal{C}}(X)$  is the category of  $X$ -stable  $\mathcal{C}$ -objects with  $X$ -stable  $\mathcal{C}$ -morphisms between them.  $C_{\mathcal{C}}(X)$  is an  $N_A(X)$ -category.

If  $X_1$  and  $X_2$  are conjugate subsets of  $A$ , then their centralizers in  $\mathcal{C}$  are isomorphic categories. Indeed,  $C_{\mathcal{C}}(X^g) = C_{\mathcal{C}}(X)^g$  for any element  $g \in A$  and any subset  $X \subseteq A$ .

For later reference we define  $A$ -adjunctions between  $A$ -categories and note that there are induced  $N_A(X)$ -adjunctions between centralizer categories.

**Definition 2.2.** A  $A$ -adjunction between the  $A$ -categories  $\mathcal{C}$  and  $\mathcal{D}$  is a quadruple  $(L, R, \eta, \varepsilon)$  consisting of  $A$ -functors,  $R$  and  $L$ , and  $A$ -natural transformations,  $\eta$  (the unit) and  $\varepsilon$  (the counit),

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{R} & \mathcal{D} & & 1_{\mathcal{C}} & \xrightarrow{\eta} & LR \\ & & & & & & \\ \mathcal{C} & \xleftarrow{L} & \mathcal{D} & & 1_{\mathcal{D}} & \xleftarrow{\varepsilon} & RL \end{array}$$

such that the  $A$ -natural transformations

$$L \xrightarrow{\eta L} LRL \xrightarrow{L\varepsilon} L \qquad R \xrightarrow{R\eta} RLR \xrightarrow{\varepsilon R} R$$

are the identity transformations.

**Proposition 2.3.** Suppose that  $(L, R, \eta, \varepsilon)$  is a  $A$ -adjunction between the  $A$ -categories  $\mathcal{C}$  and  $\mathcal{D}$ .

(1) The induced maps  $BC \xrightleftharpoons[BL]{BR} BD$  are  $A$ -homotopy equivalences between  $A$ -spaces.

(2) There is an induced  $N_A(X)$ -adjunction between the  $N_A(X)$ -categories  $C_{\mathcal{C}}(X)$  and  $C_{\mathcal{D}}(X)$  and the induced

maps  $BC_{\mathcal{C}}(X) \xrightleftharpoons[BL]{BR} BC_{\mathcal{D}}(X)$  are  $N_A(X)$ -homotopy equivalences between  $N_A(X)$ -spaces.

**Example 2.4.** Let  $C_2 = \langle \tau \rangle$  be the group of order two. The product,  $G \times G$ , of any group  $G$  with itself is a  $C_2$ -group with action given by  $(g_1, g_2)\tau = (g_2, g_1)$ , and  $\mathcal{S}_{G \times G}^p$  is then a  $C_2$ -poset. The product poset  $\mathcal{S}_G^p \times \mathcal{S}_G^p$  is a  $C_2$ -poset with action  $(H_1, H_2)\tau = (H_2, H_1)$ . The  $C_2$ -adjunction

$$\mathcal{S}_G^p \times \mathcal{S}_G^p \xrightleftharpoons[L]{R} \mathcal{S}_{G \times G}^p \qquad (H_1, H_2)R = H_1 \times H_2, \quad HL = (H\pi_1, H\pi_2)$$

restricts to an adjunction between  $C_{\mathcal{S}_G^p \times \mathcal{S}_G^p}(\tau)$  and  $C_{\mathcal{S}_{G \times G}^p}(\tau)$  and restricts further to an adjunction between the posets  $C_{\mathcal{S}_G^{p+*} \times \mathcal{S}_G^{p+*}}(\tau)$  and  $C_{\mathcal{S}_{G \times G}^{p+*}}(\tau)$ . But  $\mathcal{S}_G^p$  is isomorphic to  $C_{\mathcal{S}_G^p \times \mathcal{S}_G^p}(\tau)$  and  $\mathcal{S}_G^{p+*}$  to  $C_{\mathcal{S}_{G \times G}^{p+*}}(\tau)$ , so we conclude that  $(\mathcal{S}_G^p, C_{\mathcal{S}_{G \times G}^p}(\tau))$  and  $(\mathcal{S}_G^{p+*}, C_{\mathcal{S}_{G \times G}^{p+*}}(\tau))$  are pairs of adjoint posets.

**2.1. Euler characteristics of  $A$ -posets.** In this subsection we make a few easy observations about weightings, coweightings [23, Definition 1.10], and Euler characteristics of finite  $A$ -posets.

**Lemma 2.5.** Let  $\mathcal{S}$  be a finite  $A$ -poset. Then the Möbius function  $\mu: \mathcal{S} \times \mathcal{S} \rightarrow \mathbf{Z}$ , the weighting  $k^\bullet: \mathcal{S} \rightarrow \mathbf{Z}$  and the coweighting  $k_\bullet: \mathcal{S} \rightarrow \mathbf{Z}$  are  $A$ -invariant.

*Proof.* Note that  $\zeta(r, s) = \zeta(r^a, s^a)$  for all  $r, s \in \mathcal{S}$  and  $a \in A$ . The function  $(s, t) \rightarrow \mu(s^a, t^a)$  satisfies the defining relations

$$\sum_s \zeta(r, s) \mu(s^a, t^a) = \sum_s \zeta(r^a, s^a) \mu(s^a, t^a) = \delta_{r^a, t^a} = \delta_{r, t}$$

for  $(s, t) \rightarrow \mu(s, t)$ . It is now clear that the weighting  $k^s = \sum_t \mu(s, t)$  and the coweighting  $k_t = \sum_s \mu(s, t)$  are constant on the orbits for the  $A$ -action on  $\mathcal{S}$ .  $\square$

For any two  $A$ -orbits  $x, y \in \mathcal{S}/A$  and any two elements  $s, t \in \mathcal{S}$ , let  $\mathcal{S}(s, y) = |\{t \in y \mid s \leq t\}|$  be the number of successors of  $s$  in  $y$  and  $\mathcal{S}(x, t) = |\{s \in x \mid s \leq t\}|$  the number of predecessors of  $t$  in  $x$ .

**Definition 2.6.** A weighting on  $\mathcal{S}/A$  is a function  $k^\bullet: \mathcal{S}/A \rightarrow \mathbf{Z}$  such that  $\sum_{y \in \mathcal{S}/A} \mathcal{S}(s, y) k^y = 1$  for all  $s \in \mathcal{S}$ . A coweighting on  $\mathcal{S}/A$  is a function  $k_\bullet: \mathcal{S}/A \rightarrow \mathbf{Z}$  such that  $\sum_{x \in \mathcal{S}/A} k_x \mathcal{S}(x, t) = 1$  for all  $t \in \mathcal{S}$ .

**Lemma 2.7.** Let  $k^\bullet: \mathcal{S}/A \rightarrow \mathbf{Z}$  be a weighting and  $k_\bullet: \mathcal{S}/A \rightarrow \mathbf{Z}$  a coweighting on  $\mathcal{S}/A$ . Then

- (1)  $\mathcal{S} \rightarrow \mathcal{S}/A \xrightarrow{k^\bullet} \mathbf{Z}$  is the weighting and  $\mathcal{S} \rightarrow \mathcal{S}/A \xrightarrow{k_\bullet} \mathbf{Z}$  the coweighting for  $\mathcal{S}$
- (2) The Euler characteristic of  $\mathcal{S}$  is  $\sum_{x \in \mathcal{S}/A} k_x |x| = \chi(\mathcal{S}) = \sum_{y \in \mathcal{S}/A} |y| k^y$

*Proof.* Let  $s$  be a fixed element of  $\mathcal{S}$  representing the orbit  $x = sA$  of  $\mathcal{S}/A$ . Then

$$\sum_{t \in \mathcal{S}} |\mathcal{S}(s, t)| k^{tA} = \sum_{y \in \mathcal{S}/A} \sum_{t \in y} |\mathcal{S}(s, t)| k^{tA} = \sum_{y \in \mathcal{S}/A} \mathcal{S}(s, y) k^y = 1$$

Thus  $t \rightarrow k^{tA}$  is the weighting for  $\mathcal{S}$ . The Euler characteristic is  $\chi(\mathcal{S}) = \sum_{t \in \mathcal{S}} k^{tA} = \sum_{y \in \mathcal{S}/A} |y| k^y$ .  $\square$

**Corollary 2.8.**  $\mathcal{S}/A$  has a unique weighting and a unique coweighting induced from the unique weighting and coweighting on  $\mathcal{S}$ .

*Proof.* Let  $k^\bullet : \mathcal{S} \rightarrow \mathbf{Z}$  be the unique weighting on  $\mathcal{S}$ . Since  $k^\bullet$  is  $A$ -invariant (Lemma 2.5) it induces a function  $k^y : \mathcal{S}/A \rightarrow \mathbf{Z}$  on the  $A$ -orbits. Since

$$1 = \sum_{t \in \mathcal{S}} \mathcal{S}(s, t) k^t = \sum_{y \in \mathcal{S}/A} \sum_{t \in y} \mathcal{S}(s, t) k^y = \sum_{x \in \mathcal{S}/A} \mathcal{S}(s, y) k^y$$

the function  $k^y$  is a weighting for  $\mathcal{S}/A$ . Conversely, if  $k^y : \mathcal{S}/A \rightarrow \mathbf{Z}$  is a weighting for  $\mathcal{S}/A$  then Lemma 2.7 shows that it is induced by the weighting on the poset  $\mathcal{S}$ .  $\square$

**2.2. Equivariant Euler characteristics of  $A$ -categories.** As suggested by Atiyah and Segal [4] there is a hierarchy of equivariant Euler characteristics  $\chi_r(\mathcal{C}, A)$ , and reduced equivariant Euler characteristics  $\tilde{\chi}_r(\mathcal{C}, A)$ ,  $r \geq 0$ , given by

$$\chi_r(\mathcal{C}, A) = \frac{1}{|A|} \sum_{x \in C_r(A)} \chi(C_{\mathcal{C}}(x)), \quad \tilde{\chi}_r(\mathcal{C}, A) = \frac{1}{|A|} \sum_{x \in C_r(A)} \tilde{\chi}(C_{\mathcal{C}}(x))$$

Consult Remark 2.20 for the definition of the set  $C_r(A)$  of commuting  $r$ -tuples in  $A$  and for the function  $\varphi_r(B)$  used in Proposition 2.9 below. If  $A$  acts trivially on  $\mathcal{C}$  then  $\chi_r(\mathcal{C}, A)$  equals the usual Euler characteristic  $\chi(\mathcal{C})$  multiplied by  $|C_r(A)|/|A|$  for all  $r \geq 0$ . The relation between the equivariant Euler characteristic and the *reduced* equivariant Euler characteristic is that

$$\chi_r(\mathcal{C}, A) = \tilde{\chi}_r(\mathcal{C}, A) + \frac{|C_r(A)|}{|A|}, \quad r \geq 0$$

We are here implicitly assuming that the centralizer subcategories do have Euler characteristics in the sense of Leinster [23]. We note that the equivariant Euler characteristics are invariant under equivariant adjunctions: If the  $A$ -categories  $\mathcal{C}$  and  $\mathcal{D}$  are EI-categories and if there is an  $A$ -adjunction between them, then their equivariant Euler characteristics coincide. This follows from Proposition 2.3 and the invariance of category Euler characteristic under adjunction [23, Proposition 2.4].

**Proposition 2.9.** *The  $r$ th,  $r \geq 1$ , equivariant Euler characteristic of the  $A$ -category  $\mathcal{C}$  is*

$$\begin{aligned} \chi_r(\mathcal{C}, A) &= \frac{1}{|A|} \sum_{x \in A} \chi_{r-1}(C_{\mathcal{C}}(x), C_A(x)) |C_A(x)| = \sum_{[x] \in [A]} \chi_{r-1}(C_{\mathcal{C}}(x), C_A(x)) = \sum_{[x] \in [C_{r-1}(A)]} \chi_1(C_{\mathcal{C}}(x), C_A(x)) = \\ &= \frac{1}{|A|} \sum_{B \in \mathcal{S}_A^{\text{abe}}} \chi(C_{\mathcal{C}}(B)) \varphi_r(B) = \sum_{[B] \in [\mathcal{S}_A^{\text{abe}}]} \chi(C_{\mathcal{C}}(B)) \frac{\varphi_r(B)}{|N_A(B)|} \end{aligned}$$

*Similar formulas hold in the reduced case.*

*Proof.* This is immediate from the definition. For instance,

$$\begin{aligned} \chi_r(\mathcal{C}, A) &= \frac{1}{|A|} \sum_{(x_1, \dots, x_{r-1}, x_r) \in C_r(C_A(x))} \chi(C_{\mathcal{C}}(x_1, \dots, x_{r-1}, x_r)) \\ &= \frac{1}{|A|} \sum_{x_r \in A} \sum_{(x_1, \dots, x_{r-1}) \in C_{r-1}(C_A(x_r))} \chi(C_{C_{\mathcal{C}}(x_r)}(x_1, \dots, x_{r-1})) = \frac{1}{|A|} \sum_{x \in A} \chi_{r-1}(C_{\mathcal{C}}(x), C_A(x)) |C_A(x)| \\ &= \sum_{[x] \in [A]} \chi_{r-1}(C_{\mathcal{C}}(x), C_A(x)) \end{aligned}$$

Declare two  $r$ -tuples of  $C_r(A)$  to be equivalent if they generate the same abelian subgroup of  $A$ . The number of  $r$ -tuples in the equivalence class of the abelian subgroup  $B \leq A$  is  $\varphi_r(B)$ . Therefore we may write

$$\chi_r(\mathcal{C}, A) = \frac{1}{|A|} \sum_{B \in \mathcal{S}_A^{\text{abe}}} \chi(C_{\mathcal{C}}(B)) \varphi_r(B)$$

where the sum ranges over the abelian subgroups  $B$  of  $A$ .  $\square$

For any subgroup  $C$  of  $A$

$$|C|\chi_r(\mathcal{C}, C) = \sum_B \chi(C_C(B))\varphi_r(B)|\mathcal{S}_A(B, C)|, \quad \sum_B |B|\chi_r(\mathcal{C}, B)\mu(B, C) = \chi(C_C(C))\varphi_r(C)$$

by Möbius inversion in the poset  $\mathcal{S}_A$  of subgroups of  $A$ .

**Definition 2.10.** *The  $r$ th,  $r \geq 1$ , equivariant Euler class function  $\alpha_r(\mathcal{C}, A)$  of the  $A$ -category  $\mathcal{C}$  is the rational class function on  $A$  that takes  $x \in A$  to*

$$\alpha_r(\mathcal{C}, A)(x) = \chi_{r-1}(C_C(x), C_A(x))$$

*The reduced Euler class function  $\tilde{\alpha}_r(\mathcal{C}, A)$  is defined similarly using reduced Euler characteristics.*

For any  $x \in A$  and any  $r \geq 1$  the value at  $x$  of the  $r$ th equivariant Euler class function is

$$\alpha_r(\mathcal{C}, A)(x) = \chi_{r-1}(C_C(x), C_A(x)) = \sum_{[y] \in [C_A(x)]} \chi_{r-2}(C_C(x, y), C_A(x, y))$$

and for  $r = 2$ , in particular,

$$(2.11) \quad \alpha_2(\mathcal{C}, A)(x) = \sum_{[y] \in [C_A(x)]} \frac{\chi(C_C(x, y))}{|C_A(x, y)|}$$

because  $\chi_0(\mathcal{C}, A) = \chi(\mathcal{C})/|A|$ .

Let  $\langle \varphi, \psi \rangle_A = |A|^{-1} \sum_{x \in A} \varphi(x) \overline{\psi(x)} = \sum_{[x] \in [A]} |C_A(x)|^{-1} \varphi(x) \overline{\psi(x)}$  be the character inner product (symmetric bilinear form) in the complex class function space on  $A$  [18, Definition 2.16]. The 1-character  $1_A$  is characterized by the property that its inner product with any class function  $\varphi$  is the average  $\langle \varphi, 1_A \rangle_A = |A|^{-1} \sum_{x \in A} \varphi(x) = \sum_{[x] \in [A]} \varphi(x)|A : C_A(x)|$  of the values of  $\varphi$ . Let  $|C_A| : x \rightarrow |C_A(x)| = |C_A(x^{-1})|$  be the conjugation character: The permutation character for the conjugation action of  $A$  on itself. The conjugation character is characterized by the property that the inner product  $\langle \varphi, |C_A| \rangle_A = \sum_{[x] \in [A]} x^\varphi$  is the sum of the values of  $\varphi$  on the conjugacy classes of  $A$ . In particular, the inner product of the  $r$ th equivariant Euler class function and  $|C_A|$  is the  $r$ th equivariant Euler characteristic:

$$(2.12) \quad \langle \alpha_r(\mathcal{C}, A), |C_A| \rangle_A = \sum_{[x] \in [A]} \alpha_r(\mathcal{C}, A)(x) = \sum_{[x] \in [A]} \chi_{r-1}(C_C(x), C_A(x)) \stackrel{\text{Prop 2.9}}{=} \chi_r(\mathcal{C}, A)$$

Similarly, the inner product of *reduced* class function  $\tilde{\alpha}_r(\mathcal{C}, A)$  and  $|C_A|$  coincides with the *reduced* equivariant Euler characteristic  $\tilde{\chi}_r(\mathcal{C}, A)$ .

**2.3. Equivariant Euler characteristics of  $A$ -posets.** We now specialize from a finite  $A$ -category,  $\mathcal{C}$ , to a finite  $A$ -poset,  $\mathcal{S}$ . The order  $\Delta$ -set of  $\mathcal{S}$  is the  $A$ - $\Delta$ -set,  $\Delta\mathcal{S}$ , of all simplices in  $\mathcal{S}$  (§2.4). The equivariant Euler characteristics of a finite  $A$ -poset  $\mathcal{S}$  for  $r = 0, 1, 2$  are

$$\chi_0(\mathcal{S}, A) = \chi(\mathcal{S})/|A|, \quad \chi_1(\mathcal{S}, A) = \chi(\Delta\mathcal{S}/A), \quad \chi_2(\mathcal{S}, A) = \dim_{\mathbf{Q}}(K_A^0(B\mathcal{S}) \otimes \mathbf{Q}) - \dim_{\mathbf{Q}}(K_A^1(B\mathcal{S}) \otimes \mathbf{Q})$$

according to Proposition 2.13 below for  $r = 1$  and [4] [37, Corollary 2.2] for  $r = 2$ .  $K_A^*(B\mathcal{S})$  is the  $A$ -equivariant complex  $K$ -theory of the finite  $A$ -simplicial complex  $B\mathcal{S}$ . The corresponding *reduced* Euler characteristics are  $\tilde{\chi}_0(\mathcal{S}, A) = \chi_0(\mathcal{S}, A) - |A|^{-1}$ ,  $\tilde{\chi}_1(\mathcal{S}, A) = \chi_1(\mathcal{S}, A) - 1$ , and  $\tilde{\chi}_2(\mathcal{S}, A) = \chi_2(\mathcal{S}, A) - k(A)$  (Remark 2.20).

The next proposition shows that the first equivariant Euler characteristic of the  $A$ -poset  $\mathcal{S}$  is the usual Euler characteristic of the orbit  $\Delta$ -set of the  $A$ - $\Delta$ -set  $\Delta\mathcal{S}$ . This implies, as expressed in the following corollary, that the  $r$ th equivariant Euler characteristic for  $r \geq 2$  can be expressed by means of the usual Euler characteristics of quotients of  $\Delta$ -sets of centralizer subposets of  $\mathcal{S}$ .

**Proposition 2.13.** *For any subgroup  $H$  of  $A$*

$$\chi_1(C_{\mathcal{S}}(H), C_A(H)) = \chi(\Delta C_{\mathcal{S}}(H)/C_A(H))$$

*A similar formula holds in the reduced case.*

*Proof.* For any simplicial complex  $B$  let  $B_d$  denote set the of  $d$ -dimensional simplices in  $B$ . Then

$$\begin{aligned} \chi(\Delta C_{\mathcal{S}}(H)/C_A(H)) &= \sum_{d \geq 0} (-1)^d |\Delta C_{\mathcal{S}}(H)_d / C_A(H)| = \sum_{d \geq 0} (-1)^d \frac{1}{|C_A(H)|} \sum_{x \in C_A(H)} |C_{\Delta C_{\mathcal{S}}(H)_d}(x)| \\ &= \frac{1}{|C_A(H)|} \sum_{x \in C_A(H)} \sum_{d \geq 0} (-1)^d |\Delta C_{\mathcal{S}}(\langle H, x \rangle)_d| = \frac{1}{|C_A(H)|} \sum_{x \in C_A(H)} \chi(C_{\mathcal{S}}(\langle H, x \rangle)) = \chi_1(C_{\mathcal{S}}(H), C_A(H)) \end{aligned}$$

where the Cauchy–Frobenius Lemma 2.16 justifies the second equality above.  $\square$



**Corollary 2.14.** *The equivariant Euler characteristics of the finite  $A$ -poset  $\mathcal{S}$  are  $\chi_1(\mathcal{S}, A) = \chi(\Delta\mathcal{S}/A)$  and*

$$\chi_r(\mathcal{S}, A) = \sum_{[x] \in [C_{r-1}(A)]} \chi(\Delta C_{\mathcal{S}}(x)/C_A(x)) = \frac{1}{|A|} \sum_{B \in \mathcal{S}_A^{\text{abe}}} \chi(\Delta C_{\mathcal{S}}(B)/C_A(B)) |C_A(B)| \varphi_{r-1}(B)$$

for  $r \geq 2$ . The first sum is taken over the set  $[C_{r-1}(A)]$  of conjugacy classes of commuting  $(r-1)$ -tuples in  $A$ .

