On the coverings of Euclidian manifolds $\mathcal{B}_1$ and $\mathcal{B}_2$

G. Chelnokov
grishabenruven@yandex.ru

M. Deryagina*
Sobolev Institute of Mathematics, Russia
Moscow State University of Technologies and
Management named after K.G. Razumovskiy, Russia
madinaz@rambler.ru

A. Mednykh†
Sobolev Institute of Mathematics,
Novosibirsk, Russia
Novosibirsk State University,
Novosibirsk, Russia
Chelyabinsk State University,
Chelyabinsk, Russia
mednykh@math.nsc.ru

Abstract

There are only 10 Euclidean three dimensional forms, that is, compact locally Euclidean 3-manifold without boundary. Six of them, $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$, $\mathcal{G}_4$, $\mathcal{G}_5$, $\mathcal{G}_6$, are orientable and the other four, $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_3$, $\mathcal{B}_4$, are non-orientable manifolds. Following J. H. Conway and J. P. Rossetti, we call them platycosms. In the present paper, a new algebraic method is given to classify and enumerate n-fold coverings over platycosms. We decribe all types of n-fold coverings over $\mathcal{B}_1$ and over $\mathcal{B}_2$, and calculate the numbers of non-equivalent coverings of each type. Recall that the manifolds $\mathcal{B}_1$ and $\mathcal{B}_2$ are uniquely determined among the others non-orientable forms by their homology groups $H_1(\mathcal{B}_1) = \mathbb{Z}_2 \times \mathbb{Z}^2$ and $H_1(\mathcal{B}_2) = \mathbb{Z}^2$.

Key Words: Euclidean form, platycosm, amphicosms, flat 3-manifold, non-equivalent coverings, crystallographic group.

2010 Mathematics Subject Classification: 20H15, 57M10, 55R10.

*This work was supported by the Russian Foundation for Basic Research (grant 13-01-00513).

†This work was supported by the Laboratory of Quantum Topology, Chelyabinsk State University, under contract no. 14.Z50.31.0020 with the Ministry of Education and Science of the Russian Federation, and the Russian Foundation for Basic Research (grant 15-01-07906).

Introduction

Let $\mathcal{N}$, $\mathcal{N}'$ and $\mathcal{M}$ be connected manifolds. Two coverings $\rho : \mathcal{N} \longrightarrow \mathcal{M}$ and $\rho' : \mathcal{N}' \longrightarrow \mathcal{M}$ are *equivalent* if there exists a homeomorphism $\eta : \mathcal{N} \longrightarrow \mathcal{N}'$ such that $\rho = \rho' \circ \eta$.

From general theory of covering spaces, it follows that any n-fold covering over $\mathcal{M}$ corresponds to the unique subgroup of index n in the fundamental group $\pi_1(\mathcal{M})$. Two coverings are equivalent if and only if the corresponding subgroups are conjugate in $\pi_1(\mathcal{M})$ (see [2] p. 67). The equivalence classes of n-fold covering of N are in one-to-one correspondence with the conjugacy classes of subgroups of index n in the fundamental group $\pi_1(N)$.

The number of subgroups of a given index in the free group F_r was determined by M. Hall [3]. An explicit formula for the number of conjugacy classes of subgroups of a given index in F_r was obtained by V. A. Liskovets [4]. Both problems for the fundamental group Γ_g of a closed orientable surface of genus g were completely resolved in [7] and [8] respectively. For the fundamental group Φ_p of a closed non-orientable surface of genus p they were resolved in [10]. See paper [11] for a short proof of the above mentioned results. In the paper [5] the results were extended to some 3-dimensional manifolds which are circle bundles over a surface.

Following Conway-Rossetti ([1]), we use the term platycosm ("flat universe") for a compact locally Euclidean 3-manifold without boundary. In the present paper we suggest a new algebraic method to classify n-fold coverings over amphicosms and enumerate them. It is important to emphasize that numerical methods to solve these problems was developed by the Bilbao group [15].

Platycosms are the simplest alternative universes for us to think of living in [6]. (Jeff Weeks created the program, which allows to "fly" through platycosms and other spaces.) If you lived in a small enough platycosm, you would appear to be surrounded by images of yourself which can be arranged in one of ten essentially different ways. (See [1]).

There are six orientable platicosms, denoted in Wolfs notation $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$, $\mathcal{G}_4$, $\mathcal{G}_5$, $\mathcal{G}_6$ and four non-orientable manifolds $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_3$, $\mathcal{B}_4$. The aim of this paper is to classify types of n-fold coverings over $\mathcal{B}_1$ and over $\mathcal{B}_2$, and calculate the numbers of non-equivalent coverings of each type. We classify all types of subgroups in the fundamental groups $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively, and calculate the numbers of conjugated classes of each type of subgroups for index n.

Notations

During this paper we will use the following notations: $s_G(n)$ is the total number of subgroups of index n in the group G, $s_{H,G}(n)$ is the number of subgroups of index n in the group G, isomorphic to the group H. The same way $c_G(n)$ is the total number of conjugancy classes of subgroups of index n in the group G, $c_{H,G}(n)$ is the number conjugancy classes of subgroups of index n in the group G, isomorphic to the group H.

$$\sigma_0(n) = \sum_{k|n} 1 \quad \text{if } n \text{ is natural, } \sigma_0(n) = 0 \text{ otherwise,}$$

$$\sigma_1(n) = \sum_{k|n} k \quad \text{if } n \text{ is natural, } \sigma_1(n) = 0 \text{ otherwise,}$$

$$\sigma_2(n) = \sum_{k|n} \sigma_1(k) \quad \text{if } n \text{ is natural, } \sigma_2(n) = 0 \text{ otherwise.}$$

The epimorphism $\phi : \pi_1(\mathcal{B}_1) \to \mathbb{Z}^2$ is determined in **??**, the epimorphism $\psi : \pi_1(\mathcal{B}_2) \to \mathbb{Z}^2$ is determined in **??**.

The invariant of a subgroup $\Delta \leqslant \pi_1(\mathcal{B}_1)$, $l(\Delta)$ is determined in **??**, its analog for $\Delta \leqslant \pi_1(\mathcal{B}_2)$ is determined in **??**.

The invariants $\rho(\Delta)$ and $\varepsilon(\Delta)$, applied to non-abelian subgroup $\Delta \leqslant \pi_1(\mathcal{B}_1)$ only, are determined after **??**, analogous ones for $\Delta \leqslant \pi_1(\mathcal{B}_2)$ are determined after **??**.

In this paper we widely use the the summing $\sum\limits_{2k|n}$. We consider it equals zero if n is odd since in this case the sum is taken over the empty set of terms.

One can find the correspondence between Wolf's and Conway-Rossetti's notations of flat 3-manifold and its fundamental groups in Table 1. We use Wolf's notation.

name	Conway-Rosetti	other names	Wolf	fund.group (internatl. no name)
torocosm	c_1	3-torus	$\mathcal{G}_1$	$1.P1$
dicosm	c_2	half turn space	$\mathcal{G}_2$	$4.P2_1$
tricosm	c_3	one-third turn space	$\mathcal{G}_3$	$144.P3_1$ $145.P3_2$
tetracosm	c_4	quarter turn space	$\mathcal{G}_4$	$76.P4_1$ $78.P4_3$
hexacosm	c_6	one-sixth turn space	$\mathcal{G}_5$	$169.P6_1$ $170.P6_5$
didicosm	c_{22}	Hantzsche-Wendt space	$\mathcal{G}_6$	$19.P2_12_12_1$
first amphicosm	$+a_1$	Klein bottle times circle	$\mathcal{B}_1$	$7.Pc$
second amphicosm	$-a_1$		$\mathcal{B}_2$	$9.Cc$
first amphidicosm	$+a_2$		$\mathcal{B}_3$	$29.Pca2_1$
second amphidicosm	$-a_2$		$\mathcal{B}_4$	$33.Pa2_1$

Table 1

1 The brief overview of achieved results

Since the problem of enumeration of n-fold coverings reduces to the problem of enumeration of conjugacy classes of some subgroups, it is natural to expect that the enumeration of subgroups without respect of conjugacy would be helpful. The next theorem provides the complete solution of this problem. For brevity we use notation Pc for $\pi_1(\mathcal{B}_1)$ and Cc for $\pi_1(\mathcal{B}_2)$, see Table 1.

Theorem 1. *Every subgroup Δ of finite index n in $\pi_1(\mathcal{B}_1)$ is isomorphic to either $\mathbb{Z}^3$, or $\pi_1(\mathcal{B}_1)$, or $\pi_1(\mathcal{B}_2)$, and*

(i)
$$s_{\mathbb{Z}^3,\pi_1(\mathcal{B}_1)}(n) = \sum_{2l|n} \sigma_1(l)l,$$

(ii)
$$s_{\pi_1(\mathcal{B}_1),\pi_1(\mathcal{B}_1)}(n) = \sum_{l|n} \sigma_1(\frac{n}{l})l - \sum_{2l|n} \sigma_1(\frac{n}{2l})l,$$

(iii)
$$s_{\pi_1(\mathcal{B}_2),\pi_1(\mathcal{B}_1)}(n) = \sum_{2l|n} 2l\sigma_1(\frac{n}{2l}) - \sum_{4l|n} 2l\sigma_1(\frac{n}{4l}).$$

To prove this theorem we need the following propositions. To prove this theorem we need the following propositions. **??** presents a canonical form of elements in $\pi_1(\mathcal{B}_1)$ and introduce an invariant $\phi(\Delta)$. **??** provides a necessary and sufficient condition that the subgroup $\Delta \leqslant \pi_1(\mathcal{B}_1)$ is abelian in terms of $\phi(\Delta)$. This proposition also provides the fact that all abelian subgroups of finite index in $\pi_1(\mathcal{B}_1)$ belongs to one largest abelian subgroup (this is certainly false for abelian subgroups of infinite index). Finally, **??** show that 4-plet of introduced invariants $((l(\Delta), \phi(\Delta), \rho(\Delta), \varepsilon(\Delta))$ determine a non-abelian subgroup uniquely and **??** describes the type of subgroup in terms of $\varepsilon(\Delta)$.

