A geometric approach for the upper bound theorem for Minkowski sums of convex polytopes

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Abstract

We derive tight expressions for the maximum number of k-faces, $0 \le k \le d-1$, of the Minkowski sum, $P_1+\cdots+P_r$, of r convex d-polytopes $P_1,\ldots,P_r$ in $\mathbb{R}^d$, where $d \ge 2$ and $r < d$, as a (recursively defined) function on the number of vertices of the polytopes. Our results coincide with those recently proved by Adiprasito and Sanyal [2]. In contrast to Adiprasito and Sanyal's approach, which uses tools from Combinatorial Commutative Algebra, our approach is purely geometric and uses basic notions such as f- and h-vector calculus and shellings, and generalizes the methodology used in [15] and [14] for proving upper bounds on the f-vector of the Minkowski sum of two and three convex polytopes, respectively. The key idea behind our approach is to express the Minkowski sum $P_1+\cdots+P_r$ as a section of the Cayley polytope $\mathcal{C}$ of the summands; bounding the k-faces of $P_1+\cdots+P_r$ reduces to bounding the subset of the $(k+r-1)$-faces of $\mathcal{C}$ that contain vertices from each of the r polytopes. We end our paper with a sketch of an explicit construction that establishes the tightness of the upper bounds.

1 Introduction

Given two sets A and B in $\mathbb{R}^d$, $d \ge 2$, their Minkowski sum $A+B$ is the set $\{a+b \mid a \in A, b \in B\}$. The Minkowski sum definition can be extended naturally to any number of summands: $A_{[r]} \coloneqq A_1 + A_2 + \cdots + A_r = \{a_1 + a_2 + \cdots + a_r \mid a_i \in A_i, 1 \le i \le r\}$. Minkowski sums have a wide range of applications, including algebraic geometry, computational commutative algebra, collision detection, computer-aided design, graphics, robot motion planning and game theory, just to name a few (see also [2], [14] and the references therein).

In this paper we focus on convex polytopes, and we are interested in computing the worst-case complexity of their Minkowski sum. More precisely, given r d-polytopes $P_1,\ldots,P_r$ in $\mathbb{R}^d$, we seek tight bounds on the number of k-faces $f_k(P_{[r]})$, $0 \le k \le d-1$, of their Minkowski sum $P_{[r]} \coloneqq P_1 + P_2 + \cdots + P_r$. This problem, which can be seen as a generalization of the Upper Bound Theorem (UBT) for polytopes [18], has a history of more than 20 years. Gritzmann and Sturmfels [11] were the first to consider the problem, and gave a complete answer to it, for any number of d-polytopes in $\mathbb{R}^d$, in terms of the number of non-parallel edges of the r polytopes. More than 10 years later, Fukuda and Weibel [7] proved tight upper bounds on the number of k-faces of the Minkowski sum of two 3-polytopes, expressed either in terms on the number of vertices or number of facets of the summands. Fogel, Halperin, and Weibel [6] extended one of the results in [7], and expressed the number of facets of the Minkowski sum of r 3-polytopes in terms of the number of facets of the summands. Quite recently Weibel [21] provided a relation for the number of k-faces of the Minkowski sum of $r \geq d$ summands in terms of the k-faces of the Minkowski sums of subsets of size $d-1$ of these summands. This result should be viewed in conjunction with a result by Sanyal [19] stating that the number of vertices of the Minkowski sum of r d-polytopes, where $r \geq d$, is strictly less than the product of the vertices of the summands (whereas for $r \leq d-1$ this is indeed possible). About 3 years ago, the authors of this paper proved the first tight upper bound on the number of k-faces for the Minkowski sum of two d-polytopes in $\mathbb{R}^d$, for any $d \geq 2$ and for all $0 \leq k \leq d-1$ (cf. [15]), a result which was subsequently extended to three summands in collaboration with Konaxis (cf. [14]).

In a recent paper, Adiprasito and Sanyal [2] provide the complete resolution of the *Upper Bound Theorem for Minkowski sums (UBTM)*. In particular, they show that there exists, what they call, a *Minkowski-neighborly* family of r d-polytopes $N_1, \ldots, N_r$, with $f_0(N_i) = n_i$, $1 \leq i \leq r$, such that for any r d-polytopes $P_1, P_2, \ldots, P_r \subset \mathbb{R}^d$ with $f_0(P_i) = n_i$, $1 \leq i \leq r$, $f_k(P_{[r]})$ is bounded by above by $f_k(N_{[r]})$, for all $0 \leq k \leq d-1$. The majority of the arguments in the UBTM proof by Adiprasito and Sanyal make use of powerful tools from Combinatorial Commutative Algebra. The high-level layout of the proof is analogous to McMullen's proof of the UBT, as well as the proofs of the UBTM in [15] and [14] for two and three summands, respectively:

1. Consider the Cayley polytope $\mathcal{C} \subset \mathbb{R}^{d+r-1}$ of the r polytopes $P_1, P_2, \ldots, P_r$, and identify their Minkowski sum as a section of $\mathcal{C}$ with an appropriately defined d-flat $\overline{W}$. Let $\mathcal{F} \subset \mathbb{R}^{d+r-1}$ be the faces of $\mathcal{C}$ that intersect $\overline{W}$, and let $\mathcal{K}$ be the closure of $\mathcal{F}$ under subface inclusion ($\mathcal{K}$ is a $(d+r-1)$-polytopal complex). By the Cayley trick, there is a bijection between the faces of $\mathcal{F}$ and the faces of $P_{[r]}$; as a result, to bound the number of faces of $P_{[r]}$ it suffices to bounds the number of faces of $\mathcal{F}$.

2. Define the h-vector $\boldsymbol{h}(\mathcal{F})$ of $\mathcal{F}$, and prove the Dehn-Sommerville equations for $\boldsymbol{h}(\mathcal{F})$, relating its elements to the elements of $\boldsymbol{h}(\mathcal{K})$.

3. Prove a recurrence relation for the elements of $\boldsymbol{h}(\mathcal{F})$.

4. Use the recurrence relation above to prove upper bounds for $h_k(\mathcal{F})$, for all $0 \leq k \leq \lfloor \frac{d+r-1}{2} \rfloor$.

5. Prove upper bounds for $h_k(\mathcal{K})$, for all $0 \leq k \leq \lfloor \frac{d+r-1}{2} \rfloor$.

6. Provide necessary and sufficient conditions under which the elements of both $\boldsymbol{h}(\mathcal{F})$ and $\boldsymbol{h}(\mathcal{K})$ are maximized for all k. These conditions are conditions on the *lower half* of the h-vector of $\mathcal{F}$. Due to the relation between the f- and h-vectors of $\mathcal{F}$, these are also conditions for the maximality of the elements of $\boldsymbol{f}(\mathcal{F})$.

7. Describe a family of polytopes for which the necessary and sufficient conditions hold; clearly, such a family establishes the tightness of the upper bounds.

In Adiprasito and Sanyal's proof steps 2, 3 and 4 are proved by introducing a powerful new theory that they call the *relative Stanley-Reisner theory* for simplicial complexes. The focus of this theory is on relative simplicial complexes, and is able to reveal properties of such complexes not only under topological restrictions, but also account for their combinatorial and geometric structure. To apply their theory, Adiprasito and Sanyal consider the simplicial complex $\mathcal{K}$ and then define $\mathcal{F}$ as a relative simplicial complex (they call them the Cayley and *relative Cayley* complex, respectively). They then apply their relative Stanley-Reisner theory to $\mathcal{F}$ to establish the Dehn-Sommerville equations of step 2, the recurrence relation of step 3 and finally the upper bounds for $\boldsymbol{h}(\mathcal{F})$ in 4. Steps 5 and 6 are done by clever algebraic manipulation of the h-vectors of $\mathcal{F}$ and $\mathcal{K}$, by exploiting the geometric properties of $\mathcal{K}$, and by making use of the recurrence relation in step 3. Step 7 is reduced to results by Matschke, Pfeifle, and Pilaud [17] and Weibel [21].

Our contribution. In what follows, we provide a completely geometric proof of the UBTM, that generalizes the technique we used in [15] and [14] for two and three summands to the case of r summands, when $r < d$. Instead of relying on algebraic tools, we use basic notions from combinatorial geometry, such as stellar subdivisions and shellings. Our proof, in essence, differs from that of Adiprasito and Sanyal in steps 2, 3, 4 and 5 of the layout above (the remaining steps do not use tools from Combinatorial Commutative Algebra anyway).

In more detail, to prove the various intermediate results, towards the UBTM, we consider the Cayley polytope $\mathcal{C}$ and we perform a series of stellar subdivisions to get a simplicial polytope $\mathcal{Q}$. From the analysis of the combinatorial structure of $\mathcal{Q}$, we derive the Dehn-Sommerville equations of step 2 (see Sections 3 and 4), as well as the recurrence relation of step 3 (see Section 5). This recurrence relation is then used for establishing the upper bounds for the elements of $\boldsymbol{h}(\mathcal{F})$ and $\boldsymbol{h}(\mathcal{K})$ (see Section 6). We end with a construction similar to the one presented in [17, Theorem 2.6], that establishes the tightness of the upper bounds (see Section 7).

2 Preliminaries

Let P be a d-dimensional polytope, or d-polytope for short. Its dimension is the dimension of its affine span. The faces of P are $\varnothing, P$, and the intersections of P with its supporting hyperplanes. The $\varnothing$ and P faces are called *improper*, while the remaining faces are called *proper*. Each face of P is itself a polytope, and a face of dimension k is called a k-face. Faces of P of dimension $0, 1, d-2$ and $d-1$ are called vertices, edges, ridges, and facets, respectively.

A d-dimensional *polytopal complex* or, simply, *d-complex*, $\mathscr{C}$ is a finite collection of polytopes in $\mathbb{R}^d$ such that (i) $\varnothing \in \mathscr{C}$, (ii) if $P \in \mathscr{C}$ then all the faces of P are also in $\mathscr{C}$ and (iii) the intersection $P \cap Q$ for two polytopes P and Q in $\mathscr{C}$ is a face of both. The dimension $\dim(\mathscr{C})$ of $\mathscr{C}$ is the largest dimension of a polytope in $\mathscr{C}$. A polytopal complex is called *pure* if all its maximal (with respect to inclusion) faces have the same dimension. In this case the maximal faces are called the *facets* of $\mathscr{C}$. A polytopal complex is *simplicial* if all its faces are simplices. A polytopal complex $\mathscr{C}'$ is called a *subcomplex* of a polytopal complex $\mathscr{C}$ if all faces of $\mathscr{C}'$ are also faces of $\mathscr{C}$. For a polytopal complex $\mathscr{C}$, the *star* of v in $\mathscr{C}$, denoted by $\operatorname{star}(v, \mathscr{C})$, is the subcomplex of $\mathscr{C}$ consisting of all faces that contain v, and their faces. The *link* of v, denoted by $\mathscr{C}/v$, is the subcomplex of $\operatorname{star}(v, \mathscr{C})$ consisting of all the faces of $\operatorname{star}(v, \mathscr{C})$ that do not contain v.

A d-polytope P, together with all its faces, forms a d-complex, denoted by $\mathscr{C}(P)$. The polytope P itself is the only maximal face of $\mathscr{C}(P)$, i.e., the only facet of $\mathscr{C}(P)$, and is called the *trivial* face of $\mathscr{C}(P)$. Moreover, all proper faces of P form a pure $(d-1)$-complex, called the *boundary complex* $\mathscr{C}(\partial P)$, or simply ∂P, of P. The facets of ∂P are just the facets of P.

For a $(d-1)$-complex $\mathscr{C}$, its f-vector is defined as $\boldsymbol{f}(\mathscr{C}) = (f_{-1}, f_0, f_1, \ldots, f_{d-1})$, where $f_k = f_k(\mathscr{C})$ denotes the number of k-faces of P and $f_{-1}(\mathscr{C}) := 1$ corresponds to the empty face of $\mathscr{C}$. From the f-vector of $\mathscr{C}$ we define its h-vector as the vector $\boldsymbol{h}(\mathscr{C}) = (h_0, h_1, \ldots, h_d)$, where $h_k = h_k(\mathscr{C}) := \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{d-k} f_{i-1}(\mathscr{C})$, $0 \le k \le d$.

Denote by $\mathcal{Y}$ a generic subset of faces of a polytopal complex $\mathscr{C}$, and define its dimension $\dim(\mathcal{Y})$ as the maximum of the dimensions of its faces. Let $\dim(\mathcal{Y}) = \delta - 1$; then we may define (if not already properly defined), the h-vector $\boldsymbol{h}(\mathcal{Y})$ of $\mathcal{Y}$ as:

$$h_k(\mathcal{Y}) = \sum_{i=0}^{\delta} (-1)^{k-i} \binom{\delta - i}{\delta - k} f_{i-1}(\mathcal{Y}). \tag{2.1}$$

We can further define the m-order g-vector of $\mathcal{Y}$ according to the following recursive formula:

$$g_k^{(m)}(\mathcal{Y}) = \begin{cases} h_k(\mathcal{Y}), & m = 0, \\ g_k^{(m-1)}(\mathcal{Y}) - g_{k-1}^{(m-1)}(\mathcal{Y}), & m > 0. \end{cases} \tag{2.2}$$

Clearly, $\boldsymbol{g}^{(m)}(\mathcal{Y})$ is nothing but the backward m-order finite difference of $\boldsymbol{h}(\mathcal{Y})$; therefore:

$$g_k^{(m)}(\mathcal{Y}) = \sum_{i=0}^{m} (-1)^i \binom{m}{i} h_{k-i}(\mathcal{Y}), \qquad k, m \geq 0. \tag{2.3}$$

Observe that for $m = 0$ we get the h-vector of $\mathcal{Y}$, while for $m = 1$ we get what is typically defined as the g-vector.