*Proof.* Proposition 2.13 with  $H$  trivial shows that  $\chi_1(\mathcal{S}, A) = \chi(\Delta\mathcal{S}/A)$ . Using Proposition 2.9 we get

$$\chi_2(\mathcal{S}, A) = \frac{1}{|A|} \sum_{x \in A} \chi_1(C_{\mathcal{S}}(x), C_A(x)) |C_A(x)| = \frac{1}{|A|} \sum_{x \in A} \chi(\Delta C_{\mathcal{S}}(x)/C_A(x)) |C_A(x)|$$

so that, by induction,

$$\chi_r(\mathcal{S}, A) = \frac{1}{|A|} \sum_{x \in C_{r-1}(A)} \chi(\Delta C_{\mathcal{S}}(x)/C_A(x)) |C_A(x)| = \sum_{[x] \in [C_{r-1}(A)]} \chi(\Delta C_{\mathcal{S}}(x)/C_A(x))$$

where the last sum is taken over the set  $[C_{r-1}(A)] = C_{r-1}(A)/A$  of conjugacy classes of commuting  $r$ -tuples in  $A$  and  $r \geq 2$ .  $\square$

**Corollary 2.15.** *The equivariant Euler characteristics  $\chi_r(\mathcal{S}, A)$  are integers for  $r \geq 1$  and the equivariant Euler class functions  $\alpha_r(\mathcal{S}, A)$  take integer values for  $r \geq 2$ . The same holds in the reduced case.*

*Proof.* The first Euler characteristic  $\chi_1(\mathcal{S}, A) = \chi(\Delta\mathcal{S}/A)$  is an integer by Proposition 2.13 since it is the Euler characteristic of a finite  $\Delta$ -set. By induction, using one of the formulas of Proposition 2.9, we get that  $\chi_r(\mathcal{S}, A) \in \mathbf{Z}$  for all  $r \geq 1$ . By applying this result to  $(C_{\mathcal{S}}(x), C_A(x))$  we see that the equivariant Euler class function  $\alpha_r(\mathcal{S}, A)$  takes integer values on each  $x \in A$  when  $r \geq 2$ .  $\square$

**Lemma 2.16.** [34, Lemma 7.24.5] *Let  $S \times A \rightarrow A$  be an action of the group  $A$  on the set  $S$ . Then*

$$\sum_{a \in A} |C_S(a)| = |S/A| |A| = \sum_{s \in S} |C_A(s)|$$

and the number of orbits is  $|S/A| = \sum_{s \in S} |C_A(s)| / |A|$ .

We now specialize even further. When  $G$  is a finite group, recall that the Brown  $G$ -poset  $\mathcal{S}_G^{p+*}$  is the  $G$ -poset of nonidentity  $p$ -subgroups of  $G$  with the conjugation action.

**Remark 2.17** (Equivariant Euler characteristics of the Brown poset with conjugation action). Here are three statements about the equivariant Euler characteristics for  $r = 0, 1, 2$  of the poset  $\mathcal{S}_G^{p+*}$  with conjugation action of  $A = G$ :

- (1)  $\forall G: -\tilde{\chi}_0(\mathcal{S}_G^{p+*}, G) = 0 \iff O_p(G) \neq 1$
- (2)  $\forall G: -\tilde{\chi}_1(\mathcal{S}_G^{p+*}, G) = \begin{cases} 0 & p \mid |G| \\ 1 & p \nmid |G| \end{cases}$
- (3)  $\forall G: -\tilde{\chi}_2(\mathcal{S}_G^{p+*}, G) = z_p(G)$

Statement (1) is the strong form of Quillen's conjecture about the vanishing of the reduced Euler characteristic of the Brown poset [27, Conjecture 2.9] [19, p. 2667], (2) was proved by Webb [38, Proposition 8.2.(i)] (and sharpened by Symonds [35] and later Słomińska [31]), and (3) is equivalent (Section 5) to (the non-blockwise form of) Alperin's weight conjecture [37, Theorem 3.1].

The significance of the equivariant Euler characteristics  $\chi_r(\mathcal{S}_G^{p+*}, G)$  for  $r \geq 3$  remains unclear. In the case of the alternating and symmetric groups the 3rd reduced Euler characteristics  $-\tilde{\chi}_3(\mathcal{S}_G^{p+*}, G)$  at  $p = 2$  are

$n$	4	5	6	7	8	9	10
$-\tilde{\chi}_3(\mathcal{S}_{A_n}^{2+*}, A_n)$	0	8	24	-2	32	20	42
$-\tilde{\chi}_3(\mathcal{S}_{\Sigma_n}^{2+*}, \Sigma_n)$	0	2	12	-2	10	11	16

For  $G = \text{GL}_3(\mathbf{F}_2)$ , the Euler characteristics  $-\tilde{\chi}_3(\mathcal{S}_G^{p+*}, G)$  are 12, 24, 7 at  $p = 2, 3, 7$  and for  $G = M_{11}$  they are 29, 19, 62, 35 at  $p = 2, 3, 5, 11$ .

Similarly, the significance of the more general equivariant Euler characteristics  $\chi_r(\mathcal{S}_G^{p+*}, A)$ ,  $r \geq 1$ , for an  $A$ -group  $G$  remains unclear. (See Remark 3.19.)

Using Proposition 2.9, Webb's computation of  $\chi_1(\mathcal{S}_G^{p^{+*}}, G)$  (Remark 2.17.(2)) may be reformulated as either of three equivalent identities

$$(2.18) \quad \sum_{x \in G} \chi(C_{\mathcal{S}_G^{p^{+*}}}(x)) = |G|, \quad \sum_{[x] \in [G]} \chi(C_{\mathcal{S}_G^{p^{+*}}}(x))|G : C_G(x)| = |G|, \quad \sum_{[x] \in [G]} \tilde{\chi}(C_{\mathcal{S}_G^{p^{+*}}}(x))|G : C_G(x)| = 0$$

valid whenever  $p \mid |G|$  (and reminiscent of the class equation).

**Example 2.19.** The following two tables list the centralizer indices  $|G : C_G(x)|$  and the centralizer Euler characteristics  $\chi(C_{\mathcal{S}_G^{p^{+*}}}(x))$  as  $x$  runs through the conjugacy classes of  $G$  and  $p$  runs through the prime divisors of  $|G|$  for  $G = \mathrm{GL}_3(\mathbf{F}_2)$  of order 168 and the Mathieu group  $G = M_{11}$  of order 7920, respectively.

$ x $	1	2	3	4	7	7	·
$ G : C_G(x) $	1	21	56	42	24	24	
$\tilde{\chi}(C_{\mathcal{S}_G^{2^{+*}}}(x))$	-8	0	1	0	-1	-1	0
$\tilde{\chi}(C_{\mathcal{S}_G^{3^{+*}}}(x))$	27	3	0	-1	-1	-1	0
$\tilde{\chi}(C_{\mathcal{S}_G^{7^{+*}}}(x))$	7	-1	1	-1	0	0	0

  

$ x $	1	2	3	4	5	6	8	8	11	11	·
$ G : C_G(x) $	1	165	440	990	1584	1320	990	990	720	720	
$\tilde{\chi}(C_{\mathcal{S}_G^{2^{+*}}}(x))$	-496	0	8	0	-1	0	0	0	-1	-1	0
$\tilde{\chi}(C_{\mathcal{S}_G^{3^{+*}}}(x))$	54	6	0	2	-1	0	0	0	-1	-1	0
$\tilde{\chi}(C_{\mathcal{S}_G^{5^{+*}}}(x))$	395	11	-1	3	0	-1	-1	-1	-1	-1	0
$\tilde{\chi}(C_{\mathcal{S}_G^{11^{+*}}}(x))$	143	-1	-1	-1	3	-1	-1	-1	0	0	0

Note that all rows satisfy Webb's relation (2.18). It is no coincidence that  $\tilde{\chi}(C_{\mathcal{S}_G^{p^{+*}}}(x)) = 0$  when  $p$  divides the order of  $x$  (Lemma 3.14).

**Remark 2.20** (Commuting  $r$ -tuples). For  $r = 0, 1, 2, \dots$  let

$$C_r(A) = \{(x_1, x_2, \dots, x_r) \mid \forall i, j: [x_i, x_j] = 1\}$$

denote the set of commuting  $r$ -tuples in  $A$  (with the understanding that  $C_0(A) = \{e\}$  is the set consisting of the unit element of  $A$ ).

Then  $|C_0(A)| = 1$ ,  $|C_1(A)| = |A|$ ,  $|C_2(A)| = |A|k(A)$ , and

$$|C_r(A)| = |A| \sum_{x \in C_{r-2}(A)} k(C_A(x)), \quad r \geq 2$$

This follows from the recursive relation

$$|C_r(A)| = \sum_{x \in A} |C_{r-1}(C_A(x))| = \sum_{[x] \in A} |A : C_A(x)| |C_{r-1}(C_A(x))|, \quad r \geq 1,$$

obtained by noting that there are  $|C_{r-1}(C_A(x))|$  commuting  $r$ -tuples with first coordinate  $x$  for any  $x \in A$ .

Let also

$$\varphi_r(A) = |\{(x_1, \dots, x_r) \in C_r(A) \mid \langle x_1, \dots, x_r \rangle = A\}|$$

be the number of generating commuting  $r$ -tuples. Of course,  $\varphi_r(A) = 0$  for all  $r$  unless  $A$  is abelian. When  $A$  is abelian

$$\varphi_r(A) = \prod_p \varphi_r(O_p(A))$$

since any abelian group  $A = \prod_p O_p(A)$  is the product of its Sylow  $p$ -subgroups. Thus  $\varphi_r$  is completely determined by its values on cyclic  $p$ -groups.

Similar to one of the expressions of Proposition 2.9 we get

$$|C_r(A)| = \sum_{B \in \mathcal{S}_A^{\mathrm{abe}}} \varphi_r(B)$$

by using the partition of  $C_r(A)$  into equivalence classes.



**2.4. Orbit posets and orbit  $\Delta$ -sets.** This subsection contains remarks about orbit posets of  $A$ -posets.

**Definition 2.21.** *Let  $\mathcal{S}$  be an  $A$ -poset.*

- $A$  simplex in  $\mathcal{S}$  is a totally ordered subset of  $\mathcal{S}$
- $B(\mathcal{S})$  is the set of simplices in  $\mathcal{S}$  viewed as an  $A$ -simplicial complex
- $\Delta(\mathcal{S})$  is the set of simplices in  $\mathcal{S}$  viewed as an  $A$ - $\Delta$ -set<sup>1</sup> [15]
- $\text{sd}(\mathcal{S})$  is the set of simplices in  $\mathcal{S}$  viewed as an  $A$ -poset
- $\mathcal{S}$  is graded if  $\mathcal{S}$  admits an  $A$ -invariant rank function  $\rho: \mathcal{S} \rightarrow \{0, 1, 2, \dots\}$  [33, §3.1]
- $\mathcal{S}$  is  $A$ -regular if  $\mathcal{S}$  if  $s \leq t$ ,  $s^a \leq t \implies s = s^a$  holds for all  $s, t \in \mathcal{S}$ ,  $a \in A$  [7, p 116]

Any  $A$ -poset  $\mathcal{S}$  admits an orbit poset  $\mathcal{S}/A$  and the  $A$ - $\Delta$ -set  $\Delta(\mathcal{S})$  admits an orbit  $\Delta$ -set  $\Delta(\mathcal{S})/A$ .

**Proposition 2.22.** *Let  $\mathcal{S}$  be any  $A$ -poset.*

- (1)  $\text{sd}(\mathcal{S})$  is an  $A$ -poset and  $\chi(\Delta(\text{sd}(\mathcal{S}))/A) = \chi(\Delta(\mathcal{S})/A)$
- (2) If  $\mathcal{S}$  is  $A$ -regular then  $\text{sd}(\mathcal{S})/A = \text{sd}(\mathcal{S}/A)$  and  $\Delta(\mathcal{S})/A = \Delta(\mathcal{S}/A)$
- (3) If  $\mathcal{S}$  is graded then  $\text{sd}(\mathcal{S})$  is  $A$ -regular, and  $\chi_1(\mathcal{S}, A) = \chi(\text{sd}(\mathcal{S})/A)$
- (4) The first subdivision  $\text{sd}(\mathcal{S})$  is graded, the second subdivision  $\text{sd}^2(\mathcal{S})$  is  $A$ -regular, and  $\chi_1(\mathcal{S}, A) = \chi(\text{sd}^2(\mathcal{S})/A)$

*Proof.* (1) Since the standard homeomorphism  $|\Delta(\text{sd}(\mathcal{S}))| \rightarrow |\Delta(\mathcal{S})|$  between the topological realizations of the subdivided and the original poset [32, §3, Theorem 9] is  $A$ -equivariant by construction, it induces a homeomorphism between the  $A$ -orbit spaces.

(2) The poset morphism  $\mathcal{S} \rightarrow \mathcal{S}/A$  induces a poset morphism  $\text{sd}(\mathcal{S}) \rightarrow \text{sd}(\mathcal{S}/A)$  and a  $\Delta$ -set morphism  $\Delta(\mathcal{S}) \rightarrow \Delta(\mathcal{S}/A)$ . When  $\mathcal{S}$  is  $A$ -regular these induced morphisms induce isomorphisms  $\text{sd}(\mathcal{S})/A \rightarrow \text{sd}(\mathcal{S}/A)$  and  $\Delta(\mathcal{S})/A \rightarrow \Delta(\mathcal{S}/A)$  because  $\{s^a \mid s \in \sigma\} = \{t^a \mid t \in \tau\} \implies \sigma^A = \tau^A$  for all simplices  $\sigma$  and  $\tau$  in  $\mathcal{S}$ .

(3) We must prove that  $\sigma \subseteq \tau$ ,  $\sigma^a \subseteq \tau \implies \sigma = \sigma^a$  for all simplices  $\sigma, \tau$  in  $\mathcal{S}$  and  $a \in A$ . The elements of the totally ordered subset  $\tau$  are determined by their values under the rank function. Subsets of  $\tau$  are therefore determined by their images under the rank function. Since the rank function is  $A$ -invariant,  $\sigma$  and  $\sigma^a$  have the same images, so they must be equal. The first equivariant Euler characteristic is

$$\chi_1(\mathcal{S}, A) \stackrel{\text{Prop 2.13}}{=} \chi(\Delta(\mathcal{S})/A) \stackrel{(1)}{=} \chi(\Delta(\text{sd}(\mathcal{S}))/A) \stackrel{(2)}{=} \chi(\Delta(\text{sd}(\mathcal{S}/A))) = \chi(\text{sd}(\mathcal{S})/A)$$

because  $\text{sd}(\mathcal{S})$  is  $A$ -regular.

(4) The dimension function is an  $A$ -invariant rank function on  $\text{sd}(\mathcal{S})$ . The first equivariant Euler characteristic is

$$\chi_1(\mathcal{S}, A) \stackrel{\text{Prop 2.13}}{=} \chi(\Delta(\mathcal{S})/A) \stackrel{(1)}{=} \chi(\Delta(\text{sd}(\mathcal{S}))/A) \stackrel{(1)}{=} \chi(\Delta(\text{sd}^2(\mathcal{S}))/A) \stackrel{(2)}{=} \chi(\Delta(\text{sd}^2(\mathcal{S})/A)) = \chi(\text{sd}^2(\mathcal{S}/A))$$

since  $\text{sd}^2(\mathcal{S})$  is  $A$ -regular by (3). □

For any finite  $A$ -group  $G$  and any subset  $X$  of  $A$ , the  $X$ -centralized Brown poset  $C_{\mathcal{S}_G^{p+*}}(X)$  is a graded  $N_A(X)$ -poset with rank function  $\rho$  given by  $|H| = p^{\rho(H)}$  for any  $X$ -normalized  $p$ -subgroup  $H$  of  $G$ . Thus the first equivariant Euler characteristic of Proposition 2.13

$$\chi_1(C_{\mathcal{S}_G^{p+*}}(X), C_A(X)) = \chi(\text{sd}(C_{\mathcal{S}_G^{p+*}}(X))/C_A(X))$$

can be expressed entirely within the category of posets.

**Example 2.23.** In case  $G = \text{GL}_3(\mathbf{F}_2)$  and  $p = 2$  the 2-dimensional contractible orbit  $\Delta$ -set  $\Delta(\mathcal{S}_G^{p+*})/G$

$$\{1, \dots, 5\} \xleftarrow{\begin{pmatrix} 1 & 2 & 3 & 5 & 5 & 5 & 5 & 5 & 5 \\ 4 & 4 & 4 & 4 & 1 & 3 & 2 & 4 & 4 \end{pmatrix}} \{1, \dots, 9\} \xleftarrow{\begin{pmatrix} 5 & 7 & 6 & 5 & 7 \\ 4 & 4 & 4 & 8 & 9 \\ 1 & 2 & 3 & 1 & 2 \end{pmatrix}} \{1, \dots, 5\}$$

is not a simplicial complex as three 1-simplices connect the 0-simplices 4 and 5. In the realization, 2-simplices 1, 4 and 2,5 form two cones joined along the 1-simplex 4 to which also one edge of 2-simplex 3 is attached.

<sup>1</sup>A  $\Delta$ -set is a functor  $\Delta_{<} \rightarrow \mathbf{SET}$  where  $\Delta_{<}$  is the category of standard simplices  $n_+ = \{0, \dots, n\}$ ,  $n = 0, 1, 2, \dots$ , with strictly increasing maps.

3. THE POSET OF  $p$ -SUBGROUPS

Write  $\mathcal{S}_G$  for the poset of all subgroups of  $G$  ordered by inclusion,  $\mathcal{S}_G^p$  for the poset of all  $p$ -subgroups, and  $\mathcal{S}_G^{p+*}$  for the Brown poset of all nonidentity  $p$ -subgroups of  $G$ .

For any  $p$ -subgroup  $K$  of  $G$ , the poset  $\mathcal{S}_K$  of subgroups of  $K$  has both an initial, 1, and a terminal element,  $K$ . We write  $\mathcal{S}_K^{(1,K)}$  for the poset of nonidentity and proper subgroups of  $K$ . The Möbius function on  $\mathcal{S}_K$  is the restriction of the Möbius function  $\mu$  for  $\mathcal{S}_G$  [19, Remark 2.6]. It is well-known that  $\tilde{\chi}(\mathcal{S}_K^{(1,K)}) = \mu(1, K)$  when  $K$  is nontrivial. In fact, if  $\mathcal{S}$  is any finite poset with Möbius function  $\mu$ ,  $a \in \mathcal{S}$ ,  $b \in \mathcal{S}$  with  $a < b$ , and  $(a, b)$  and  $[a, b]$  are the open and closed intervals from  $a$  to  $b$ , then

$$\begin{aligned} 1 = \chi([a, b]) &= \sum_{a \leq s, t \leq b} \mu(s, t) = \sum_{a < s, t < b} \mu(s, t) + \sum_{a \leq t \leq b} \mu(a, t) + \sum_{a \leq s \leq b} \mu(s, b) - \mu(a, b) \\ &= \chi(a, b) + \delta(a, b) + \delta(a, b) - \mu(a, b) = \chi(a, b) - \mu(a, b) \end{aligned}$$

by the defining properties of Möbius functions [33, 3.3.7].

For any  $p$ -subgroup  $H$  of  $G$  we write  $\mathcal{O}_G(H)$  for  $N_G(H)/H$  (the automorphism group of  $H$  in the orbit category  $\mathcal{O}_G^p$ ).

**3.1. Euler characteristic of the Brown poset  $\mathcal{S}_G^{p+*}$ .** The arguably most basic fact about the Brown poset is its contractibility when  $\mathcal{O}_p(G)$  is nontrivial.

**Lemma 3.1.** [27, Proposition 2.4]  $\mathcal{O}_p(G) \neq 1 \implies \mathcal{S}_G^{p+*} \simeq * \implies \chi(\mathcal{S}_G^{p+*}) = 1$

The weighting and the coweighting in the sense of Leinster [23, Definition 1.10] for the Brown poset are known.

**Proposition 3.2.** [19, Theorems 1.1 and 1.3] *The functions*

$$k^H = -\tilde{\chi}(H//\mathcal{S}_G^p) = -\tilde{\chi}(\mathcal{S}_{\mathcal{O}_G(H)}^{p+*}), \quad k_K = -\tilde{\chi}(\mathcal{S}_K^{(1,K)})$$

are a weighting for  $\mathcal{S}_G^p$  and a coweighting for  $\mathcal{S}_G^{p+*}$ . The Euler characteristic of  $\mathcal{S}_G^{p+*}$  is

$$\sum_{H \in \mathcal{S}_G^{p+*+\text{rad}}} -\tilde{\chi}(\mathcal{S}_{\mathcal{O}_G(H)}^{p+*}) = \chi(\mathcal{S}_G^{p+*}) = \sum_{K \in \mathcal{S}_G^{p+*+\text{eab}}} -\tilde{\chi}(\mathcal{S}_K^{(1,K)})$$

The weighting is concentrated on the nonidentity  $G$ -radical  $p$ -subgroups (Lemma 3.1) and the coweighting on the elementary abelian  $p$ -subgroups [19, Lemma 3.2]. We observed above that the coweighting can also be viewed as a Möbius function. Let  $K$  be any  $p$ -subgroup of  $G$ . Since the weighting for  $\mathcal{S}_G^p$  restricts to a weighting for the contractible left ideal  $\mathcal{S}_G^{p \geq K}$  of  $p$ -subgroups of  $G$  containing  $K$  [19, Remark 2.6] there is a relation

$$(3.3) \quad \sum_{K \in H/\mathcal{S}_G^{p+\text{rad}}} -\tilde{\chi}(\mathcal{S}_{\mathcal{O}_G(K)}^{p+*}) = 1$$

between the Euler characteristics of the Brown posets of  $\mathcal{O}_G(K) = N_G(K)/K$  for  $K \in [H, G]$ .