Theorem 2 provides the total number of conjugacy classes of subgroups of index n in $\pi_1(\mathcal{B}_1)$, thus the total number of non-equivalent n-fold coverings of $\mathcal{B}_1$.

Theorem 2. *The total number of non-equivalent n-fold coverings over $\mathcal{B}_1$ is*

$$c_{\pi_1(\mathcal{B}_1)}(n) = \frac{1}{n} \sum_{\substack{l|n \\ lm=n}} (\varphi_3(l) \sum_{2k|m} \sigma_1(k)k + \sum_{d|l} \mu\left(\frac{l}{d}\right)(2,d)d^2 \sum_{k|m} (\sigma_1(\frac{m}{k}) - \sigma_1(\frac{m}{2k}))k +$$
$$+\varphi_2(l) \sum_{2k|m} (\sigma_1(\frac{m}{2k}) - \sigma_1(\frac{m}{4k}))2k)\,,$$

where $\mu(k)$ is the Möbius function, $\varphi_3(l)$ and $\varphi_2(l)$ are Jordan totient functions,$(2,d)$ is a greater common divisor of numbers 2 and d.

The next theorem provides the number of conjugacy classes of subgroups of index n in $\pi_1(\mathcal{B}_1)$ with respect of the isomorphism type of a subgroup, thus the number of n-fold coverings of $\mathcal{B}_1$ with respect of the isomorphism type of a cover.

Theorem 3. *Let $\mathcal{N} \to \mathcal{B}_1$ be an n-fold covering over $\mathcal{B}_1$. If n is odd then $\mathcal{N}$ is homeomorphic to $\mathcal{B}_1$. If n is even then $\mathcal{N}$ is homeomorphic to $\mathcal{G}_1$ or $\mathcal{B}_1$ or $\mathcal{B}_2$. The corresponding numbers of nonequivalent coverings are given by the following formulas:*

(i)
$$c_{\mathbb{Z}^3,\pi_1(\mathcal{B}_1)}(n) = \sum_{2l|n} \sum_{m|\frac{n}{2l}} \Big(l^2 + \frac{5}{2} + \frac{3}{2}(-1)^l\Big)m,$$

(ii)
$$c_{\pi_1(\mathcal{B}_1),\pi_1(\mathcal{B}_1)}(n) = \sum_{l|n} \Big(\frac{3}{2} + \frac{1}{2}(-1)^l\Big)\sigma_1(\frac{n}{l}) - \sum_{2l|n} \Big(\frac{3}{2} + \frac{1}{2}(-1)^l\Big)\sigma_1(\frac{n}{2l}),$$

(iii)
$$c_{\pi_1(\mathcal{B}_2),\pi_1(\mathcal{B}_1)}(n) = 2\Big(\sum_{2l|n} \sigma_1(\frac{n}{2l}) - \sum_{4l|n} \sigma_1(\frac{n}{4l})\Big).$$

Despite **??** covers **??** we present both, since the proof of **??** can be also obtained from other considerations, see **??**.

The results towards the manifold $\mathcal{B}_2$ follows much the same way. **??**, **??** and **??** are similar to **??**, **??** and **??** respectively.

Theorem 4. *Every subgroup Δ of finite index n in $\pi_1(\mathcal{B}_2)$ is isomorphic to either $\mathbb{Z}^3$, or $\pi_1(\mathcal{B}_2)$, or $\pi_1(\mathcal{B}_1)$, and*

(i)
$$s_{\mathbb{Z}^3,\pi_1(\mathcal{B}_2)}(n) = \sum_{2l|n} \sigma_1(l)l,$$

(ii)
$$s_{\pi_1(\mathcal{B}_2),\pi_1(\mathcal{B}_2)} = \begin{cases} \sum_{4k|n} 2k\big(\sigma_1(\frac{n}{2k}) - \sigma_1(\frac{n}{4k})\big) & \textit{if } n \textit{ is even} \\ \sum_{k|n} \sigma_1(\frac{n}{k})k & \textit{if } n \textit{ is odd} \end{cases}$$

(iii)
$$s_{\pi_1(\mathcal{B}_1),\pi_1(\mathcal{B}_2)} = \sum_{2k|n} 2k\sigma_1(\frac{n}{2k}) - \sum_{4k|n} 2k\sigma_1(\frac{n}{4k}).$$

Theorem 5. *The total number of non-equivalent n-fold coverings over $\mathcal{B}_2$ is*

$$c_{\pi_1(\mathcal{B}_2)} = \frac{1}{n} \sum_{\substack{l|n \\ lm=n}} \Big(\varphi_3(l) \sum_{2k|m} \sigma_1(k)k + s_{\pi_1(\mathcal{B}_1),\pi_1(\mathcal{B}_2)}(m) \sum_{d|l} \mu\left(\frac{l}{d}\right)(2,d)d^2 + \varphi_2(l)s_{\pi_1(\mathcal{B}_2),\pi_1(\mathcal{B}_2)}(m)\Big),$$

where $\mu(t)$ is the Möbius function, $\varphi_3(l)$ and $\varphi_2(l)$ are Jordan totient functions,$(2,d)$ is a greater common divisor of numbers 2 and d.

Theorem 6. *Let $\mathcal{N} \to \mathcal{B}_2$ be an n-fold covering over $\mathcal{B}_2$. If n is odd then $\mathcal{N}$ is homeomorphic to $\mathcal{B}_2$. If n is even then $\mathcal{N}$ is homeomorphic to $\mathcal{G}_1$ or $\mathcal{B}_1$ or $\mathcal{B}_2$. The corresponding numbers of nonequivalent coverings are given by the following formulas:*

(i) $$c_{\mathbb{Z}^3,\pi_1(\mathcal{B}_2)}(n) = \frac{1}{2}\sum_{2l|n}\sum_{m|\frac{n}{2l}}\Big(l^2+\frac{3}{2}+\frac{1}{2}(-1)^l+(-1)^{\frac{n}{2lm}}+(-1)^{l+\frac{n}{2lm}}\Big)m,$$

(ii) $$c_{\pi_1(\mathcal{B}_2),\pi_1(\mathcal{B}_2)}(n) = \begin{cases} \sum\limits_{4k|n}(\sigma_1(\frac{n}{2k})-\sigma_1(\frac{n}{4k})) & \textit{if } n \textit{ is even} \\ \sum\limits_{l|n}\sigma_1(\frac{n}{l}) & \textit{if } n \textit{ is odd} \end{cases}$$

(iii) $$c_{\pi_1(\mathcal{B}_1),\pi_1(\mathcal{B}_2)}(n) = 2\Big(\sum_{2k|n}\sigma_1(\frac{n}{2k})-\sum_{4k|n}\sigma_1(\frac{n}{4k})\Big).$$

2 Preliminaries

Consider manifold, referred as $\mathcal{B}_1$ and $\mathcal{B}_2$ in [17], also as $+a_1$ and $-a_1$ in [1].

Let us remark that $\mathcal{B}_1$ can be considered as a Seifert fiber space. $\mathcal{B}_1$ is the trivial S^1-bundle over Klein bottle $\mathcal{K}$, so $\mathcal{B}_1 = \mathcal{K} \times S^1$. Its fundamental group $\pi_1(\mathcal{B}_1)$ is

$$\pi_1(\mathcal{B}_1) = \pi_1(\mathcal{K}) \times \pi_1(S^1) = \Lambda \times \mathbb{Z}, \tag{2.1}$$

where Λ is a fundamental group of Klein bottle.

From [1] the fundamental group $\pi_1(\mathcal{B}_1)$, can be represented in the form

$$\pi_1(\mathcal{B}_1) = \langle W, X, Z : X^{-1}ZX = W^{-1}ZW = Z^{-1}, W^{-1}Z^{-1}WZ = 1\rangle. \tag{2.2}$$

It is also a crystallographic group Pc.

The fundamental group $\pi_1(\mathcal{B}_2)$, can be represented in the form

$$\pi_1(\mathcal{B}_2) = \langle W', X', Z' : X'^{-1}Z'X' = W'^{-1}Z'W' = Z'^{-1}, W'^{-1}Z'^{-1}W'Z' = Z'\rangle. \tag{2.3}$$

It is also a crystallographic group Cc.

For our convenience we replace $X = a$, $W = ca$, $Z = b$ and obtain

$$\pi_1(\mathcal{B}_1) = \langle a, b, c : cac^{-1}a^{-1} = cbc^{-1}b^{-1} = 1, aba^{-1}b = 1\rangle. \tag{2.4}$$

Substituting $X' = \alpha$, $Z' = \beta^{-1}$ and $W' = \alpha\gamma$, we get

$$\pi_1(\mathcal{B}_2) = \langle \alpha, \beta, \gamma : \gamma\beta\gamma^{-1} = \alpha\gamma\alpha^{-1}\gamma^{-1} = \beta, \alpha\beta\alpha^{-1} = \beta^{-1}\rangle. \tag{2.5}$$

A 3-torus is denoted by $\mathcal{G}_1$ in our paper. Note that the fundamental group $\pi_1(\mathcal{G}_1)$ is represented in the form

$$\pi_1(\mathcal{G}_1) = \mathbb{Z}^3 \tag{2.6}$$

It is also a crystallographic group $P1$.

3 On the coverings of $\mathcal{B}_1$

3.1 The structure of the group $\pi_1(\mathcal{B}_1)$

The following proposition provides the canonical form of an element in $\pi_1(\mathcal{B}_1)$.

Proposition 1. *(i) Each element of $\pi_1(\mathcal{B}_1)$ can be represented in the canonical form $a^x b^y c^z$ for some integer x, y, z.*

(ii) The product of two canonical forms is given by the formula

$$a^x b^y c^z \cdot a^{x'} b^{y'} c^{z'} = a^{x+x'} b^{(-1)^{x'} y + y'} c^{z+z'}. \tag{3.7}$$

(iii) The canonical epimorphism $\phi : \pi_1(\mathcal{B}_1) \to \pi_1(\mathcal{B}_1)/\langle b \rangle \cong \mathbb{Z}^2$, given by the formula $a^x b^y c^z \to (x, z)$ is well-defined.