The relation between the f- and h-vector of $\mathcal{Y}$ is better manipulated using generating functions. We define the f-polynomial and h-polynomial of $\mathcal{Y}$ as follows:

$$\mathtt{f}(\mathcal{Y};t) = \sum_{i=0}^{\delta} f_{i-1} t^{\delta-i} = f_{\delta-1} + f_{\delta-2} t + \cdots + f_{-1} t^{\delta}, \quad \mathtt{h}(\mathcal{Y};t) = \sum_{i=0}^{\delta} h_i t^{\delta-i} = h_\delta + h_{\delta-1} t + \cdots + h_0 t^{\delta},$$

where, we simplified $f_i(\mathcal{Y})$ and $h_i(\mathcal{Y})$ to f_i and h_i. In this set-up, the relation between the f-vector and h-vector (cf. (2.1)) can be expressed as:

$$\mathtt{f}(\mathcal{Y};t) = \mathtt{h}(\mathcal{Y};t+1), \qquad \text{or, equivalently, as} \qquad \mathtt{h}(\mathcal{Y};t) = \mathtt{f}(\mathcal{Y};t-1). \tag{2.4}$$

2.1 The Cayley embedding, the Cayley polytope and the Cayley trick

Let $P_1, P_2, \ldots, P_r$ be r d-polytopes with vertex sets $\mathscr{V}_1, \mathscr{V}_2, \ldots, \mathscr{V}_r$, respectively. Let $\boldsymbol{e}_0, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_{r-1}$ be an affine basis of $\mathbb{R}^{r-1}$ and call $\mu_i : \mathbb{R}^d \to \mathbb{R}^{r-1} \times \mathbb{R}^d$ the affine inclusion given by $\mu_i(\boldsymbol{x}) = (\boldsymbol{e}_i, \boldsymbol{x})$. The *Cayley embedding* $\mathcal{C}(\mathscr{V}_1, \mathscr{V}_2, \ldots, \mathscr{V}_r)$ of the point sets $\mathscr{V}_1, \mathscr{V}_2, \ldots, \mathscr{V}_r$ is defined as $\mathcal{C}(\mathscr{V}_1, \mathscr{V}_2, \ldots, \mathscr{V}_r) = \bigcup_{i=1}^{r} \mu_i(\mathscr{V}_i)$. The polytope corresponding to the convex hull $\mathrm{conv}\big(\mathcal{C}(\mathscr{V}_1, \mathscr{V}_2, \ldots, \mathscr{V}_r)\big)$ of the Cayley embedding $\mathcal{C}(\mathscr{V}_1, \mathscr{V}_2, \ldots, \mathscr{V}_r)$ of $\mathscr{V}_1, \mathscr{V}_2, \ldots, \mathscr{V}_r$ is typically referred to as the *Cayley polytope* of $P_1, P_2, \ldots, P_r$.

The following lemma, known as *the Cayley trick for Minkowski sums*, relates the Minkowski sum of the polytopes $P_1, P_2, \ldots, P_r$ with their Cayley polytope.

Lemma 2.1 ([12, Lemma 3.2])**.** *Let $P_1, P_2, \ldots, P_r$ be r d-polytopes with vertex sets $\mathscr{V}_1, \mathscr{V}_2, \ldots, \mathscr{V}_r \subset \mathbb{R}^d$. Moreover, let $\overline{W}$ be the d-flat defined as $\{\frac{1}{r}\boldsymbol{e}_1 + \cdots + \frac{1}{r}\boldsymbol{e}_r\} \times \mathbb{R}^d \subset \mathbb{R}^{r-1} \times \mathbb{R}^d$. Then, the Minkowski sum $P_{[r]}$ has the following representation as a section of the Cayley embedding $\mathcal{C}(\mathscr{V}_1, \mathscr{V}_2, \ldots, \mathscr{V}_r)$ in $\mathbb{R}^{r-1} \times \mathbb{R}^d$:*

$$\begin{aligned} P_{[r]} &\cong \mathcal{C}(\mathscr{V}_1, \mathscr{V}_2, \ldots, \mathscr{V}_r) \cap \overline{W} \\ &:= \Big\{ \mathrm{conv}\{(\boldsymbol{e}_i, \boldsymbol{v}_i) \mid 1 \leq i \leq r\} \cap \overline{W} : (\boldsymbol{e}_i, \boldsymbol{v}_i) \in \mathcal{C}(\mathscr{V}_1, \mathscr{V}_2, \ldots, \mathscr{V}_r), 1 \leq i \leq r \Big\}. \end{aligned}$$

Moreover, F is a facet of $P_{[r]}$ if and only if it is of the form $F = F' \cap \overline{W}$ for a facet F' of $\mathcal{C}(\mathscr{V}_1, \mathscr{V}_2, \ldots, \mathscr{V}_r)$ containing at least one point $(\boldsymbol{e}_i, \boldsymbol{v}_i)$ for all $1 \leq i \leq r$.

Let $\mathcal{C}_{[r]}$ be the Cayley polytope of $P_1, P_2, \ldots, P_r$, and call $\mathcal{F}_{[r]}$ the set of faces of $\mathcal{C}_{[r]}$ that have non-empty intersection with the d-flat $\overline{W}$. A direct consequence of Lemma 2.1 is a bijection between the $(k-1)$-faces of $\overline{W}$ and the $(k-r)$-faces of $\mathcal{F}_{[r]}$, for $r \leq k \leq d+r-1$. This further implies that:

$$f_{k-1}(\mathcal{F}_{[r]}) = f_{k-r}(P_{[r]}), \quad \text{for all } r \leq k \leq d+r-1. \tag{2.5}$$

In what follows, to keep the notation lean, we identify V_i with its pre-image $\mathscr{V}_i$. For any $\varnothing \subset R \subseteq [r]$, we denote by $\mathcal{C}_R$ the Cayley polytope of the polytopes P_i where $i \in R$. In particular, if $R = \{i\}$ for some $i \in [r]$, then $\mathcal{C}_{\{i\}} \equiv P_i$. We shall assume below that $\mathcal{C}_{[r]}$ is *"as simplicial* *as possible"*. This means that we consider all faces of $\mathcal{C}_{[r]}$ to be simplicial, except possibly for the trivial faces $\{\mathcal{C}_R\}$[1], $\varnothing \subset R \subseteq [r]$. Otherwise, we can employ the so called *bottom-vertex triangulation* [16, Section 6.5, pp. 160–161] to triangulate all proper faces of $\mathcal{C}_{[r]}$ except for the trivial ones, i.e., $\{\mathcal{C}_R\}$, $\varnothing \subset R \subseteq [r]$. The resulting complex is polytopal (cf. [4]) with all of its faces being simplicial, except possibly for the trivial ones. Moreover, it has the same number of vertices as $\mathcal{C}_{[r]}$, while the number of its k-faces is never less than the number of k-faces of $\mathcal{C}_{[r]}$.

For each $\varnothing \subset R \subseteq [r]$, we denote by $\mathcal{F}_R$ the set of faces of $\mathcal{C}_R$ having at least one vertex from each V_i, $i \in R$ and we call it the set of *mixed faces of* $\mathcal{C}_R$. We trivially have that $\mathcal{F}_{\{i\}} \equiv \partial P_i$. We define the dimension of $\mathcal{F}_R$ to be the maximum dimension of the faces in $\mathcal{F}_R$, i.e., $\dim(\mathcal{F}_R) = \max_{F\in\mathcal{F}_R}\dim(F) = d + |R| - 2$. Under the *"as simplicial as possible"* assumption above, the faces in $\mathcal{F}_R$ are simplicial. We denote by $\mathcal{K}_R$ the *closure*, under subface inclusion, of $\mathcal{F}_R$. By construction, $\mathcal{K}_R$ contains: (1) all faces in $\mathcal{F}_R$, (2) all faces that are subfaces of faces in $\mathcal{F}_R$, and (3) the empty set. It is easy to see that $\mathcal{K}_R$ does not contain any of the trivial faces $\{\mathcal{C}_S\}$, $\varnothing \subset S \subseteq R$, and thus, $\mathcal{K}_R$ is a pure simplicial $(d+|R|-2)$-complex. It is also easy to verify that

$$f_k(\mathcal{K}_R) = \sum_{\varnothing\subset S\subseteq R} f_k(\mathcal{F}_S), \qquad -1 \le k \le d+|R|-2, \tag{2.6}$$

where in order for the above equation to hold for $k = -1$, we set $f_{-1}(\mathcal{F}_S) = (-1)^{|S|-1}$ for all $\varnothing \subset S \subseteq R$. In what follows we use the convention that $f_k(\mathcal{F}_R) = 0$, for any $k < -1$ or $k > d+|R|-2$.

A general form of the Inclusion-Exclusion Principle states that if f and g are two functions defined over the subsets of a finite set A, such that $f(A) = \sum_{\varnothing\subset B\subseteq A} g(B)$, then $g(A) = \sum_{\varnothing\subset B\subseteq A}(-1)^{|A|-|B|} f(B)$ [9, Theorem 12.1]. Applying this principle in (2.6), we deduce that:

$$f_k(\mathcal{F}_R) = \sum_{\varnothing\subset S\subseteq R} (-1)^{|R|-|S|} f_k(\mathcal{K}_S), \qquad -1 \le k \le d+|R|-2. \tag{2.7}$$

In the majority of our proofs that involve evaluation of f- and h-vectors, we use generating functions as they significantly simplify calculations. The starting point is to evaluate $\mathtt{f}(\mathcal{K}_R;t)$ (resp., $\mathtt{f}(\mathcal{F}_R;t)$) in terms of the generating functions $\mathtt{f}(\mathcal{F}_S;t)$ (resp., $\mathtt{f}(\mathcal{K}_S;t)$), $\varnothing \subset S \subseteq R$, for each fixed choice of $\varnothing \subset R \subseteq [r]$. Then, using (2.4) we derive the analogous relations between their h-vectors.

Recalling that $\dim(\mathcal{K}_R) = d + |R| - 2$ and $\dim(\mathcal{F}_S) = d + |S| - 2$ we have:

$$\begin{aligned}\mathtt{f}(\mathcal{K}_R;t) &= \sum_{k=0}^{d+|R|-1} f_{k-1}(\mathcal{K}_R)t^{d+|R|-1-k} \overset{(2.6)}{=} \sum_{k=0}^{d+|R|-1}\ \sum_{\varnothing\subset S\subseteq R} f_{k-1}(\mathcal{F}_S)t^{d+|R|-1-k} \\ &= \sum_{S\subseteq R} t^{|R|-|S|} \sum_{k=0}^{d+|R|-1} f_{k-1}(\mathcal{F}_S)t^{d+|S|-1-k} = \sum_{\varnothing\subset S\subseteq R} t^{|R|-|S|}\mathtt{f}(\mathcal{F}_S;t).\end{aligned} \tag{2.8}$$

Rewriting the above relation as $t^{-|R|}\mathtt{f}(\mathcal{K}_R;t) = \sum_{\varnothing\subset S\subseteq R} t^{-|S|}\mathtt{f}(\mathcal{F}_S;t)$ and using Möbious inversion, we get:

$$\mathtt{f}(\mathcal{F}_R;t) = \sum_{\varnothing\subset S\subseteq R} (-1)^{|R|-|S|} t^{|R|-|S|}\mathtt{f}(\mathcal{K}_S;t). \tag{2.9}$$

Setting $t := t-1$ in (2.8) we have:

$$\begin{aligned}\mathtt{h}(\mathcal{K}_R;t) = \mathtt{f}(\mathcal{K}_R;t-1) &= \sum_{\varnothing\subset S\subseteq R} (t-1)^{|R|-|S|}\mathtt{f}(\mathcal{F}_S;t-1) \\ &= \sum_{\varnothing\subset S\subseteq R} (t-1)^{|R|-|S|}\mathtt{h}(\mathcal{F}_S;t) = \sum_{\varnothing\subset S\subseteq R} \mathtt{g}^{(|R|-|S|)}(\mathcal{F}_S;t).\end{aligned} \tag{2.10}$$

[1]We denote by $\{\mathcal{C}_R\}$ the polytope $\mathcal{C}_R$ as a trivial face itself (without its non-trivial faces).

And similarly, from (2.9) we obtain:

$$\mathtt{h}(\mathcal{F}_R;t) = \sum_{\varnothing\subset S\subseteq R} (-1)^{|R|-|S|}\mathtt{g}^{(|R|-|S|)}(\mathcal{K}_S;t). \tag{2.11}$$

Comparing coefficients in the above generating functions, we deduce that:

$$h_k(\mathcal{K}_R) = \sum_{\varnothing\subset S\subseteq R} g_k^{(|R|-|S|)}(\mathcal{F}_S), \qquad \text{for all } 0 \le k \le d+|R|-1, \text{ and} \tag{2.12}$$

$$h_k(\mathcal{F}_R) = \sum_{\varnothing\subset S\subseteq R} (-1)^{|R|-|S|} g_k^{(|R|-|S|)}(\mathcal{K}_S), \qquad \text{for all } 0 \le k \le d+|R|-1. \tag{2.13}$$

3 The construction of the auxiliary simplicial polytope $\mathcal{Q}_{[r]}$.

The non-trivial faces of the Cayley polytope $\mathcal{C}_{[r]}$ of $P_1,\dots,P_r$ are the faces in each $\mathcal{F}_R$, $\varnothing\subset R\subseteq[r]$ as well as all trivial faces $\{\mathcal{C}_R\}$ with $\varnothing\subset R\subset[r]$. Since the latter are not necessarily simplices, the Cayley polytope $\mathcal{C}_{[r]}$ may not be simplicial. In order to exploit the combinatorial structure of $\mathcal{C}_{[r]}$, we add auxiliary points on $\mathcal{C}_{[r]}$ so that the resulting polytope, denoted by $\mathcal{Q}_{[r]}$, is simplicial.

The main tool for describing our construction is *stellar subdivisions*. Let $P\subset\mathbb{R}^d$ be a d-polytope, and consider a point y_F in the relative interior of a face F of ∂P. The *stellar subdivision* $\mathrm{st}(y_F,\partial P)$ of ∂P over F, replaces F by the set of faces $\{y_F, F'\}$ where F' is a non-trivial face of F. It is a well-known fact that stellar subdivisions preserve polytopality (cf. [5, pp. 70–73]), in the sense that the newly constructed complex is combinatorially equivalent to a polytope each facet of which lies on a distinct supporting hyperplane.