**Proposition 3.4.** *The weighting  $k^H: \mathcal{S}_G^p \rightarrow \mathbf{Z}$  for  $\mathcal{S}_G^p$  is given by*

$$k^H = \begin{cases} k^{[H]} & H \text{ is } G\text{-radical} \\ 0 & \text{otherwise} \end{cases}$$

where  $k^{[H]}: \mathcal{S}_G^{p+\text{rad}}/G \rightarrow \mathbf{Z}$  is the weighting for  $\mathcal{S}_G^{p+\text{rad}}/G$  (Definition 2.6). The coweighting  $k_K: \mathcal{S}_G^{p+*} \rightarrow \mathbf{Z}$  for  $\mathcal{S}_G^{p+*}$  is given by

$$k_K = \begin{cases} k_{[K]} & K \text{ is elementary abelian} \\ 0 & \text{otherwise} \end{cases}$$

where  $k_{[K]}: \mathcal{S}_G^{p+*+\text{eab}} \rightarrow \mathbf{Z}$  is the coweighting for  $\mathcal{S}_G^{p+*+\text{eab}}/G$ .

*Proof.* We noted above that the weighting  $k^\bullet$  for  $\mathcal{S}_G^p$  vanishes off the subposet  $\mathcal{S}_G^{p+\text{rad}}$  of  $G$ -radical  $p$ -subgroups. If  $H$  is any  $G$ -radical  $p$ -subgroup of  $G$ , then the inclusion  $H//\mathcal{S}_G^{p+\text{rad}} \hookrightarrow H//\mathcal{S}_G^p$  is a homotopy equivalence by Bouc's theorem [6] [12, Theorem 4.2] because

$$J//\mathcal{S}_G^p \not\cong * \iff J//\mathcal{S}_G^p \not\cong * \iff \mathcal{S}_{N_G(J)/J}^{p+*} \not\cong * \xrightarrow{\text{Lemma 3.1}} J \in H//\mathcal{S}_G^{p+\text{rad}}$$

for any  $J \in H//\mathcal{S}_G^p$ . Thus the weighting for  $\mathcal{S}_G^p$  on  $H$ ,  $k^H = -\tilde{\chi}(H//\mathcal{S}_G^p) = -\tilde{\chi}(H//\mathcal{S}_G^{p+\text{rad}})$ , is the weighting for  $\mathcal{S}_G^{p+\text{rad}}$  on  $H$ . The weighting on  $\mathcal{S}_G^{p+\text{rad}}$  is invariant under conjugation by  $G$  so it induces the unique weighting  $k^{[H]}: \mathcal{S}_G^{p+\text{rad}}/G \rightarrow \mathbf{Z}$  defined on the set of conjugacy classes of  $G$ -radical  $p$ -subgroups  $[H]$  of  $G$  (Corollary 2.8).

The part about coweightings is clear as the nonidentity elementary abelian  $p$ -subgroups form a right ideal in the poset of nonidentity  $p$ -subgroups (§3.3).  $\square$

**Definition 3.5.** When  $H$  and  $K$  are subgroups of  $G$ ,  $\mathcal{S}_G(H, [K])$  is the number of conjugates of  $K$  containing  $H$ , and  $\mathcal{S}_G([H], K)$  is the number of conjugates of  $H$  contained in  $K$ .

In particular,  $\mathcal{S}_G(1, [K]) = |G : N_G(K)|$  is the length of  $K$ ,  $\mathcal{S}_G([H], G) = |G : N_G(H)|$  is the length of  $H$ , and  $\mathcal{S}_G(H, [G]) = 1 = \mathcal{S}_G([1], K)$ .

**Lemma 3.6.** For any two subgroups  $H, K$  of  $G$ ,

$$\mathcal{S}_G(H, [K]) = \frac{|\mathcal{O}_G(H, K)|}{|\mathcal{O}_G(K)|}, \quad \mathcal{S}_G([H], K) = \frac{|K|}{|H|} \frac{|\mathcal{O}_G(H, K)|}{|\mathcal{O}_G(H)|} = \frac{|\mathcal{F}_G(H, K)|}{|\mathcal{F}_G(H)|}, \quad \frac{\mathcal{S}_G(H, [K])}{\mathcal{S}_G([H], K)} = \frac{|N_G(H)|}{|N_G(K)|}$$

*Proof.* There is a surjection

$$N_G(H, K) \rightarrow \{J \in [H] \mid J \leq K\}: g \rightarrow H^g$$

and the fibre over  $J = H^x$  is  $\{g \in G \mid H^g = H^x\} = N_G(H)x$ . Since thus

$$N_G(H) \backslash N_G(H, K) = \{J \in [H] \mid J \leq K\}, \quad N_G(H, K)/N_G(K) = \{L \in [K] \mid H \leq L\}$$

we have

$$|N_G(H)|\mathcal{S}_G([H], K) = |N_G(H, K)| = \mathcal{S}_G(H, [K])|N_G(K)|$$

and also

$$\begin{aligned} |\mathcal{O}_G(H, K)| &= |N_G(H, K)||K|^{-1} = \mathcal{S}_G(H, [K])|\mathcal{O}_G(K)| \\ |\mathcal{F}_G(H, K)| &= |C_G(H)|^{-1}|N_G(H, K)| = |\mathcal{F}_G(H)|\mathcal{S}_G([H], K) \end{aligned}$$

This finishes the proof.  $\square$

The point of Proposition 3.4 is that the weighting  $k^H$  for  $\mathcal{S}_G^p$  restricts to the weighting for the  $G$ -radical subposet  $\mathcal{S}_G^{p+\text{rad}}$  and that the weighting for  $\mathcal{S}_G^{p+\text{rad}}/G$  can be computed by solving the linear equation

$$(3.7) \quad \left( \mathcal{S}_G^{p+\text{rad}}(H, [K]) \right)_{H, [K] \in \mathcal{S}_G^{p+\text{rad}}/G} \begin{pmatrix} \vdots \\ k^{[K]} \\ \vdots \end{pmatrix}_{[K] \in \mathcal{S}_G^{\text{rad}}/G} = \begin{pmatrix} \vdots \\ 1 \\ \vdots \end{pmatrix}$$

Similarly, the coweighting for  $\mathcal{S}_G^{p+*}$  can be computed entirely inside the subposet  $\mathcal{S}_G^{p+*+\text{eab}}$  by solving the linear equation

$$(3.8) \quad (\dots \ k_{[H]} \ \dots) \left( \mathcal{S}_G^{p+\text{eab}}([H], K) \right)_{H, K \in \mathcal{S}_G^{p+\text{eab}}/G} = (\dots \ 1 \ \dots)$$

If  $k^{[K]}$  is the weighting for  $(\mathcal{S}_G^p(H, [K]))_{H, K \in \mathcal{S}_G^p/G}$  and  $k_{[H]}$  the coweighting for  $(\mathcal{S}_G^{p+*}(H, [K]))_{H, K \in \mathcal{S}_G^{p+*}/G}$ , then  $k^{[K]}/|\mathcal{O}_G^p(K)|$  is the weighting for  $[\mathcal{O}_G^p]$  and  $k_{[H]}/|\mathcal{F}_G^{p+*}(H)|$  the coweighting for  $[\mathcal{F}_G^{p+*}]$  [19, §2.4].

**Example 3.9.** The radical 2-subgroups of  $\text{GL}_3(\mathbf{F}_2)$

$$U_\emptyset = \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}, \quad U_{\{1\}} = \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}, \quad U_{\{2\}} = \begin{pmatrix} 1 & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad U_\Pi = 1$$

are indexed up to conjugacy by subsets of the set  $\Pi = \{\alpha_1, \alpha_2\}$  of fundamental roots for the root system  $A_2$  according to the Borel–Tits theorem [14, Theorem 3.1.3, Corollary 3.1.5]. The weighting  $k^{[U_J]} = -\tilde{\chi}(U_J//\mathcal{S}_{\text{GL}_3(\mathbf{F}_2)}^{\text{rad}}) = \tilde{\chi}(\mathcal{S}_{L_J}^{2+*})$  for  $\mathcal{S}_{\text{GL}_3(\mathbf{F}_2)}^{\text{rad}}/\text{GL}_3(\mathbf{F}_2)$  lists the reduced Euler characteristics of the Levi complements  $L_J = N_{\text{GL}_3(\mathbf{F}_2)}(U_J)/U_J$  [14, Theorem 2.6.5] and is the solution to the linear equation (3.7)

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 21 & 7 & 7 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ -2 \\ 8 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

where the matrix is the table  $(\mathcal{S}_{\text{GL}_3(\mathbf{F}_2)}^{2+\text{rad}}(U_J, [U_K]))_{J, K \subset \Pi}$ . In particular,  $-\tilde{\chi}(1//\mathcal{S}_{\text{GL}_3(\mathbf{F}_2)}^2) = -\tilde{\chi}(\mathcal{S}_{\text{GL}_3(\mathbf{F}_2)}^{2+*}) = 8$ .

The situation is completely different in the cross characteristic case. With computer assistance one finds that  $\mathrm{SL}_3(\mathbf{F}_3)$  contains four  $\mathrm{SL}_3(\mathbf{F}_3)$ -radical subgroups of orders 16, 8, 4, 1. The table  $(\mathcal{S}_{\mathrm{SL}_3(\mathbf{F}_3)}^{2+\mathrm{rad}}(H, [K])$  and the weighting  $k^{[K]}$  for  $\mathcal{S}_{\mathrm{SL}_3(\mathbf{F}_3)}^{2+\mathrm{rad}}/G$  satisfy the linear equation

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 351 & 117 & 234 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ -2 \\ 352 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

In particular,  $-\tilde{\chi}(\mathcal{S}_{\mathrm{SL}_3(\mathbf{F}_3)}^{2+*}) = 352$ .

**3.2. Euler characteristics of centralized subposets of  $\mathcal{S}_G^{p+*}$ .** Suppose now that the group  $A$  acts on  $G$ . The centralizer of  $A$  in  $\mathcal{S}_G^{p+*}$  is the  $N_G(A)$ -subposet

$$C_{\mathcal{S}_G^{p+*}}(A) = \{H \in \mathcal{S}_G^{p+*} \mid [H, A] \leq H\}$$

of nontrivial  $A$ -normalized  $p$ -subgroups  $H$  of  $G$  (Definition 2.1).

As in the non-equivariant case (Lemma 3.1) this centralizer poset is contractible as soon as  $O_p(G)$  is nontrivial.

**Lemma 3.10.**  $O_p(G) \neq 1 \implies C_{\mathcal{S}_G^{p+*}}(A) \simeq * \implies \chi(C_{\mathcal{S}_G^{p+*}}(A)) = 1$

*Proof.* The characteristic subgroup  $O_p(G)$  is a nonidentity  $A$ -normalized normal  $p$ -subgroup of  $G$ . The natural transformations  $K \leq KO_p(G) \geq O_p(G)$  link the identity functor to a constant endofunctor of  $C_{\mathcal{S}_G^{p+*}}(A)$ .  $\square$

We can easily determine the unique weighting and coweighting in the sense of Leinster [23] on  $C_{\mathcal{S}_G^{p+*}}(A)$ .

**Lemma 3.11.** *The functions*

$$k^H = -\tilde{\chi}(C_{H//\mathcal{S}_G^p}(A)) = -\tilde{\chi}(C_{\mathcal{S}_{O_G(H)}^{p+*}}(A)), \quad k_K = -\tilde{\chi}(C_{\mathcal{S}_K^{(1,K)}}(A))$$

are a weighting for  $C_{\mathcal{S}_G^p}(A)$  and a coweighting for  $C_{\mathcal{S}_G^{p+*}}(A)$  and the Euler characteristic of  $C_{\mathcal{S}_G^p}(A)$  is

$$\sum_{H \in C_{\mathcal{S}_G^{p+*+\mathrm{rad}}}(A)} -\tilde{\chi}(C_{\mathcal{S}_{O_G(H)}^{p+*}}(A)) = \chi(C_{\mathcal{S}_G^{p+*}}(A)) = \sum_{K \in C_{\mathcal{S}_G^{p+*+\mathrm{eab}}}(A)} -\tilde{\chi}(C_{\mathcal{S}_K^{(1,K)}}(A))$$

where the sum to the left runs over the nonidentity  $G$ -radical  $A$ -normalized  $p$ -subgroups of  $G$  and the sum to the right over the nonidentity elementary abelian  $A$ -normalized  $p$ -subgroups of  $G$ .

*Proof.* The adjunctions of [12, §6],  $KR = N_K(H)/H$  and  $\bar{K}L = K$  where  $\bar{K} = K/H$ ,

$$H/\mathcal{S}_G^{p+*} \begin{matrix} \xrightarrow{R} \\ \xleftarrow{L} \end{matrix} \mathcal{S}_{O_G(H)}^p \quad H//\mathcal{S}_G^{p+*} \begin{matrix} \xrightarrow{R} \\ \xleftarrow{L} \end{matrix} \mathcal{S}_{O_G(H)}^{p+*}$$

are  $A$ -adjunctions (Definition 2.2) inducing adjunctions

$$H/C_{\mathcal{S}_G^{p+*}}(A) \begin{matrix} \xrightarrow{R} \\ \xleftarrow{L} \end{matrix} C_{\mathcal{S}_{O_G(H)}^p}(A) \quad H//C_{\mathcal{S}_G^{p+*}}(A) \begin{matrix} \xrightarrow{R} \\ \xleftarrow{L} \end{matrix} C_{\mathcal{S}_{O_G(H)}^{p+*}}(A)$$

between centralizer posets (Proposition 2.3). In the notation of [12, Theorem 3.7], the weighting for  $C_{\mathcal{S}_G^{p+*}}(A)$  is  $k^H = -\tilde{\chi}(H//C_{\mathcal{S}_G^{p+*}}(A)) = -\tilde{\chi}(C_{\mathcal{S}_{O_G(H)}^{p+*}}(A))$ .

Similarly, the coweighting for  $C_{\mathcal{S}_G^{p+*}}(A)$  is  $k_K = -\tilde{\chi}(C_{\mathcal{S}_G^{p+*}}(A)//K)$ . Since  $\mathcal{S}_G^{p+*}/K = \mathcal{S}_K^{(1,K)}$  it is clear that  $C_{\mathcal{S}_G^{p+*}}(A)//K = C_{\mathcal{S}_G^{p+*}/K}(A) = C_{\mathcal{S}_K^{(1,K)}}(A)$ .

The Euler characteristic of  $C_{\mathcal{S}_G^{p+*}}(A)$  is the sum of the values of the weighting (coweighting). By Lemma 3.10, the weighting of Lemma 3.11 vanishes off the  $G$ -radical  $p$ -subgroups.

The argument at the very beginning of this section shows that  $k_K = \mu_A(1, K)$  where  $\mu_A$  stands for the Möbius function of the poset  $C_{\mathcal{S}_G^{p+*}}(A)$ . An equivariant version of the argument from [21, Proposition 2.4] shows that the Möbius function  $k_K = \mu_A(1, K)$  for  $C_{\mathcal{S}_G^{p+*}}(A)$  is nonzero exactly when  $K$  is a nonidentity elementary abelian  $p$ -subgroup.  $\square$

Let  $\mathcal{S}_G^{p^{++}\text{rad}}$  ( $\mathcal{S}_G^{p^{++}\text{eab}}$ ) be the subposet of nontrivial  $G$ -radical  $p$ -subgroups (nontrivial elementary abelian  $p$ -subgroups). Lemma 3.11 shows that the centralizer posets  $C_{\mathcal{S}_G^{p^{++}\text{rad}}}(A)$  and  $C_{\mathcal{S}_G^{p^{++}\text{eab}}}(A)$  have the same Euler characteristic as  $C_{\mathcal{S}_G^{p^{++}}}(A)$ . They are, in fact, homotopy equivalent:

**Proposition 3.12.** *The inclusions  $C_{\mathcal{S}_G^{p^{++}\text{eab}}}(A) \hookrightarrow C_{\mathcal{S}_G^{p^{++}}}(A) \hookleftarrow C_{\mathcal{S}_G^{p^{++}\text{rad}}}(A)$  are homotopy equivalences.*

*Proof.* This proof is similar to that of [12, Theorem 6.1] and goes back to Bouc [6] and Quillen's Theorem A [26, Theorem A]. It suffices to show that only the elementary abelian or the  $G$ -radical  $p$ -subgroups contribute to the homotopy type of  $C_{\mathcal{S}_G^{p^{++}}}(A)$ . More precisely, it is enough to show that

$$K \text{ is not an elementary abelian } p\text{-subgroup of } G \implies C_{\mathcal{S}_G^{p^{++}}}(A)//K \text{ is contractible}$$

$$H \text{ is not a } G\text{-radical } p\text{-subgroup of } G \implies H//C_{\mathcal{S}_G^{p^{++}}}(A) \text{ is contractible}$$

when  $H$  and  $K$  are nonidentity  $A$ -normalized subgroups of  $G$ . In the first case, we use that  $C_{\mathcal{S}_G^{p^{++}}}(A)//K = C_{\mathcal{S}_K^{(1,K)}}(A)$ . When  $K$  is not elementary abelian, the Frattini subgroup  $\Phi(K)$  is a nontrivial  $A$ -normalized subgroup of  $K$  and  $H \leq H\Phi(K) \geq \Phi(K)$  defines a retraction of  $C_{\mathcal{S}_K^{(1,K)}}(A)$ . In the second case, we use the adjunction between  $H//C_{\mathcal{S}_G^{p^{++}}}(A)$  and  $C_{\mathcal{S}_{\mathcal{O}_G(H)}^{p^{++}}}(A)$  and recall that the latter poset is contractible when  $H$  is not  $G$ -radical by Lemma 3.10.  $\square$

**Proposition 3.13.** *The weighting  $k^H: C_{\mathcal{S}_G^p}(A) \rightarrow \mathbf{Z}$  for  $C_{\mathcal{S}_G^p}(A)$  is given by*

$$k^H = \begin{cases} k^{[H]} & K \text{ is } G\text{-radical} \\ 0 & \text{otherwise} \end{cases}$$

where  $k^{[H]}: C_{\mathcal{S}_G^{p^{++}\text{rad}}}(A)/N_G(A) \rightarrow \mathbf{Z}$  is the weighting for  $C_{\mathcal{S}_G^{p^{++}\text{rad}}}(A)/N_G(A)$ . The coweighting  $k_K: C_{\mathcal{S}_G^{p^{++}}}(A) \rightarrow \mathbf{Z}$  for  $C_{\mathcal{S}_G^{p^{++}}}(A)$  is given by

$$k_K = \begin{cases} k_{[K]} & K \text{ is elementary abelian} \\ 0 & \text{otherwise} \end{cases}$$

where  $k_{[K]}: C_{\mathcal{S}_G^{p^{++}\text{eab}}}(A)/N_G(A) \rightarrow \mathbf{Z}$  is the coweighting for  $C_{\mathcal{S}_G^{p^{++}\text{eab}}}(A)/N_G(A)$ .

We omit the proof of Proposition 3.13 which is similar to that of Proposition 3.4 with Lemma 3.10 replacing Lemma 3.1.

*Proof of Theorem 1.1.(2).* Let  $K$  be an  $A$ -normalized  $p$ -subgroup of  $G$ . The weighting for  $C_{\mathcal{S}_G^p}(A)$  (Lemma 3.11) restricts to the weighting for the contractible left ideal  $C_{K//\mathcal{S}_G^p}(A)$  (§3.3) of  $A$ -normalized  $p$ -subgroups containing  $K$ . This means that

$$1 = \chi(C_{K//\mathcal{S}_G^p}(A)) = \sum_{H \geq K} k^H = \sum_{H \geq K} -\tilde{\chi}(C_{\mathcal{S}_{\mathcal{O}_G(H)}^{p^{++}}}(A))$$

generalizing the nonequivariant Equation (3.3).  $\square$

The next lemma applies in the special case where  $A$  is a subgroup of  $G$  acting on  $\mathcal{S}_G$  by conjugation.

**Lemma 3.14.** [38, §4][39, Lemma 2.1.2] *If  $A$  is a subgroup of  $G$ ,*

$$O_p(A) \neq 1 \implies C_{\mathcal{S}_G^{p^{++}}}(A) \simeq * \implies \chi(C_{\mathcal{S}_G^{p^{++}}}(A)) = 1$$

*Proof.* The natural transformations  $K \leq KO_p(A) \geq O_p(A)$  link the identity endofunctor to a constant endofunctor of  $C_{\mathcal{S}_G^{p^{++}}}(A)$ . The product  $KO_p(A)$  is a subgroup as  $A$  normalizes  $K$ .  $\square$

**Corollary 3.15.** *Consider the Brown poset  $\mathcal{S}_G^{p^{++}}$  as a  $G$ -poset with the conjugation action.*

(1) *The  $r$ th,  $r \geq 1$ , reduced equivariant Euler characteristic is a weighted sum of reduced Euler characteristics*

$$\tilde{\chi}_r(\mathcal{S}_G^{p^{++}}, G) = \frac{1}{|G|} \sum_{A \in \mathcal{S}_G^{p^{++}\text{abe}}} \tilde{\chi}((C_{\mathcal{S}_G^{p^{++}}}(A))) \varphi_r(A) = \sum_{[A] \in \mathcal{S}_G^{p^{++}\text{abe}}/G} \tilde{\chi}((C_{\mathcal{S}_G^{p^{++}}}(A))) \frac{\varphi_r(A)}{|N_G(A)|}$$

*with contributions only from the  $p$ -regular abelian subgroups  $A$  of  $G$ .*

(2) *The  $r$ th,  $r \geq 2$ , reduced equivariant Euler class function  $\tilde{\alpha}_r(\mathcal{S}_G^{p^{++}}, G)$  (Definition 2.10) vanishes off the  $p$ -regular classes.*

(3) The  $r$ th,  $r \geq 2$ , reduced equivariant Euler class function has the form

$$\tilde{\alpha}_r(\mathcal{S}_G^{p^{++}}, G) = \sum_{[C] \in \mathcal{S}_G^{p'+\text{cyc}}/G} \frac{\tilde{\alpha}_r(\mathcal{S}_G^{p^{++}}, G)([C])}{|N_G(C) : C|} 1_C^G$$

where the sum runs over the set of conjugacy classes  $[C]$  of  $p$ -regular cyclic subgroups and where the uniquely determined Artin coefficients  $\tilde{\alpha}_r(\mathcal{S}_G^{p^{++}}, G)([C])$  are integers.