(iv) The representation in the canonical form $a^x b^y c^z$ for each element is unique.

Proof. The part (i) follows from the part (ii), part (ii) can be verified directly. The part (iii) is equivalent the fact that the subgroup $\langle b \rangle$ is a normal subgroup of $\pi_1(\mathcal{B}_1)$, which fact immediately follows from the representation **??** of $\pi_1(\mathcal{B}_1)$. For the proof of the part (iv) consider arbitrary element $\pi_1(\mathcal{B}_1)$ and its an arbitrary representation of this element in the canonical form. The values of x and z are uniquely defined by (iii). Thus y is also uniquely defined, since b is an element of finite order in Pc otherwise. $\square$

Lemma 1. *Let Δ be a subgroup of index n in $\pi_1(\mathcal{B}_1)$. By $l(\Delta)$ denote the minimal positive integer, such that $b^{l(\Delta)} \in \Delta$. Such an $l(\Delta)$ exists, and satisfy the relation $l(\Delta) \cdot [\pi_1(\mathcal{B}_1) : \phi(\Delta)] = n$.*

Proof. If $l(\Delta)$ does not exists then all elements $b, b^2, b^3, \dots$ belong to mutually different cosets of Δ in $\pi_1(\mathcal{B}_1)$, thus the index of Δ is infinite, which is a contradiction.

Let $g_1, \dots g_k$ be such elements of $\pi_1(\mathcal{B}_1)$, that $\phi(g_1), \dots \phi(g_k)$ is a complete system of right coset representatives of $\phi(\Delta)$ in $\mathbb{Z}^2$. Then $\{g_i b^j \mid 1 \le i \le k, 0 \le j \le l(\Delta) - 1\}$ is a complete system of right coset representatives of Δ in $\pi_1(\mathcal{B}_1)$. Thus $l(\Delta) \cdot [\pi_1(\mathcal{B}_1) : \phi(\Delta)] = [\pi_1(\mathcal{B}_1) : \Delta] = n$. $\square$

The next proposition shows that the introduced above invariant $\phi(\Delta)$ is sufficient to determine whether Δ is abelian or not.

Proposition 2. *Let Δ be a subgroup of finite index in $\pi_1(\mathcal{B}_1)$. Then Δ is abelian if and only if $\phi(\Delta) \leqslant \{(2x, z) | \ x, z \in \mathbb{Z}\}$.* [1]

Proof. Since

$$a^{2x} b^y c^z \cdot a^{2x'} b^{y'} c^{z'} = a^{2x+2x'} b^{y+y'} c^{z+z'} = a^{2x'} b^{y'} c^{z'} \cdot a^{2x} b^y c^z,$$

[1]In other words, Δ is abelian iff $\Delta \leqslant \langle a^2, b, c \rangle$

the if part is obvious. The inequality $a^{2x+1}b^yc^z \cdot b^{l(\Delta)} \neq b^{l(\Delta)} \cdot a^{2x+1}b^yc^z$ proves the only if part. Indeed, by **??** $a^{2x+1}b^yc^z \cdot b^{l(\Delta)} = a^{2x+1}b^{y+l(\Delta)}c^z$ and $b^{l(\Delta)} \cdot a^{2x+1}b^yc^z = a^{2x+1}b^{y-l(\Delta)}c^z$, this expressions are not equal since the order of b is infinite. $\square$

As a corollary of **??** we obtain the value of $s_{\mathbb{Z}^3,\pi_1(\mathcal{B}_1)}(n)$.

Corollary 1. *The number of subgroups of index n in $\pi_1(\mathcal{B}_1)$, which are isomorphic to $\mathbb{Z}^3$, is given by the formula:*

$$s_{\mathbb{Z}^3,\pi_1(\mathcal{B}_1)}(n) = \sum_{2l|n} \sigma_1(l)l.$$

Proof. Let Δ be a subgroup of index n in $\pi_1(\mathcal{B}_1)$ isomorphic to $\mathbb{Z}^3$, then by **??** $\phi(\Delta) \leqslant \{(2x, z)|\ x, z \in \mathbb{Z}\}$. Thus from the definition of ϕ it follows that $\Delta \leqslant < a^2, b, c > \cong \mathbb{Z}^3$. Since $[\pi_1(\mathcal{B}_1) :< a^2, b, c >] = 2$, we have $[< a^2, b, c >: \Delta] = n/2$. The number of subgroups in $\mathbb{Z}^3$ of index $n/2$ is well known (see, for instance, [5] Corollary 4.4) and equals to $\sum_{k|\frac{n}{2}} \sigma_1(k)k$. $\square$

The following lemmas are technical statements, needed to introduce the important invariant ν.

Lemma 2. *Let Δ be a subgroup of a finite index in $\pi_1(\mathcal{B}_1)$. Then $\phi(\Delta) \cong \mathbb{Z}^2$.*

Proof. The index $[\phi(\pi_1(\mathcal{B}_1)) : \phi(\Delta)]$ divides $[\pi_1(\mathcal{B}_1) : \Delta]$, thus it is finite. Any subgroup of finite index in $\mathbb{Z}^2$ is necessarily isomorphic to $\mathbb{Z}^2$. $\square$

Notation. By $\overline{v} = (x_v, z_v)$ and $\overline{u} = (x_u, z_u)$ denote a pair of generators of $\phi(\Delta)$, where $\phi(\Delta)$ is considered as a subgroup of $\{(x, z) \mid x \in \mathbb{Z}, z \in \mathbb{Z}\}$.

Lemma 3. *Any subgroup of finite index in $\langle a^2, b, c\rangle$ is isomorphic to $\mathbb{Z}^3$.*

Proof. The subgroup of finite index in $\mathbb{Z}^3$ is isomorphic to $\mathbb{Z}^3$. $\square$

Lemma 4. *Let $(x, z) \in \phi(\Delta)$. Then there exist an integer number $\mu(x, z)$, $0 \leq \mu(x, z) \leq l(\Delta) - 1$, such that for all $a^xb^yc^z \in \Delta$ we have $y \equiv \mu(x, z) \mod l(\Delta)$.*

Proof. Assume the converse is true. This means that there exist $g = a^xb^yc^z \in \Delta$ and $h = a^xb^{y'}c^z \in \Delta$, such that $y \not\equiv y' \mod l(\Delta)$. Since $h^{-1}g \in \Delta$, $b^{y-y'} \in \Delta$, that contradicts the minimality of $l(\Delta)$. $\square$

Lemma 5. *Assume $\phi(\Delta) \nleqslant \{(2x, z)|\ x, z \in \mathbb{Z}\}$, then one can choose the generators $\overline{v} = (x_v, z_v)$ and $\overline{u} = (x_u, z_u)$ in such a way that x_v is odd and x_u is even.*

Proof. At least one x_v and x_u is odd, otherwise the group Δ, generated by $\overline{v} = (x_v, z_v)$ and $\overline{u} = (x_u, z_u)$ is a subgroup of $\{(2x, z)|\ x, z \in \mathbb{Z}\}$. Without loss of generality suppose x_v is odd. If x_u is odd replace $\overline{u}$ with $\overline{u} + \overline{v}$. $\square$

From now on we fix some $\overline{v}$ and $\overline{u}$, chosen in this way.

Notation. Let $p, q \in \mathbb{Z}$, put $(x, z) = p\overline{v} + q\overline{u}$. Denote $\nu(p, q) = \mu(x, z)$.

Remark. Consider an arbitrary element of $\phi(\Delta)$, $w = (x, z) = p\overline{v} + q\overline{u}$. Let $a^{px_v+qx_u}b^yc^{pz_v+qz_u} \in \Delta$ be an arbitrary preiamge of w under ϕ . Then by definition $\nu(p, q) \equiv y \mod l(\Delta)$.

Lemma 6 (almost additivity)**.** $\nu(s+2p, t+q) \equiv \nu(s,t) + \nu(2p; q) \mod l(\Delta)$.

Proof. Let $g, h \in \Delta$ be a preimages of elements $s\overline{v} + t\overline{u}$ and $2p\overline{v} + q\overline{u}$ respectively under the homomorphism ϕ. In other words, $g = a^{sx_v+tx_u}b^{\nu(s,t)+kl}c^{sz_v+tz_u}$ and $h = a^{2px_v+qx_u}b^{k'l}c^{2pz_v+qz_u}$. Then $gh = a^{(s+2p)x_v+(t+s)x_u}b^{\nu(s,t)+\nu(2p,q)+(k+k')l(\Delta)}c^{(s+2p)z_v+(t+s)z_u}$ by the formula **??**. Thus $\nu(s+2p, t+q) \equiv \nu(s,t) + \nu(2p; q) \mod l(\Delta)$ by the definition of $\nu(s+2p, t+q)$. □

Lemma 7. $\nu(2p, 2q) = 0$.

Proof. Let $g \in \Delta$ be a preimage of element $\overline{v}$ under the homomorphism ϕ. In other words, $g = a^{x_v}b^y c^{z_v}$. Then

$$g^2 = a^{2x_v}b^{(-1)^{x_v}y+y}c^{2z_v} = a^{2x_v}c^{2z_v}.$$

The last equality holds since x_v is odd. Thus $\nu(2,0) = 0$.

Analogously, consider $h \in \Delta$ a preimage of $\overline{v} + \overline{u}$ under ϕ to conclude $\nu(2,2) = 0$. Use **??** to finish the proof. □

In other words to define the function $\nu(s,t)$ it is sufficient to determine $\nu(0,0)$, $\nu(0,1)$, $\nu(1,0)$ and $\nu(1,1)$, also **??** gives $\nu(0,0) = 0$.

Lemma 8. *The following holds:*

(i) $\nu(1,1) \equiv \nu(0,1) + \nu(1,0) \mod l(\Delta)$

(ii) $2\nu(0,1) \equiv 0 \mod l(\Delta)$.