Our goal is to triangulate each face $\{\mathcal{C}_R\}$, $\varnothing\subset R\subset[r]$, of $\mathcal{C}_{[r]}$ so that the boundaries of the resulting complexes, denoted by $\mathcal{Q}_S$, $\varnothing\subset S\subseteq[r]$, are simplicial polytopes. We obtain this by performing a series of stellar subdivisions. First set $\mathcal{Q}_S \coloneqq \mathcal{C}_S$, for all $\varnothing\subset S\subseteq[r]$. Then, we add auxiliary vertices as follows:

$$\begin{aligned}
&\text{for } s \text{ from } 1 \text{ to } r-1\\
&\quad\text{for all } S\subseteq[r] \text{ with } |S| = s\\
&\qquad\text{choose } y_S \in \mathrm{relint}(\mathcal{Q}_S)\\
&\qquad\text{for all } T \text{ with } S\subset T\subseteq[r]\\
&\qquad\quad \mathcal{Q}_T \coloneqq \mathrm{st}(y_S,\mathcal{Q}_T)
\end{aligned} \tag{3.1}$$

The recursive step of the previous definition is well defined due to the fact that for any fixed s, the order in which we add the auxiliary points y_S is independent of the S chosen, since the relative interiors of all $\mathcal{Q}_S$ with $|S| = s$ are pairwise disjoint. At the end of the s-th iteration, the faces of each $\mathcal{Q}_T$ of dimension less than $d+s-1$ are simplices. At the end of the iterative procedure above, and in view of the fact that stellar subdivisions preserve polytopality, the above construction results in simplicial $(d+|R|-1)$-polytopes $\mathcal{Q}_R$, for all $\varnothing\subset R\subseteq[r]$.

The following two lemmas express the faces of $\partial\mathcal{Q}_R$ in terms of the sets $\mathcal{F}_S$, $\mathcal{K}_S$, $\varnothing\subset S\subseteq R$, and the auxiliary vertices added. Unless otherwise stated, all set unions are disjoint.

Lemma 3.1. *For $\varnothing\subset R\subseteq[r]$, the non-trivial faces of the simplicial polytope $\mathcal{Q}_R$ are:*

$$\partial\mathcal{Q}_R = \bigcup_{\varnothing\subset S\subseteq R}\mathcal{F}_S \bigcup_{\substack{\varnothing\subset S\subset R\\ S\subseteq S_1\subset S_2\subset\cdots\subset S_\ell\subset R}} \{y_{S_1},y_{S_2},\dots,y_{S_\ell},\mathcal{F}_S\} \bigcup_{\varnothing\subset S_1\subset S_2\subset\cdots\subset S_\ell\subset R} \{y_{S_1},y_{S_2},\dots,y_{S_\ell}\}, \tag{3.2}$$

where $\{y_{S_1},y_{S_2},\dots,y_{S_\ell},\mathcal{F}_S\}$ is the set of faces formed by the vertices $y_{S_1},\dots,y_{S_\ell}$ and a face in $\mathcal{F}_S$.

Proof. We use induction on the size of $|R|$, the case $|R| = 1$ being trivial. We next assume that our result holds true for $|R| = \rho$ and we prove it for $|R| = \rho + 1$.

When $|R| = \rho + 1$ the recursion in (3.1) coincides with that of the case $|R| = \rho$, until the last but one step, i.e., when $s = \rho - 1$. Thus, before doing the last recursion, we have:

(a) By induction:

$$\partial \mathcal{Q}_{S'} = \bigcup_{\varnothing \subset S \subseteq S'} \mathcal{F}_S \bigcup_{\substack{\varnothing \subset S \subset S' \\ S \subseteq S_1 \subset S_2 \subset \cdots \subset S_\ell \subset S'}} \{y_{S_1}, y_{S_2}, \dots, y_{S_\ell}, \mathcal{F}_S\} \bigcup_{\varnothing \subset S_1 \subset S_2 \subset \cdots \subset S_\ell \subset S'} \{y_{S_1}, y_{S_2}, \dots, y_{S_\ell}\},$$

for all $\varnothing \subset S' \subseteq R$ with $|S'| \leq \rho - 1$.

(b) By our construction, the faces in $\partial \mathcal{Q}_R$ are:

1) faces in each $\partial \mathcal{Q}_{R'}, |R'| = \rho - 1$,
2) the (trivial) faces $\{\mathcal{Q}_{R'}\}$ for $|R'| = \rho - 1$, and
3) faces in $\mathcal{F}_R$.

The faces in (b.1)-(b.3) are not necessarily disjoint. However, using (a) we can write them disjointly as follows:

$$\partial \mathcal{Q}_R = \bigcup_{\varnothing \subset S \subseteq R} \mathcal{F}_S \bigcup_{\substack{\varnothing \subset S \subset R \\ S \subseteq S_1 \subset S_2 \subset \cdots \subset S_\ell \subset R \\ |S| \leq \rho - 1}} \{y_{S_1}, y_{S_2}, \dots, y_{S_\ell}, \mathcal{F}_S\} \bigcup_{\substack{\varnothing \subset S_1 \subset S_2 \subset \cdots \subset S_\ell \subset R \\ |S| \leq \rho - 1}} \{y_{S_1}, y_{S_2}, \dots, y_{S_\ell}\} \bigcup_{\substack{S \subset R \\ |S| = \rho - 1}} \{\mathcal{Q}_S\}. \tag{3.3}$$

The faces in (3.3) that will be stellarly subdivided in the last recursion of (3.1) are all in some $\{\mathcal{Q}_S\}$ with $|S| = \rho - 1$. These, will be replaced by:

$$\bigcup_{\substack{S \subseteq R \\ |S| = \rho}} \left(\bigcup_{\varnothing \subset X \subseteq S} \{y_S, \mathcal{F}_X\} \bigcup_{\substack{\varnothing \subset X \subset S \\ X \subseteq S_1 \subset S_2 \subset \cdots \subset S_\ell \subset S}} \{y_{S_1}, y_{S_2}, \dots, y_{S_\ell}, y_S, \mathcal{F}_X\} \bigcup_{\varnothing \subset S_1 \subset S_2 \subset \cdots \subset S_\ell \subset S} \{y_{S_1}, y_{S_2}, \dots, y_{S_\ell}, y_S\} \right). \tag{3.4}$$

Combining (3.3) and (3.4) and recalling that $|R| = \rho + 1$ we conclude that indeed

$$\partial \mathcal{Q}_R = \bigcup_{\varnothing \subset S \subseteq R} \mathcal{F}_S \bigcup_{\substack{\varnothing \subset S \subset R \\ S \subseteq S_1 \subset S_2 \subset \cdots \subset S_\ell \subset R}} \{y_{S_1}, y_{S_2}, \dots, y_{S_\ell}, \mathcal{F}_S\} \bigcup_{\varnothing \subset S_1 \subset S_2 \subset \cdots \subset S_\ell \subset R} \{y_{S_1}, y_{S_2}, \dots, y_{S_\ell}\}.$$

□

Lemma 3.2. *For $\varnothing \subset R \subseteq [r]$, the non-trivial faces of the simplicial polytope $\mathcal{Q}_R$ are:*

$$\partial \mathcal{Q}_R = \mathcal{K}_R \bigcup_{\substack{\varnothing \subset S \subset R \\ S = S_1 \subset S_2 \subset \cdots \subset S_\ell \subset R}} \{y_{S_1}, y_{S_2}, \dots, y_{S_\ell}, \mathcal{K}_S\}. \tag{3.5}$$

Proof. Recall that the faces of $\mathcal{K}_R$ are all faces in $\bigcup_{\varnothing \subset S \subseteq R} \mathcal{F}_S$ together with the empty set. We can therefore write the right-hand side of (3.5) as:

$$\begin{aligned}
&\bigcup_{\varnothing \subset S \subseteq R} \mathcal{F}_S \bigcup_{\substack{\varnothing \subset S \subset R \\ S = S_1 \subset S_2 \subset \cdots \subset S_\ell \subset R}} \{y_{S_1}, y_{S_2}, \dots, y_{S_\ell}, \bigcup_{\varnothing \subset S' \subseteq S} \mathcal{F}_{S'}\} \bigcup_{\substack{\varnothing \subset S \subset R \\ S = S_1 \subset S_2 \subset \cdots \subset S_\ell \subset R}} \{y_{S_1}, y_{S_2}, \dots, y_{S_\ell}\} \\
&= \bigcup_{\varnothing \subset S \subseteq R} \mathcal{F}_S \bigcup_{\substack{\varnothing \subset S' \subseteq S \subset R \\ S' \subseteq S = S_1 \subset S_2 \subset \cdots \subset S_\ell \subset R}} \{y_{S_1}, y_{S_2}, \dots, y_{S_\ell}, \mathcal{F}_{S'}\} \bigcup_{\substack{\varnothing \subset S \subset R \\ S = S_1 \subset S_2 \subset \cdots \subset S_\ell \subset R}} \{y_{S_1}, y_{S_2}, \dots, y_{S_\ell}\} \\
&= \bigcup_{\varnothing \subset S \subseteq R} \mathcal{F}_S \bigcup_{\substack{\varnothing \subset S' \subset R \\ S' \subseteq S_1 \subset S_2 \subset \cdots \subset S_\ell \subset R}} \{y_{S_1}, y_{S_2}, \dots, y_{S_\ell}, \mathcal{F}_{S'}\} \bigcup_{\substack{\varnothing \subset S \subset R \\ S = S_1 \subset S_2 \subset \cdots \subset S_\ell \subset R}} \{y_{S_1}, y_{S_2}, \dots, y_{S_\ell}\},
\end{aligned}$$

which is precisely the quantity in (3.2) and thus equal to the set of faces of $\partial \mathcal{Q}_R$. □

The next lemma shows how the iterated stellar subdivisions performed in (3.1) are captured in the enumerative structure of $\mathcal{Q}_R$.

Lemma 3.3. *For any $\varnothing \subset R \subseteq [r]$ and $-1 \leq k \leq d + |R| - 2$, we have:*

$$f_k(\partial\mathcal{Q}_R) = f_k(\mathcal{F}_R) + \sum_{\varnothing\subset S\subset R}\left(\sum_{i=0}^{|R|-|S|} i!\, S^{i+1}_{|R|-|S|+1}\, f_{k-i}(\mathcal{F}_S)\right), \tag{3.6}$$

$$f_k(\partial\mathcal{Q}_R) = f_k(\mathcal{K}_R) + \sum_{\varnothing\subset S\subset R}\left(\sum_{i=0}^{|R|-|S|-1} (i+1)!\, S^{i+1}_{|R|-|S|}\, f_{k-1-i}(\mathcal{K}_S)\right), \tag{3.7}$$

where S^k_m are the Stirling numbers of the second kind [20]:

$$S^k_m = \frac{1}{k!}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i} i^m, \qquad m \geq k \geq 0.$$

Proof. To prove (3.6), we count the $(k+1)$-element subsets of the set in relation (3.2) of Lemma 3.1. This gives:

$$\begin{aligned}
f_k(\partial\mathcal{Q}_R) &= \sum_{\varnothing\subset S\subseteq R} f_k(\mathcal{F}_R) + \sum_{\varnothing\subset S\subset R}\sum_{i=1}^{|R|-|S|} |\{S \subseteq S_1 \subset S_2 \subset \cdots \subset S_i \subset R\}|\, f_{k-i}(\mathcal{F}_S) &(3.8)\\
&= \sum_{\varnothing\subset S\subseteq R} f_k(\mathcal{F}_R) + \sum_{\varnothing\subset S\subset R}\sum_{i=1}^{|R|-|S|} \mathsf{A}_{|R|-|S|}(\varnothing, i) f_{k-i}(\mathcal{F}_S) &(3.9)\\
&= \sum_{\varnothing\subset S\subseteq R} f_k(\mathcal{F}_R) + \sum_{\varnothing\subset S\subset R}\sum_{i=1}^{|R|-|S|} i!\, S^{i+1}_{|R|-|S|+1} f_{k-i}(\mathcal{F}_S) &(3.10)\\
&= f_k(\mathcal{F}_R) + \sum_{\varnothing\subset S\subset R}\sum_{i=0}^{|R|-|S|} i!\, S^{i+1}_{|R|-|S|+1} f_{k-i}(\mathcal{F}_S), &(3.11)
\end{aligned}$$

where,

- the value $i = k+1$ in (3.8) combined with the fact that $f_{-1}(\mathcal{F}_S) = (-1)^{|S|-1}$, counts precisely the elements in $\bigcup_{\varnothing\subset S_1\subset S_2\subset\cdots\subset S_\ell\subset R}\{y_{S_1}, y_{S_2}, \ldots, y_{S_\ell}\}$ in relation (3.2) of Lemma 3.1 via inclusion exclusion,
- to go from (3.9) to (3.10) we used Lemma A.1(ii), and
- from (3.10) to (3.11) we used the fact that $S^1_m = 1$ for all $m \geq 1$.

To prove (3.7), we utilize Lemma 3.2:

$$\begin{aligned}
f_k(\partial\mathcal{Q}_R) &= f_k(\mathcal{K}_R) + \sum_{\varnothing\subset S\subset R}\sum_{i=1}^{|R|-|S|} |\{S = S_1 \subset S_2 \subset \cdots \subset S_i \subset R\}|\, f_{k-i}(\mathcal{K}_S) &\\
&= f_k(\mathcal{K}_R) + \sum_{\varnothing\subset S\subset R}\sum_{i=1}^{|R|-|S|} \mathsf{B}_{|R|-|S|}(\varnothing, i)\, f_{k-i}(\mathcal{K}_S) &(3.12)\\
&= f_k(\mathcal{K}_R) + \sum_{\varnothing\subset S\subset R}\sum_{i=1}^{|R|-|S|} i!\, S^{i}_{|R|-|S|}\, f_{k-i}(\mathcal{K}_S) &(3.13)\\
&= f_k(\mathcal{K}_R) + \sum_{\varnothing\subset S\subset R}\sum_{i=0}^{|R|-|S|-1} (i+1)!\, S^{i+1}_{|R|-|S|}\, f_{k-i-1}(\mathcal{K}_S), &
\end{aligned}$$

where, to go from (3.12) to (3.13) we used Lemma A.1(i). □

Restating relations (3.6) and (3.7) in terms of generating functions, we arrive at Lemma 3.4. These relations will be used to transform (3.6) and (3.7) in their h-vector equivalents.