(4) When  $r = 2$  the Artin coefficients satisfy the equation

$$0 = \sum_{[C] \in \mathcal{S}_G^{p'+\text{cyc}}} \tilde{\alpha}_2(\mathcal{S}_G^{p^{++}}, G)(C) \frac{1}{|N_G(C)|}$$

for all prime divisors  $p$  of  $|G|$ .

*Proof.* (1) Use two of the reformulations of the equivariant Euler characteristic from Proposition 2.9. Lemma 3.14 implies that only the  $p$ -regular abelian subgroups of  $G$  contribute to the sum.

(2) The  $r$ th equivariant reduced class function  $\tilde{\alpha}_r(\mathcal{S}_G^{p^{++}}, G)$  takes  $x \in G$  to

$$\tilde{\chi}_{r-1}(C_{\mathcal{S}_G^{p^{++}}}(x), C_G(x)) = \frac{1}{|C_G(x)|} \sum_{y \in C_{r-1}(C_G(x))} \tilde{\chi}(C_{\mathcal{S}_G^{p^{++}}}(\langle x, y \rangle))$$

If  $p$  divides the order of  $x$ ,  $O_p(\langle x, y \rangle)$  is nontrivial and  $\tilde{\chi}(C_{\mathcal{S}_G^{p^{++}}}(\langle x, y \rangle)) = 0$  for all  $y$  by Lemma 3.14.

(3) Since the value of  $\tilde{\alpha}_r(\mathcal{S}_G^{p^{++}}, G)$  at  $x \in G$  only depends on the subgroup  $\langle x \rangle$  generated by  $x$  (Definition 2.10), the proof of Artin's induction theorem [18, Theorem 5.21] shows that  $\tilde{\alpha}_r(\mathcal{S}_G^{p^{++}}, G)$  can be decomposed as claimed. Since  $\tilde{\alpha}_r(\mathcal{S}_G^{p^{++}}, G)$  is supported on the  $p$ -regular classes, as we saw in item (2), only  $p$ -regular cyclic subgroups are used in this decomposition. The coefficients are uniquely determined since the class functions  $1_C^G$  as  $C$  runs through the set of conjugacy classes of cyclic subgroups of  $G$  are a basis for the vector space  $\mathbf{Q} \otimes R_{\mathbf{Q}}(G)$  of  $\mathbf{Q}$ -linear combinations of rational characters [29, §13.1].

(4) The virtual degree of the right hand side of the equation in item (3) is

$$\sum_{[C] \in \mathcal{S}_G^{p'+\text{cyc}}/G} \frac{\tilde{\alpha}_2(\mathcal{S}_G^{p^{++}}, G)(C)}{|N_G(C) : C|} |G : C| = |G| \sum_{[C] \in \mathcal{S}_G^{p'+\text{cyc}}/G} \tilde{\alpha}_2(\mathcal{S}_G^{p^{++}}, G)(C) \frac{1}{|N_G(C)|}$$

and on the left side it is  $\tilde{\alpha}_2(\mathcal{S}_G^{p^{++}}, G)(1) = \tilde{\chi}_1(\mathcal{S}_G^{p^{++}}, G) = 0$  by Webb's theorem [38, Proposition 8.2.(i)].  $\square$

The Artin coefficients  $\tilde{\alpha}_r(\mathcal{S}_G^{p^{++}}, G)(C)$  of Corollary 3.15.(3) determine the  $r$ th equivariant reduced Euler characteristic as

$$\begin{aligned} (3.16) \quad \tilde{\chi}_r(\mathcal{S}_G^{p^{++}}, G) &= \langle \tilde{\alpha}_r(\mathcal{S}_G^{p^{++}}, G), |C_G| \rangle_G = \sum_{[C] \in \mathcal{S}_G^{p'+\text{cyc}}/G} \tilde{\alpha}_r(\mathcal{S}_G^{p^{++}}, G)(C) \sum_{x \in C} \frac{|C_G(x)|}{|N_G(C)|} \\ &= \sum_{[C] \in \mathcal{S}_G^{p'+\text{cyc}}/G} \tilde{\alpha}_r(\mathcal{S}_G^{p^{++}}, G)(C) \sum_{\substack{x \in C \\ x \neq 1}} \frac{|C_G(x)|}{|N_G(C)|} \end{aligned}$$

because  $\langle 1_C^G, |C_G| \rangle_G = \langle 1_C, |C_G| \rangle_C = \sum_{x \in C} |C_G(x) : C|$  by Frobenius reciprocity. Use Corollary 3.15.(4) to get the final equality. It is perhaps worth noting that the factor  $\sum_{x \in C} |C_G(x)|/|N_G(C)|$  from Equation (3.16) is a natural number.

**Proposition 3.17.**  $|N_G(H)| \mid \sum_{x \in H} |C_G(x)|$  for any subgroup  $H \leq G$ .

*Proof.* Note that the function  $x \rightarrow |C_G(x)|$  is constant over the orbits for the  $N_G(H)$ -action on  $H$ . For any element  $x_0 \in H$ , the contribution to the sum  $\sum_{x \in H} |C_G(x)|$  from the orbit through  $x_0$ ,  $|N_G(H) : N_G(H) \cap C_G(x_0)| |C_G(x_0)| = |N_G(H)| |C_G(x_0) : C_G(x_0) \cap N_G(H)|$ , is a multiple of  $|N_G(H)|$ .  $\square$

**Remark 3.18.** If  $A \leq G$  with  $O_p(A)$  and  $O_p(C_G(A))$  both nontrivial then  $C_{\mathcal{S}_G^{p^{++}}}(A) \simeq * \simeq \mathcal{S}_{C_G(A)}^{p^{++}}$  are both contractible. In particular,

$$A \leq G, p \mid |A|, A \text{ abelian} \implies \chi(C_{\mathcal{S}_G^{p^{++}}}(A)) = 1 = \chi(\mathcal{S}_{C_G(A)}^{p^{++}})$$

However, when  $p \mid |A|$  and  $A$  is nonabelian, the two Euler characteristics may not be equal: Let  $p = 2$ ,  $G = \Sigma_7$  the symmetric group, and  $A = \Sigma_3$  the subgroup  $A = \langle (1, 2), (1, 2, 3) \rangle \leq \Sigma_7$ . Then  $C_G(A) = \Sigma_4$ ,  $|C_G(A)|_2 = 8$ , and

$\tilde{\chi}(\mathcal{S}_{C_G(A)}^{2+*}) = 0$  since  $O_2(\Sigma_4) \neq 1$ . However, it turns out that  $\tilde{\chi}(C_{\mathcal{S}_G^{2+*}}(A)) = -8$ . Thus  $\chi(C_{\mathcal{S}_G^{2+*}}(A)) \neq \chi(\mathcal{S}_{C_G(A)}^{2+*})$  in this case.

**Remark 3.19.** The  $r$ th,  $r \geq 1$ , reduced equivariant Euler characteristic is an integer valued function  $\tilde{\chi}_r(\mathcal{S}_G^{p+*}, -)$  in the second variable (Corollary 2.15) on the set of conjugacy classes of subgroups of  $G$ . Proposition 2.9 shows that for any subgroup  $K$  of  $G$

$$\tilde{\chi}_r(\mathcal{S}_G^{p+*}, K) = \frac{1}{|K|} \sum_{[A] \in \mathcal{S}_G^{p'+\text{eab}}/G} \tilde{\chi}(C_{\mathcal{S}_G^{p+*}}(A)) \varphi_r(A) \mathcal{S}_G([A], K)$$

or, in matrix form,

$$\left( \dots \quad \chi(C_{\mathcal{S}_G^{p+*}}(A)) \varphi_r(A) \quad \dots \right)_{A \in \mathcal{S}_G^{p'+\text{abe}}/G} \left( \mathcal{S}_G([A], K) \right)_{\substack{A \in \mathcal{S}_G^{p'+\text{abe}}/G \\ K \in \mathcal{S}_G^p/G}} = \left( \dots \quad \tilde{\chi}_r(\mathcal{S}_G^{p+*}, K) \quad \dots \right)_{K \in \mathcal{S}_G^p/G}$$

It suffices to let the sum range over the  $p$ -singular abelian subgroups of  $G$  by Lemma 3.14 and we may replace  $C_{\mathcal{S}_G^{p+*}}(A)$  by  $C_{\mathcal{S}_G^{p+*+\text{rad}}}(A)$  or  $C_{\mathcal{S}_G^{p+*+\text{eab}}}(A)$  by Proposition 3.12. When  $K = 1$  is the trivial subgroup we get the usual reduced Euler characteristic  $\tilde{\chi}(\mathcal{S}_G^{p+*})$  and when  $K = G$  we get the  $r$ th reduced equivariant Euler characteristic  $\tilde{\chi}_r(\mathcal{S}_G^{p+*}, G)$  that we have focused on until now. For the alternating group  $G = A_5$  at  $p = 2$  the Euler characteristics  $-\chi_r(\mathcal{S}_G^{2+*}, K)$  for  $r = 1, 2, 3$  and for the subgroups  $K$  of  $G$  are

$ K $	1	2	3	5	4	6	10	12	60
$-\tilde{\chi}_1(\mathcal{S}_G^{2+*}, K)$	-4	-2	-2	0	-1	-1	0	-1	0
$-\tilde{\chi}_2(\mathcal{S}_G^{2+*}, K)$	-4	-2	-4	4	-1	-2	2	-3	1
$-\tilde{\chi}_3(\mathcal{S}_G^{2+*}, K)$	-4	-2	-10	24	-1	-5	12	-9	8

and the simple group  $G = \text{GL}_3(\mathbf{F}_2)$ ,  $p = 2$  they are

$ K $	1	2	3	7	4	4	4	6	21	8	12	12	24	24	168
$-\tilde{\chi}_1(\mathcal{S}_G^{2+*}, K)$	8	4	2	2	2	2	2	1	0	1	0	0	0	0	0
$-\tilde{\chi}_2(\mathcal{S}_G^{2+*}, K)$	8	4	0	8	2	2	2	0	0	1	-2	-2	-1	-1	1
$-\tilde{\chi}_3(\mathcal{S}_G^{2+*}, K)$	8	4	-6	50	2	2	2	-3	8	1	-8	-8	-4	-4	12

No interpretations similar to those of Remark 2.17 of the intermediate Euler characteristics  $\chi_r(\mathcal{S}_G^{p+*}, K)$ ,  $r = 1, 2, 3$ , with nontrivial and proper subgroup  $K$  are known.

**3.3. Ideals in the centralized poset  $C_{\mathcal{S}_G^{p+*}}(A)$ .** In this subsection we discuss the relation between the centralizer subposet  $C_{\mathcal{S}_G^{p+*}}(A)$  and the poset  $\mathcal{S}_{C_G(A)}^{p+*}$  of the centralizer subgroup for an  $A$ -group  $G$ .

Let  $\mathcal{S}_-$  and  $\mathcal{S}_+$  be subposets of a (finite) poset  $\mathcal{S}$ . By definition,  $\mathcal{S}_-$  is a left ideal and  $\mathcal{S}_+$  a right ideal if

$$\forall a, b \in \mathcal{S} : \mathcal{S}_- \ni a < b \implies b \in \mathcal{S}_-, \quad \forall a, b \in \mathcal{S} : a < b \in \mathcal{S}_+ \implies a \in \mathcal{S}_+$$

The weighting on the left ideal  $\mathcal{S}_-$ ,  $k^a = -\tilde{\chi}(a//\mathcal{S}_-) = -\tilde{\chi}(a//\mathcal{S})$ ,  $a \in \mathcal{S}_-$ , is the restriction of the weighting on  $\mathcal{S}$ , and the coweighting on the right ideal  $\mathcal{S}_+$ ,  $k_b = -\tilde{\chi}(\mathcal{S}_+//b) = -\tilde{\chi}(\mathcal{S}//b)$ ,  $b \in \mathcal{S}_+$ , is the restriction of the coweighting [12, Theorem 3.7] [19, Remark 2.6].  $\mathcal{S}_-$  is a left ideal if and only if the complement  $\mathcal{S}_+ = \mathcal{S} - \mathcal{S}_-$  is a right ideal.

**Proposition 3.20.** *Let  $\mathcal{S}$  be a finite poset.*

(1) *Suppose that  $\mathcal{S} = \mathcal{S}_- + \mathcal{S}_+$  where  $\mathcal{S}_-$  is a left ideal and  $\mathcal{S}_+$  a right ideal. Then*

$$\chi(\mathcal{S}) = \chi(\mathcal{S}_-) + \chi(\mathcal{S}_+) - \sum_{\mathcal{S}_- \ni a > b \in \mathcal{S}_+} \tilde{\chi}(a//\mathcal{S}) \tilde{\chi}(\mathcal{S}//b)$$

(2) *Suppose that  $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$  is the union of two left or right ideals. Then*

$$\chi(\mathcal{S}) = \chi(\mathcal{S}_1) + \chi(\mathcal{S}_2) - \chi(\mathcal{S}_1 \cap \mathcal{S}_2)$$

*Proof.* (1) Simplices in  $\mathcal{S}$  not entirely contained in  $\mathcal{S}_-$  or  $\mathcal{S}_+$  contain  $b < a$  for some  $b \in \mathcal{S}_+$  and some  $a \in \mathcal{S}_-$ . They have the form  $b < a$ ,  $b < a < a_0 < \dots < a_d$ ,  $b_0 < \dots < b_e < b < a$ , or  $b_0 < \dots < b_e < b < a < a_0 < \dots < a_d$ . Their contribution to the Euler characteristic of  $\mathcal{S}$  is  $-1 + \chi(a//\mathcal{S}) + \chi(\mathcal{S}//b) - \chi(a//\mathcal{S})\chi(\mathcal{S}//b) = -(\chi(a//\mathcal{S}) - 1)(\chi(\mathcal{S}//b) - 1) = -\tilde{\chi}(a//\mathcal{S})\tilde{\chi}(\mathcal{S}//b)$ .

(2) Mayer–Vietoris. □



More generally, in case  $\mathcal{S} = \bigcup_{j \in J} \mathcal{S}_j$  is the union of finitely many left or right ideals  $\mathcal{S}_j$  then there is an inclusion-exclusion principle

$$\chi(\mathcal{S}) = \sum_{K \subset J} (-1)^{|K|-1} \chi\left(\bigcap_{k \in K} \mathcal{S}_k\right)$$

as we see by induction from Proposition 3.20.(2).

Suppose that  $A$  is a  $p$ -regular group acting on  $G$  and let  $H$  be an  $A$ -normalized  $p$ -subgroup of  $G$ . Then  $H = [H, A]C_H(A)$  is the product of the normal commutator subgroup  $[H, A]$  and centralizer subgroup  $C_H(A)$  [13, Theorem 5.3.5]. Because

- $H \leq K \iff H \leq [K, A]$  if  $[H, A] = H$
- $H \leq K \iff H \leq C_K(A)$  if  $[H, A] = 1$

there are adjunctions

$$\begin{aligned} \mathcal{S}_1 &= \{K \mid [K, A] \neq 1\} \begin{array}{c} \xrightarrow{K \rightarrow [K, A]} \\ \xleftarrow{H \leftarrow H} \end{array} \{H \mid [H, A] = H\} \\ \mathcal{S}_2 &= \{K \mid C_K(A) \neq 1\} \begin{array}{c} \xrightarrow{K \rightarrow C_K(A)} \\ \xleftarrow{H \leftarrow H} \end{array} \{H \mid [H, A] = 1\} \end{aligned}$$

between subsets of  $C_{\mathcal{S}_G^{p+*}}(A)$ . To set up the first adjunction we use that  $[K, A, A] = [K, A]$  [13, Theorem 5.3.6]. We note that

$$\{K \mid [K, A] \neq 1\} = \{K \mid C_K(A) \neq K\}, \quad \{H \mid [H, A] = 1\} = \{H \mid C_H(A) = H\} = \mathcal{S}_{C_G(A)}^{p+*}$$

where we use that  $[K, A] = 1 \iff C_K(A) = K$ .

**Corollary 3.21.** *Let  $A$  be a  $p$ -regular group acting on  $G$ .*

- (1) *The left ideal  $\mathcal{S}_1 = \{K \in C_{\mathcal{S}_G^{p+*}}(A) \mid [K, A] \neq 1\}$  and the subposet  $\{H \in C_{\mathcal{S}_G^{p+*}}(A) \mid [H, A] = H\}$  have the same Euler characteristics.*
- (2) *The left ideal  $\mathcal{S}_2 = \{K \in C_{\mathcal{S}_G^{p+*}}(A) \mid C_K(A) \neq 1\}$  and the poset  $\mathcal{S}_{C_G(A)}^{p+*}$  of the centralizer subgroup have the same Euler characteristics.*
- (3) *The difference between the Euler characteristics of the centralizer subposet,  $C_{\mathcal{S}_G^{p+*}}(A)$ , and the centralizer subgroup poset,  $\mathcal{S}_{C_G(A)}^{p+*}$ , is*

$$\chi(C_{\mathcal{S}_G^{p+*}}(A)) - \chi(\mathcal{S}_{C_G(A)}^{p+*}) = \chi(\{K \in C_{\mathcal{S}_G^{p+*}}(A) \mid C_K(A) \not\leq K\}) - \chi(\{K \in C_{\mathcal{S}_G^{p+*}}(A) \mid 1 \not\leq C_K(A) \leq K\})$$

*Proof.* (1) – (2) Adjoint finite posets have identical Euler characteristics [23, Proposition 2.4].

(3) Apply Proposition 3.20.(2) with  $\mathcal{S} = C_{\mathcal{S}_G^{p+*}}(A) = \mathcal{S}_1 \cup \mathcal{S}_2$  where  $\mathcal{S}_1$  and  $\mathcal{S}_2$  are the two left ideals from (1) – (2) and use that  $\chi(\mathcal{S}_2) = \chi(\mathcal{S}_{C_G(A)}^{p+*})$  by (2).  $\square$

The Mayer–Vietoris relation of Corollary 3.21.(3) may also be written as

$$\chi(C_{\mathcal{S}_G^{p+*}}(A)) = \chi(\{H \mid [H, A] = 1\}) + \chi(\{H \mid [H, A] = H\}) - \chi(\{H \mid 1 \not\leq C_H(A) \leq H\})$$

This identity is tautological when  $A$  is trivial.

Since the weighting (coweighting) of  $C_{\mathcal{S}_G^{p+*}}(A)$  is concentrated on the  $G$ -radical  $p$ -subgroups (elementary abelian  $p$ -subgroups) (Lemma 3.11), the left (right) ideals of Corollary 3.21.(1)–(3) have the same Euler characteristics as the corresponding left ideals in  $C_{\mathcal{S}_G^{p+*+\text{rad}}}(A)$  ( $C_{\mathcal{S}_G^{p+*+\text{eab}}}(A)$ ).

**Corollary 3.22.** *Let  $A$  be a  $p$ -regular group acting on  $G$ . Suppose that  $C_K(A) \neq 1$  for all nonidentity  $G$ -radical  $p$ -subgroups  $K$  of  $G$ . Then  $\chi(\mathcal{S}_{C_G(A)}^{p+*}) = \chi(C_{\mathcal{S}_G^{p+*}}(A))$ .*

*Proof.* The subsets  $\{K \in C_{\mathcal{S}_G^{p+*+\text{rad}}}(A) \mid C_K(A) \not\leq K\}$  and  $\{K \in C_{\mathcal{S}_G^{p+*+\text{rad}}}(A) \mid 1 \not\leq C_K(A) \leq K\}$  are identical so  $\chi(\mathcal{S}_{C_G(A)}^{p+*}) = \chi(C_{\mathcal{S}_G^{p+*}}(A))$  by Corollary 3.21.(3).  $\square$

**Example 3.23.** When  $G = \Sigma_7$ ,  $p = 2$ , and  $A$  is any of the six abelian 2-regular subgroups of  $G$ , the reduced Euler characteristics of Corollary 3.21.(3) are

A	1	3	3	5	7	3 × 3
$\tilde{\chi}(C_{\mathcal{S}_G^{p+*}}(A))$	160	-8	4	0	-1	1
$\tilde{\chi}(\mathcal{S}_{C_G(A)}^{p+*})$	160	0	2	0	-1	-1
$\tilde{\chi}(\{K \mid C_K(A) \not\leq K\})$	-1	3	4	-1	-1	1
$\tilde{\chi}(\{K \mid 1 \not\leq C_K(A) \not\leq K\})$	-1	11	2	-1	-1	-1
$ C_G(A) _p$	16	8	2	2	1	1

We note that  $\tilde{\chi}(C_{\mathcal{S}_G^{p+*}}(A))$  and  $\tilde{\chi}(\mathcal{S}_{C_G(A)}^{p+*})$  may not be equal, that both reduced Euler characteristics are divisible by the  $p$ -part  $|C_G(A)|_p$ , and that, in each column, the differences of Corollary 3.21.(3) between the first two numbers and the next two numbers are equal.