Proof. Immediate corollary of lemmas 6 and 7. □

Notation. Put $\rho(\Delta) = \nu(1,0)$ and $\varepsilon(\Delta) = \nu(0,1)$.

Definition. A 4-plet $(l(\Delta), \phi(\Delta), \rho(\Delta), \varepsilon(\Delta))$ is called *n-essential* if the following conditions holds:

(i) $l(\Delta)$ is a positive divisor of n,

(ii) $\phi(\Delta)$ is a subgroup of index $n/l(\Delta)$ in $\mathbb{Z}^2$, but not a subgroup of $\{(2p, q \mid p \in \mathbb{Z},\ q \in \mathbb{Z})\}$,

(iii) $\rho(\Delta), \varepsilon(\Delta) \in \{0, 1, \ldots, l(\Delta)-1\}$, and $2\varepsilon(\Delta) \equiv 0 \mod l(\Delta)$.

Proposition 3. *There is a bijection between the set of n-essential 4-plets* $(l(\Delta), \phi(\Delta), \rho(\Delta), \varepsilon(\Delta))$ *and non-abelian subgroups* Δ *of index* n *in* $\pi_1(\mathcal{B}_1)$.

Proof. Let Δ be a non-abelian subgroup of index n in $\pi_1(\mathcal{B}_1)$. Since $(l(\Delta)$, $\phi(\Delta)$, $\rho(\Delta)$ and $\varepsilon(\Delta))$ are well-defined there is an injection from the set of considered subgroups to the set of 4-plets. This 4-plets are n-essential in virtue of **??** and **??**. Thus we have to show that every 4-plet is achieved in this way.

Consider an n-essential 4-plet $(l(\Delta), \phi(\Delta), \rho(\Delta), \varepsilon(\Delta))$. Choose two generating vectors $\overline{v}$ and $\overline{u}$ for the group $\phi(\Delta)$ as described in **??**. Direct verification shows that the set

$$\Delta = \{a^{2px_v+2qx_u} b^{kl(\Delta} c^{2pz_v+2qz_u} |\ p,q,k \in \mathbb{Z}\} \bigcup \{a^{(2p+1)x_v+2qx_u} b^{\rho(\Delta+kl(\Delta} c^{(2p+1)z_v+2qz_u} |\ p,q,k \in \mathbb{Z}\} \bigcup$$
$$\bigcup \{a^{2px_v+(2q+1)x_u} b^{\varepsilon(\Delta+kl(\Delta} c^{2pz_v+(2q+1)z_u} |\ p,q,k \in \mathbb{Z}\} \bigcup$$
$$\bigcup \{a^{(2p+1)x_v+(2q+1)x_u} b^{\rho(\Delta+\varepsilon(\Delta+kl(\Delta} c^{(2p+1)z_v+(2q+1)z_u} |\ p,q,k \in \mathbb{Z}\}$$

is a subgroup in $\pi_1(\mathcal{B}_1)$, again Δ have the index n in $\pi_1(\mathcal{B}_1)$ in virtue of **??**. $\square$

Proposition 4. *The type of nonabelian subgroup Δ of $\pi_1(\mathcal{B}_1)$ is uniquely determined by the value of $\varepsilon(\Delta)$. More precisely, if $\varepsilon(\Delta) = 0$ then $\Delta \cong Pc$, if $l(\Delta)$ is even and $\varepsilon(\Delta) = l(\Delta)/2$ then $\Delta \cong Cc$.*

Proof. In the case $\varepsilon(\Delta) = 0$ denote $a' = a^{x_v} b^{\rho(\Delta)} c^{z_v}$, $b' = b^l(\Delta)$ and $c' = a^{x_u} c^{z_u}$. Direct verification shows that the relations $c'a'(c')^{-1}(a')^{-1} = c'b'(c')^{-1}b'^{-1} = e$ and $a'b'(a')^{-1}b' = e$ holds. Further we call this relations *the proper relations of the subgroup* Δ. Thus the map $a \to a',\ b \to b',\ c \to c'$ can be extended to an epimorphism $\pi_1(\mathcal{B}_1) \to \Delta$. To prove that this epimorphism is really an isomorphism we need to show that each relation in Δ is a corollary of proper relations. We call a relation, that is not a corollary of proper relations an *improper relation*.

Assume the contrary. Since in Δ the proper relations holds, each element can be represented in the canonical form, given by **??** in terms of a', b', c', by using just the proper relations. I.e. each element g can be represented as

$$g = a'^x b'^y c'^z.$$

If there is an improper relation then there is an equality

$$a'^p b'^q c'^r = a'^{p'} b'^{q'} c'^{r'} \tag{3.8}$$

where at least one of the inequalities $p \neq p'$, $q \neq q'$, $r \neq r'$ holds. Substitute $a' = a^{x_v} b^{\rho(\Delta)} c^{z_v}$, $b' = b^l(\Delta)$ and $c' = a^{x_u} c^{z_u}$ to **??** and apply the homomorphism ϕ to both left and right parts. We get

$$\begin{cases} px_v + rx_u = p'x_v + r'x_u \\ pz_v + rz_u = p'z_v + r'z_u \end{cases} \tag{3.9}$$

Since the vectors $\overline{v} = (x_v, z_v)$ and $\overline{u} = (x_u, z_u)$ generate a subgroup of finite index in $\mathbb{Z}^2$, the matrix $\begin{pmatrix} x_v & z_v \\ x_u & z_u \end{pmatrix}$ is nonsingular. Thus **??** implies $p = p'$ and $r = r'$. That means $q \neq q'$.

So **??** can be simplified to $b^{l(\Delta)(q-q')} = e$, which is a contradiction since $\pi_1(\mathcal{B}_1)$ have no elements of finite order.

In the second case denote $\alpha' = a^{x_v}b^{\rho}c^{z_v}$, $\beta' = b^l$ and $\gamma' = a^{x_u}b^{-\varepsilon}c^{z_u}$. Direct verification shows that the relations $\alpha'\beta'(\alpha')^{-1} = (\beta')^{-1}$, $\gamma'\beta'(\gamma')^{-1} = \beta'$ and $\alpha'\gamma'(\alpha')^{-1}(\gamma')^{-1} = \beta'$ holds. Thus the map $\alpha \to \alpha'$, $\beta \to \beta'$, $\gamma \to \gamma'$ can be extended to an isomorphism $\pi_1(\mathcal{B}_1) \to \Delta$. The proof is analogous to the previous case. $\square$

Remark. The groups Pc and Cc are not isomorphic since they have different homologies: $H_1(Pc) = \mathbb{Z}_2 \times \mathbb{Z}^2$ and $H_1(Cc) = \mathbb{Z}^2$, see [17] or [1].

3.2 The proof of Theorem 1

Proceed to the proof of **??**. First of all we show that there exist only 3 types of subgroups in $\pi_1(\mathcal{B}_1)$. Consider a subgroup Δ of index n in $\pi_1(\mathcal{B}_1)$. Then either $\phi(\Delta) \leqslant \{(2p, q \mid p \in \mathbb{Z},\ q \in \mathbb{Z})\}$ or $\phi(\Delta) \nleqslant \{(2p, q \mid p \in \mathbb{Z},\ q \in \mathbb{Z})\}$. In the first case the **??** states that Δ is abelian and $\Delta \leqslant \langle a^2, b, c\rangle$, thus the group $\Delta \cong \mathbb{Z}^3$ as a subgroup of finite index in $\mathbb{Z}^3$.

If $\phi(\Delta) \nleqslant \{(2p, q \mid p \in \mathbb{Z},\ q \in \mathbb{Z})\}$ then Δ is bijectively determined by an n-essential 4-plet $(l(\Delta), \phi(\Delta), \rho(\Delta), \varepsilon(\Delta))$ in virtue of **??**. Recall that $2\varepsilon(\Delta) \equiv 0 \mod l(\Delta)$. Thus there are only two cases: $\varepsilon(\Delta) = 0$ and $\varepsilon(\Delta) = l(\Delta)/2$ (the latter one is possible only if $l(\Delta)$ is even).

In case $\varepsilon(\Delta) = 0$ **??** claims that $\Delta \cong Pc$. In case $\varepsilon(\Delta) = l(\Delta)/2$ **??** yields $\Delta \cong Cc$. Thus we proved that Δ is isimorphic to one of the groups Pc, $\mathbb{Z}^3$ and Cc, and the latter two cases are possible only if n is even. Consider all three cases separately.

Case (i). The number $s_{\mathbb{Z}^3,\pi_1(\mathcal{B}_1)}(n)$ is calculated in **??**.