Lemma 3.4. *For all $\varnothing \subset R \subseteq [r]$ we have:*

$$\mathtt{f}(\partial\mathcal{Q}_R;t) = \mathtt{f}(\mathcal{F}_R;t) + \sum_{\varnothing\subset S\subset R}\sum_{i=0}^{|R|-|S|} i!\,S^{i+1}_{|R|-|S|+1} t^{|R|-|S|-i}\mathtt{f}(\mathcal{F}_S;t), \tag{3.14}$$

$$\mathtt{f}(\partial\mathcal{Q}_R;t) = \mathtt{f}(\mathcal{K}_R;t) + \sum_{\varnothing\subset S\subset R}\sum_{i=0}^{|R|-|S|-2} (i+1)!\,S^{i+1}_{|R|-|S|} t^{|R|-|S|-i}\mathtt{f}(\mathcal{K}_S;t). \tag{3.15}$$

Proof. Using relation (3.6) and recalling that $\dim(\partial\mathcal{Q}_R) = d + |R| - 2$, we have:

$$\begin{aligned}
\mathtt{f}(\partial\mathcal{Q}_R;t) &= \sum_{k=0}^{d+|R|-1} f_{k-1}(\partial\mathcal{Q}_R)t^{d+|R|-1-k} \\
&= \sum_{k=0}^{d+|R|-1} f_{k-1}(\mathcal{F}_R)t^{d+|R|-1-k} + \sum_{k=0}^{d+|R|-1}\sum_{\varnothing\subset S\subset R}\left(\sum_{i=0}^{|R|-|S|} i!\,S^{i+1}_{|R|-|S|+1} f_{k-1-i}(\mathcal{F}_S)\right)t^{d+|R|-1-k} \\
&= \mathtt{f}(\mathcal{F}_R;t) + \sum_{\varnothing\subset S\subset R}\sum_{i=0}^{|R|-|S|} i!\,S^{i+1}_{|R|-|S|+1}t^{|R|-|S|-i}\sum_{k=0}^{d+|R|-1} f_{k-i-1}(\mathcal{F}_S)\,t^{d+|S|-1-k+i} \\
&= \mathtt{f}(\mathcal{F}_R;t) + \sum_{\varnothing\subset S\subset R}\sum_{i=0}^{|R|-|S|} i!\,S^{i+1}_{|R|+|S|+1}t^{|R|-|S|-i}\sum_{k=i}^{d+|S|-1+i} f_{k-i-1}(\mathcal{F}_S)\,t^{d+|S|-1-k+i} \\
&= \mathtt{f}(\mathcal{F}_R;t) + \sum_{\varnothing\subset S\subset R}\sum_{i=0}^{|R|-|S|} i!\,S^{i+1}_{|R|-|S|+1}t^{|R|-|S|-i}\sum_{k=0}^{d+|S|-1} f_{k-1}(\mathcal{F}_S)\,t^{d+|S|-1-k} \\
&= \mathtt{f}(\mathcal{F}_R;t) + \sum_{\varnothing\subset S\subset R}\sum_{i=0}^{|R|-|S|} i!\,S^{i+1}_{|R|-|S|+1}t^{|R|-|S|-i}\mathtt{f}(\mathcal{F}_S;t).
\end{aligned}$$

Analogously, converting (3.7) into its generating function equivalent, we get:

$$\begin{aligned}
&\mathtt{f}(\partial\mathcal{Q}_R;t) \\
&\quad= \sum_{k=0}^{d+|R|-1} f_{k-1}(\mathcal{K}_R)t^{d+|R|-1-k} + \sum_{k=0}^{d+|R|-1}\sum_{\varnothing\subset S\subset R}\left(\sum_{i=0}^{|R|-|S|-1} (i+1)!\,S^{i+1}_{|R|-|S|} f_{k-1-i}(\mathcal{K}_S)\right)t^{d+|R|-1-k} \\
&\quad= \mathtt{f}(\mathcal{K}_R;t) + \sum_{\varnothing\subset S\subset R}\sum_{i=0}^{|R|-|S|-1} (i+1)!\,S^{i+1}_{|R|-|S|}t^{|R|-|S|-i}\sum_{k=0}^{d+|R|-1} f_{k-i-1}(\mathcal{K}_S)\,t^{d+|S|-1-k+i} \\
&\quad= \mathtt{f}(\mathcal{K}_R;t) + \sum_{\varnothing\subset S\subset R}\sum_{i=0}^{|R|-|S|-1} (i+1)!\,S^{i+1}_{|R|-|S|}t^{|R|-|S|-i}\sum_{k=0}^{d+|S|-1} f_{k-1}(\mathcal{K}_S)\,t^{d+|S|-1-k} \\
&\quad= \mathtt{f}(\mathcal{K}_R;t) + \sum_{\varnothing\subset S\subset R}\sum_{i=0}^{|R|-|S|-1} (i+1)!\,S^{i+1}_{|R|-|S|}t^{|R|-|S|-i}\mathtt{f}(\mathcal{K}_S;t),
\end{aligned}$$

where, in order to go from the third to the fourth line, we changed variables (in the last sum) and we used the fact that $f_{k-1}(\mathcal{K}_S) = 0$ for $k > d + |S| - 1$. ◻

The h-vector relations stemming from the f-vector relations above are the subject of the following lemma.

Lemma 3.5. *For all $\varnothing \subset R \subseteq [r]$ we have:*

$$\mathtt{h}(\partial\mathcal{Q}_R;t) = \mathtt{h}(\mathcal{F}_R;t) + \sum_{\varnothing\subset S\subset R}\sum_{j=0}^{|R|-|S|-1} E^{j}_{|R|-|S|}\,t^{j+1}\,\mathtt{h}(\mathcal{F}_S;t), \tag{3.16}$$

$$\mathtt{h}(\partial\mathcal{Q}_R;t) = \mathtt{h}(\mathcal{K}_R;t) + \sum_{\varnothing\subset S\subset R}\sum_{j=0}^{|R|-|S|-1} E^{j}_{|R|-|S|}\,t^{j}\,\mathtt{h}(\mathcal{K}_S;t), \tag{3.17}$$

where E_m^k are the Eulerian numbers [1, 10]:

$$E_m^k = \sum_{i=0}^{k}(-1)^i\binom{m+1}{i}(k+1-i)^m, \qquad m \geq k+1 > 0.$$

Proof. Using (2.4), (3.14) and the symmetry of Eulerian numbers, we get:

$$\begin{aligned}
\mathtt{h}(\partial\mathcal{Q}_R;t) &= \mathtt{f}(\partial\mathcal{Q}_R;t-1)\\
&= \mathtt{f}(\mathcal{F}_R;t-1) + \sum_{\varnothing\subset S\subset R}\underbrace{\sum_{i=0}^{|R|-|S|-1} i!\,S_{|R|-|S|+1}^{i+1}(t-1)^{|R|-|S|-i}}_{\text{cf. (A.3)}}\mathtt{f}(\mathcal{F}_S;t-1)\\
&= \mathtt{h}(\mathcal{F}_R;t) + \sum_{\varnothing\subset S\subset R}\sum_{j=0}^{|R|-|S|-1} E_{|R|-|S|}^{j}t^{|R|-|S|-j}\mathtt{h}(\mathcal{F}_S;t)\\
&= \mathtt{h}(\mathcal{F}_R;t) + \sum_{\varnothing\subset S\subset R}\sum_{j=0}^{|R|-|S|-1} E_{|R|-|S|}^{|R|-|S|-1-j}t^{|R|-|S|-j}\mathtt{h}(\mathcal{F}_S;t)\\
&= \mathtt{h}(\mathcal{F}_R;t) + \sum_{\varnothing\subset S\subset R}\sum_{j=0}^{|R|-|S|-1} E_{|R|-|S|}^{j}t^{j+1}\mathtt{h}(\mathcal{F}_S;t).
\end{aligned}$$

Analogously, using (2.4), (3.15) and the symmetry of Eulerian numbers, we deduce that:

$$\begin{aligned}
\mathtt{h}(\partial\mathcal{Q}_R;t) &= \mathtt{f}(\partial\mathcal{Q}_R;t-1)\\
&= \mathtt{f}(\mathcal{K}_R;t-1) + \sum_{\varnothing\subset S\subset R}\sum_{i=0}^{|R|-|S|-2} (i+1)!\,S_{|R|-|S|}^{i+1}t^{|R|-|S|-i-1}\mathtt{f}(\mathcal{K}_S;t-1)\\
&= \mathtt{h}(\mathcal{K}_R;t) + \sum_{\varnothing\subset S\subset R}\underbrace{\sum_{i=0}^{|R|-|S|-2} (i+1)!\,S_{|R|-|S|}^{i+1}(t-1)^{|R|-|S|-i}}_{\text{cf. (A.4)}}\mathtt{h}(\mathcal{K}_S;t)\\
&= \mathtt{h}(\mathcal{K}_R;t) + \sum_{\varnothing\subset S\subset R}\sum_{i=0}^{|R|-|S|-2} E_{|R|-|S|}^{i}t^{|R|-|S|-i-1}\mathtt{h}(\mathcal{K}_S;t)\\
&= \mathtt{h}(\mathcal{K}_R;t) + \sum_{\varnothing\subset S\subset R}\sum_{i=0}^{|R|-|S|-2} E_{|R|-|S|}^{|R|-|S|-1-i}t^{|R|-|S|-i-1}\mathtt{h}(\mathcal{K}_S;t)\\
&= \mathtt{h}(\mathcal{K}_R;t) + \sum_{\varnothing\subset S\subset R}\sum_{i=0}^{|R|-|S|-2} E_{|R|-|S|}^{i}t^{i}\mathtt{h}(\mathcal{K}_S;t).
\end{aligned}$$

□

4 The Dehn-Sommervile equations

A very important structural property of the Cayley polytope $\mathcal{C}_R$ is, what we call, the *Dehn-Sommervile equations*. For a single polytope they reduce to the well-known Dehn-Sommerville equations, whereas for two or more summands they relate the h-vectors of the sets $\mathcal{F}_R$ and $\mathcal{K}_R$. The Dehn-Sommerville equations for $\mathcal{C}_R$ are one of the major key ingredients for establishing our upper bounds, as they permit us to reason for the maximality of the elements of $\boldsymbol{h}(\mathcal{F}_R)$ and $\boldsymbol{h}(\mathcal{K}_R)$ by considering only the lower halves of these vectors.

Theorem 4.1 (Dehn-Sommerville equations)**.** *Let $\mathcal{C}_R$ be the Cayley polytope of the d-polytopes $P_i, i \in R$. Then, the following relations hold:*

$$t^{d+|R|-1}\mathtt{h}(\mathcal{F}_R;\tfrac{1}{t}) = \mathtt{h}(\mathcal{K}_R;t) \tag{4.1}$$

or, equivalently,

$$h_{d+|R|-1-k}(\mathcal{F}_R) = h_k(\mathcal{K}_R), \qquad 0 \le k \le d+|R|-1. \tag{4.2}$$

Proof. We prove our claim by induction on the size of R, the case $|R|=1$ being the Dehn-Somerville equations for a d-polytope. We next assume that our claim holds for all $\varnothing \subset S \subset R$ and prove it for R. The ordinary Dehn-Somerville relations, written in generating function form, for the (simplicial) $(d+|R|-1)$-polytope $\mathcal{Q}_R$ imply that:

$$\mathtt{h}(\partial\mathcal{Q}_R;t) = t^{d+|R|-1}\mathtt{h}(\partial\mathcal{Q}_R;\tfrac{1}{t}). \tag{4.3}$$

In view of relation (3.16) of Lemma 3.5, the right-hand side of (4.3) becomes:

$$t^{d+|R|-1}\mathtt{h}(\mathcal{F}_R;\tfrac{1}{t}) + t^{d+|R|-1}\sum_{\varnothing\subset S\subset R}\sum_{j=0}^{|R|-|S|-1} E^{j}_{|R|-|S|}t^{-j-1}\mathtt{h}(\mathcal{F}_S;\tfrac{1}{t}). \tag{4.4}$$

Using relation (3.17), along with the induction hypothesis, the left-hand side of (4.3) becomes:

$$\mathtt{h}(\mathcal{K}_R;t) + \sum_{\varnothing\subset S\subset R}\sum_{j=0}^{|R|-|S|-1} E^{j}_{|R|-|S|}t^{j}\mathtt{h}(\mathcal{K}_S;t) \tag{4.5}$$

$$= \mathtt{h}(\mathcal{K}_R;t) + \sum_{\varnothing\subset S\subset R}\sum_{j=0}^{|R|-|S|-1} E^{j}_{|R|-|S|}t^{|R|-|S|-j-1}\mathtt{h}(\mathcal{K}_S;t) \tag{4.6}$$

$$= \mathtt{h}(\mathcal{K}_R;t) + \sum_{\varnothing\subset S\subset R}\sum_{j=0}^{|R|-|S|-1} E^{j}_{|R|-|S|}t^{|R|-|S|-j-1}t^{d+|S|-1}\mathtt{h}(\mathcal{F}_S;\tfrac{1}{t})$$

$$= \mathtt{h}(\mathcal{K}_R;t) + \sum_{\varnothing\subset S\subset R}\sum_{j=0}^{|R|-|S|-1} E^{j}_{|R|-|S|}t^{d+|R|-j-2}\mathtt{h}(\mathcal{F}_S;\tfrac{1}{t}), \tag{4.7}$$

where to go from (4.5) to (4.6) we changed variables and used the well-known symmetry of the Eulerian numbers, namely, $E^k_m = E^{m-k-1}_m$, for all $m \ge k+1 > 0$.

Now, substituting (4.4) and (4.7) in (4.3), we deduce that $t^{d+|R|-1}\mathtt{h}(\mathcal{F}_R;\frac{1}{t}) = \mathtt{h}(\mathcal{K}_R;t)$, which is, coefficient-wise, equivalent to (4.2). $\square$

5 The recurrence relation for $\boldsymbol{h}(\mathcal{F}_R)$

The subject of this section is the generalization, for the h-vector of $\mathcal{F}_R$, $\varnothing \subset R \subseteq [r]$, of the recurrence relation

$$(k+1)h_{k+1}(\partial P) + (d-k)h_k(\partial P) \le n\,h_k(\partial P), \quad 0 \le k \le d-1, \tag{5.1}$$

that holds true for any simplicial d-polytope $P \subset \mathbb{R}^d$. This is the content of the next theorem. Its proof is postponed until Section 5.6. In the next five subsections we build upon the necessary intermediate results for proving this theorem.