#### 4. THE ORBIT CATEGORY OF $p$ -SUBGROUPS

Write  $\mathcal{O}_G$  for the category of subgroups,  $H$  and  $K$ , of  $G$  with morphism sets and automorphism groups

$$\mathcal{O}_G(H, K) = N_G(H, K)/K, \quad \mathcal{O}_G(H) = N_G(H)/H$$

where  $N_G(H, K) = \{x \in G \mid H^x \leq K\}$  is the transporter set. In other words,  $\mathcal{O}_G$  is the finite EI-category whose objects are the transitive  $G$ -orbits  $G/H$  and whose morphisms are the left  $G$ -maps  $G/H \rightarrow G/K$  between the orbits: The effect of  $x \in N_G(H, K)/K$  is  $G/H \ni gH \rightarrow gxK \in G/K$  for any  $g \in G$ .  $\mathcal{O}_G^p$  is the full subcategory of  $\mathcal{O}_G$  generated by all the  $p$ -subgroups of  $G$ .

**4.1. Euler characteristic of the orbit category  $\mathcal{O}_G^p$ .** Frobenius proved in 1907 that the number  $|G_p|$  of  $p$ -singular elements in  $G$  is divisible by  $|G|_p$  [11, 17] [10, Corollary 41.11] [29, 11.2, Corollary 2].

**Theorem 4.1** (Frobenius 1907).  $|G|_p \mid |G_p|$

K.S. Brown proved in 1975 that the reduced Euler characteristic of the Brown poset  $\mathcal{S}_G^{p+*}$  is divisible by  $|G|_p$  [8, Corollary 2]. This was later reproved with new arguments by Quillen [27, Corollary 4.2] or Webb [38, Theorem 8.1].

**Theorem 4.2** (Brown 1975).  $|G|_p \mid \tilde{\chi}(\mathcal{S}_G^{p+*})$

In this section we give yet another argument for Brown's theorem using the Euler characteristic of the orbit category  $\mathcal{O}_G^p$ . This argument will show that the two theorems of Frobenius and Brown are, in some sense, equivalent (and thus equivalent to the Sylow theorem). We finish the section by proving Theorem 1.1.

**Lemma 4.3.** *The number of  $p$ -singular elements in  $G$  is*

$$|G_p| = 1 + \sum_{1 < C \leq G} (1 - p^{-1})|C| = p^{-1} + \sum_{1 \leq C \leq G} (1 - p^{-1})|C|$$

where the sum is over all cyclic  $p$ -subgroups  $C$  of  $G$ .

*Proof.* Declare two  $p$ -singular elements to be equivalent if they generate the same cyclic subgroup. The set of equivalence classes is the set of cyclic  $p$ -subgroups  $C$  of  $G$ . The number of elements in the equivalence class  $C$  is the number of generators of  $C$ : 1 if  $|C| = 1$  and  $|C| - p^{-1}|C|$  if  $|C| > 1$ .  $\square$

Let  $V$  be an elementary abelian  $p$ -group (or a finite dimensional vector space over the finite field  $\mathbf{F}_p$  with  $p$  elements).

**Proposition 4.4.** *The function*

$$k_U = \begin{cases} |V|^{-1} & \dim U = 0 \\ (p-1)|V|^{-1} & \dim U = 1 \\ 0 & \text{otherwise} \end{cases}$$

is a coweighting for the orbit category  $\mathcal{O}_V$ .

*Proof.* The assertion is that

$$\sum_{U_1 \leq U_2} k_{U_1} \frac{|V|}{|U_2|} = 1$$

for any subspace  $U_2$  of  $V$ . This is easily verified.  $\square$

Let  $\mathcal{O}_V^{[1, V]}$  be the full subcategory of  $\mathcal{O}_V$  generated by all objects but the final object  $V$ .

**Corollary 4.5.** *The Euler characteristic of  $\mathcal{O}_V^{[1,V]}$  is*

$$\chi(\mathcal{O}_V^{[1,V]}) = \begin{cases} 0 & \dim V = 0 \\ p^{-1} & \dim V = 1 \\ 1 & \dim V > 1 \end{cases}$$

*Proof.*  $\mathcal{O}_V^{[1,V]}$  is the 0-object category when  $\dim V = 0$  and the 1-object category given by the group  $V$  when  $\dim V = 1$ . When  $\dim V > 1$ ,  $\chi(\mathcal{O}_V^{[1,V]}) = \chi(\mathcal{O}_V)$ , because the coweighting for  $\mathcal{O}_V$  restricts to a coweighting for the right ideal  $\mathcal{O}_V^{[1,V]}$  [19, Remark 2.6] and it has value 0 at the deleted object  $V$ . Now note that as the category  $\chi(\mathcal{O}_V)$  has a final object, its Euler characteristic is 1 [23, Examples 2.3.(d)].  $\square$

**Proposition 4.6.** [19, Theorem 4.1] *The function*

$$k_K = \begin{cases} |G|^{-1} & K = 1 \\ |G|^{-1}(1-p^{-1})|K| & K > 1 \text{ cyclic} \\ 0 & \text{otherwise} \end{cases}$$

*is a coweighting for  $\mathcal{O}_G^p$  and the Euler characteristic*

$$\chi(\mathcal{O}_G^p) = \frac{|G_p|}{|G|}$$

*is the density of the  $p$ -singular elements in  $G$ .*

*Proof.* The coweighting for  $\mathcal{O}_G^p$  is the function

$$k_K = \frac{-\tilde{\chi}(\mathcal{O}_G^p//K)}{|[K]||\mathcal{O}_G(K)|} = \frac{-\tilde{\chi}(\mathcal{O}_G^p//K)}{|G:K|}$$

defined for any  $p$ -subgroup  $K$  of  $G$  [12, Theorem 3.7]. There are equivalences of categories

$$i_K: \mathcal{O}_K \rightarrow \mathcal{O}_G/K, \quad i_K: \mathcal{O}_K^{[1,K]} \rightarrow \mathcal{O}_G//K$$

On objects,  $Hi_K = 1K \in N_G(H, K)/K = \mathcal{O}_G(H, K)$ , for any subgroup  $H$  of  $K$ . We observe that there is an obvious identification of morphism sets,

$$\mathcal{O}_K(H_1, H_2) = (\mathcal{O}_G/K)(H_1i_K, H_2i_K)$$

and we use this identification to define  $i_K$  on morphism sets. By construction,  $i_K$  is full and faithful, and as it is also essentially surjective on objects,  $i_K$  is an equivalence of categories. Finally, we have that

$$\chi(\mathcal{O}_K^{[1,K]}) = \chi(\mathcal{O}_V^{[1,V]})$$

where  $V = K/\Phi(K)$  is the Frattini quotient of  $K$ : There is, by the the proof of [12, Lemma 5.1.(c)], an adjunction between these two categories so that they must have the same Euler characteristics [23, Proposition 2.4]. To arrive at the description of the coweighting we recall that the  $p$ -group  $K$  is cyclic if and only if its Frattini quotient is 1-dimensional [13, Corollary 5.1.2] and use Corollary 4.5. Now the Euler characteristic, the sum of the values of the coweighting, can be computed thanks to the counting formula of Lemma 4.3.  $\square$

**Proposition 4.7.** [19, Theorem 1.3.(4)] *The function*

$$k^H = \frac{-\tilde{\chi}(\mathcal{S}_{\mathcal{O}_G(H)}^{p+*})}{|G:H|}$$

*is a weighting for  $\mathcal{O}_G^p$  and the Euler characteristic is*

$$\chi(\mathcal{O}_G^p) = \sum_H \frac{-\tilde{\chi}(\mathcal{S}_{\mathcal{O}_G(H)}^{p+*})}{|G:H|} = \sum_{[H]} \frac{-\tilde{\chi}(\mathcal{S}_{\mathcal{O}_G(H)}^{p+*})}{|\mathcal{O}_G(H)|}$$

*where the first sum is over the set of  $p$ -subgroups  $H$  of  $G$  and the second one over the set of conjugacy classes of such subgroups.*

*Proof.* Lemma 3.6 implies that the weightings for  $[\mathcal{O}_G^p]$  and  $\mathcal{O}_G^p$  are

$$\frac{k^{[K]}}{|\mathcal{O}_G(K)|}, \quad \frac{k^{[K]}}{|\mathcal{O}_G(K)|} \frac{1}{|G:N_G(K)|} = \frac{k^{[K]}}{|G:K|}$$

where  $k^{[K]} = -\tilde{\chi}(\mathcal{S}_{\mathcal{O}_G(K)}^{p+*})$  is the weighting for  $\mathcal{S}_G^p/G$ .  $\square$

By comparing the two expressions from Propositions 4.6 and 4.7 for the Euler characteristic of  $\mathcal{O}_G^p$  we obtain the global formula

$$(4.8) \quad \sum_{H \in \mathcal{S}_G^{p+\text{rad}}} -\tilde{\chi}(\mathcal{S}_{\mathcal{O}_G(H)}^{p+*})|H| = |G_p|$$

For our purposes it will be convenient to isolate the contribution from the trivial subgroup and rewrite Equation (4.8) on the form

$$(4.9) \quad |G_p| + \tilde{\chi}(\mathcal{S}_G^{p+*}) + \sum_{[H] \neq 1} \frac{\tilde{\chi}(\mathcal{S}_{\mathcal{O}_G(H)}^{p+*})}{|\mathcal{O}_G(H)|_p} \frac{|G|}{|\mathcal{O}_G(H)|_{p'}} = 0$$

that will allow us to verify that the theorems of Frobenius and Brown are equivalent.

**Theorem 4.10.** *Given relation (4.9), Frobenius' Theorem 4.1 and Brown's Theorem 4.2 are equivalent.*

*Proof.* Assume first that Frobenius' Theorem 4.1 holds. In Equation (4.9), we may assume that

- $\tilde{\chi}(\mathcal{S}_{\mathcal{O}_G(H)}^{p+*})/|\mathcal{O}_G(H)|_p$  is an integer when  $H$  is nontrivial (as part of an inductual argument)
- $|G|/|\mathcal{O}_G(H)|_{p'}$  is an integer divisible by  $|G|_p$  (as  $|\mathcal{O}_G(H)|$  divides  $|G|$ )

Thus every term in the sum is divisible by  $|G|_p$  and so is  $|G_p|$  by assumption. We conclude that  $\tilde{\chi}(\mathcal{S}_G^{p+*})$  is divisible by  $|G|_p$ . This is Brown's Theorem 4.2.

Next assume that Brown's Theorem 4.2 holds. In Equation (4.9)

- $|G|/|\mathcal{O}_G(H)|_{p'}$  is an integer divisible by  $|G|_p$
- $\tilde{\chi}(\mathcal{S}_{\mathcal{O}_G(H)}^{p+*})/|\mathcal{O}_G(H)|_p$  is an integer
- $\tilde{\chi}(\mathcal{S}_G^{p+*})$  is divisible by  $|G|_p$

and thus  $|G|_p$  divides for  $|G_p|$ . This is Frobenius' Theorem 4.1.  $\square$

See [16, Theorem 6.3] for one direction of Theorem 4.10 in an even more general context.

**Example 4.11.** Let  $K \in \text{Lie}(p)$  be a finite group of Lie type in defining characteristic  $p$ . Then  $K = O^{p'}C_{\bar{K}}(\sigma)$  is the subgroup of  $C_{\bar{K}}(\sigma)$  generated by its  $p$ -singular elements where  $(\bar{K}, \sigma)$  is a  $\sigma$ -setup for  $K$  [14, Definitions 2.1.1–2.2.2]. We shall assume that  $K = \Sigma(q)$  is an untwisted group of Lie type [14, Definition 2.2.4] and investigate Equations (4.8) and (3.3) in this case.

Let  $(\Sigma, \Pi)$  be the root system for the simple algebraic group  $\bar{K}$  over  $\bar{\mathbf{F}}_p$ . Write  $r(\Sigma)$  for the rank of  $\Sigma$  and  $\Sigma^+$  for the set of positive roots. For any subset  $J$  of the set  $\Pi$  of fundamental roots, there is an associated parabolic subgroup  $P_J$  [14, Definition 2.6.4] and  $P_J = U_J \rtimes L_J$ ,  $U_J = O_p(P_J)$ ,  $P_J = N_K(U_J)$  where  $L_J$  is the Levi complement [14, Theorem 2.6.5].

We first note that

$$(4.12) \quad -\tilde{\chi}(\mathcal{S}_{\Sigma(q)}^{p+*}) = (-1)^{r(\Sigma)} q^{|\Sigma^+|}$$

because  $\mathcal{S}_{\Sigma(q)}^{p+*}$  is homotopy equivalent to the building of  $\Sigma(q)$  [27, Theorem 3.1] which, by the Solomon–Tits theorem [1, Theorem 4.73], has the homotopy type of a wedge of  $q^{|\Sigma^+|}$  spheres of dimension  $r(\Sigma) - 1$ . Since the Borel–Tits theorem tells us that any  $\Sigma(q)$ -radical  $p$ -subgroup of  $\Sigma(q)$  is conjugate to precisely one of the subgroups  $U_J$  [14, Theorem 3.1.3, Corollary 3.1.5], the identity

$$|\Sigma(q)_p| = \sum_{J \subset \Pi} -\tilde{\chi}(\mathcal{S}_{L_J}^{p+*})|U_J||\Sigma(q) : P_J|$$

is the manifestation of Equation (4.8) for  $G = \Sigma(q)$ . Here,

$$-\tilde{\chi}(\mathcal{S}_{L_J}^{p+*})|U_J| = (-1)^{|J|}|L_J|_p|U_J| = (-1)^{|J|}|P_J|_p = (-1)^{|J|}q^{|\Sigma^+|}$$

where  $\Sigma^+$  is the set of positive roots of  $\Sigma$ . This follows from Equation (4.12) because  $M_J = O^{r'}(L_J)$  is a central product of groups from  $\text{Lie}(p)$  [14, Theorem 2.6.5.(f)] and  $-\tilde{\chi}(\mathcal{S}_{L_J}^{p+*})$  is a multiplicative function [19, Theorem 6.1]. Since also,

$$|\Sigma(q) : P_J| = \frac{|P_\Pi : P_\emptyset|}{|P_J : P_\emptyset|} = \frac{\text{KB}(\Sigma, \Pi)}{\text{KB}(\Sigma, J)}$$

we conclude that Equation (4.8) for  $\Sigma(q)$  has the form

$$(4.13) \quad |\Sigma(q)_p| = q^{|\Sigma^+|} \sum_{J \subset \Pi} (-1)^{|J|} \frac{\text{KB}(\Sigma, \Pi)}{\text{KB}(\Sigma, J)}$$

Lemma 4.15 now implies that  $|\Sigma(q)_p| = q^{|\Sigma|} = |\Sigma(q)|_p^2$ . (See below for the notation used here.) Similar arguments show that Equation (3.3) with  $G = \Sigma(q)$  specializes to

$$(4.14) \quad \sum_{J \subset \Pi} (-1)^{|J|} \frac{\text{KB}(\Sigma, \Pi)}{\text{KB}(\Sigma, J)} q^{|\Sigma_J^+|} = 1$$

where we used that  $-\tilde{\chi}(\mathcal{S}_{L_J}^{p+*}) = (-1)^{|J|} q^{|\Sigma_J^+|}$ .

Let  $(\Sigma, \Pi)$  be a reduced crystallographic root system with fundamental roots  $\Pi$ . Put  $\text{KB}(\Sigma) = \prod [d]$  where  $d$  runs through the set of degrees for the Weyl group  $W(\Sigma)$  [9, §9.3, Proposition 10.2.5] and  $[m] \in \mathbf{Z}[X]$ ,  $m \geq 1$ , is the polynomial

$$[m] = \frac{X^m - 1}{X - 1} = 1 + X + \cdots + X^{m-1}$$

Thus  $\text{KB}(\Sigma) = |K : B|$  where  $B$  is the Borel subgroup of  $K = \Sigma(q)$  and  $\text{KB}(A_m) = [2] \cdots [m+1]$ , for instance. More generally, for any set  $J \subset \Pi(\Sigma)$  of fundamental roots, set  $\text{KB}(\Sigma, J) = \prod \text{KB}(\Sigma_i)$  when the orthogonal decomposition of the root system  $\Sigma_J = \Sigma \cap \mathbf{R}J$  spanned by the fundamental roots in  $J$  is  $\bigcup \Sigma_i$  for irreducible root systems  $\Sigma_i$  [14, Definition 1.8.4]. Thus  $\text{KB}(\Sigma, \Pi) = \text{KB}(\Sigma)$ . We let  $\text{KB}(\Sigma, \emptyset) = 1$  by convention.

**Lemma 4.15.** [9, Theorem 9.4.5] *For any reduced crystallographic root system  $\Sigma$*

$$\sum_{J \subset \Pi} (-1)^{|J|} \frac{\text{KB}(\Sigma, \Pi)}{\text{KB}(\Sigma, J)} = X^{|\Sigma^+|}$$

In the concrete case where the root system  $\Sigma = A_{m-1}$ , Lemma 4.15 and Equation (4.14) give two prototypical combinatorial identities involving Gaussian multinomial coefficients [33, §1.7]

$$\sum_{(m_1, \dots, m_k) \in \text{OP}(m)} (-1)^k \binom{[m]}{[m_1], \dots, [m_k]} = X^{\binom{m}{2}}, \quad \sum_{(m_1, \dots, m_k) \in \text{OP}(m)} (-1)^k \binom{[m]}{[m_1], \dots, [m_k]} X^{\Sigma \binom{m_i}{2}} = 1$$

where the sums run over the set  $\text{OP}(m)$  of the  $2^{m-1}$  ordered partitions of  $m$ .

Experiments indicate that Equation (4.12) holds also for Steinberg and Suzuki–Ree groups when we replace  $r(\Sigma)$  with the twisted rank  $r(\tilde{\Sigma})$  [14, Proposition 2.3.2, Remark 2.3.3].

**Example 4.16.** The exponential generating function for the integer sequence  $n \rightarrow |(\Sigma_n)_p|$  counting the number of  $p$ -singular permutations of  $n$  things is [34, Example 5.2.10] is the Artin–Hasse exponential

$$\sum_{n=1}^{\infty} |(\Sigma_n)_p| \frac{x^n}{n!} = \sum_{n=1}^{\infty} \chi(\mathcal{O}_{\Sigma_n}^p) x^n = \exp\left(x + \frac{x^p}{p} + \frac{x^{p^2}}{p^2} + \cdots + \frac{x^{p^m}}{p^m} + \cdots\right)$$

The sequence  $|(\Sigma_n)_2|$  begins with 1, 2, 4, 16, 56, 256, 1072, 11264, 78976, 672256 for  $1 \leq n \leq 10$  (OEIS A005388).

The  $p$ -radical subgroups of the symmetric group are described in [3, §2] and [24, Lemma 2.2] as follows: Let  $r = (c_1, c_2, \dots)$  be a finite sequence of natural numbers  $c_i \geq 0$ . Define the degree at  $p$  of the sequence  $r$  to be

$$\deg_p r = p^{\sum r}$$

The set  $\mathcal{S}_{\Sigma_n}^{p+*+\text{rad}}/\Sigma_n$  of conjugacy classes of nonidentity  $\Sigma_n$ -radical  $p$ -subgroups is in bijective correspondence with the set of nonempty multisets  $R = \{r_1, r_2, \dots\}$  of finite sequences  $r_i$  such that  $\sum_i \deg_p r_i \leq n$ . There are extra restrictions, somewhat ignored in [3, 24], in case  $p = 2$  or  $p = 3$  since  $O_2(\Sigma_n)$  is nontrivial (only) for  $n = 2, 4$  and  $O_3(\Sigma_n)$  is nontrivial (only) for  $n = 3$ . In case  $p = 2$  we must also demand that

- $n - \sum_i \deg_p r_i$  does not equal 2 or 4
- none of the sequences  $(1), (1, 1), \dots$  has multiplicity 2 or 4 in the multiset  $R$

and when  $p = 3$  we must also demand that

- $n - \sum_i \deg_p r_i$  does not equal 3
- none of the sequences  $(1), (1, 1), \dots$  has multiplicity 3 in the multiset  $R$

The correspondence of [3, §2] sends the multiset  $R = \{r_1, r_2, \dots\}$  to the  $p$ -radical subgroup  $A_R = 1 \times A_{r_1} \times A_{r_2} \times \cdots$  where 1 is the trivial subgroup of the symmetric group on  $n - \sum_i \deg_p r_i$  elements and the  $A_{r_i}$ s are certain (iterated) wreath products of elementary abelian subgroups. We can now fill out the table

$R$	$A_R$	$ A_R $	$N_{\Sigma_n}(A_R)/A_R$	$k^{[A_R]}$	$ \Sigma_n : N_{\Sigma_n}(A_R) $
$\{(1, 1, 1)\}$	$A_1 \wr A_1 \wr A_1$	128	$\mathrm{GL}_1(\mathbf{F}_2) \times \mathrm{GL}_1(\mathbf{F}_2) \times \mathrm{GL}_1(\mathbf{F}_2)$	1	315
$\{(1, 2)\}$	$A_1 \wr A_2$	64	$\mathrm{GL}_1(\mathbf{F}_2) \times \mathrm{GL}_2(\mathbf{F}_2)$	-2	105
$\{(2, 1)\}$	$A_2 \wr A_1$	32	$\mathrm{GL}_2(\mathbf{F}_2) \times \mathrm{GL}_1(\mathbf{F}_2)$	-2	210
$\{(2), (1, 1)\}$	$A_2 \times A_1 \wr A_1$	32	$\mathrm{GL}_2(\mathbf{F}_2) \times \mathrm{GL}_1(\mathbf{F}_2) \times \mathrm{GL}_1(\mathbf{F}_2)$	-2	210
$\{(2), (2)\}$	$A_2 \times A_2$	16	$\mathrm{GL}_2(\mathbf{F}_2) \wr \Sigma_2$	16	35
$\{(3)\}$	$A_3$	8	$\mathrm{GL}_3(\mathbf{F}_2)$	8	30
$\{(1)\}$	$1_6 \times A_1$	2	$\Sigma_6 \times \mathrm{GL}_1(\mathbf{F}_2)$	16	28
$\emptyset$	$1_8$	1	$\Sigma_8$	-512	1

of 2-radical subgroups of  $\Sigma_8$ . With computer assistance it is possible to determine the table  $(\mathcal{S}_{\Sigma_8}(A_R, [A_S]))$  and then read off the weighting  $k^{[K]}$  for  $\mathcal{S}_{\Sigma_8}^{2+\mathrm{rad}}/\Sigma_8$  (Definition 2.6) as the solution to the linear equation (3.7)

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 9 & 0 & 6 & 6 & 1 & 0 & 0 & 0 \\ 21 & 7 & 0 & 7 & 0 & 1 & 0 & 0 \\ 45 & 15 & 15 & 0 & 0 & 0 & 1 & 0 \\ 315 & 105 & 210 & 210 & 35 & 30 & 28 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ -2 \\ -2 \\ 16 \\ 8 \\ 16 \\ -512 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

In particular,  $-\tilde{\chi}(\mathcal{S}_{\Sigma_8}^{2+*}) = -512$  and  $-\tilde{\chi}(\mathcal{S}_{\mathrm{GL}_3(\mathbf{F}_2)}^{2+*}) = 8$  (agreeing with Equation (4.12)) and

$$|\Sigma_8| |\chi(\mathcal{O}_{\Sigma_8}^2)| = \sum_{[A_R] \geq 1} k^{[A_R]} |\Sigma_8 : N_{\Sigma_8}(A_R)| |A_R| = 11264$$

correctly counts the number  $|(\Sigma_8)_2|$  of 2-singular elements in  $\Sigma_8$  (Propositions 3.2, 4.7).