Case (ii). To find the number of subgroups, isomorphic to $\pi_1(\mathcal{B}_1)$ by Propositions 3 and 4 we need to calculate the cardinality of the set of n-essential 4-plets with $\varepsilon(\Delta) = 0$, i.e.

$$\{(l(\Delta), \phi(\Delta), \rho(\Delta), 0) \mid (l(\Delta), \phi(\Delta), \rho(\Delta), 0) \text{ is an } n\text{-essential 4-plet}\}.$$

Keeping in mind the definition of an n-essential 4-plet we see that $l(\Delta)$ is an arbitrary factor of n. The amount of possible $\phi(\Delta)$ depending of $l(\Delta)$ may be calculated the following way. By definition of n-essential 4-plet $\phi(\Delta) \leqslant \mathbb{Z}^2, \phi(\Delta) \nleqslant \{(2p, q \mid p \in \mathbb{Z},\ q \in \mathbb{Z})\}$ and $\left[\mathbb{Z}^2 : \phi(\Delta)\right] = n/l(\Delta)$. The total amount of $\phi(\Delta)$, such that $\left[\mathbb{Z}^2 : \phi(\Delta)\right] = n/l(\Delta)$ is $\sigma_1(\frac{n}{l(\Delta)})$, (see [5] Corollary 4.4). Analogously the amount of $\phi(\Delta)$, such that $\phi(\Delta) \leqslant \{(2p, q \mid p \in \mathbb{Z},\ q \in \mathbb{Z})\}$ and $\left[\mathbb{Z}^2 : \phi(\Delta)\right] = n/l(\Delta)$ is $\sigma_1(\frac{n}{2l(\Delta)})$. Thus amount of required $\phi(\Delta)$ is $\sigma_1(\frac{n}{l(\Delta)}) - \sigma_1(\frac{n}{2l(\Delta)})$. The amount of possible $\rho(\Delta)$ does not depends on a choice of $\phi(\Delta)$ and equals $l(\Delta)$. Thus for every fixed value of $l(\Delta)$ the amount of n-essential 4-plets with this $l(\Delta)$ and $\varepsilon(\Delta) = 0$ is $\big(\sigma_1(\frac{n}{l(\Delta)}) - \sigma_1(\frac{n}{2l(\Delta)})\big)l(\Delta)$. Summing this amount over all possible values of $l(\Delta)$ we get

$$\sum_{l|n}(\sigma_1(\frac{n}{l}) - \sigma_1(\frac{n}{2l}))l = \sum_{l|n}\sigma_1(\frac{n}{l})l - \sum_{2l|n}\sigma_1(\frac{n}{2l})l.$$

Case (iii). Arguing similarly we get that the amount of subgroups, isomorphic to Cc is

$$\sum_{2l|n} 2l\sigma_1(\frac{n}{2l}) - \sum_{4l|n} 2l\sigma_1(\frac{n}{4l}).$$

3.3 The total number of subgroups of index n in $\pi_1(\mathcal{B}_1)$

As an immediate consequence of **??** we get

$$s_{\pi_1(\mathcal{B}_1)}(n) = s_{\mathbb{Z}^3,\pi_1(\mathcal{B}_1)}(n) + s_{\pi_1(\mathcal{B}_1),\pi_1(\mathcal{B}_1)}(n) + s_{\pi_1(\mathcal{B}_2),\pi_1(\mathcal{B}_1)}(n).$$

By the way there are at least two different considerations leading to this result, we present them here.

Proposition 5.

$$s_{\pi_1(\mathcal{B}_1)}(n) = \sum_{m\,|\,n} m\, c_{\pi_1(\mathcal{K})}(m), \tag{3.10}$$

where

$$c_{\pi_1(\mathcal{K})}(m) = \begin{cases} \sigma_0(m), \text{ if } m \text{ is odd,} \\ \frac{3}{2}\sigma_0(m) + \frac{1}{2}\sum_{d\,|\,\frac{m}{2}}(d-1), \text{ if } m \text{ is even.} \end{cases} \tag{3.11}$$

Proof. In [14] p. 112, in Equation 5.125, Stanley proves that if G is a finitely generated group then [2]

$$s_{G\times\mathbb{Z}}(n) = \sum_{m\,|\,n} m\, c_G(m). \tag{3.12}$$

The formula **??** is proven in [9] (see Theorem 2). Since $\pi_1(\mathcal{B}_1) = \pi_1(\mathcal{K}) \times \mathbb{Z}$, to finish the proof substitute **??** to **??**. □

Some simple calculations, omitted here, show that the expressions for $s_{\pi_1(\mathcal{B}_1)}(n)$, obtained by **??** and by **??** are equal.

A sequence $s_{\pi_1(\mathcal{B}_1)}(n)$ coincides with the sequence A027844 in the 'On-Line Encyclopedia of Integer Sequences' ([12]).

Remark 1. *Note that the other formula for $s_{\pi_1(\mathcal{B}_1)}(n)$ was obtained by different method by M.N.Shmatkov in PhD thesis [13] (see p. 150–151).*

3.4 The proof Theorem 2

To obtain the total number of n-coverings over $\mathcal{B}_1$ we use following theorem from [11]:

Theorem (Mednykh)**.** *Let Γ be a finitely generated group. Then the number of conjugated classes of subgroups of index n in the group Γ, is given by the formula*

$$c_\Gamma(n) = \frac{1}{n} \sum_{\substack{l|n \\ lm=n}} \sum_{K<_m\Gamma} |Epi(K, \mathbb{Z}_l)|,$$

where the sum $\sum_{K<_m\Gamma}$ is taken over all subgroups K of index m in the group Γ and $Epi(K,\mathbb{Z}_l)$ is the set of epimorphisms of the group K onto the cyclic group $\mathbb{Z}_l$ of order l.

[2]In Stanley notations: $j_G(n) = c_G(n)$ and $u_G(n) = s_G(n)$.

Since **??** classifies all subgroups of finite index in $\pi_1(\mathcal{B}_1)$, we just have to calculate $|Epi(\mathbb{Z}^3,\mathbb{Z}_l)|$, $|Epi(Pc,\mathbb{Z}_l)|$ and $|Epi(Cc,\mathbb{Z}_l)|$.

Lemma 9. *(i)* $H_1(\mathbb{Z}^3,\mathbb{Z})=\mathbb{Z}^3$,

(ii) $H_1(Pc,\mathbb{Z})=\mathbb{Z}_2\oplus\mathbb{Z}^2$,

(iii) $H_1(Cc,\mathbb{Z})=\mathbb{Z}^2$,
where $H_1(\Delta,\mathbb{Z})$ is a first homology group.

Proof. See [1] section 7. □

The previous lemma and Lemma 4 of [11] yield the following result.

Lemma 10. *We have*

(i) $|Epi(\mathbb{Z}^3,\mathbb{Z}_l)|=\sum_{d|l}\mu\left(\frac{l}{d}\right)d^3:=\varphi_3(l)$,

(ii) $|Epi(Pc,\mathbb{Z}_l)|=\sum_{d|l}\mu\left(\frac{l}{d}\right)(2,d)d^2$,

(iii) $|Epi(Cc,\mathbb{Z}_l)|=\sum_{d|l}\mu\left(\frac{l}{d}\right)d^2:=\varphi_2(l)$,

where $\mu(n)$ is the Möbius function, $\varphi_3(l)$ and $\varphi_2(l)$ are Jordan totient functions,$(2,d)$ is a greater common divisor of numbers 2 and d .

Substituting the formulas from **??** to Mednykh's Theorem we get the statement of **??**.

3.5 The proof Theorem 3

The isomorphism types of subgroups are already provided by **??**. Thus we have to calculate the number of conjugacy classes for each type separately.

Lemma 11.

$$c_{\mathbb{Z}^3,\pi_1(\mathcal{B}_1)}(n)=\frac{1}{2}\sum_{2l|n}\sum_{m|\frac{n}{2l}}\Big(l^2+\frac{5}{2}+\frac{3}{2}(-1)^l\Big)m.$$

Proof. Let Δ be a subgroup of index n in $\pi_1(\mathcal{B}_1)$, isomorphic to $\mathbb{Z}^3$. Then $\Delta\leqslant\langle a^2,b,c\rangle$ by **??**. Thus $\Delta^{a^2}=\Delta^b=\Delta^c=\Delta$, so the conjugacy class of Δ in $\pi_1(\mathcal{B}_1)$ contains at most two subgroups: Δ and Δ^a. Thus we have to find out whether $\Delta=\Delta^a$.

The arguments, analogous to **??** shows that there is a bijection between the set of isomorphic to $\mathbb{Z}^3$ subgroups $\Delta\leqslant\pi_1(\mathcal{B}_1)$ and the setb 4-plets $(l(\Delta),\phi(\Delta),y_v(\Delta),y_u(\Delta))$, such that

(i) $l(\Delta)$ is a positive divisor of n,

(ii) $\phi(\Delta)\leqslant\langle a^2,b,c\rangle$ and $[\langle a^2,b,c\rangle:\phi(\Delta)]=\frac{n}{2l(\Delta)}$,

(iii) $y_v(\Delta)$ and $y_u(\Delta)$ are arbitrary residues modulo $l(\Delta)$.

Obviously, $l(\Delta)=l(\Delta^a)$, $\phi(\Delta)=\phi(\Delta^a)$, $y_v(\Delta)=-y_v(\Delta^a)$, $y_u(\Delta)=-y_u(\Delta^a)$. Thus we have to find the number of pairs (y_v,y_u) of residues modulo $l(\Delta)$, such that $y_v=-y_v$ and $y_u=-y_u$. If $l(\Delta)$ is odd there is 1 pair $(0,0)$, if $l(\Delta)$ is even there are 4 pairs: $(0,0)$, $(0,l(\Delta)/2)$, $(l(\Delta)/2,0)$ and $(l(\Delta)/2,l(\Delta)/2)$. □

Lemma 12.

$$c_{\pi_1(\mathcal{B}_1),\pi_1(\mathcal{B}_1)}(n)=\sum_{l|n}\Big(\frac{3}{2}+\frac{1}{2}(-1)^l\Big)\sigma_1(\frac{n}{l})-\sum_{2l|n}\Big(\frac{3}{2}+\frac{1}{2}(-1)^l\Big)\sigma_1(\frac{n}{2l}).$$

Proof. Let Δ be a subgroup of index n in $\pi_1(\mathcal{B}_1)$ isomorphic to $\pi_1(\mathcal{B}_1)$. Recall that by **??** there is a bijection between the set of isomorphic to $\pi_1(\mathcal{B}_1)$ subgroups $\Delta\leqslant\pi_1(\mathcal{B}_1)$ and the set of n-essential 4-plets $(l(\Delta),\phi(\Delta),\rho(\Delta),\varepsilon(\Delta))$.

Obviously, $l(\Delta)=l(\Delta^d)$ and $\phi(\Delta)=\phi(\Delta^d)$ for any $d\in\pi_1(+a_1)$. Also $\rho(\Delta)=-\rho(\Delta^a)$, $\rho(\Delta)=\rho(\Delta^b)+2$ and $\rho(\Delta)=\rho(\Delta^c)$. Thus the parity of $\rho(\Delta)$ is the only invariant for a conjugacy class with fixed $l(\Delta)$ and $\phi(\Delta)$.