Theorem 5.1 (Recurrence inequality)**.** *For any $\varnothing \subset R \subseteq [r]$ we have:*

$$h_{k+1}(\mathcal{F}_R) \le \frac{n_R - d - |R| + 1 + k}{k+1}h_k(\mathcal{F}_R) + \sum_{i\in R}\frac{n_i}{k+1}g_k(\mathcal{F}_{R\setminus\{i\}}), \qquad 0 \le k \le d+|R|-2, \tag{5.2}$$

where: (1) $n_R = \sum_{i\in R} n_i$, $n_\varnothing = \varnothing$, and, (2) $g_k(\mathcal{F}_\varnothing) = g_k(\varnothing) = 0$, for all k.

5.1 Relating the h-vector of $\mathcal{Q}_R/v$ with the h-vectors of $\mathcal{F}_R/v$ and $\mathcal{K}_R/v$

For any $\varnothing \subset R \subseteq [r]$, let $V_R := \cup_{i\in R} V_i$. We define the link of a vertex $v \in V_R$ in $\mathcal{F}_R$ as the intersection of the link $\mathcal{K}_R/v$ with $\mathcal{F}_R$. The following lemma relates the h-vector of $\mathcal{Q}_R/v$ with the h-vectors of $\mathcal{F}_R/v$ and $\mathcal{K}_R/v$.

Lemma 5.2. *For any $v \in V_R$ we have:*

$$\mathtt{h}(\partial\mathcal{Q}_R/v;t) = \mathtt{h}(\mathcal{F}_R/v;t) + \sum_{\{i\}\subseteq S\subset R} \sum_{j=0}^{|R|-|S|-1} E^j_{|R|-|S|} t^{j+1} \mathtt{h}(\mathcal{F}_S/v;t), \tag{5.3}$$

and

$$\mathtt{h}(\partial\mathcal{Q}_R/v;t) = \mathtt{h}(\mathcal{K}_R/v;t) + \sum_{\{i\}\subseteq S\subset R} \sum_{j=0}^{|R|-|S|-1} E^j_{|R|-|S|} t^{j} \mathtt{h}(\mathcal{K}_S/v;t). \tag{5.4}$$

Proof. Let us fix some $v \in V_j$, $j \in R$. In view of relation (3.2) in Lemma 3.1 we can write:

$$\partial\mathcal{Q}_R/v = \bigcup_{\varnothing\subset S\subseteq R} \mathcal{F}_S/v \bigcup_{\substack{\varnothing\subset S\subset R \\ S\subseteq S_1\subset S_2\subset\cdots\subset S_\ell\subset R}} \{y_{S_1}, y_{S_2}, \ldots, y_{S_\ell}, \mathcal{F}_S/v\}, \tag{5.5}$$

where it is understood that both $\mathcal{F}_S/v$ and $\{y_{S_1}, y_{S_2}, \ldots, y_{S_\ell}, \mathcal{F}_S/v\}$ are empty if $v \notin V_S$. Taking this into account, we simplify (5.5) as follows:

$$\partial\mathcal{Q}_R/v = \bigcup_{\{j\}\subseteq S\subseteq R} \mathcal{F}_S/v \bigcup_{\substack{\{j\}\subseteq S\subset R \\ S\subseteq S_1\subset S_2\subset\cdots\subset S_\ell\subset R}} \{y_{S_1}, y_{S_2}, \ldots, y_{S_\ell}, \mathcal{F}_S/v\}. \tag{5.6}$$

Since each auxiliary point of a face in $\{y_{S_1}, y_{S_2}, \ldots, y_{S_\ell}, \mathcal{F}_S/v\}$ increases the dimension by one, from (5.6) we can write :

$$f_k(\partial\mathcal{Q}_R/v) = \sum_{\{j\}\subseteq S\subseteq R} f_k(\mathcal{F}_S/v) + \sum_{\{j\}\subseteq S\subset R} \sum_{i=1}^{|R|-|S|} \sum_{S\subseteq S_1\subset S_2\subset\cdots\subset S_i\subset R} f_{k-i}(\mathcal{F}_S/v).$$

In view of Lemma A.1(i) the above can be written as:

$$\begin{aligned} f_k(\partial\mathcal{Q}_R/v) &= \sum_{\{j\}\subseteq S\subseteq R} f_k(\mathcal{F}_S/v) + \sum_{\{j\}\subseteq S\subset R} \sum_{i=1}^{|R|-|S|} i!\, S^{i+1}_{|R|-|S|+1} f_{k-i}(\mathcal{F}_S/v) \\ &= f_k(\mathcal{F}_R/v) + \sum_{\{j\}\subseteq S\subset R} \sum_{i=0}^{|R|-|S|} i!\, S^{i+1}_{|R|-|S|+1} f_{k-i}(\mathcal{F}_S/v), \end{aligned}$$

where in the last step we used the fact that $S^1_m = 1$ for all $m \geq 1$.

Recalling that $\dim(\mathcal{F}_S/v) = d + |S| - 3$ and converting the above relation into generating function we get:

$$\mathtt{f}(\partial\mathcal{Q}_R/v;t) = \mathtt{f}(\mathcal{F}_R/v;t) + \sum_{\{j\}\subseteq S\subset R} \sum_{i=0}^{|R|-|S|} i!\, S^{i+1}_{|R|-|S|+1} t^{|R|-|S|-i} \mathtt{f}(\mathcal{F}_S/v;t). \tag{5.7}$$

We thus have:

$$\begin{aligned} \mathtt{h}(\partial\mathcal{Q}_R/v;t) &= \mathtt{f}(\partial\mathcal{Q}_R/v;t-1) \\ &= \mathtt{f}(\mathcal{F}_R/v;t-1) + \sum_{\{j\}\subseteq S\subset R} \sum_{i=0}^{|R|-|S|} i!\, S^{i+1}_{|R|-|S|+1} t^{|R|-|S|-i} \mathtt{f}(\mathcal{F}_S/v;t-1) \\ &= \mathtt{h}(\mathcal{F}_R/v;t) + \sum_{\{j\}\subseteq S\subset R} \sum_{i=0}^{|R|-|S|} i!\, S^{i+1}_{|R|-|S|+1} (t-1)^{|R|-|S|-i} \mathtt{f}(\mathcal{F}_S/v;t-1) \end{aligned}$$

$$= \mathtt{h}(\mathcal{F}_R/v;t) + \sum_{\{j\}\subseteq S\subset R} \sum_{i=0}^{|R|-|S|} i!\, S^{i+1}_{|R|-|S|+1} (t-1)^{|R|-|S|-i} \mathtt{h}(\mathcal{F}_S/v;t) \tag{5.8}$$

$$= \mathtt{h}(\mathcal{F}_R/v;t) + \sum_{\{j\}\subseteq S\subset R} \sum_{j=0}^{|R|-|S|-1} E^{j}_{|R|-|S|} t^{|R|-|S|-j} \mathtt{h}(\mathcal{F}_S/v;t) \tag{5.9}$$

$$= \mathtt{h}(\mathcal{F}_R/v;t) + \sum_{\{j\}\subseteq S\subset R} \sum_{j=0}^{|R|-|S|-1} E^{|R|-|S|-1-j}_{|R|-|S|} t^{|R|-|S|-j} \mathtt{h}(\mathcal{F}_S/v;t)$$

$$= \mathtt{h}(\mathcal{F}_R/v;t) + \sum_{\{j\}\subseteq S\subset R} \sum_{j=0}^{|R|-|S|-1} E^{j}_{|R|-|S|} t^{j+1} \mathtt{h}(\mathcal{F}_S/v;t),$$

where to go from (5.8) to (5.9) we used relation (A.3) from Lemma A.2.

Let us now turn our attention to relation (5.4) . In view of (3.5) of Lemma 3.2 we have:

$$\partial\mathcal{Q}_R/v = \mathcal{K}_R/v \bigcup_{\substack{\{j\}\subseteq S\subset R\\ S=S_1\subset S_2\subset\cdots\subset S_\ell\subset R}} \{y_{S_1}, y_{S_2}, \ldots, y_{S_\ell}, \mathcal{K}_S/v\},$$

which in turn gives

$$\begin{aligned} f_k(\partial\mathcal{Q}_R/v) &= f_k(\mathcal{K}_R/v) + \sum_{\{j\}\subseteq S\subset R} \sum_{i=1}^{|R|-|S|} \sum_{S=S_1\subset S_2\subset\cdots\subset S_i\subset R} f_{k-i}(\mathcal{K}_S/v) \\ &= f_k(\mathcal{K}_R/v) + \sum_{\{j\}\subseteq S\subset R} \sum_{i=1}^{|R|-|S|} i!\, S^{i}_{|R|-|S|} f_{k-i}(\mathcal{K}_S/v) \\ &= f_k(\mathcal{K}_R/v) + \sum_{\{j\}\subseteq S\subset R} \sum_{i=0}^{|R|-|S|-1} (i+1)!\, S^{i+1}_{|R|-|S|} f_{k-i-1}(\mathcal{K}_S/v). \end{aligned}$$

Recalling that $\dim(\mathcal{K}_S/v) = d+|S|-3$ and converting the above relation into generating function, we get:

$$\mathtt{f}(\partial\mathcal{Q}_R/v;t) = \mathtt{f}(\mathcal{K}_R/v;t) + \sum_{\{j\}\subseteq S\subset R} \sum_{i=0}^{|R|-|S|-1} (i+1)!\, S^{i+1}_{|R|-|S|} t^{|R|-|S|-i}\, \mathtt{f}(\mathcal{K}_S/v;t), \tag{5.10}$$

which further implies that

$$\begin{aligned} \mathtt{h}(\partial\mathcal{Q}_R/v;t) &= \mathtt{f}(\partial\mathcal{Q}_R/v;t-1) \\ &= \mathtt{f}(\mathcal{K}_R/v;t-1) + \sum_{\{j\}\subseteq S\subset R} \sum_{i=0}^{|R|-|S|-1} (i+1)!\, S^{i+1}_{|R|-|S|} (t-1)^{|R|-|S|-1-i} \mathtt{f}(\mathcal{K}_S/v;t-1) \end{aligned} \tag{5.11}$$

$$= \mathtt{h}(\mathcal{K}_R/v;t) + \sum_{\{j\}\subseteq S\subset R} \sum_{j=0}^{|R|-|S|-1} E^{i}_{|R|-|S|} t^{|R|-|S|-1-i} \mathtt{h}(\mathcal{K}_S/v;t) \tag{5.12}$$

$$= \mathtt{h}(\mathcal{K}_R/v;t) + \sum_{\{j\}\subseteq S\subset R} \sum_{j=0}^{|R|-|S|-1} E^{|R|-|S|-1-j}_{|R|-|S|} t^{|R|-|S|-1-j}\, \mathtt{h}(\mathcal{K}_S/v;t)$$

$$= \mathtt{h}(\mathcal{K}_R/v;t) + \sum_{\{j\}\subseteq S\subset R} \sum_{j=0}^{|R|-|S|-1} E^{j}_{|R|-|S|} t^{j}\, \mathtt{h}(\mathcal{K}_S/v;t),$$

where to go from (5.11) to (5.12) we used (A.4) from Lemma A.2. □

5.2 The link of y_S in $\partial\mathcal{Q}_R$

Our next goal is to find an expression analogous to those of Lemma 5.2, but now involving links of type $\partial\mathcal{Q}_R/y_S$, where $\varnothing \subset S \subseteq R$. To do this, we first need to express $f_k(\partial\mathcal{Q}_R/y_S)$ in terms of sums of $f_i(\mathcal{F}_X)$ with $i \le k$ and $X \subseteq S$. This is the content of the next Lemma. In order to state it we need to introduce a new set. Let $X \subseteq T \subset R$ and ℓ be a positive integer. We define the set

$$\mathcal{D}(R,T,X,\ell) := \{(S_1,\dots,S_\ell) : X \subseteq S_1 \subset S_2 \subset \cdots \subset S_\ell \subset R \text{ and } S_i = T \text{ for some } 1 \le i \le \ell\}, \tag{5.13}$$

and denote by $\mathsf{D}(R,T,X,\ell)$ its cardinality.

Lemma 5.3. *For every $\varnothing \subset S \subset R$ we have:*

$$\mathtt{f}(\partial\mathcal{Q}_R/y_S;t) = \sum_{\varnothing\subset X\subseteq S}\sum_{\ell=1}^{|R|-|X|} \mathsf{D}(R,S,X,\ell)\, t^{|R|-|X|-\ell}\mathtt{f}(\mathcal{F}_X;t). \tag{5.14}$$

Proof. First of all, notice that, in view of relation (3.2), if we denote by $y_S * \partial\mathcal{Q}_R$ the set of all faces in $\partial\mathcal{Q}_R$ containing y_S, we have:

$$y_S * \partial\mathcal{Q}_R = \bigcup_{\substack{\varnothing\subset X\subseteq S\\ X\subseteq S_1\subset S_2\subset\cdots\subset S_\ell\subset R\\ S_i=S \text{ for some } 1\le i\le\ell}} \{y_{S_1}, y_{S_2},\dots,y_{S_\ell},\mathcal{F}_X\}.$$

Then clearly,

$$\begin{aligned}
f_k(\partial\mathcal{Q}_R/y_S) = f_{k+1}(y_S * \partial\mathcal{Q}_R) &= \sum_{\varnothing\subset X\subseteq S}\sum_{\ell=1}^{|R|-|X|}\sum_{\substack{X\subseteq S_1\subset S_2\subset\cdots\subset S_\ell\subset R\\ S_i=S \text{ for some } 1\le i\le\ell}} f_{k-\ell+1}(\mathcal{F}_X)\\
&= \sum_{\varnothing\subset X\subseteq S}\sum_{\ell=1}^{|R|-|X|} \mathsf{D}(R,S,X,\ell)\, f_{k-\ell+1}(\mathcal{F}_X).
\end{aligned}$$