It is, however, still unclear how to continue the computer generated sequence

$n$	3	4	5	6	7	8	9	10	11	12	13	14	15
$\chi(\mathcal{S}_{\Sigma_n}^{2+*})$	3	1	-15	-15	161	513	-639	-7935	-20735	235521	3244033	2232321	-190068735

of Euler characteristics of the Brown posets at  $p = 2$  for the symmetric groups. It is possible that the methods of [25] can be used to extend the sequence a little longer. The generating functions  $\sum_{n=1}^{\infty} \chi(\mathcal{S}_{\Sigma_n}^{2+*}) x^n$  are unknown.

**4.2. Euler characteristics of centralized subcategories of  $\mathcal{O}_G^p$ .** Let  $A$  be a group acting on  $G$  from the right. The centralized subcategory  $C_{\mathcal{O}_G}(A)$  of  $\mathcal{O}_G$  has objects, morphism sets, and automorphism groups

$$\begin{aligned} \mathrm{Ob}(C_{\mathcal{O}_G}(A)) &= C_{\mathrm{Ob}(\mathcal{O}_G)}(A) = \{K \leq G \mid [K, A] \leq K\} \\ C_{\mathcal{O}_G}(A)(H, K) &= C_{\mathcal{O}_G(H, K)}(A) = \{g \in N_G(H, K) \mid [g, A] \leq K\} / K \\ C_{\mathcal{O}_G(A)}(K) &= C_{\mathcal{O}_G(K)}(A) = \{g \in N_G(K) \mid [g, A] \leq K\} / K \end{aligned}$$

The category  $\mathcal{O}_{(G, A)}$  is defined to be the subcategory of the centralized subcategory with the same objects but with morphism sets, and automorphism groups

$$\mathcal{O}_{(G, A)}(H, K) = C_{N_G(H, K)}(A) / C_K(A), \quad \mathcal{O}_{(G, A)}(H) = C_{N_G(H)}(A) / C_H(A)$$

and with composition  $C_{N_G(H, K)}(A) / C_K(A) \times C_{N_G(K, L)}(A) / C_L(A) \rightarrow C_{N_G(H, L)}(A) / C_L(A)$  induced by composition in  $G$ . This is well-defined for if  $k \in C_K(A)$ ,  $y \in C_{N_G(K, L)}(A)$  then  $k^{y^a} = k^{ay} = k^y$  so  $k^y \in C_L(A)$ .

Let  $[K]$  be the set of objects in  $C_{\mathcal{O}_G}(A)$  isomorphic to the object  $K$ . There is a bijection

$$\{g \in N_G(K) \mid [g, A] \leq K\} \setminus \{g \in G \mid [g, A] \leq K\} \xrightarrow{K \rightarrow K^g} [K]$$

between  $[K]$  and the orbit set for the free left action of the group  $\{g \in N_G(K) \mid [g, A] \leq K\}$  on the set  $\{g \in G \mid [g, A] \leq K\}$ . Indeed, if  $g \in G$  and conjugation by  $g$  is an isomorphism  $K \rightarrow K^g$  in  $C_{\mathcal{O}_G}(A)$ , then  $K^g$  is normalized by  $A$  and  $[g, A] \leq K$  as we have the commutative square

$$\begin{array}{ccc} K & \xrightarrow{g} & K^g \\ a \downarrow & & \downarrow a \\ K & \xrightarrow{g} & K^g \end{array}$$

in  $\mathcal{O}_G$ . Conversely, if  $g \in G$  and  $[g, A] \leq K$ , then  $K^g$  is normalized by any  $a \in A$  as  $K^{ga} = K^{ag} = K^g$  and the above diagram commutes.

The free right action of  $N_G(K)/K = \mathcal{O}_G(K)$  on  $G/K$ ,  $gK \cdot xK = gxK$  for  $g \in G$ ,  $x \in N_G(K)$ , restricts to a free right action of  $C_{N_G(K)/K}(A) = C_{\mathcal{O}_G(K)}(A)$  on  $C_{G/K}(A)$ . The number of orbits for this action equals the number  $||[K]||$  of objects of  $C_{\mathcal{O}_G^p}(A)$  isomorphic to  $K$ . To see this we note that

$$|C_{G/K}(A)| = \frac{|\{g \in G \mid [g, A] \leq K\}|}{|K|}, \quad |C_{N_G(K)/K}(A)| = \frac{|\{g \in N_G(K) \mid [g, A] \leq K\}|}{|K|}$$

and thus

$$(4.17) \quad |[K]| = \frac{|\{g \in G \mid [g, A] \leq K\}|}{|\{g \in N_G(K) \mid [g, A] \leq K\}|} = \frac{|C_{G/K}(A)|}{|C_{N_G(K)/K}(A)|} = \frac{|C_{G/K}(A)|}{|C_{\mathcal{O}_G(K)}(A)|}$$

For later use we record that

$$(4.18) \quad |[K]| |C_{\mathcal{O}_G}(A)(K)| = |[K]| |C_{\mathcal{O}_G(K)}(A)| = |C_{G/K}(A)|$$

for any object  $K$  of  $C_{\mathcal{O}_G}(A)$ . Under the additional assumptions that  $A$  centralizes  $K$  and the order of  $A$  is prime to the order of  $K$ ,  $[K, A] = 1$  and  $(|K|, |A|) = 1$ , we have that  $[g, a] \leq K \iff [g, a] = 1$ . To see this, note that if  $g \in G$  and  $[g, a] \leq K$  then, in fact,  $[g, a] = 1$  is the neutral element of  $G$  as the order of  $[g, a]$  divides  $|K|$  since it is an element of  $K$  and also divides the order of  $A$  as  $[g, a^j] = [g, a]^j$  for any  $j \geq 0$  [28, 5.1.5.(ii)]. We conclude that

$$(4.19) \quad |[K]| |C_{\mathcal{O}_G}(A)(K)| = |C_{G/K}(A)| = \frac{|\{g \in G \mid [g, A] \leq K\}|}{|K|} = \frac{|C_G(A)|}{|K|} = |C_G(A) : K|$$

when  $[K, A] = 1$  and  $(|K|, |A|) = 1$ .

By using nonabelian cohomology groups we can get a similar result in a slightly different situation.

**Corollary 4.20.**  $\mathcal{O}_{(G,A)}^p = C_{\mathcal{O}_G^p}(A)$  when  $A$  is  $p$ -regular.

*Proof.* The map  $C_{N_G(H,K)}(A) \rightarrow C_{N_G(H,K)/K}(A)$  is surjective: Let  $x \in N_G(H, K)$  and suppose that the image of  $x$  in  $N_G(H, K)/K$  is centralized by  $A$ . Define a function  $A \rightarrow K : a \rightarrow k_a$  by  $x^a = xk_a$ . As  $k_{a_1 a_2} = k_{a_2} k_{a_1}^{a_2}$  this is a nonabelian 1-cocycle [30, §5]. But  $H^1(A; K) = \{*\}$  [30, Exercises p 59] so there is  $\ell \in K$  such that  $k_a = \ell \ell^{-a}$ . Then  $x\ell$  is centralized by  $A$ . This implies that  $\mathcal{O}_{(G,A)}(H, K) = C_{N_G(H,K)}(A)/C_K(A) = C_{N_G(H,K)/K}(A) = C_{\mathcal{O}_G}(A)(H, K)$  for any two  $A$ -normalized  $p$ -subgroups  $H, K \leq G$ .  $\square$

As a special case, suppose that  $V$  is a finite dimensional  $\mathbf{F}_p$ -vector space  $V$  and  $A \leq \mathrm{GL}(V)$ . The objects of the category  $\mathcal{O}_{(V,A)}$  are the  $\mathbf{F}_p[A]$ -submodules of the  $\mathbf{F}_p[\mathrm{GL}(V)]$ -module  $V$ . The set of morphisms between two  $\mathbf{F}_p[A]$ -submodules,  $U_1$  and  $U_2$ , is

$$(4.21) \quad C_{\mathcal{O}_V}(A)(U_1, U_2) = \begin{cases} C_V(A)/C_{U_2}(A) & U_1 \leq U_2 \\ \emptyset & U_1 \not\leq U_2 \end{cases}$$

$V$  is the terminal object in  $\mathcal{O}_{(V,A)}$ .

**Proposition 4.22.** Let  $A$  be a finite group acting on the finite dimensional  $\mathbf{F}_p$ -vector space  $V$ . The function

$$k_U = \begin{cases} |C_V(A)|^{-1} & \dim U = 0 \\ (p-1)|C_V(A)|^{-1} & \dim U = 1 = \dim C_U(A) \\ 0 & \text{otherwise} \end{cases}$$

is a coweighting for  $\mathcal{O}_{(V,A)}$ .

*Proof.* Let  $U_2$  be an  $\mathbf{F}_p[A]$ -submodule of  $V$ . Then

$$\sum_{\substack{U_1 \leq U_2 \\ [A, U_1] \leq U_1}} k_{U_1} |\mathcal{O}_{(V,A)}(U_1, U_2)| = \sum_{U_1 \leq C_{U_2}(A)} k_{U_1} |\mathcal{O}_{(V,A)}(U_1, U_2)| = \sum_{U_1 \leq C_{U_2}(A)} k_{U_1} \frac{|C_V(A)|}{|C_{U_2}(A)|} = \sum_{U_1 \leq C_{U_2}(A)} k_{U_1}^{C_{U_2}(A)} = 1$$

since

- $k_{U_1} \neq 0 \implies U_1 = C_{U_1}(A)$
- if  $U_1 \leq C_{U_2}(A) \leq U_2$  then  $C_{\mathcal{O}_V}(A)(U_1, U_2) = C_V(A)/C_{U_2}(A) = \mathcal{O}_{C_V(A)}(U_1, C_{U_2}(A))$  (4.21)
- $k_{U_1} \frac{|C_V(A)|}{|C_{U_2}(A)|}$  is the coweighting  $k_{U_1}^{C_{U_2}(A)}$  on  $\mathcal{O}_{C_{U_2}(A)}$  from Proposition 4.4

This shows that the function  $k_\bullet$  is a coweighting on  $\mathcal{O}_{(V,A)}$ .  $\square$



Let  $\mathcal{O}_{(V,A)}^{[1,V]}$  be the full subcategory of  $\mathcal{O}_{(V,A)}$  generated by all its object but the final object  $V$ .

**Lemma 4.23.** *Let  $A$  be a finite group acting on the finite dimensional  $\mathbf{F}_p$ -vector space  $V$ . Then*

$$\chi(\mathcal{O}_{(V,A)}^{[1,V]}) = \begin{cases} 0 & \dim V = 0 \\ p^{-1} & \dim V = 1 = \dim C_V(A) \\ 1 & \text{otherwise} \end{cases}$$

*Proof.* If  $\dim V = 0$ ,  $\mathcal{O}_{(V,A)}^{[1,V]}$  is the 0-object category which has Euler characteristics  $\chi = 0$ . If  $\dim V = 1$ ,  $\mathcal{O}_{(V,A)}^{[1,V]}$  is the 1-object category of the group  $C_V(A)$  so its Euler characteristic is  $p^{-1}$  if  $A$  acts trivially and 1 if not. If  $\dim V > 1$ ,  $\chi(\mathcal{O}_{(V,A)}^{[1,V]}) = \chi(\mathcal{O}_{(V,A)})$  because the coweighting for  $\mathcal{O}_{(V,A)}$  (Proposition 4.22) restricts to a coweighting for the right ideal  $\mathcal{O}_{(V,A)}^{[1,V]}$  [19, Remark 2.6] and it has value 0 at the deleted object  $V$ . But clearly,  $\chi(\mathcal{O}_{(V,A)}) = 1$  as this category has a final object.  $\square$

**Example 4.24.** When  $A$  is  $p$ -regular,  $\mathcal{O}_{(V,A)} = C_{\mathcal{O}_V}(A)$  by Corollary 4.20. However, when  $p = 2$ ,  $V = \mathbf{F}_2^2$  has dimension 2, and  $A = \Sigma_2 \leq \text{GL}(V)$  has order 2, then  $\chi(C_{\mathcal{O}_V^{[1,v]}}(A)) = \frac{1}{2}$  and  $\chi(\mathcal{O}_{(V,A)}) = 1$ . Thus these two categories are not equivalent.

**Proposition 4.25.** *Consider the centralized subcategory  $C_{\mathcal{O}_G^p}(A)$  and its subcategory  $\mathcal{O}_{(G,A)}^p$ .*

(1) *The function*

$$k_K = \frac{-\tilde{\chi}(C_{\mathcal{O}_V^{[1,V]}}(A))}{|C_{G/K}(A)|}, \quad V = K/\Phi(K)$$

*is a coweighting for the centralized orbit category  $C_{\mathcal{O}_G^p}(A)$ .*

(2) *The function*

$$k_K = \begin{cases} |C_G(A)|^{-1} & K = 1 \\ |C_G(A)|^{-1}(1 - p^{-1})|K| & K > 1 \text{ cyclic} \\ 0 & \text{otherwise} \end{cases}$$

*is a coweighting for  $\mathcal{O}_{(G,A)}^p$  and the Euler characteristic*

$$\chi(\mathcal{O}_{(G,A)}^p) = \frac{|C_G(A)_p|}{|C_G(A)|}$$

*is the density of  $p$ -singular elements in  $C_G(A)$ .*

*Proof.* Let  $K$  be an  $A$ -normalized  $p$ -subgroup of  $G$ . Any morphism  $xK \in C_{N_G(H,K)/K}(A) = C_{\mathcal{O}_G^p}(A)(H, K)$  is isomorphic in  $C_{\mathcal{O}_G^p}(A)//K$  to  $K \in C_{N_G(H^x,K)/K}(A) = C_{\mathcal{O}_G^p}(A)(H^x, K)$ . This observation shows that there are category equivalences

$$C_{\mathcal{O}_K}(A) \rightarrow C_{\mathcal{O}_G^p}(A)/K, \quad C_{\mathcal{O}_K^{[1,K]}}(A) \rightarrow C_{\mathcal{O}_G^p}(A)//K$$

and, using (4.18), it follows that

$$k_K = \frac{-\tilde{\chi}(C_{\mathcal{O}_K^{[1,K]}}(A))}{|[K]||C_{\mathcal{O}_G}(A)(K)|} = \frac{-\tilde{\chi}(C_{\mathcal{O}_V^{[1,v]}}(A))}{|[K]||C_{\mathcal{O}_G}(A)(K)|} = \frac{-\tilde{\chi}(C_{\mathcal{O}_V^{[1,v]}}(A))}{|C_{G/K}(A)|}, \quad V = V(K),$$

is a coweighting for  $C_{\mathcal{O}_G^p}(A)$  according to (equivariant versions of) [12, Theorem 3.7, Lemma 5.1.(c)]. Here,  $V = V(K) = K/\Phi(K)$  is the Frattini quotient with its inherited  $A$ -action.

Similarly, there are equivalences of categories

$$\mathcal{O}_{(K,A)} \rightarrow \mathcal{O}_{(G,A)}/K, \quad \mathcal{O}_{(K,A)}^{[1,K]} \rightarrow \mathcal{O}_{(G,A)}//K$$

and we get the coweighting for  $\mathcal{O}_{(G,A)}^p$

$$k_K = \frac{-\tilde{\chi}(\mathcal{O}_{(K,A)}^{[1,K]})}{|[K]||\mathcal{O}_{(G,A)}(K)|} = \frac{-\tilde{\chi}(\mathcal{O}_{(V,A)}^{[1,V]})}{|C_G(A) : C_K(A)|}$$

as there are  $|[K]| = |C_G(A) : C_{N_G(K)}(A)|$  objects of  $\mathcal{O}_{(G,A)}^p$  isomorphic to  $K$ . Since  $K$  is cyclic if and only if its Frattini quotient  $V = V(K)$  is cyclic [13, Corollary 5.1.2] we see from Lemma 4.23 that  $k_K$  is concentrated on the cyclic  $p$ -subgroups normalized by  $A$ , in fact on the cyclic  $p$ -subgroups centralized by  $A$ . Here we use that if  $A$

centralizes  $V(K)$  then  $A$  centralizes  $K$  [13, Theorem 5.3.5]. The coweighting takes value  $k_1 = |C_G(A)|^{-1}$  at the trivial subgroup and value

$$k_C = \frac{1 - p^{-1}}{|C_G(A) : C|} = \frac{1}{|C_G(A)|} (1 - p^{-1}) |C|$$

at a nontrivial cyclic subgroup  $C$  centralized by  $A$  by Equation (4.19). The counting formula of Lemma 4.3 now finishes the proof.  $\square$

**Example 4.26.** Let  $p = 2$ ,  $G = \Sigma_4$ , and  $A = \langle (1, 2)(3, 4) \rangle = C_2$ . Then  $A$  is *not*  $p$ -regular, the coweighting for  $C_{\mathcal{O}_G^p}(A)$  is *not* concentrated on the cyclic subgroups, and  $\chi(C_{\mathcal{O}_G^p}(A)) = 2/3$  does *not* equal the density, 1, of the  $p$ -singular elements in  $C_G(A) = D_8$ . Thus  $C_{\mathcal{O}_G^p}(A)$  and  $\mathcal{O}_{(G,A)}^p$  are not equivalent.

It is a curious fact, in contrast to the situation for Brown posets (Corollary 3.21.(3)), that the centralizer subcategory of the orbit category,  $C_{\mathcal{O}_G^p}(A)$ , and the orbit category of centralizer subgroup,  $\mathcal{O}_{C_G(A)}^p$ , have the same Euler characteristics when  $A$  is  $p$ -regular (Propositions 4.6 and 4.25).