Summarizing the above considerations. $l(\Delta)$ can be any positive divisor of n, thus we get the sum over all divisors the amount of corresponding pairs $(\phi(\Delta),\rho(\Delta) \mod 2)$. The amount of all $\phi(\Delta)$, such that $[\mathbb{Z}^2:\phi(\Delta)]=n/l(\Delta)$ is $\sigma_1(\frac{n}{l(\Delta)})$ the amount of $\phi(\Delta)$, such that $\phi(\Delta)$ is a subgroup of index $\frac{n}{2l(\Delta)}$ in $\{(2p,q\mid p\in\mathbb{Z},\ q\in\mathbb{Z})\}$ is $\sigma_1(\frac{n}{2l(\Delta)})$ (again, we consider $\sigma_1(\frac{n}{2l(\Delta)})=0$ if $\frac{n}{2l(\Delta)}$ is not integer). Thus the amount $\phi(\Delta)$, satisfying the condition of n-essential 4-plet is $\sigma_1(\frac{n}{l(\Delta)})-\sigma_1(\frac{n}{2l(\Delta)})$. We multiply it by the amoumt of possible parities of $\rho(\Delta)$, i.e. 2 for even $l(\Delta)$ and 1 for odd, in other words by $\Big(\frac{3}{2}+\frac{1}{2}(-1)^{l(\Delta)}\Big)$. Thus we get

$$c_{\pi_1(\mathcal{B}_1),\pi_1(\mathcal{B}_1)}(n)=\sum_{l|n}\Big(\frac{3}{2}+\frac{1}{2}(-1)^l\Big)\Big(\sigma_1(\frac{n}{l})-\sigma_1(\frac{n}{2l})\Big)=$$
$$=\sum_{l|n}\Big(\frac{3}{2}+\frac{1}{2}(-1)^l\Big)\sigma_1(\frac{n}{l})-\sum_{2l|n}\Big(\frac{3}{2}+\frac{1}{2}(-1)^l\Big)\sigma_1(\frac{n}{2l})$$

□

Lemma 13.

$$c_{\pi_1(\mathcal{B}_2),\pi_1(\mathcal{B}_1)}(n)=2\Big(\sum_{2k|n}\sigma_1(\frac{n}{2k})-\sum_{4k|n}\sigma_1(\frac{n}{4k})\Big).$$

Proof. The proof is analogous to **??**. □

4 On the coverings of $\mathcal{B}_2$

Most of the statements and proofs in this section are similar to corresponding parts of section 3. The proofs are given only in case of significant difference.

4.1 The structure of the group $\pi_1(\mathcal{B}_2)$

The following proposition provides the canonical form of an element in $\pi_1(\mathcal{B}_2)$.

Proposition 6. *(i) Each element of $\pi_1(\mathcal{B}_2)$ can be represented in the canonical form $\alpha^x\beta^y\gamma^z$ for some integer x, y, z.*

(ii) The product of two canonical forms is given by the formula

$$\alpha^x\beta^y\gamma^z \cdot \alpha^{x'}\beta^{y'}\gamma^{z'} = \begin{cases} \alpha^{x+x'}\beta^{y+y'}\gamma^{z+z'} \text{if } x' \text{is even} \\ \alpha^{x+x'}\beta^{-y-z+y'}\gamma^{z+z'} \text{if } x' \text{is odd} \end{cases} \tag{4.13}$$

(iii) The canonical epimorphism $\psi : \pi_1(\mathcal{B}_2) \to \pi_1(\mathcal{B}_2)/\langle\beta\rangle \cong \mathbb{Z}^2$, given by the formula $\alpha^x\beta^y\gamma^z \to (x, z)$ is well-defined.

(iv) The representation in the canonical form $\alpha^x\beta^y\gamma^z$ for each element is unique.

Lemma 14. *Let Δ be a subgroup of index n in $\pi_1(\mathcal{B}_2)$. By $l(\Delta)$ denote the minimal positive integer, such that $\beta^{l(\Delta)} \in \Delta$. Such an $l(\Delta)$ exists, and satisfy the relation $l(\Delta) \cdot [\pi_1(\mathcal{B}_2) : \psi(\Delta)] = n$.*

The next proposition shows that the introduced above invariant $\psi(\Delta)$ is sufficient to determine whether Δ is abelian or not.

Proposition 7. *Let Δ be a subgroup of finite index in $\pi_1(\mathcal{B}_2)$. Then Δ is abelian if and only if $\psi(\Delta) \leqslant \{(2x, z)|\ x, z \in \mathbb{Z}\}$.* [3]

As a corollary of **??** we obtain the value of $s_{\mathbb{Z}^3,\pi_1(\mathcal{B}_2)}(n)$.

Corollary 2. *The number of subgroups of index n in $\pi_1(\mathcal{B}_2)$, which are isomorphic to $\mathbb{Z}^3$, is given by the formula:*

$$s_{\mathbb{Z}^3,\pi_1(\mathcal{B}_1)}(n) = \sum_{2l|n} \sigma_1(l)l.$$

The following lemmas are technical statements, needed to introduce the important invariant ν.

Lemma 15. *Let Δ be a subgroup of a finite index in $\pi_1(\mathcal{B}_2)$. Then $\psi(\Delta) \cong \mathbb{Z}^2$.*

Notation. By $\overline{v} = (x_v, z_v)$ and $\overline{u} = (x_u, z_u)$ denote a pair of generators of $\psi(\Delta)$, where $\psi(\Delta)$ is considered as a subgroup of $\{(x, z) \mid x \in \mathbb{Z}, z \in \mathbb{Z}\}$.

Lemma 16. *Any subgroup of finite index in $\langle\alpha^2, \beta, \gamma\rangle$ is isomorphic to $\mathbb{Z}^3$.*

Lemma 17. *Let $(x, z) \in \psi(\Delta)$. Then there exist an integer number $\mu(x, z)$, $0 \leq \mu(x, z) \leq l(\Delta) - 1$, such that for all $\alpha^x\beta^y\gamma^z \in \Delta$ we have $y \equiv \mu(x, z) \mod l(\Delta)$.*

[3]In other words, Δ is abelian iff $\Delta \leqslant \langle\alpha^2, \beta, \gamma\rangle$

Lemma 18. *Assume $\psi(\Delta) \nleq \{(2x, z)|\ x, z \in \mathbb{Z}\}$, then one can choose the generators $\overline{v} = (x_v, z_v)$ and $\overline{u} = (x_u, z_u)$ in such a way that x_v is odd and x_u is even.*

From now on we fix some $\overline{v}$ and $\overline{u}$, chosen in this way.

Lemma 19. $z_u \equiv [\mathbb{Z}^2 : \psi(\Delta)] \mod 2$.

Proof. Follows from $[\mathbb{Z}^2 : \psi(\Delta)] = |x_v z_u - x_u z_v|$, x_v is odd and x_u is even. □

Lemma 20 (almost additivity)**.** $\nu(s + 2p, t + q) \equiv \nu(s, t) + \nu(2p; q) \mod l(\Delta)$.

Lemma 21. $\nu(2p, 2q) = -pz_v - qz_u$.

Proof. Consider $g \in \Delta,\ \psi(g) = \overline{v}$, i.e. $g = \alpha^{x_v}\beta^{\nu(1,0)+kl}\gamma^{z_v}$. Since x_v is odd,

$$g^2 = \alpha^{2x_v}\beta^{-\nu(1,0)-kl-z_v+\nu(1,0)+kl}\gamma^{2z_v} = \alpha^{2x_v}\beta^{-z_v}\gamma^{2z_v}.$$

Thus $\nu(2, 0) = -z_v$. Use **??** to finish the proof. Also note that in terms of μ the statement of **??** is much shorter: $\mu(2x, 2z) = -z$ if determined. □

Lemma 22. *The following holds:*

- $\nu(1, 1) \equiv \nu(0, 1) + \nu(1, 0) \mod l(\Delta)$
- $2\nu(0, 1) \equiv \nu(0, 2) \mod l(\Delta)$.

Notation. Denote $\rho(\Delta) = \nu(1, 0)$ and $\varepsilon(\Delta) = \nu(0, 1)$.
Summing up Lemmas 20–22 we state the following.

$$\begin{cases} \nu(2p, 2q) \equiv -pz_v - qz_u \mod l(\Delta), \\ \nu(2p, 2q + 1) \equiv \nu(0, 1) - pz_v - qz_u \mod l(\Delta), \\ \nu(2p + 1, 2q) \equiv \nu(1, 0) - pz_v - qz_u \mod l(\Delta), \\ \nu(2p + 1, 2q + 1) \equiv \nu(1, 0) + \nu(0, 1) - pz_v - qz_u \mod l(\Delta). \end{cases}$$

Definition. A 4-plet $(l(\Delta), \phi(\Delta), \rho(\Delta), \varepsilon(\Delta))$ is called *n-essential* if the following conditions holds:

(i) $l(\Delta)$ is a positive divisor of n,

(ii) $\psi(\Delta)$ is a subgroup of index $n/l(\Delta)$ in $\mathbb{Z}^2$, but not a subgroup of $\{(2p, q \mid p \in \mathbb{Z},\ q \in \mathbb{Z})\}$,

(iii) $\rho(\Delta), \varepsilon(\Delta) \in \{0, 1, \ldots, l(\Delta) - 1\}$, and $2\varepsilon(\Delta) \equiv -z_u \mod l(\Delta)$.