Using the fact that $\dim(\partial\mathcal{Q}/y_S) = d+|R|-3$ and rewriting in terms of generating functions, the above becomes:

$$\begin{aligned}
\mathtt{f}(\partial\mathcal{Q}_R/y_S;t) &= \sum_{k=0}^{d+|R|-|S|-1}\sum_{\varnothing\subset X\subseteq S}\sum_{\ell=1}^{|R|-|X|} \mathsf{D}(R,S,X,\ell)\, f_{k-\ell+1}(\mathcal{F}_X)t^{d+|R|-2-k}\\
&= \sum_{k=0}^{d+|R|-|S|-1}\sum_{\varnothing\subset X\subseteq S}\sum_{\ell=1}^{|R|-|X|} t^{|R|-|X|-\ell}\mathsf{D}(R,S,X,\ell)\, f_{k-\ell+1}(\mathcal{F}_X)t^{d+|X|-1-(k-\ell+1)}\\
&= \sum_{\varnothing\subset X\subseteq S}\sum_{\ell=1}^{|R|-|X|}\mathsf{D}(R,S,X,\ell)t^{|R|-|X|-\ell}\sum_{k-\ell+1=|X|-|R|+1}^{d+|R|-|S|-1} f_{k-\ell+1}(\mathcal{F}_X)t^{d+|X|-1-(k-\ell+1)}\\
&= \sum_{\varnothing\subset X\subseteq S}\sum_{\ell=1}^{|R|-|X|} \mathsf{D}(R,S,X,\ell)t^{|R|-|X|-\ell}\sum_{j=0}^{d+|R|-|S|-1} f_j(\mathcal{F}_X)t^{d+|X|-1-j}\\
&= \sum_{\varnothing\subset X\subseteq S}\sum_{\ell=1}^{|R|-|X|} \mathsf{D}(R,S,X,\ell)t^{|R|-|X|-\ell}\sum_{j=0}^{d+|R|-|X|-1} f_j(\mathcal{F}_X)t^{d+|X|-1-j}\\
&= \sum_{\varnothing\subset X\subseteq S}\sum_{\ell=1}^{|R|-|X|} \mathsf{D}(R,S,X,\ell)t^{|R|-|X|-\ell}\mathtt{f}(\mathcal{F}_X;t).
\end{aligned}$$

□

Converting relation (5.14) of the above lemma to its h-vector equivalent we get:

$$\begin{aligned}
\mathtt{h}(\partial\mathcal{Q}_R/y_S;t) &= \mathtt{f}(\partial\mathcal{Q}_R/y_S;t-1)\\
&= \sum_{\varnothing\subset X\subseteq S}\sum_{\ell=1}^{|R|-|X|} \mathsf{D}(R,S,X,\ell)(t-1)^{|R|-|X|-j}\mathtt{h}(\mathcal{F}_X;t).
\end{aligned} \tag{5.15}$$

The following lemma expresses the sum of the h-vectors of the links $\mathcal{Q}_R/y_S$ to the h-vectors of the sets $\mathcal{F}_X$.

Lemma 5.4. *For every $\emptyset \subset R \subseteq [r]$ we have:*

$$\sum_{\emptyset \subset S \subset R} \mathtt{h}(\partial \mathcal{Q}_R / y_S; t) = \sum_{\emptyset \subset X \subset R} \sum_{j=0}^{|R|} \big(E^j_{|R|-|X|+1} - E^j_{|R|-|X|} \big) t^{|R|-|X|-j} \mathtt{h}(\mathcal{F}_X; t). \tag{5.16}$$

Proof. By means of relation (5.15), the sum $\sum_{\emptyset \subset S \subset R} \mathtt{h}(\partial \mathcal{Q}_R / y_S; t)$ is equal to:

$$\sum_{\emptyset \subset X \subseteq S \subset R} \sum_{\ell=1}^{|R|-|X|} \mathsf{D}(R,S,X,\ell)(t-1)^{|R|-|X|-l} \mathtt{h}(\mathcal{F}_X; t)$$

$$= \sum_{\ell=1}^{|R|} \sum_{\emptyset \subset X \subset R} \sum_{X \subseteq S \subset R} \mathsf{D}(R,S,X,\ell)(t-1)^{|R|-|X|-l} \mathtt{h}(\mathcal{F}_X; t) \tag{5.17}$$

$$= \sum_{\ell=1}^{|R|} \sum_{\emptyset \subset X \subset R} \ell\, \ell!\, S^{\ell+1}_{|R|-|X|+1} (t-1)^{|R|-|X|-l} \mathtt{h}(\mathcal{F}_X; t) \tag{5.18}$$

$$= \sum_{\emptyset \subset X \subset R} \left(\sum_{\ell=0}^{|R|} \ell\, \ell!\, S^{\ell+1}_{|R|-|X|+1} (t-1)^{|R|-|X|-\ell} \right) \mathtt{h}(\mathcal{F}_X; t)$$

$$= \sum_{\emptyset \subset X \subset R} \left(\sum_{\ell=0}^{|R|} (\ell+1)\, \ell!\, S^{\ell+1}_{|R|-|X|+1} (t-1)^{|R|-|X|-\ell} - \sum_{\ell=1}^{|R|} S^{\ell+1}_{|R|-|X|+1} (t-1)^{|R|-|X|-l} \right) \mathtt{h}(\mathcal{F}_X; t) \tag{5.19}$$

$$= \sum_{\emptyset \subset X \subset R} \left(\sum_{j=0}^{|R|} E^j_{|R|-|X|+1} (t-1)^{|R|-|X|-j} - \sum_{j=0}^{|R|} E^j_{|R|-|X|} (t-1)^{|R|-|X|-j} \right) \mathtt{h}(\mathcal{F}_X; t) \tag{5.20}$$

$$= \sum_{\emptyset \subset X \subset R} \sum_{j=0}^{|R|} \big(E^j_{|R|-|X|+1} - E^j_{|R|-|X|} \big) (t-1)^{|R|-|X|-j} \mathtt{h}(\mathcal{F}_X; t),$$

where, to go from (5.17) to (5.18) and from (5.19) to (5.20) we used Lemma B.2 and Lemma A.2, respectively. □

5.3 Links and non-links

The following theorem generalizes Lemma B.1 in the context of Cayley polytopes.

Theorem 5.5. *For any $\emptyset \subset R \subseteq [r]$,*

$$(d + |R| - 1)\mathtt{h}(\mathcal{F}_R; t) + (1-t)\mathtt{h}'(\mathcal{F}_R; t) = \sum_{v \in V_R} \mathtt{h}(\mathcal{F}_R / v; t), \tag{5.21}$$

where $V_R = \cup_{i \in R} V_i$.

Proof. We proceed by induction on the size of R. The case $|R| = 1$ is considered in Lemma B.1. Assume now that (5.21) holds for all $\emptyset \subset S \subset R$. By applying Lemma B.1 to the simplicial polytope $\mathcal{Q}_R$ we have:

$$(d + |R| - 1)\, \mathtt{h}(\partial \mathcal{Q}_R; t) + (1-t)\mathtt{h}'(\partial \mathcal{Q}_R; t) = \sum_{v \in \mathsf{vert}(\partial \mathcal{Q}_R)} \mathtt{h}(\partial \mathcal{Q}_R / v; t). \tag{5.22}$$

Recall from Lemma 3.5 that:

$$\mathtt{h}(\partial \mathcal{Q}_R; t) = \mathtt{h}(\mathcal{F}_R; t) + \sum_{\emptyset \subset S \subset R} \sum_{j=0}^{|R|-|S|-1} E^j_{|R|-|S|} t^{|R|-|S|-j} \mathtt{h}(\mathcal{F}_S; t). \tag{5.23}$$

Multiplying both sides of (5.23) by $d + |R| - 1$ we get:

$$(d + |R| - 1)\, \mathtt{h}(\partial \mathcal{Q}; t) =$$

$$(d+|R|-1)\mathtt{h}(\mathcal{F}_R;t) + (d+|R|-1)\sum_{\varnothing\subset S\subset R}\sum_{j=0}^{|R|-|S|-1} E^j_{|R|-|S|} t^{|R|-|S|-j}\mathtt{h}(\mathcal{F}_S;t)$$

Differentiating both sides of (5.23) and multiplying by $(1-t)$ we get:

$$\begin{aligned}(1-t)\mathtt{h}'(\partial\mathcal{Q}_R;t) &= (1-t)\mathtt{h}'(\mathcal{F}_R;t)\\ &\quad + (1-t)\sum_{\varnothing\subset S\subset R}\sum_{j=0}^{|R|-|S|-1}(|R|-|S|-j)E^j_{|R|-|S|}t^{|R|-|S|-j-1}\mathtt{h}(\mathcal{F}_S;t)\\ &\quad + (1-t)\sum_{\varnothing\subset S\subset R}\sum_{j=0}^{|R|-|S|-1}E^j_{|R|-|S|}t^{|R|-|S|-j}\mathtt{h}'(\mathcal{F}_S;t).\end{aligned}$$

Summing up the above two relations and using Lemma B.2 for the $(d+|R|-1)$-polytope $\mathcal{Q}_R$, we conclude that the right-hand side of (5.22) is equal to:

$$\begin{aligned}&(d+|R|-1)\,\mathtt{h}(\partial\mathcal{Q}_R;t) + (1-t)\mathtt{h}'(\partial\mathcal{Q}_R;t)\\ &= (d+|R|-1)\mathtt{h}(\mathcal{F}_R;t) + (d+|R|-1)\sum_{\varnothing\subset S\subset R}\sum_{j=0}^{|R|-|S|-1}E^j_{|R|-|S|}t^{|R|-|S|-j}\mathtt{h}(\mathcal{F}_S;t)\\ &\quad + (1-t)\mathtt{h}'(\mathcal{F}_R;t) + (1-t)\sum_{\varnothing\subset S\subset R}\sum_{j=0}^{|R|-|S|-1}(|R|-|S|-j)E^j_{|R|-|S|}t^{|R|-|S|-j-1}\mathtt{h}(\mathcal{F}_S;t)\\ &\quad + (1-t)\sum_{\varnothing\subset S\subset R}\sum_{j=0}^{|R|-|S|-1}E^j_{|R|-|S|}t^{|R|-|S|-j}\\ &= A + \sum_{\varnothing\subset S\subset R}\sum_{j=0}^{|R|-|S|-1}(d+|S|-1)E^j_{|R|-|S|}t^{|R|-|S|-j}\mathtt{h}(\mathcal{F}_S;t)\\ &\quad + \sum_{\varnothing\subset S\subset R}\sum_{j=0}^{|R|-|S|-1} j\,E^j_{|R|-|S|}t^{|R|-|S|-j}\mathtt{h}(\mathcal{F}_S;t)\\ &\quad + \sum_{\varnothing\subset S\subset R}\sum_{j=0}^{|R|-|S|-1}(|R|-|S|-j)E^j_{|R|-|S|}t^{|R|-|S|-j-1}\mathtt{h}(\mathcal{F}_S;t)\\ &\quad + (1-t)\sum_{\varnothing\subset S\subset R}\sum_{j=0}^{|R|-|S|-1}E^j_{|R|-|S|}t^{|R|-|S|-j}\mathtt{h}'(\mathcal{F}_S;t),\end{aligned}$$

where $A = (d+|R|-1)\mathtt{h}(\mathcal{F}_R;t) + (1-t)\mathtt{h}'(\mathcal{F}_R;t)$. In order to use our induction hypothesis, we regroup the terms of the above expression as follows:

$$\begin{aligned}&A + \sum_{\varnothing\subset S\subset R}\sum_{j=0}^{|R|-|S|-1}E^j_{|R|-|S|}\Big((d+|S|-1)\mathtt{h}(\mathcal{F}_S;t) - (1-t)\mathtt{h}'(\mathcal{F}_S;t)\Big)t^{|R|-|S|-j}\\ &\quad + \sum_{\varnothing\subset S\subset R}\sum_{j=0}^{|R|-|S|-1}\Big((j+1)E^{j+1}_{|R|-|S|} + (|R|-|S|-j)E^j_{|R|-|S|}\Big)t^{|R|-|S|-j-1}\mathtt{h}(\mathcal{F}_S;t).\end{aligned}$$

Using the well known recurrence relation for the Eulerian numbers (cf. [10]):

$$E^i_m = (m-i)\,E^i_{m-1} + (i+1)\,E^i_{m-1},$$

and the induction hypothesis, the above expression simplifies to:

$$\begin{aligned}&A + \sum_{\varnothing\subset S\subset R}\sum_{j=0}^{|R|-|S|-1}\sum_{v\in V_S}E^j_{|R|-|S|}t^{|R|-|S|-j}\mathtt{h}(\mathcal{F}_S/v;t)\\ &\quad + \sum_{\varnothing\subset S\subset R}\sum_{j=0}^{|R|-|S|-1}\Big(E^{j+1}_{|R|-|S|+1} - E^{j+1}_{|R|-|S|}\Big)t^{|R|-|S|-j-1}\mathtt{h}(\mathcal{F}_S;t).\end{aligned} \tag{5.24}$$

Since the vertices of $\mathcal{Q}_R$ are either vertices of some polytope $P_i, i \in R$, or auxiliary points y_S, $\varnothing \subset S \subseteq R$, we split the sum in the right-hand side of (5.22) as follows:

$$\sum_{v \in \mathrm{vert}(\partial \mathcal{Q}_R)} \mathtt{h}(\partial \mathcal{Q}_R/v;t) = \sum_{v \in V_R} \mathtt{h}(\partial \mathcal{Q}_R/v;t) + \sum_{S \subset R} \mathtt{h}(\partial \mathcal{Q}_R/y_S;t)$$

Using relations (5.16) and (5.3), the right-hand side of the above equation is equal to:

$$\begin{aligned}
&\underbrace{\sum_{v \in V_R} \mathtt{h}(\mathcal{F}_R/v;t)}_{B} + \sum_{\substack{i \in R \\ v \in V_i}} \sum_{\{i\} \subseteq S \subset R} \sum_{j=0}^{|R|-|S|-1} E^j_{|R|-|S|} t^{j+1} \mathtt{h}(\mathcal{F}_S/v;t) \\
&\qquad\qquad + \sum_{\varnothing \subset X \subset R} \sum_{\ell=0}^{|R|} \big(E^\ell_{|R|-|X|+1} - E^\ell_{|R|-|X|}\big) t^{|R|-|X|-\ell} \mathtt{h}(\mathcal{F}_X;t) \\
&= B + \sum_{\varnothing \subset S \subset R} \sum_{v \in V_S} \sum_{j=0}^{|R|-|S|-1} E^j_{|R|-|S|} t^{j+1} \mathtt{h}(\mathcal{F}_S/v;t) \\
&\qquad + \sum_{\varnothing \subset X \subset R} \sum_{\ell=0}^{|R|} \big(E^\ell_{|R|-|X|+1} - E^\ell_{|R|-|X|}\big) t^{|R|-|X|-l} \mathtt{h}(\mathcal{F}_X;t) \\
&= B + \sum_{\varnothing \subset S \subset R} \sum_{v \in V_S} \sum_{j=0}^{|R|-|S|-1} \underbrace{E^{|R|-|S|-j-1}_{|R|-|S|}}_{E^j_{|R|-|S|}} t^{|R|-|S|-j} \mathtt{h}(\mathcal{F}_S/v;t) \\
&\qquad + \sum_{\varnothing \subset X \subset R} \sum_{\ell=0}^{|R|-|S|} \big(E^{\ell+1}_{|R|-|X|+1} - E^{\ell+1}_{|R|-|X|}\big) t^{|R|-|X|-\ell-1} \mathtt{h}(\mathcal{F}_X;t).
\end{aligned} \tag{5.25}$$

Equating (5.24) and (5.25) we conclude that $A = B$, which is precisely relation (5.21). $\square$

Comparing coefficients in (5.21) we conclude the following:

Corollary 5.6. *For any $\varnothing \subset R \subseteq [r]$ and all $0 \leq k \leq d + |R| - 2$ we have:*

$$(k+1)h_{k+1}(\mathcal{F}_R) + (d + |R| - 1 - k)h_k(\mathcal{F}_R) = \sum_{v \in V_R} h_k(\mathcal{F}_R/v), \tag{5.26}$$

or equivalently

$$(k+1)h_{k+1}(\mathcal{F}_R) + (d + |R| - 1 - k)h_k(\mathcal{F}_R) = \sum_{\varnothing \subset S \subseteq R} (-1)^{|R|-|S|} \sum_{v \in V_S} g_k^{(|R|-|S|)}(\mathcal{K}_S/v), \tag{5.27}$$

where $V_S = \cup_{i \in S} V_i$.