**Proposition 4.27.** *Consider the centralized subcategory  $C_{\mathcal{O}_G^p}(A)$  and its subcategory  $\mathcal{O}_{(G,A)}^p$ .*

(1) *The function*

$$k^H = \frac{-\tilde{\chi}(C_{\mathcal{S}_{\mathcal{O}_G^p(H)}^{p+*}}(A))}{|C_{G/H}(A)|}$$

*is a weighting for  $C_{\mathcal{O}_G^p}(A)$  and the Euler characteristic is*

$$\chi(C_{\mathcal{O}_G^p}(A)) = \sum_H \frac{-\tilde{\chi}(C_{\mathcal{S}_{\mathcal{O}_G^p(H)}^{p+*}}(A))}{|C_{G/H}(A)|} = \sum_{[H]} \frac{-\tilde{\chi}(C_{\mathcal{S}_{\mathcal{O}_G^p(H)}^{p+*}}(A))}{|C_{\mathcal{O}_G(H)}(A)|}$$

*where the first sum runs over the set of objects  $H$  of  $C_{\mathcal{O}_G^{p+\text{rad}}}(A)$  and the second one over the set isomorphism classes of such objects.*

(2) *The function*

$$k^H = \frac{-\tilde{\chi}(C_{\mathcal{S}_{\mathcal{O}_G^p(H)}^{p+*}}(A))}{|C_G(A) : C_H(A)|}$$

*is a weighting for  $\mathcal{O}_{(G,A)}^p$  and the Euler characteristic is*

$$\chi(\mathcal{O}_{(G,A)}^p) = \sum_H \frac{-\tilde{\chi}(C_{\mathcal{S}_{\mathcal{O}_G^p(H)}^{p+*}}(A))}{|C_G(A) : C_H(A)|} = \sum_{[H]} \frac{-\tilde{\chi}(C_{\mathcal{S}_{\mathcal{O}_G^p(H)}^{p+*}}(A))}{|\mathcal{O}_{(G,A)}(H)|}$$

*where the first sum runs over the set of objects  $H$  of  $C_{\mathcal{O}_G^{p+\text{rad}}}(A)$  and the second one over the set isomorphism classes of such objects.*

*Proof.* As in the nonequivariant case [12, §8] there are homotopy equivalence of categories

$$H/C_{\mathcal{O}_G^p}(A) \rightarrow C_{\mathcal{S}_{\mathcal{O}_G^p(H)}^p}(A), \quad H//C_{\mathcal{O}_G^p}(A) \rightarrow C_{\mathcal{S}_{\mathcal{O}_G^p(H)}^{p+*}}(A)$$

which takes  $gK \in C_{N_G(H,K)/K}(A)$  to the nonidentity [28, 5.2.4]  $A$ -normalized subgroup  $N_{gK}(H)/H$  of  $N_G(H)/H$ . This subgroup is indeed  $A$ -normalized because if  $[g, A] \subseteq K$  then  $g^{-1}a = kag^{-1}$  for some  $k \in K$ . Then  $({}^gK)^a = K^{g^{-1}a} = K^{kag^{-1}} = K^{ag^{-1}} = K^{g^{-1}} = {}^gK$  for any  $a \in A$ . It now follows from [12, Theorem C] and Equation 4.18 that the function

$$k^H = \frac{-\tilde{\chi}(C_{\mathcal{S}_{\mathcal{O}_G^p(H)}^p}(A))}{|[H]||C_{\mathcal{O}_G(H)}(A)|} = \frac{-\tilde{\chi}(C_{\mathcal{S}_{\mathcal{O}_G^p(H)}^{p+*}}(A))}{|C_{G/H}(A)|}$$

is a weighting for  $C_{\mathcal{O}_G^p}(A)$ . Lemma 3.10 shows that the  $k^H = 0$  unless  $H$  is  $G$ -radical  $p$ -subgroup of  $G$ .

Similarly, there are homotopy equivalence of categories

$$H/\mathcal{O}_{(G,A)}^p \rightarrow C_{\mathcal{S}_{\mathcal{O}_G^p(H)}^p}(A), \quad H//\mathcal{O}_{(G,A)}^p \rightarrow C_{\mathcal{S}_{\mathcal{O}_G^p(H)}^{p+*}}(A)$$

and the function

$$k^H = \frac{-\tilde{\chi}(C_{\mathcal{S}_{\mathcal{O}_G^p(H)}^p}(A))}{|[H]||\mathcal{O}_{(G,A)}(H)|} = \frac{-\tilde{\chi}(C_{\mathcal{S}_{\mathcal{O}_G^p(H)}^{p+*}}(A))}{|C_G(A) : C_H(A)|}$$

is a coweighting for  $\mathcal{O}_{(G,A)}^p$ .  $\square$

We can now generalize the global formula below Proposition 4.7 to an equivariant situation.

*Proof of Theorem 1.1.(1).* The equation

$$|C_G(A)_p| + \sum_H \tilde{\chi}(C_{\mathcal{S}_{\mathcal{O}_G(H)}^{p+*}}(A)) |C_H(A)| = 0$$

simply states that the Euler characteristic of  $\mathcal{O}_{(G,A)}^p$ , the density of the  $p$ -singular elements in  $C_G(A)$  (Proposition 4.25.(2)), is the sum of the values of the weighting from Proposition 4.27.(2).  $\square$

The equation of Theorem 1.1.(1) can be reformulated as

$$\sum_{K \in C_{\mathcal{S}_G^{p+\text{rad}}(A)/N_G(A)}} k^{[K]} C_{\mathcal{S}_G^{p+\text{rad}}(A)}(1, [K]) |C_K(A)| = |C_G(A)_p|$$

where by Proposition 3.13

$$\left( C_{\mathcal{S}_G^{p+\text{rad}}(A)}(H, [K]) \right)_{H, K \in C_{\mathcal{S}_G^{p+\text{rad}}(A)/N_G(A)}} \begin{pmatrix} \vdots \\ k^{[K]} \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ 1 \\ \vdots \end{pmatrix}$$

and  $C_{\mathcal{S}_G^{p+\text{rad}}(A)}(1, [K])$  is the length of the  $N_G(A)$ -orbit through  $K \in C_{\mathcal{S}_G^{p+\text{rad}}(A)}$ . The weighting values are  $k^{[K]} = -\tilde{\chi}(C_{\mathcal{S}_{N_G(K)/K}^{p+*}}(A))$  by Lemma 3.11; in particular,  $k^{[1]} = -\tilde{\chi}(C_{\mathcal{S}_G^{p+*}}(A))$ .

**Example 4.28.** Let  $p = 2$  and  $G = \Sigma_5$  the symmetric group on 5 things considered as an  $A = \langle (1, 2, 3) \rangle$ -group with conjugation action. The centralizer  $C_G(A) = A \times \langle (4, 5) \rangle$  is cyclic of order 6 and contains  $2 = |C_G(A)_2|$  2-singular elements. The poset of  $A$ -normalized 2-subgroups

$$C_{\mathcal{S}_G^{2+\text{rad}}(A)} = \{H_4, H_3, H_2, H_1\} = \{ \langle (1, 4)(2, 3), (1, 3)(2, 4) \rangle, \langle (1, 2)(3, 5), (1, 3)(2, 5) \rangle, \langle (4, 5) \rangle, \langle 1 \rangle \}$$

consists of four subgroups:  $H_3$  and  $H_4$  are four-groups,  $H_2$  is  $O_2 C_G(A)$ , and  $H_1$  is the trivial subgroup. The orbits for the  $N_G(A)$ -action on  $C_{\mathcal{S}_G^{2+\text{rad}}(A)}$  are  $\{H_3, H_4\}, \{H_2\}, \{H_1\}$ . The weighting  $k^{[H]} = -\tilde{\chi}(C_{\mathcal{S}_{N_G(H)/H}^{2+*}}(A))$  for  $C_{\mathcal{S}_G^{2+\text{rad}}(A)/N_G(A)}$  is the solution to the linear equation

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

and the vector  $(|C_{H_i}(A)|)_{i=3,2,1}$  is  $(1, 2, 1)$ . The sum on the left hand side of the equation of Theorem 1.1.(1) is

$$k^{[H_3]} \cdot |[H_3]| \cdot |C_{H_3}(A)| + k^{[H_2]} \cdot |[H_2]| \cdot |C_{H_2}(A)| + k^{[H_1]} \cdot |[H_1]| \cdot |C_{H_1}(A)| = 1 \cdot 2 \cdot 1 + 1 \cdot 1 \cdot 2 + (-2) \cdot 1 \cdot 1 = 2$$

which indeed equals the number  $|C_G(A)_2|$  of 2-singular elements of  $C_G(A)$ . Alternatively, Equation (1.2) applied directly to  $C_G(A) = C_6$  has the form

$$\sum_{H \in \mathcal{S}_{C_G(A)}^{2+\text{rad}}} -\chi(\mathcal{S}_{C_G(A)/H}^{2+*}) |H| = -\chi(\mathcal{S}_{C_3}^{2+*}) |C_2| = 1 \cdot 2 = |C_G(A)_2|$$

as  $\mathcal{S}_{C_6}^{2+\text{rad}} = \{O_2 C_6\}$  has but one element. This shows that these two formulas are quite different from one another.

**Example 4.29.** Let  $K = \text{SL}_m(\mathbf{F}_q)$ ,  $m \geq 2$ , and  $\bar{K} = \text{SL}_m(\bar{\mathbf{F}}_q)$  where  $q$  is a power of  $p$ . The involutory graph automorphism  $\gamma$  of  $K$  or  $\bar{K}$  maps  $g$  to  $A^{-1}(g^t)^{-1}A$  where  $A$  is the  $(m \times m)$ -matrix with  $(+1, -1, \dots, \pm 1)$  in the diagonal from upper right to lower left corner [14, §2.7]. Let  $\Pi$  denote the set of fundamental roots of the root system  $\Sigma$  for the algebraic group  $\text{SL}_m(\bar{\mathbf{F}}_p)$ . Let  $\tau$  be the linear automorphism of  $\mathbf{R}^m$  given by  $a_i^\tau = -a_{m+1-i}$  where  $a_i$  is the standard basis vector. Then  $\alpha_i^\tau = \alpha_{m-i}$  and  $x_\alpha(t)^\gamma = x_{\alpha^\tau}(t)$  where  $\alpha \in \Pi$  is any fundamental root and  $x_\alpha$  the root group parameterization [14, Definition 1.9.4].

The centralizer  $C_{\text{SL}_m(\mathbf{F}_q)}(\gamma)$  of  $\gamma$  in  $A_{m-1}(q) = \text{SL}_m(\mathbf{F}_q)$  is  $B_{(m-1)/2}(q) = \text{SO}_m(\mathbf{F}_q) \leq \text{SL}_m(\mathbf{F}_q)$  if  $m$  is odd and  $C_{m/2}(q) = \text{Sp}_m(\mathbf{F}_q) \leq \text{SL}_m(\mathbf{F}_q)$  if  $m$  is even [14, §2.7]. It seems to be well-known, although I have not been able to find an explicit reference in the literature, that in fact there is an isomorphism of posets

$$C_{\text{SL}_m(\mathbf{F}_q)}^{p+\text{rad}}(\gamma) \rightarrow \mathcal{S}_{\text{SL}_m(\mathbf{F}_q)}^{p+\text{rad}}(\gamma)$$

induced by the map  $P_J \rightarrow C_{P_J}(\gamma)$  defined on the set of standard  $\gamma$ -invariant parabolic subgroups  $P_J$  where  $J \subset \Pi$  is any  $\tau$ -invariant set of fundamental roots. Consequently,

$$\chi(C_{\mathcal{S}_{\text{SL}_m(\mathbb{F}_q)}^{p^{**}+\text{rad}}}(\gamma)) = \begin{cases} \chi(\mathcal{S}_{\mathcal{C}_{\text{SO}_m(\mathbb{F}_q)}^{p^{**}+\text{rad}}}^{p^{**}+\text{rad}}) & m \text{ odd} \\ \chi(\mathcal{S}_{\mathcal{C}_{\text{Sp}_m(\mathbb{F}_q)}^{p^{**}+\text{rad}}}^{p^{**}+\text{rad}}) & m \text{ even} \end{cases}$$

and the equivariant relations of Theorem 1.1.(1)–(2) for the centralized subposet  $C_{\mathcal{S}_{\text{SL}_m(\mathbb{F}_q)}^{p^{**}+\text{rad}}}(\gamma)$  are simply the corresponding nonequivariant relations for  $\mathcal{S}_{\mathcal{C}_{\text{SO}_m(\mathbb{F}_q)}^{p^{**}}}^{p^{**}}$  and  $\mathcal{S}_{\mathcal{C}_{\text{Sp}_m(\mathbb{F}_q)}^{p^{**}}}^{p^{**}}$  studied in Example 4.11.

This section ends with two proofs of Theorem 1.1.(3). The first proof is similar to that of Theorem 4.10, and the second one is simply a rewriting of Quillen's proof of Brown's theorem [27, Corollary 4.2].

*Proof of Theorem 1.1.(3) using 1.1.(1).* According to Theorem 1.1.(1),

$$|C_G(A)_p| + \tilde{\chi}(C_{\mathcal{S}_G^{p^{**}}}(A)) + \sum_{H \neq 1} \tilde{\chi}(C_{\mathcal{S}_{\mathcal{O}_G(H)}^{p^{**}}}(A)) |C_H(A)| = 0$$

with summation over all nonidentity  $A$ -normalized  $p$ -subgroups of  $G$ , or, equivalently,

$$|C_G(A)_p| + \tilde{\chi}(C_{\mathcal{S}_G^{p^{**}}}(A)) + \sum_{[H] \neq [1]} \frac{\tilde{\chi}(C_{\mathcal{S}_{\mathcal{O}_G(H)}^{p^{**}}}(A))}{|\mathcal{O}_{(G,A)}(H)|} |C_G(A)| = 0$$

with summation over all isomorphism classes of objects of  $\mathcal{O}_{(G,A)}^{p^{**}}$ . Write each term of the sum as a product

$$\frac{\tilde{\chi}(C_{\mathcal{S}_{\mathcal{O}_G(H)}^{p^{**}}}(A))}{|\mathcal{O}_{(G,A)}(H)|} |C_G(A)| = \frac{\tilde{\chi}(C_{\mathcal{S}_{\mathcal{O}_G(H)}^{p^{**}}}(A))}{|\mathcal{O}_{(G,A)}(H)|_p} \frac{|C_G(A)|}{|\mathcal{O}_{(G,A)}(H)|_{p'}}$$

of two factors. Both these factors are integers. The first factor is an integer because, by induction, we may assume that  $|C_{\mathcal{O}_G(H)}(A)|_p = |C_{\mathcal{O}_G}(A)(H)|_p$  divides the numerator. Then also  $|\mathcal{O}_{(G,A)}(H)|_p$  divides the numerator as the automorphism group of  $H$  in  $\mathcal{O}_{(G,A)}$  is a subgroup of the automorphism group of  $H$  in  $C_{\mathcal{O}_G}(A)$ . The second factor is an integer because the automorphism group of  $H$  in  $\mathcal{O}_{(G,A)}$  is a subquotient of  $C_G(A)$ . This second factor is clearly also divisible by  $|C_G(A)|_p$ .

We now know that  $|C_G(A)|_p$  divides the sum. As it also divides the number  $|C_G(A)_p|$  of  $p$ -singular elements in  $C_G(A)$  by Frobenius' Theorem 4.1, we conclude that it divides the reduced Euler characteristic of the  $A$ -centralized Brown poset  $C_{\mathcal{S}_G^{p^{**}}}(A)$ .  $\square$

*Proof of Theorem 1.1.(3) following Quillen [27].* Let  $B$  be any nonidentity  $p$ -subgroup of  $C_G(A)$ . Consider the  $B$ -poset  $Y = \text{sd } C_{\mathcal{S}_G^{p^{**}}}(A)$  that is the subdivision of the  $A$ -centralized Brown poset. Then  $\tilde{\chi}(Y) = \tilde{\chi}(C_{\mathcal{S}_G^{p^{**}}}(A))$ . Define

$$Y' = \bigcup_{X \in \mathcal{S}_B^*} C_Y(X)$$

be the subposet of  $Y$  consisting of all elements  $y \in Y$  with a nontrivial isotropy subgroup  $B_y$ . Define  $Z$  to be the subposet of  $\mathcal{S}_B^* \times Y'$  given by

$$Z = \{(X, y) \in \mathcal{S}_B^* \times Y' \mid C_Y(X) \ni y\} = \{(X, y) \in \mathcal{S}_B^* \times Y' \mid X \leq B_y\}$$

of pairs  $(X, y)$ ,  $1 \leq X \leq B$ ,  $y \in Y'$ , such that  $X$  fixes  $y$ , or, equivalently,  $X$  is contained in the isotropy subgroup at  $y$ . When  $1 \leq X \leq B$ ,

$$\{y \in Y' \mid C_Y(X) \ni y\} = C_Y(X) = \text{sd } C_{C_{\mathcal{S}_G^{p^{**}}}(A)}(X)$$

is conically contractible:  $H \leq XH \geq X$ . When  $y \in Y'$ ,

$$\{X \in \mathcal{S}_B^* \mid (X, y) \in Z\} = \{X \in \mathcal{S}_B^* \mid X \leq B_y\} = \mathcal{S}_{B_y}^*$$

is contractible because  $B_y$  is the largest element. Now [27, Corollary 1.8] (a variant of Quillen's Theorem A) shows that  $Y'$ ,  $Z$ , and  $\mathcal{S}_B^*$  are homotopy equivalent; in fact, they are contractible as  $\mathcal{S}_B^*$ , with largest element  $B$ , is so. It follows that the reduced Euler characteristic  $\tilde{\chi}(Y) = \chi(Y, *) = \chi(Y, Y')$ . This relative Euler characteristic is divisible by  $|B|$  because  $B$  acts freely on  $Y - Y'$ .  $\square$

**Corollary 4.30.** *Let  $A$  be a  $p$ -regular group acting on  $G$ . Then*

$$\chi(\{K \in C_{\mathcal{S}_G^{p^{**}}}(A) \mid C_K(A) \neq K\}) \equiv \chi(\{K \in C_{\mathcal{S}_G^{p^{**}}}(A) \mid 1 \not\leq C_K(A) \leq K\}) \pmod{|C_G(A)|_p}$$

*Proof.* By Corollary 3.21.(3), the difference between these two integers is the difference between the reduced Euler characteristics of the centralizer subposet and the poset of the centralizer subgroup, both of which are divisible by  $|C_G(A)|_p$  (Theorems 4.2 and 1.1.(3)).  $\square$

## 5. CONJECTURES OF ALPERIN AND KNÖRR–ROBINSON

This section contains a brief comment on the Alperin Weight Conjecture.

**Definition 5.1** (KRC $_p$  and AWC $_p$ ). *The finite group  $G$  satisfies the*

- *Knörr–Robinson conjecture at  $p$  if [20, 37]*

$$\chi_2(\mathcal{S}_G^{p+*}, G) = k(G) - z_p(G), \quad -\tilde{\chi}_2(\mathcal{S}_G^{p+*}, G) = z_p(G)$$

- *Alperin weight conjecture at  $p$  if [2]*

$$k_{p'}(G) = \sum_{[P] \in \mathcal{S}_G^{p+\text{rad}}/G} z_p(N_G(P)/P)$$

where the sum runs over the set  $\mathcal{S}_G^{p+\text{rad}}/G$  of conjugacy classes  $[P]$  of  $G$ -radical  $p$ -subgroups  $P$  of  $G$ .

The two conjectures are equivalent in the sense that [20] [37, Thm 3.1]

$$\forall H \in \mathcal{S}_G^p: \text{KRC}_p(N_G(H)/H) \iff \forall H \in \mathcal{S}_G^p: \text{AWC}_p(N_G(H)/H)$$

for the given finite group  $G$ .

It is sometimes possible to verify the Knörr–Robinson conjecture by machine computations as we have seen that  $\text{KRC}_p(G)$  is equivalent to any of the following three equivalent conditions

$$(5.2) \quad \sum_{[x] \in [G]} \tilde{\alpha}_2(\mathcal{S}_G^{p+*}, G)(x) = -z_p(G)$$

$$(5.3) \quad \sum_{[C] \in \mathcal{S}_G^{p'+\text{cyc}}/G} \tilde{\alpha}_2(\mathcal{S}_G^{p+*}, G)(C) \frac{\sum_{x \in C - \{1\}} |C_G(x)|}{|N_G(C)|} = -z_p(G)$$

$$(5.4) \quad \sum_{[A] \in \mathcal{S}_G^{p'+\text{abe}}/G} \tilde{\chi}(C_{\mathcal{S}_G^{p+*}}(A)) \frac{\varphi_2(A)}{|N_G(A)|} = -z_p(G)$$

where

- In Equation (5.2), quoting Equation (2.12), the value at  $x \in G$  of the reduced class function  $\tilde{\alpha}_2(\mathcal{S}_G^{p+*}, G)$  is

$$\tilde{\alpha}_2(\mathcal{S}_G^{p+*}, G)(x) = \sum_{[y] \in [C_G(x)]} \frac{\tilde{\chi}(C_{\mathcal{S}_G^{p+*}}(\langle x, y \rangle))}{|C_G(\langle x, y \rangle)|} = \tilde{\chi}_1(C_{\mathcal{S}_G^{p+*}}(x), C_G(x)) = \tilde{\chi}(\text{sd}(C_{\mathcal{S}_G^{p+*}}(x))/C_G(x))$$

and  $\mathcal{S}_G^{p+*}$  can be replaced by the smaller posets  $\mathcal{S}_G^{p+*+\text{rad}}$  of  $G$ -radical or  $\mathcal{S}_G^{p+*+\text{eab}}$  of elementary abelian  $p$ -subgroups (Proposition 3.12, Definition 2.10, Equation (2.11), Corollary 2.14)

- Equation (5.3), repeating Equation (3.16), uses the Artin coefficients,  $\tilde{\alpha}_2(\mathcal{S}_G^{p+*}, G)(C)$ , introduced in Corollary 3.15.(3)
- In Equation (5.4) the sum runs over all conjugacy classes of  $p$ -regular abelian subgroups  $A$  of  $G$  (Corollary 3.15.(1)) and  $C_{\mathcal{S}_G^{p+*}}(A)$  may be replaced by  $C_{\mathcal{S}_G^{p+*+\text{eab}}}(A)$  or  $C_{\mathcal{S}_G^{p+*+\text{rad}}}(A)$  (Proposition 3.12).

**Example 5.5.**  $\text{KRC}_p(G)$  is true when  $G$  has a cyclic Sylow  $p$ -subgroup  $S$  [37, Example 1.4]. The discrete  $G$ -poset  $\mathcal{S}_G^{p+*+\text{rad}}$  is the right  $G$ -set  $N_G(S) \backslash G$  of  $G$ -conjugates of  $S$ , the discrete poset  $C_{\mathcal{S}_G^{p+*+\text{rad}}}(x)$  is the set  $C_{N_G(S) \backslash G}(x)$  for any  $x \in G$ , and (the unreduced form of) Equation (5.2) takes the form

$$\sum_{[x] \in [G]} |C_{N_G(S) \backslash G}(x)/C_G(x)| = k(G) - z_p(G)$$

in this special case.