Proposition 8. *There is a bijection between the set of n-essential 4-plets $(l(\Delta), \phi(\Delta), \rho(\Delta), \varepsilon(\Delta))$ and the set of non-abelian subgroups Δ of index n in $\pi_1(\mathcal{B}_2)$.*

Proposition 9. *The type of nonabelian subgroup Δ of $\pi_1(\mathcal{B}_2)$ is uniquely determined by the value of $\varepsilon(\Delta)$. More precisely, if $2\varepsilon(\Delta) \equiv -z_u \mod 2l(\Delta)$ then $\Delta \cong \pi_1(\mathcal{B}_1)$, if $2\varepsilon(\Delta) \equiv -z_u + l(\Delta) \mod 2l(\Delta)$ then $\Delta \cong \pi_1(\mathcal{B}_2)$.*

4.2 The proof of Theorem 4

Proceed to the proof of **??**. First of all we show that there exist only 3 types of subgroups in $\pi_1(\mathcal{B}_2)$. This proof follows the lines of the proof of **??**. Consider a subgroup Δ of index n in $\pi_1(\mathcal{B}_2)$. Then either $\psi(\Delta) \leqslant \{(2p, q \mid p \in \mathbb{Z},\ q \in \mathbb{Z})\}$ or $\phi(\Delta) \nleqslant \{(2p, q \mid p \in \mathbb{Z},\ q \in \mathbb{Z})\}$. In the first case the **??** states that Δ is abelian and $\Delta \leqslant \langle a^2, b, c\rangle$, thus the group $\Delta \cong \mathbb{Z}^3$ as a subgroup of finite index in $\mathbb{Z}^3$.

If $\psi(\Delta) \nleqslant \{(2p, q \mid p \in \mathbb{Z},\ q \in \mathbb{Z})\}$ then Δ is bijectively determined by an n-essential 4-plet $(l(\Delta), \phi(\Delta), \rho(\Delta), \varepsilon(\Delta))$ in virtue of **??**. Recall that $2\varepsilon(\Delta) \equiv -z_u \mod l(\Delta)$. Thus there are only two cases: $2\varepsilon(\Delta) \equiv -z_u + l(\Delta) \mod 2l(\Delta)$ and $2\varepsilon(\Delta) \equiv -z_u \mod 2l(\Delta)$ (the latter one is possible only if $l(\Delta)$ is even).

In case $2\varepsilon(\Delta) \equiv -z_u + l(\Delta) \mod 2l(\Delta)$ **??** claims that $\Delta \cong Cc$. In case $2\varepsilon(\Delta) \equiv -z_u \mod 2l(\Delta)$ **??** yields $\Delta \cong Pc$. Thus we proved that Δ is isomorphic to one of the groups Cc, $\mathbb{Z}^3$ and Pc, and the latter two cases are possible only if n is even. Consider all three cases separately.

Case (i). The number $s_{\mathbb{Z}^3, \pi_1(\mathcal{B}_2)}(n)$ is calculated in **??**.

Case (ii). To find the number of subgroups, isomorphic to $\pi_1(\mathcal{B}_2)$ by Propositions 8 and 9 we need to calculate the cardinality of the set of n-essential 4-plets with $2\varepsilon(\Delta) \equiv -z_u + l(\Delta) \mod 2l(\Delta)$, i.e.

$$\{(l(\Delta), \phi(\Delta), \rho(\Delta), 0) \mid (l(\Delta), \phi(\Delta), \rho(\Delta), \frac{-z_u + l(\Delta)}{2}) \text{ is an n-essential 4-plet}\}.$$

Keeping in mind the definition of an n-essential 4-plet we see that $l(\Delta)$ is an arbitrary factor of n. The amount of possible $\psi(\Delta)$ depending of $l(\Delta)$ may be calculated the following way. By definition of n-essential 4-plet $\psi(\Delta) \leqslant \mathbb{Z}^2, \psi(\Delta) \nleqslant \{(2p, q \mid p \in \mathbb{Z},\ q \in \mathbb{Z})\}$ and $\left[\mathbb{Z}^2 : \psi(\Delta)\right] = n/l(\Delta)$. The total amount of $\psi(\Delta)$, such that $\left[\mathbb{Z}^2 : \psi(\Delta)\right] = n/l(\Delta)$ is $\sigma_1(\frac{n}{l(\Delta)})$, (see [5] Corollary 4.4). Analogously the amount of $\psi(\Delta)$, such that $\psi(\Delta) \leqslant \{(2p, q \mid p \in \mathbb{Z},\ q \in \mathbb{Z})\}$ and $\left[\mathbb{Z}^2 : \psi(\Delta)\right] = n/l(\Delta)$ is $\sigma_1(\frac{n}{2l(\Delta)})$ (we consider $\sigma_1(\frac{n}{2l(\Delta)}) = 0$ if $\frac{n}{2l(\Delta)}$ is not integer). Thus amount of required $\psi(\Delta)$ is $\sigma_1(\frac{n}{l(\Delta)}) - \sigma_1(\frac{n}{2l(\Delta)})$. The amount of possible $\rho(\Delta)$ does not depends on a choice of $\psi(\Delta)$ and equals $l(\Delta)$. For every fixed 3-plet $l(\Delta), \psi(\Delta), \rho(\Delta)$ either exists a unique $\varepsilon(\Delta)$, or none.

Unique $\varepsilon(\Delta)$ exists if both $l(\Delta)$ and z_u are odd, or both are even, by **??** this means $l(\Delta)$ and $\frac{n}{l(\Delta)}$ are both odd or both even. First case is equivalent the statement that n is odd, second case means $l(\Delta) = 2k$ and $4k \mid n$ for some integer k. Thus we get the formula.

$$s_{\pi_1(\mathcal{B}_2), \pi_1(\mathcal{B}_2)} = \begin{cases} \sum\limits_{4k|n} 2k\big(\sigma_1(\dfrac{n}{2k}) - \sigma_1(\dfrac{n}{4k})\big) & \text{if } n \text{ is even} \\ \sum\limits_{k|n} \sigma_1(\dfrac{n}{k})k & \text{if } n \text{ is odd} \end{cases}$$

Case (iii). Arguing similarly we get that the amount of subgroups, isomorphic to Cc is

$$\sum_{2l|n} 2l\sigma_1(\frac{n}{2l}) - \sum_{4l|n} 2l\sigma_1(\frac{n}{4l}).$$

Thus the proof of **??** is completed.

Remark 2. *Notice that* $s_{\pi_1(\mathcal{B}_2)}$ *was obtained by different method by M.N.Shmatkov in PhD thesis [13] (see p. 156–157) and can be calculated by the following formula*

$$s_{\pi_1(\mathcal{B}_2)} = \begin{cases} \sum\limits_{k|n} \sigma_1(\frac{n}{k})k & \text{if } n \text{ is odd} \\ \sum\limits_{k|n,\,(2,\frac{n}{k})=(2,n)} ((2,k)(\sigma_1(\frac{n}{k}) - \sigma_1(\frac{n}{2k})) + k\sigma_1(\frac{n}{2k}))k & \text{if } n \text{ is even} \end{cases} \tag{4.14}$$

4.3 The proof Theorem 5

The proof of **??** follows the same way as the proof of **??**.

Substituting the formulas from **??** and **??** to Mednykh's Theorem we get the statement of **??**.

4.4 The proof Theorem 6

The isomorphism types of subgroups are already provided by **??**. Thus we have to calculate the number of conjugacy classes for each type separately.

Lemma 23.

$$c_{\mathbb{Z}^3,\pi_1(\mathcal{B}_2)}(n) = \frac{1}{2}\sum_{2l|n}\sum_{m|\frac{n}{2l}} \Big(l^2 + \frac{3}{2} + \frac{1}{2}(-1)^l + (-1)^{\frac{n}{2lm}} + (-1)^{l+\frac{n}{2lm}}\Big)m.$$

Proof. Let Δ be a subgroup of index n in $\pi_1(\mathcal{B}_2)$, isomorphic to $\mathbb{Z}^3$. Then $\Delta \leqslant \langle \alpha^2, \beta, \gamma\rangle$ by **??**. Thus $\Delta^{\alpha^2} = \Delta^\beta = \Delta^\gamma = \Delta$, so the conjugacy class of Δ in $\pi_1(\mathcal{B}_2)$ contains at most two subgroups: Δ and Δ^α. Thus we have to find out whether $\Delta = \Delta^\alpha$.

The arguments, analogous to **??** shows that there is a bijection between an isomorphic to $\mathbb{Z}^3$ subgroups $\Delta \leqslant \pi_1(\mathcal{B}_2)$ and 4-plets $l(\Delta), \psi(\Delta), y_v(\Delta), y_u(\Delta)$, such that

- $l(\Delta)$ is a positive divisor of n
- $\phi(\Delta) \leqslant \langle a^2, b, c\rangle$ and $[\langle a^2, b, c\rangle : \phi(\Delta)] = \frac{n}{2l(\Delta)}$,
- Choose two residues modulo $l(\Delta)$, $y_v(\Delta)$ and $y_u(\Delta)$. Here $\alpha^{2l_1}\beta^{y_v} \in \Delta$ and $\alpha^{2x_u}\beta^{y_u}\gamma^{z_u} \in \Delta$ are preimages of $\overline{v}$ and $\overline{u}$ for the homomorphism ψ.

Obviously, $l(\Delta) = l(\Delta^\alpha)$, $\psi(\Delta) = \psi(\Delta^\alpha)$, $y_v(\Delta) = -y_v(\Delta^\alpha)$, $y_u(\Delta) = -y_u(\Delta^\alpha)$. Thus we have to find the number of pairs (y_v, y_u) of residues modulo $l(\Delta)$, such that $y_v \equiv -y_v$ and $y_u \equiv z_u - y_u \mod l(\Delta)$. If $l(\Delta)$ is odd there is 1 pair $(0, \frac{z_u}{2})$, if $l(\Delta)$ is even and $\frac{n}{2l_1(\Delta)l(\Delta)}$ odd then there no such pairs, if both $l(\Delta)$ and $\frac{n}{2l_1(\Delta)l(\Delta)}$ are even then there are 4 pairs: $(0, \frac{z_u}{2})$, $(0, \frac{z_u+l(\Delta)}{2})$, $(\frac{l(\Delta)}{2}, \frac{z_u}{2})$ and $(\frac{l(\Delta)}{2}, \frac{z_u+l(\Delta)}{2})$. So the amount of this pairs as a function of the parities of $l(\Delta)$ and $\frac{n}{2l_1(\Delta)l(\Delta)}$ is given by the formula $\frac{3}{2} + \frac{1}{2}(-1)^l + (-1)^{\frac{n}{2lm}} + (-1)^{l+\frac{n}{2lm}}$.
$\square$

Lemma 24.

$$(ii) \qquad c_{\pi_1(\mathcal{B}_2),\pi_1(\mathcal{B}_2)}(n) = \begin{cases} \sum\limits_{4k|n}(\sigma_1(\frac{n}{2k}) - \sigma_1(\frac{n}{4k})) & \textit{if } n \textit{ is even} \\ \sum\limits_{l|n}\sigma_1(\frac{n}{l}) & \textit{if } n \textit{ is odd} \end{cases}$$

Proof. Let Δ be a subgroup of index n in $\pi_1(\mathcal{B}_2)$ isomorphic to $\pi_1(\mathcal{B}_2)$. Recall that by **??** and **??** there is a bijection between an isomorphic to $\pi_1(\mathcal{B}_2)$ subgroups $\Delta \leqslant \pi_1(\mathcal{B}_2)$ and n-essential 4-plets $(l(\Delta), \phi(\Delta), \rho(\Delta), \varepsilon(\Delta))$, such that $2\varepsilon(\Delta) \equiv -z_u + l(\Delta) \mod 2l(\Delta)$.