Proof. Relation (5.26) is immediate from (5.21); it suffices to compare the coefficients of the generating functions of left- and right-hand sides of (5.21).

To go from (5.26) to (5.27) we use the Inclusion-Exclusion principle, and notice that $\mathcal{K}_S/v$ is the empty set for $v \notin \mathcal{K}_S$:

$$\begin{aligned}
\sum_{v \in V_R} h_k(\mathcal{F}_R/v) &= \sum_{v \in V_R} \sum_{\varnothing \subset S \subseteq R} (-1)^{|R|-|S|} g_k^{(|R|-|S|)}(\mathcal{K}_S/v) \\
&= \sum_{\varnothing \subset S \subseteq R} (-1)^{|R|-|S|} \sum_{v \in V_R} g_k^{(|R|-|S|)}(\mathcal{K}_S/v) \\
&= \sum_{\varnothing \subset S \subseteq R} (-1)^{|R|-|S|} \sum_{v \in V_S} g_k^{(|R|-|S|)}(\mathcal{K}_S/v).
\end{aligned}$$

$\square$

5.4 Using shellings to bound the g-vectors of links

The main result of this subsection is Theorem 5.13, which is essential for proving the recursive relation in Theorem 5.1. Before proving it, some more lemmas are in order. The first two (Lemmas 5.10 and 5.11) concern inequalities of h-vectors, which are proved using their interpretation as in-degrees of the dual graph of shellable simplicial complexes (cf. [13]). The third (Lemma 5.12) shows that there exists a particular shelling of the polytope $\partial \mathcal{Q}_R$, for which the previous two lemmas are applicable.

We start with some definitions.

Definition 5.7. *Let $\mathcal{C}$ be a pure d-dimensional complex. A shelling of $\mathcal{C}$ is a linear ordering $F_1, \dots, F_s$ of its facets such that either $\mathcal{C}$ is 0-dimensional, or it satisfies the following conditions:*

(a) the boundary complex ∂F_1 of the first facet has a shelling,

(b) for $1 < j \leq s$ the intersection of the facet F_j with the previous facets is nonempty and is a beginning segment of a shelling of the $(d-1)$-dimensional boundary complex ∂F_j, that is $F_j \bigcap_{i=1}^{j-1} F_i = G_1 \cup G_2 \cup \dots \cup G_r$ for some shelling $G_1, \dots, G_r, \dots, G_t$ of ∂F_j.

A complex is shellable *if it is pure and has a shelling.*

Definition 5.8. *The dual graph $\mathscr{V}^{\triangle}(\mathcal{C})$ of a shellable simplicial d-complex $\mathcal{C}$ is the graph whose vertices are the maximal simplices (i.e., facets) and whose edges correspond to adjacent facets. If, in addition, we consider a linear ordering $F_1, \dots, F_\ell$ of the facets of $\mathcal{C}$, we can impose an orientation on the graph $\mathscr{V}^{\triangle}(\mathcal{C})$ as follows: an edge connecting two facets F_i, F_j is oriented from F_i to F_j if F_i precedes F_j in the above order.*

In the case where $\mathcal{C}$ is shellable, the h-vector of $\mathcal{C}$ encodes information about the in-degrees of the dual graph $\mathscr{V}^{\triangle}(\mathcal{C})$. This is the content of the next theorem.

Theorem 5.9. [13] *Let $\mathcal{C}$ be a shellable simplicial d-complex and consider the dual graph $\mathscr{V}^{\triangle}(\mathcal{C})$ of $\mathcal{C}$ oriented according to a shelling order of the facets of $\mathcal{C}$. Then, $h_k(\mathcal{C})$, $0 \leq k \leq d$, counts the number of vertices of the dual graph of $\mathcal{C}$ with in-degree k (and is independent of the shelling chosen).*

Let S be a shellable simplicial complex and assume that $F_1, \dots, F_\ell, F_{\ell+1}, \dots, F_s$ is a shelling order of its facets. Let A be the subcomplex of S whose facets are $F_1, \dots, F_\ell$. Clearly, A is shellable as an initial segment of a shelling of S. Consider now the set B containing all faces in $\mathsf{S} \setminus \mathsf{A}$. Notice that B has no complex structure since it contains the facets $F_{\ell+1}, \dots, F_s$ but not all their subfaces. We can however naturally define its f-vector and, since all its maximal faces are facets of S, make the convention that $\dim(\mathsf{B}) = \dim(\mathsf{S})$. Moreover, as the following lemma suggests, the h-vector of B admits a combinatorial interpretation.

Lemma 5.10. *$h_k(\mathsf{B})$ counts the number of vertices in $\mathscr{V}^{\triangle}(\mathsf{S}) \setminus \mathscr{V}^{\triangle}(\mathsf{A})$ of in-degree k.*

Proof. In view of Theorem 5.9 we have that: (i) $h_k(\mathsf{S})$ counts the number of vertices of the dual graph $\mathscr{V}^{\triangle}(\mathsf{S})$ of S with in-degree k and (ii) $h_k(\mathsf{A})$ counts the number of vertices of the dual graph $\mathscr{V}^{\triangle}(\mathsf{A})$ of A with in-degree k. However, since the facets in A are an initial segment of a shelling of S, their in-degree in $\mathscr{V}^{\triangle}(\mathsf{A})$ as well as in $\mathscr{V}^{\triangle}(\mathsf{S})$ is the same (the out-degrees of vertices in $\mathscr{V}^{\triangle}(\mathsf{A})$ might be greater when seen as vertices in $\mathscr{V}^{\triangle}(\mathsf{S})$). Thus, the difference $h_k(\mathsf{B}) = h_k(\mathsf{S}) - h_k(\mathsf{A})$ counts the vertices in $\mathscr{V}^{\triangle}(\mathsf{S}) \setminus \mathscr{V}^{\triangle}(\mathsf{A})$ with in-degree k. □

In the case where S is a simplicial polytope, A a beginning segment of its shelling and B the set theoretical difference of their faces, the above interpretation helps us compare the h-vector of B with that of its link B/v on v, for any vertex v in B.

Lemma 5.11. *$h_k(\mathsf{S}/v) - h_k(\mathsf{A}/v) \leq h_k(\mathsf{S}) - h_k(\mathsf{A})$ or equivalently, $h_k(\mathsf{B}/v) \leq h_k(\mathsf{B})$.*

Proof. To prove our claim, we use the fact that for any vertex v of a polytope S there exists a shelling such that the facets that contain v, i.e., the facets in $\mathrm{star}(\mathsf{S}, v)$, appear first in this shelling [22, Corollary 8.13]. Applying Lemma 5.10 for S as well as for S/v we have that:

- $h_k(\mathsf{B})$ counts the number of vertices in $\mathscr{V}^\triangle(\mathsf{S}) \setminus \mathscr{V}^\triangle(\mathsf{A})$ of in-degree k,
- $h_k(\mathsf{B}/v)$ counts the number of vertices in $\mathscr{V}^\triangle(\mathsf{S}/v) \setminus \mathscr{V}^\triangle(\mathsf{A}/v)$ of in-degree k.

Moreover, since in the above mentioned shellings the link is shelled first, the in-degree of a vertex in $\mathscr{V}^\triangle(\mathsf{S}) \setminus \mathscr{V}^\triangle(\mathsf{A})$ can only but be greater with respect to its in-degree in $\mathscr{V}^\triangle(\mathsf{S}/v) \setminus \mathscr{V}^\triangle(\mathsf{A}/v)$. This immediately implies the statement of the lemma. □

Using the machinery developed above, we may now show that $\partial\mathcal{Q}_R$ admits a particular shelling, as stated in the following lemma.

Lemma 5.12. *There exists a shelling of $\partial\mathcal{Q}_R$ starting from facets in $\bigcup_{j\in R\setminus\{1\}} \mathrm{star}(y_{R\setminus\{j\}}, \partial\mathcal{Q}_R)$, and finishing with facets in $\mathrm{star}(y_{R\setminus\{1\}}, \partial\mathcal{Q}_R)$.*

Proof. Let us start with some definitions: we denote by $\mathcal{Z}$ the $(d+|R|-1)$-complex we get by performing the recursion in (3.1) until the last but one step, i.e., after having added all the auxiliary vertices y_S with $|S| \le |R|-2$. Clearly, the facets of $\mathcal{Z}$ are the $(d+|R|-2)$-polytopes $\mathcal{Q}_{R\setminus\{i\}}$, $i \in R$, as well as all facets in $\mathcal{F}_R$. Since $\mathcal{Z}$ is polytopal, each line in general position induces a shelling order of its facets (cf. [22, Section 8.2]). We will chose a line in such a way, so that the induced line shelling of $\mathcal{Z}$ leads us (after adding all vertices $y_{R\setminus\{i\}}$) to the sought-for shelling of $\partial\mathcal{Q}_R$.

Notice that, by the definition of the Cayley embedding, there exists a hyperplane in $\mathbb{R}^{d+|R|-1}$ containing P_1 and being parallel to $\mathcal{C}_{R\setminus\{1\}}$ (and thus to $\mathcal{Q}_{R\setminus\{1\}}$). We can therefore choose a line ℓ beyond y_1 in $\mathcal{Z}$ and intersecting $\mathcal{Q}_{R\setminus\{1\}}$ in its interior. This line ℓ yields a shelling $\mathsf{S}(\mathcal{Z})$ of $\mathcal{Z}$ starting from facets in $\mathrm{star}(y_1, \mathcal{Z})$ and finishing with $\mathcal{Q}_{R\setminus\{1\}}$. Since the facets in $\mathrm{star}(y_1, \mathcal{Z})$ are nothing but the polytopes $\mathcal{Q}_{R\setminus\{i\}}$, $i \in R\setminus\{1\}$, the shelling $\mathsf{S}(\mathcal{Z})$ starts with all $\mathcal{Q}_{R\setminus\{i\}}$, $i \in R\setminus\{1\}$, (continues with the facets in $\mathcal{F}_R$) and ends with $\mathcal{Q}_{R\setminus\{1\}}$. Our next goal is to replace each facet $\mathcal{Q}_{R\setminus\{i\}}$ in $\mathsf{S}(\mathcal{Z})$ by all facets in $\mathrm{star}(y_{R\setminus\{i\}}, \partial\mathcal{Q}_{R\setminus\{i\}})$, ordered so that the conditions in Definition 5.7 are satisfied.

We do this by induction. If $\mathcal{Q}_{R\setminus\{2\}}$ is the first facet in the shelling order $\mathsf{S}(\mathcal{Z})$ then we can replace it by the facets in $\mathrm{star}(y_{R\setminus\{2\}}, \partial\mathcal{Q}_{R\setminus\{2\}})$, in any order "inherited" from a shelling of $\partial\mathcal{Q}_{R\setminus\{2\}}$. Without loss of generality, we assume that the facets $\mathcal{Q}_{R\setminus\{j\}}$ with $2 \le j < i$ are those preceding $\mathcal{Q}_{R\setminus\{i\}}$ in the shelling order $\mathsf{S}(\mathcal{Z})$. By our induction hypothesis, we have replaced all $\mathcal{Q}_{R\setminus\{j\}}$ by $\mathrm{star}(y_{R\setminus\{j\}}, \partial\mathcal{Q}_{R\setminus\{j\}})$ in a way that the conditions of our claim are satisfied; we want to prove the same for $j = i$.

Indeed, notice that the intersection of $\mathcal{Q}_{R\setminus\{i\}}$ with the union of the previous facets, is the union of all $\mathcal{Q}_{R\setminus\{i,j\}}$ with $2 \le j < i$, whether we consider "previous" in the shelling $\mathsf{S}(\mathcal{Z})$ or in the shelling until the current inductive step (i.e., when each $\mathcal{Q}_{R\setminus\{j\}}$ with $2 \le j < i$ is stellarly subdivided). As a result, the second condition of Definition 5.7, namely that that there exists a shelling order of the facets of $\partial\mathcal{Q}_{R\setminus\{i\}}$ starting with all facets of $\bigcup_{2\le j<i} \partial\mathcal{Q}_{R\setminus\{i,j\}}$, holds. It suffices to choose a shelling order of $\partial\mathcal{Q}_{R\setminus\{j\}}$ that respects the common shelling order with $\bigcup_{2\le j<i} \partial\mathcal{Q}_{R\setminus\{i,j\}}$. Using this shelling order, we may replace the facet $\mathcal{Q}_{R\setminus\{i\}}$ by those in $\mathrm{star}(y_{R\setminus\{i\}}, \partial\mathcal{Q}_{R\setminus\{i\}})$ (the shelling orders of each $\mathrm{star}(y_S, \partial\mathcal{Q}_S)$ are inherited from those for $\partial\mathcal{Q}_S$) and arrive at a shelling order of $\mathcal{Q}_R$ with the desired properties. The last facet $\mathcal{Q}_{R\setminus\{1\}}$ can be replaced by $\mathrm{star}(y_{R\setminus\{1\}}, \partial\mathcal{Q}_{R\setminus\{1\}})$ without any further concern, since the shelling conditions are already fulfilled from the shelling $\mathsf{S}(\mathcal{Z})$. □

Exploiting Lemmas 5.10, 5.11 and 5.12 we arrive at the following theorem, where we bound the right-hand side of (5.27) by an expression that does not involve the links $\mathcal{K}_S/v$.