$\text{KRC}_p(G)$  is true also if  $O_p(G) \neq 1$  [37, Example 1.4]. This is because  $z_p(G) = 0$  by Ito's theorem [10, Corollary 53.18] and  $\tilde{\chi}(C_{\mathcal{S}_G^{p+*}}(A)) = 0$  for all  $A$  in Equation (5.4) by Lemma 3.10. In particular the Knörr–Robinson conjecture at  $p$  is true for  $\text{GL}_n(\mathbf{F}_q)$  when  $p$  divides  $q - 1$  as these groups have a central element of order  $p$ . The conjecture also holds for  $\text{GL}_n(\mathbf{F}_q)$  when  $q$  is a power of  $p$  [36, Proposition 4.1] and for all solvable groups [37, p 195].

$ x $	1	2	3	4	5	6	8	8	11	11	$\Sigma$
$\tilde{\alpha}_2(\mathcal{S}_G^{2+*}, G)$	0	0	1	0	-1	0	0	0	-1	-1	-2
$\tilde{\alpha}_2(\mathcal{S}_G^{3+*}, G)$	0	1	0	1	-1	0	0	0	-1	-1	-1
$\tilde{\alpha}_2(\mathcal{S}_G^{5+*}, G)$	0	0	-1	1	0	-1	-1	-1	-1	-1	-5
$\tilde{\alpha}_2(\mathcal{S}_G^{11+*}, G)$	0	-1	-1	-1	3	-1	-1	-1	0	0	-3

TABLE 1. The reduced class functions  $\tilde{\alpha}_2(\mathcal{S}_G^{p+*}, G)$  for  $G = M_{11}$ 

$ C $	1	2	3	4	5	6	8	11	$\cdot$
$\sum_{x \neq 1}  C_G(x) / N_G(C) $	0	1	1	4	1	8	6	2	
$\tilde{a}_2(\mathcal{S}_G^{2+*}, G)$	320	0	1	0	-1	0	0	-1	-2
$\tilde{a}_2(\mathcal{S}_G^{3+*}, G)$	375	-2	0	1	-1	0	0	-1	-1
$\tilde{a}_2(\mathcal{S}_G^{5+*}, G)$	-296	1	2	2	0	-1	-1	-1	-5
$\tilde{a}_2(\mathcal{S}_G^{11+*}, G)$	-1463	6	2	0	3	-1	-1	0	-3

TABLE 2. The reduced Artin coefficients  $\tilde{a}_2(\mathcal{S}_G^{p+*}, G)$  for  $G = M_{11}$ 

We now use a computer to check Equations (5.2)–(5.4) in case of the Mathieu group  $G = M_{11}$ . The computations of Example 5.6 were carried out using Magma [5].

**Example 5.6.** The Mathieu group  $G = M_{11}$  of order  $|G| = 7,920 = 2^4 \cdot 3^2 \cdot 5^1 \cdot 11^1$  contains  $k(G) = 10$  conjugacy classes of orders  $(1, 2, 3, 4, 5, 6, 8, 8, 11, 11)$  and 8 classes of cyclic groups of orders  $(1, 2, 3, 4, 5, 6, 8, 11)$ . There are  $z_p(G) = (2, 1, 5, 3)$  irreducible complex representations of  $p$ -defect 0 for each of the prime divisors  $p(G) = (2, 3, 5, 11)$  of the group order. We now verify Equations (5.2)–(5.4) at all primes  $p \in p(G)$ .

Table 1 displays the values of the reduced class functions  $\tilde{\alpha}_2(\mathcal{S}_G^{p+*}, G)$  at each of the 10 conjugacy classes  $x$ . The column to the right contains the row sum. By Equation (5.2),  $\text{KRC}_p(G)$  is true (as it is for  $p = 5, 11$  by Example 5.5) if and only if this column contains  $(-z_p(G))_{p \in p(G)}$ .

Table 2 shows the reduced Artin coefficients  $\tilde{a}_2(\mathcal{S}_G^{p+*}, G)(C)$  of Corollary 3.15.(3) at each of the 8 classes of cyclic subgroups  $C$ . The top row contains the vector with coordinates  $\sum_{x \in C - \{1\}} |C_G(x)|/|N_G(C)|$  (all nonnegative integers by Proposition 3.17). The column to the right shows the standard inner product of the row with the top row. For instance, the second row means that

$$\tilde{a}_2(\mathcal{S}_G^{2+*}, G) = \frac{4}{99} \cdot 1_1^G + \frac{1}{12} 1_3^G - \frac{1}{4} 1_5^G - \frac{1}{5} 1_{11}^G, \quad 0 \cdot 320 + 1 \cdot 0 + 1 \cdot 1 + 4 \cdot 0 + 1 \cdot (-1) + 8 \cdot 0 + 6 \cdot 0 + 2 \cdot (-1) = -2$$

at the prime  $p = 2$ . (Recall from Corollary 3.15.(3) that the reduced Euler class function  $\tilde{\alpha}_2(\mathcal{S}_G^{p+*}, G)$  is the linear combination of the Artin coefficients at  $[C]$  divided by  $|N_G(C) : C|$  which here equals  $(7920, 24, 12, 4, 4, 2, 2, 5)$ .) By Equation (5.3),  $\text{KRC}_p(G)$  is true (as it is for  $p = 5, 11$  by Example 5.5) if and only if the rightmost column contains  $(-z_p(G))_{p \in p(G)}$ .

Thirdly, the four tables of Figure 1 list, for each prime divisor  $p \in p(G)$ , the  $p$ -part of the order of  $C_G(A)$ , the Euler characteristic  $-\tilde{\chi}(C_{\mathcal{S}_G^{p+*}}(A))$  of the  $A$ -centralized subposet, the number  $\varphi_2(A)$  of generating pairs of  $A$ , the length  $|G : N_G(A)|$  of  $A$ , and, in the bottom row, the product of these last three numbers for each conjugacy class of abelian  $p$ -regular subgroups  $A$  of  $G$ ;  $\text{KRC}_p(G)$  is true if and only if the sum of the integers of the bottom row equals  $z_p(G)|G|$ . (The row containing  $|C_G(A)|_p$  plays no role here but serves to illustrate the divisibility statement of Theorem 1.1.(3).)

In each of these four tables the sum of the numbers of the bottom row is  $z_p(M_{11})|M_{11}|$  and thus Equation (5.3) holds for  $G = M_{11}$  for all primes  $p \in p(G)$ .

Similar machine computations demonstrate that  $\text{KRC}_p(G)$  is true when  $G = M_{11}, M_{12}, M_{22}, M_{23}, M_{24}$  is one of the five Mathieu groups or  $G = J_1, J_2, J_3$  one of the three first Janko groups and  $p$  is any prime divisor of  $|G|$ .

## 6. THE ORBIT CATEGORY $\pi$ -SUBGROUPS

Let  $\pi$  is a nonempty set of primes. In this short section we consider the orbit category  $\mathcal{O}_G^\pi$  of  $\pi$ -singular subgroups of  $G$ . The arguments here will be minor variations of the ones used in the case considered above where  $\pi$  consisted of a single prime.

$A$	1	3	5	11	$3 \times 3$				
$ C_G(A) _2$	16	2	1	1	1				
$-\tilde{\chi}(C_{S_G^{2+*}}(A))$	496	-8	1	1	1				
$\varphi_2(A)$	1	8	24	120	48				
$ G : N_G(A) $	1	220	396	144	55				
	496	-14080	9504	17280	2640				
$A$	1	2	5	11	4	8	$2 \times 2$		
$ C_G(A) _3$	9	3	1	1	1	1	1		
$-\tilde{\chi}(C_{S_G^{3+*}}(A))$	-54	-6	1	1	-2	0	-2		
$\varphi_2(A)$	1	3	24	120	12	48	6		
$ G : N_G(A) $	1	165	396	144	495	495	330		
	-54	-2970	9504	17280	-11880	0	-3960		
$A$	1	2	3	4	6	8	11	$2 \times 2$	$3 \times 3$
$ C_G(A) _5$	5	1	1	1	1	1	1	1	1
$-\tilde{\chi}(C_{S_G^{5+*}}(A))$	-395	-11	1	-3	1	1	1	1	1
$\varphi_2(A)$	1	3	8	12	24	48	120	6	48
$ G : N_G(A) $	1	165	220	495	660	495	144	330	55
	-395	-5445	1760	-17820	15840	23760	17280	1980	2640
$A$	1	2	3	4	5	6	8	$2 \times 2$	$3 \times 3$
$ C_G(A) _{11}$	11	1	1	1	1	1	1	1	1
$-\tilde{\chi}(C_{S_G^{11+*}}(A))$	-143	1	1	1	-3	1	1	1	1
$\varphi_2(A)$	1	3	8	12	24	24	48	6	48
$ G : N_G(A) $	1	165	220	495	396	660	495	330	55
	-143	495	1760	5940	-28512	15840	23760	1980	-640

FIGURE 1. Tables for Equation (5.4) for  $G = M_{11}$  at  $p = 2, 3, 5, 11$ 

A left ideal in an EI-category  $\mathcal{C}$  is a full subcategory  $\mathcal{I}$  of  $\mathcal{C}$  if  $a \in \mathcal{I}$ ,  $\mathcal{C}(a, b) \neq \emptyset \implies b \in \mathcal{I}$  for all objects  $a, b$  of  $\mathcal{C}$ . Similarly, a right ideal is a full subcategory  $\mathcal{J}$  of  $\mathcal{C}$  if  $b \in \mathcal{J}$ ,  $\mathcal{C}(a, b) \neq \emptyset \implies a \in \mathcal{J}$  for all objects  $a, b$  of  $\mathcal{C}$ . If  $\mathcal{I}$  be a left and  $\mathcal{J}$  a right ideal in  $\mathcal{C}$ . Then

$$(6.1) \quad \chi(\mathcal{C}) - \chi(\mathcal{I}) = \sum_{a \in \mathcal{C} - \mathcal{I}} k^a, \quad \chi(\mathcal{C}) - \chi(\mathcal{J}) = \sum_{a \in \mathcal{C} - \mathcal{J}} k_b$$

where  $k^\bullet$  and  $k_\bullet$  are the unique isomorphism invariant weighting and coweighting on  $\mathcal{C}$  [12, Theorem 3.7]. If the weighting is concentrated on  $\mathcal{I}$  then  $\chi(\mathcal{C}) = \chi(\mathcal{I})$  and if the coweighting is concentrated on  $\mathcal{J}$  then  $\chi(\mathcal{C}) = \chi(\mathcal{J})$ .

With more sophisticated methods it is possible to generalize Corollary 4.5.

**Lemma 6.2.** *For any finite group  $K$*

$$-\tilde{\chi}(\mathcal{O}_K^{[1, K]}) = \begin{cases} \phi(|K|)/|K| & K \text{ cyclic} \\ 0 & K \text{ noncyclic} \end{cases}$$

where  $\phi$  is Eulers  $\phi$ -function.

*Proof.* The coweighting on the contractible category  $\mathcal{O}_K$  is  $k_B = |K|^{-1} \sum_{A \leq B} |A| \mu(A, B)$  [19, Theorem 2.18.(3)]. The subcategory  $\mathcal{O}_K^{[1, K]}$  of proper subgroups is a right ideal so by Equations (6.1)

$$1 = \chi(\mathcal{O}_K) = \chi(\mathcal{O}_K^{[1, K]}) + k_K$$

When  $K$  is not cyclic then  $k_K = 0$  by [22, Proposition 2.8] and the formula follows in this case.

Next, assume that  $K$  is cyclic. For any subgroup  $H$  of  $K$ ,  $H//\mathcal{O}_K^{[1, K]}$  is the interval  $(1, K/H)$  in the poset  $\mathcal{S}_K$ ,  $-\tilde{\chi}(H//\mathcal{O}_K^{[1, K]}) = -\tilde{\chi}(1, K/H) = -\mu(|K|/|H|)$  and

$$-|K| \tilde{\chi}(\mathcal{O}_K^{[1, K]}) = |K| - |K| \chi(\mathcal{O}_K^{[1, K]}) = \sum_{1 \leq H \leq K} \mu(|K|/|H|) |H| = \phi(|K|)$$

by classical Möbius inversion.  $\square$



**Lemma 6.3.** *The function*

$$k_K = \frac{-\tilde{\chi}(\mathcal{O}_K^{[1,K]})}{|G : K|}$$

is a coweighting for  $\mathcal{O}_G^\pi$  and the Euler characteristic

$$\chi(\mathcal{O}_G^\pi) = \frac{|G_\pi|}{|G|}$$

is the density of the  $\pi$ -singular elements in  $G$ .

*Proof.* As we saw in the proof of Proposition 4.6, the category  $\mathcal{O}_G^\pi//K = \mathcal{O}_G//K$  is equivalent to  $\mathcal{O}_K^{[1,K]}$  for any object  $K$  of  $\mathcal{O}_G^\pi$ . This gives the coweighting by [12, Theorem 3.7] as  $|\mathcal{O}_G^\pi(K)||[K]| = |N_G(K) : K||G : N_G(K)| = |G : K|$ . Now apply Lemma 6.2 to compute the Euler characteristic as the sum of the values of the coweighting.  $\square$

**Lemma 6.4.** *The function*

$$k^H = \frac{-\tilde{\chi}(H//\mathcal{S}_G^\pi)}{|G : H|}$$

is a weighting for  $\mathcal{O}_G^\pi$  and the Euler characteristic is

$$\chi(\mathcal{O}_G^\pi) = \sum_{H \in \mathcal{S}_G^\pi} \frac{-\tilde{\chi}(H//\mathcal{S}_G^\pi)}{|G : H|} = \sum_{[H] \in \mathcal{S}_G^\pi/G} \frac{-\tilde{\chi}(H//\mathcal{S}_G^\pi)}{|N_G(H) : H|}$$

*Proof.* The objects of the category  $H//\mathcal{O}_G^\pi$  are left cosets  $gK$  where  $g$  is an element of  $G$ ,  $K$  is a  $\pi$ -singular subgroup of  $G$ , and  $H \leq {}^gK$ . The functor  $H//\mathcal{O}_G^\pi \rightarrow H//\mathcal{S}_G^\pi$  taking  $gK$  to  ${}^gK$  is an equivalence of categories since it is surjective on objects and bijective on morphism sets as  $(H//\mathcal{O}_G^\pi)(g_1K_1, g_2K_2) = \mathcal{S}_G^\pi(g_1K_1, g_2K_2)$ . This gives the weighting on  $\mathcal{O}_G^\pi$  by [12, Theorem 3.7].  $\square$

**Lemma 6.5.** *The function  $k^H = -\tilde{\chi}(H//\mathcal{S}_G^\pi) = \sum_{H \leq K \in \mathcal{S}_G^\pi} \mu(H, K)$  is the weighting on  $\mathcal{S}_G^\pi$ .*

*Proof.* The Möbius function  $\mu$  for  $\mathcal{S}_G$  restricts to the Möbius function for the right ideal  $\mathcal{S}_G^\pi$ . Thus the weighting is  $k^H = \sum_{H \leq K \in \mathcal{S}_G^\pi} \mu(H, K)$ . The weighting may also be expressed as  $-\chi(H//\mathcal{S}_G^\pi)$  [12, Theorem 3.7].  $\square$

We can now generalize Equations (3.3) and (4.8) to sets  $\pi$  of maybe more than one prime.

**Corollary 6.6.** *For any finite group  $G$ , any set  $\pi$  of primes, and any  $\pi$ -subgroup  $H \in \mathcal{S}_G^\pi$  of  $G$ ,*

$$\sum_{K \in H/\mathcal{S}_G^\pi} -\tilde{\chi}(K//\mathcal{S}_G^\pi) = 1, \quad \sum_{H \in \mathcal{S}_G^\pi} -\tilde{\chi}(H//\mathcal{S}_G^\pi)|H| = |G_\pi|$$

*Proof.* The weighting for  $\mathcal{S}_G^\pi$  restricts to a weighting for the contractible left ideal  $H/\mathcal{S}_G^\pi$ . Thus  $1 = \sum_{K \in H/\mathcal{S}_G^\pi} k^K = \sum_{K \in H/\mathcal{S}_G^\pi} -\tilde{\chi}(K//\mathcal{S}_G^\pi) = \sum_{[K] \in \mathcal{S}_G^\pi/K} \mathcal{S}_G^\pi(H, [K])(-\tilde{\chi}(K//\mathcal{S}_G^\pi))$ . This proves the first equation. Combine Lemmas 6.3 and 6.4 to prove the second equation by computing the Euler characteristic of  $\mathcal{O}_G^\pi$  in two different ways.  $\square$

Corollary 6.6 shows that the linear equation

$$(6.7) \quad (\mathcal{S}_G^\pi(H, [K]))_{H, [K] \in \mathcal{S}_G^\pi/G} \begin{pmatrix} \vdots \\ -\tilde{\chi}(K//\mathcal{S}_G^\pi) \\ \vdots \end{pmatrix}_{K \in \mathcal{S}_G^\pi/G} = \begin{pmatrix} \vdots \\ 1 \\ \vdots \end{pmatrix}$$

determines the weighting for  $k^{[K]} = -\tilde{\chi}(K//\mathcal{S}_G^\pi) : \mathcal{S}_G^\pi/G \rightarrow \mathbf{Z}$  for  $\mathcal{S}_G^\pi/G$  (Definition 2.6). The next proposition offers a sufficient condition for the vanishing of  $k^{[K]}$ .

**Proposition 6.8.** *If  $H$  is a Sylow  $\pi$ -subgroup in  $G$  then  $H//\mathcal{S}_G^\pi$  is empty and  $-\tilde{\chi}(H//\mathcal{S}_G^\pi) = 1$ . If  $H$  is not a Sylow  $\pi$ -subgroup of  $O(G, H) = \bigcap_{K \in H//\mathcal{S}_G^\pi} N_G(K)$  then  $H//\mathcal{S}_G^\pi$  is contractible and  $-\tilde{\chi}(H//\mathcal{S}_G^\pi) = 0$ .*

*Proof.* The first assertion is clear. In the second case, the assumption is that  $G$  possesses a  $\pi$ -subgroup  $L > H$  normalizing all  $\pi$ -subgroups  $K \geq H$ . But then  $K \leq KL \geq L$  is a contraction of  $H//\mathcal{S}_G^\pi$ .  $\square$

**Example 6.9.** The simple group  $G = \mathrm{GL}_3(\mathbf{F}_2)$  of order 168 contains 12 conjugacy classes of  $\pi$ -subgroups where  $\pi = \{2, 3\}$ . The table

$ H $	24	24	12	12	8	6	4	4	4	3	2	1
$-\tilde{\chi}(H//\mathcal{S}_G^\pi)$	1	1	0	0	-1	-1	0	0	0	0	4	-48
$ G : N_G(H) $	7	7	7	7	21	28	7	7	21	28	21	1
$ N_G(H) : H _\pi$	1	1	2	2	1	1	6	6	2	2	4	24

shows the orders and lengths of the conjugacy classes of  $\pi$ -subgroups  $H$  of  $G$  together with the weighting values  $k^H = -\tilde{\chi}(H//\mathcal{S}_G^\pi)$  and the  $\pi$ -part of  $|N_G(H) : H|$ . The criterion of Proposition 6.8 accounts for four of the six 0s in the first row of the table and for the two 1s at the Sylow  $\pi$ -subgroups (abstractly isomorphic to  $\Sigma_4$ ). These Euler characteristics were determined as the solution to the linear equation (6.7)

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 0 & 1 & 3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 1 & 1 & 0 & 3 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 3 & 3 & 1 & 1 & 5 & 4 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 7 & 7 & 7 & 7 & 21 & 28 & 7 & 7 & 21 & 28 & 21 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ -1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 4 \\ -48 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

where the  $(12 \times 12)$ -matrix is the table  $(\mathcal{S}_G^\pi(H, [K]))_{H, K \in \mathcal{S}_G^{\pi+\text{rad}}/G}$ . In particular,  $-\tilde{\chi}(1//\mathcal{S}_G^\pi) = -\tilde{\chi}(\mathcal{S}_G^{\pi+*}) = -48$  and  $\chi(\mathcal{S}_G^{\pi+*}) = 1 - (-48) = 49$ . Since

$$\sum_{[H] \in \mathcal{S}_G^\pi/G} -\tilde{\chi}(H//\mathcal{S}_G^\pi) |G : N_G(H)| |H| = 1 \cdot 7 \cdot 24 + \dots + (-48) \cdot 1 \cdot 1 = 120$$

and  $G$  indeed contains 120  $\pi$ -singular elements the numbers of this table agrees with Corollary 6.6.

An even more general form of the second formula of Corollary 6.6 was first proved in [16, Theorem 6.3]. In the table of Example 6.9 it happens that  $-\tilde{\chi}(H//\mathcal{S}_G^\pi)$  is always divisible by the  $\pi$ -part of  $|N_G(H) : H|$ . This is no coincidence as shown by the following generalization of Brown's theorem.

**Theorem 6.10.** [16, Corollary 3.9]  $|N_G(H) : H|_\pi \mid -\tilde{\chi}(H//\mathcal{S}_G^\pi)$  for any  $\pi$ -subgroup  $H$  of  $G$ .

When  $\pi$  consists of just one prime the Euler characteristic of  $H//\mathcal{S}_G^\pi$  is a function of the quotient group  $N_G(H)/H$  (Proposition 4.7) but this is not true when  $\pi$  has more than one element.

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INSTITUT FOR MATEMATISKE FAG, UNIVERSITETSPARKEN 5, DK-2100 KØBENHAVN

E-mail address: [moller@math.ku.dk](mailto:moller@math.ku.dk)

URL: <http://www.math.ku.dk/~moller>