Obviously, $l(\Delta) = l(\Delta^d)$ and $\phi(\Delta) = \phi(\Delta^d)$ for any $d \in \pi_1(\mathcal{B}_2)$. Also $\rho(\Delta^\alpha) = -\rho(\Delta) + z_v$, $\rho(\Delta^\beta) = \rho(\Delta) + 2$ and $\rho(\Delta^\gamma) = \rho(\Delta) + 1$. Thus subgroups that differ only in the parameter ρ corresponds to one class of conjugancy. Also the required $\varepsilon(\Delta)$ exists iff $l(\Delta)$ is odd and z_v is odd or $l(\Delta)$ is even and z_v is even. By **??** the $z_u \equiv \frac{n}{l(\Delta)} \mod 2$, thus $\varepsilon(\Delta)$ exists if n is odd or $l(\Delta) \mid \frac{n}{2}$ and $l(\Delta)$ is even. So, we get the amount

$$\begin{cases} \sum\limits_{l|n}\sigma_1(\frac{n}{l}) & \text{if } n \text{ is odd} \\ \sum\limits_{l|\frac{n}{2},\, 2|l}(\sigma_1(\frac{n}{l}) - \sigma_1(\frac{n}{2l})) & \text{if } n \text{ is even} \end{cases}$$

Summarizing the above considerations. $l(\Delta)$ can be any positive divisor of n, thus we get the sum over all divisors the amount of corresponding pairs $(\phi(\Delta), \rho(\Delta) \mod 2)$. The amount of all $\phi(\Delta)$, such that $[\mathbb{Z}^2 : \phi(\Delta)] = n/l(\Delta)$ is $\sigma_1(\frac{n}{l(\Delta)})$ the amount of $\phi(\Delta)$, such that $\phi(\Delta)$ is a subgroup of index $\frac{n}{2l(\Delta)}$ in $\{(2p, q \mid p \in \mathbb{Z},\, q \in \mathbb{Z})\}$ is $\sigma_1(\frac{n}{2l(\Delta)})$ (again, we consider $\sigma_1(\frac{n}{2l(\Delta)}) = 0$ if $\frac{n}{2l(\Delta)}$ is not integer). Thus the amount $\phi(\Delta)$, satisfying the condition of n-essential 4-plet is $\sigma_1(\frac{n}{l(\Delta)}) - \sigma_1(\frac{n}{2l(\Delta)})$. We multiply it by the amoumt of possible parities of $\rho(\Delta)$, i.e. 2 for even $l(\Delta)$ and 1 for odd, in other words by $\Big(\frac{3}{2} + \frac{1}{2}(-1)^{l(\Delta)}\Big)$. Thus we get

$$c_{\pi_1(\mathcal{B}_1),\pi_1(\mathcal{B}_1)}(n) = \sum_{l|n}\Big(\frac{3}{2} + \frac{1}{2}(-1)^l\Big)\Big(\sigma_1(\frac{n}{l}) - \sigma_1(\frac{n}{2l})\Big) =$$
$$= \sum_{l|n}\Big(\frac{3}{2} + \frac{1}{2}(-1)^l\Big)\sigma_1(\frac{n}{l}) - \sum_{2l|n}\Big(\frac{3}{2} + \frac{1}{2}(-1)^l\Big)\sigma_1(\frac{n}{2l})$$

□

Lemma 25.
$$c_{\pi_1(\mathcal{B}_2),\pi_1(\mathcal{B}_1)}(n) = 2\Big(\sum_{2k|n}\sigma_1(\frac{n}{2k}) - \sum_{4k|n}\sigma_1(\frac{n}{4k})\Big).$$

Proof. The proof is analogous to **??**. □

5 Acknowledgment

The authors are grateful to V. A. Liskovets, R. Nedela, M. Shmatkov for helpful discussions during the work of this paper.

6 Appendix

Here we put some tables which illustrated our formulas. Note that numerical results for n from 1 to 9 were previously obtained in [15].

6.1 Total number $c_{\pi_1(\mathcal{B}_1)}(n)$ of n-coverings over the first amphicosm

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$c_{\pi_1(\mathcal{B}_1)}(n)$	1	7	5	23	7	39	9	65	18	61	13	143	15	87	35	183

Table 2

6.2 The number $c_{\mathbb{Z}^3,\pi_1(\mathcal{B}_1)}(2n)$ of 3-torus $2n$-coverings over the first amphicosm

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$c_{\mathbb{Z}^3,\pi_1(\mathcal{B}_1)}(2n)$	1	7	9	29	19	63	33	107	74	133	73	285	99	231	219	393

Table 3

6.3 The number $c_{Cc,\pi_1(\mathcal{B}_1)}(2n)$ of the second amphicosm $2n$-coverings over the first amphicosm

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$c_{Cc,\pi_1(\mathcal{B}_1)}(2n)$	2	6	10	14	14	30	18	30	36	42	26	70	30	54	70	62

Table 4

6.4 The number $c_{Pc,\pi_1(\mathcal{B}_1)}(n)$ of the first amphicosm n-coverings over the first amphicosm

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$c_{Pc,\pi_1(\mathcal{B}_1)}(n)$	1	4	5	10	7	20	9	22	18	28	13	50	15	36	35	46

Table 5

6.5 Total number $c_{\pi_1(\mathcal{B}_2)}(n)$ of n-coverings over the second amphicosm

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$c_{\pi_1(\mathcal{B}_2)}(n)$	1	3	5	13	7	19	9	43	18	33	13	93	15	51	35	137

Table 6

6.6 The number $c_{\mathbb{Z}^3,\pi_1(\mathcal{B}_2)}(2n)$ of 3-torus $2n$-coverings over the second amphicosm

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$c_{\mathbb{Z}^3,\pi_1(\mathcal{B}_2)}(2n)$	1	5	9	23	19	53	33	93	74	119	73	255	99	213	219	363

Table 7

6.7 The number $c_{Pc,\pi_1(\mathcal{B}_2)}(2n)$ of the first amphicosm $2n$-coverings over the second amphicosm

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$c_{Pc,\pi_1(\mathcal{B}_2)}(2n)$	2	6	10	14	14	30	18	30	36	42	26	70	30	54	70	62

Table 8

6.8 The number $c_{Cc,\pi_1(\mathcal{B}_2)}(n)$ of the second amphicosm n-coverings over the second amphicosm

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$c_{Cc,\pi_1(\mathcal{B}_2)}(n)$	1	0	5	2	7	0	9	6	18	0	13	10	15	0	35	14

Table 8

References

[1] Conway J. H. and Rossetti J. P., Describing the Platycosms // arXiv:math.DG/0311476v1.

[2] Hatcher, Allen (2002), Algebraic Topology, Cambridge: Cambridge University Press, ISBN 0-521-79540-0.

[3] Hall M., Jr., Subgroups of finite index in free groups. *Canad. J. Math.*, **1**, No 1 (1949), 187–190.

[4] V. A. Liskovets, Towards the enumeration of subgroups of the free group. *Dokl. AN BSSR*, **15**, No 1 (1971), 6–9 (in Russian).

[5] Liskovets V., Mednykh A., Enumeration of subgroups in the fundamental groups of orientable circle bundles over surfaces, Commun. Algebra, **28**, No. 4 (2000), 1717–1738.

[6] Luminet J. P., Starkman G. D., Weeks J. R., Is Space Finite? // Scientific American (1999), no. April, 90–97.

[7] Mednykh A. D., Determination of the number of nonequivalent coverings over a compact Riemann surface. *Soviet Math. Dokl.*, **19**, No 2 (1978), 318–320.

[8] Mednykh A. D., On unramified coverings of compact Riemann surfaces. *Soviet Math. Dokl.*, **20**, No 1 (1979), 85–88.

[9] Mednykh A. D., On the number of subgroups in the fundamental group of a closed surfaces, Comm. in Algebra, **16**, No.10, (1988), 2137–2148.

[10] Mednykh A. D. and Pozdnyakova G. G., Number of nonequivalent coverings over a nonorientable compact surface. *Siber. Math. J.*, **27**, No 1 (1986), 99–106.

[11] Mednykh A., Counting conjugacy classes of subgroups in a finitely generated group //Journal of Algebra. – 2008. – V. 320. – No. 6. – P. 2209–2217.

[12] On-Line Encyclopedia of Integer Sequences, http://oeis.org/

[13] Shmatkov M. N, Enumeration of coverings over 3-manifolds. Ph.D.Thesis (in Russian), Sobolev Institute of Mathematics, Novosibirsk, 2004.

[14] Stanley R. J., Enumerative Combinatorics, vol. 2, Cambridge University Press, Cambridge, 1999.

[15] The Bilbao Crystallographic Server (http://www.cryst.ehu.es/cryst/cellsub.html)

[16] Weeks J. R., Real-Time Rendering in Curved Spaces.// IEEE Computer Graphics and Applications 22 (2002), no 6, 90–99.

[17] Wolf J. A., Spaces of Constant Curvature, Mc Graw-Hill Book Comp., New York, 1972.