Theorem 5.13. *For all $v \in V_R$ and all $k \geq 0$ we have:*

$$\sum_{\varnothing \subset S \subseteq R} (-1)^{|R|-|S|} \sum_{v \in V_S} g_k^{(|R|-|S|)}(\mathcal{K}_S/v) \leq \sum_{\varnothing \subset S \subseteq R} (-1)^{|R|-|S|} \sum_{v \in V_S} g_k^{(|R|-|S|)}(\mathcal{K}_S), \tag{5.28}$$

where $V_S = \cup_{i \in S} V_i$.

Proof. Let us first observe that, by rearranging terms, we can rewrite relation (5.28) as:

$$\sum_{i \in R} \sum_{v \in V_i} \sum_{\{i\} \subseteq S \subseteq R} (-1)^{|R|-|S|} g_k^{(|R|-|S|)}(\mathcal{K}_S/v) \leq \sum_{i \in R} \sum_{v \in V_i} \sum_{\{i\} \subseteq S \subseteq R} (-1)^{|R|-|S|} g_k^{(|R|-|S|)}(\mathcal{K}_S). \tag{5.29}$$

Clearly, to show that relation (5.29) holds, it suffices to prove that:

$$\sum_{\{i\} \subseteq S \subseteq R} (-1)^{|R|-|S|} g_k^{(|R|-|S|)}(\mathcal{K}_S/v) \leq \sum_{\{i\} \subseteq S \subseteq R} (-1)^{|R|-|S|} g_k^{(|R|-|S|)}(\mathcal{K}_S), \tag{5.30}$$

for any arbitrary fixed $i \in R$.

Without loss of generality we may assume that $i = 1$. Define $\mathcal{G}_1 = \mathcal{F}_R \cup \mathcal{F}_{R \setminus \{1\}}$. Since $\mathcal{F}_R$ and $\mathcal{F}_{R \setminus \{1\}}$ are disjoint, we can write:

$$\begin{aligned} f_k(\mathcal{G}_1) &= f_k(\mathcal{F}_R) + f_k(\mathcal{F}_{R \setminus \{1\}}) \\ &= \sum_{S \subseteq R} (-1)^{|R|-|S|} f_k(\mathcal{K}_S) + \sum_{S \subseteq R \setminus \{1\}} (-1)^{|R|-1-|S|} f_k(\mathcal{K}_S) \\ &= \sum_{S \subseteq R} (-1)^{|R|-|S|} f_k(\mathcal{K}_S) - \sum_{S \subseteq R \setminus \{1\}} (-1)^{|R|-|S|} f_k(\mathcal{K}_S) \\ &= \sum_{\{1\} \subseteq S \subseteq R} (-1)^{|R|-|S|} f_k(\mathcal{K}_S). \end{aligned} \tag{5.31}$$

Similarly, for all $v \in V_1$:

$$f_k(\mathcal{G}_1/v) = \sum_{\{1\} \subseteq S \subseteq R} (-1)^{|R|-|S|} f_k(\mathcal{K}_S/v). \tag{5.32}$$

Converting the above relations into h-vector relations (using generating functions and comparing coefficients) we deduce that:

$$h_k(\mathcal{G}_1) = \sum_{\{1\} \subseteq S \subseteq R} (-1)^{|R|-|S|} g_k^{(|R|-|S|)}(\mathcal{K}_S), \tag{5.33}$$

and

$$h_k(\mathcal{G}_1/v) = \sum_{\{1\} \subseteq S \subseteq R} (-1)^{|R|-|S|} g_k^{(|R|-|S|)}(\mathcal{K}_S/v). \tag{5.34}$$

Thus, in view of (5.33) and (5.34), proving (5.30) reduces to showing that $h_k(\mathcal{G}_1/v) \leq h_k(\mathcal{G}_1)$. Define $\partial\mathcal{Q}'_R$ to be the polytopal $(d + |R| - 1)$-complex whose facets are the facets of $\partial\mathcal{Q}_R$ not incident to $y_{R \setminus \{1\}}$. To understand the face structure of $\partial\mathcal{Q}'_R$, we use Lemma 3.1 to rewrite $\partial\mathcal{Q}_R$ as the union:

$$\mathcal{F}_R \bigcup_{i \in R} \operatorname{star}(y_{R \setminus \{i\}}, \partial\mathcal{Q}_R)$$

of, not necessarily disjoint, faces. After removing all faces of $\partial\mathcal{Q}_R$ incident to $y_{R \setminus \{1\}}$ we are left with the following set of faces:

$$\partial\mathcal{Q}'_R = \overbrace{\bigcup_{i \in R \setminus \{1\}} \operatorname{star}(y_{R \setminus \{i\}}, \partial\mathcal{Q}_R)}^{\text{A}} \cup \overbrace{\mathcal{F}_R \cup \mathcal{F}_{R \setminus \{1\}}}^{\text{B}}.$$

Although the face sets in the above union are not disjoint, the face sets A and B are. This further implies that the facets of $\partial\mathcal{Q}'_R$ are the facets in A and those in B. We next claim that $\partial\mathcal{Q}'_R$ is shellable and that there exists a shelling of $\partial\mathcal{Q}'_R$ in which all facets in A come first.

Indeed, according to Lemma 5.12, there exists a shelling of $\partial\mathcal{Q}_R$ starting from facets in $\bigcup_{i\in R\smallsetminus\{1\}}\mathrm{star}(y_{R\smallsetminus\{i\}},\partial\mathcal{Q}_R)$, continuing with those in $\mathcal{F}_R$ and ending with facets in $\mathrm{star}(y_{R\smallsetminus\{1\}},\partial\mathcal{Q}_R)$. Discarding the facets in $\mathrm{star}(y_{R\smallsetminus\{1\}},\partial\mathcal{Q}_R)$ we obtain a shelling of $\partial\mathcal{Q}'_R$ starting from facets in $\bigcup_{i\in R\smallsetminus\{1\}}\mathrm{star}(y_{R\smallsetminus\{i\}},\partial\mathcal{Q}_R)$ and ending with facets in $\mathcal{F}_R$. We then apply Lemma 5.11 with $\mathsf{S} := \partial\mathcal{Q}'_R$ and $\mathsf{A} := \bigcup_{i\in R\smallsetminus\{1\}}\mathrm{star}(y_{R\smallsetminus\{i\}},\partial\mathcal{Q}_R)$ and we deduce that $h_k(\mathsf{B}/v)\leq h_k(\mathsf{B})$, or equivalently that $h_k(\mathcal{G}_1/v)\leq h_k(\mathcal{G}_1)$. This completes our proof. □

5.5 The last step towards the recurrence relation

The last step for proving Theorem 5.1, is the following lemma that involves calculations which simplify the right-hand side of (5.28).

Lemma 5.14. *Let $\varnothing\subset R\subseteq[r]$, and $V_S=\cup_{i\in S}V_i$, for all $\varnothing\subset S\subseteq R$. Then, for all $k\geq 0$ we have:*

$$\sum_{\varnothing\subset S\subseteq R}(-1)^{|R|-|S|}\sum_{v\in V_S}g_k^{(|R|-|S|)}(\mathcal{K}_S)=n_R\,h_k(\mathcal{F}_R)+\sum_{i\in R}n_i\,g_k(\mathcal{F}_{R\smallsetminus\{i\}}),\tag{5.35}$$

where $n_R=\sum_{i\in R}n_i$ and $n_\varnothing=0$.

Proof. From relation (2.12) and the definition of the m-order g-vector (cf. (2.2)), we can easily show that, for any $\varnothing\subset R\subseteq[r]$,

$$g_k^{(m)}(\mathcal{K}_R)=\sum_{\varnothing\subset S\subseteq R}g_k^{(|R|-|S|+m)}(\mathcal{F}_S).$$

Hence, for all $0\leq k\leq d+|R|-1$, we get:

$$g_k^{(|R|-|S|)}(\mathcal{K}_S)=\sum_{\varnothing\subset X\subseteq S}g_k^{(|S|-|X|+(|R|-|S|))}(\mathcal{F}_X)=\sum_{\varnothing\subset X\subseteq S}g_k^{(|R|-|X|)}(\mathcal{F}_X).$$

Thus, the left-hand side of (5.35) becomes:

$$\begin{aligned}\sum_{\varnothing\subset S\subseteq R}(-1)^{|R|-|S|}\sum_{v\in V_S}g_k^{(|R|-|S|)}(\mathcal{K}_S)&=\sum_{\varnothing\subset S\subseteq R}(-1)^{|R|-|S|}n_S\,g_k^{(|R|-|S|)}(\mathcal{K}_S)\\&=\sum_{\varnothing\subset S\subseteq R}(-1)^{|R|-|S|}n_S\sum_{\varnothing\subset X\subseteq S}g_k^{(|R|-|X|)}(\mathcal{F}_X)=\sum_{\varnothing\subset X\subseteq R}\Big(\sum_{X\subseteq S\subseteq R}(-1)^{|R|-|S|}n_S\Big)g_k^{(|R|-|X|)}(\mathcal{F}_X).\end{aligned}\tag{5.36}$$

We next evaluate the coefficient of $g_k^{(|R|-|X|)}(\mathcal{F}_X)$ in (5.36), i.e., the quantity

$$\sum_{X\subseteq S\subseteq R}(-1)^{|R|-|S|}n_S.\tag{5.37}$$

We separate cases:

(a) If $X=R$ the sum in (5.37) simplifies to n_R.

(b) If $|X|=|R|-1$, then $X=R\smallsetminus\{i\}$ for some $i\in R$ and the sum in (5.37) simplifies to

$$\sum_{X\subseteq S\subseteq R}(-1)^{|R|-|S|}n_S=(-1)^{|R|-(|R|-1)}n_{R\smallsetminus\{i\}}+(-1)^{|R|-|R|}n_R=n_R-n_{R\smallsetminus\{i\}}=n_i.$$

(c) If $|X|<|R|-1$ then for every $i\in R\smallsetminus X$ and every $0\leq j\leq|R|-|X|-1$ there exist $\binom{|R|-|X|-1}{j}$ sets of size $|X|+j+1$ containing i. We therefore have:

$$\sum_{X\subseteq S\subseteq R}(-1)^{|R|-|S|}n_S=\sum_{i\in R\smallsetminus X}\sum_{j=0}^{|R|-|X|-1}(-1)^{|R|-|X|-j-1}\binom{|R|-|X|-1}{j}n_i=\sum_{i\in R\smallsetminus X}0^{|R|-|X|-1}n_i=0.$$

From (a)-(c) we deduce that the only non-zero coefficients of $g_k^{(|R|-|X|)}(\mathcal{F}_X)$ in (5.36) are those for which $|X| = |R|$ or $|R|-1$. Thus, the sum in (5.36) simplifies to

$$n_R h_k(\mathcal{F}_R) + \sum_{i\in R} n_i\, g_k(\mathcal{F}_{R\setminus\{i\}}),$$

which is precisely the right-hand side of (5.35). □

5.6 The proof of Theorem 5.1

Proof of Theorem 5.1. To prove the inequality in the statement of the theorem, we generalize McMullen's steps in the proof of his Upper Bound theorem [18].

Our starting point is relation (5.1) applied to the simplicial $(d+|R|-1)$-polytope $\mathcal{Q}_R$, expressed in terms of generating functions:

$$(d+|R|-1)\,\mathtt{h}(\partial\mathcal{Q}_R;t) + (1-t)\mathtt{h}'(\partial\mathcal{Q}_R;t) = \sum_{v\in\mathrm{vert}(\partial\mathcal{Q}_R)} \mathtt{h}(\partial\mathcal{Q}_R/v;t). \tag{5.38}$$

Exploiting the combinatorial structure of $\mathcal{Q}_R$ in order to express: (1) $\boldsymbol{h}(\partial\mathcal{Q}_R)$ in terms of $\boldsymbol{h}(\mathcal{F}_S)$, $\varnothing\subset S\subset R$, and (2) $\boldsymbol{h}(\partial\mathcal{Q}_R/v)$ in terms of $\boldsymbol{h}(\mathcal{F}_R/v)$ and $\boldsymbol{h}(\mathcal{F}_S)$, $\varnothing\subset S\subset R$, relation (5.38) yields (see Sections 5.1–5.3):

$$(d+|R|-1)\mathtt{h}(\mathcal{F}_R;t) + (1-t)\mathtt{h}'(\mathcal{F}_R;t) = \sum_{v\in V_R} \mathtt{h}(\mathcal{F}_R/v;t),$$

the element-wise form of which is:

$$(k+1)h_{k+1}(\mathcal{F}_R) + (d+|R|-1-k)h_k(\mathcal{F}_R) = \sum_{v\in V_R} h_k(\mathcal{F}_R/v), \quad 0\le k\le d+|R|-2.$$

Noticing that $h_k(\mathcal{F}_R/v)$ is equal to $\sum_{\varnothing\subset S\subseteq R}(-1)^{|R|-|S|}\sum_{v\in V_S} g_k^{(|R|-|S|)}(\mathcal{K}_S/v)$ (by the Inclusion-Exclusion Principle), we have that (see Section 5.4):

$$\sum_{\varnothing\subset S\subseteq R}(-1)^{|R|-|S|}\sum_{v\in V_S} g_k^{(|R|-|S|)}(\mathcal{K}_S/v) \le \sum_{\varnothing\subset S\subseteq R}(-1)^{|R|-|S|}\sum_{v\in V_S} g_k^{(|R|-|S|)}(\mathcal{K}_S).$$

The right-hand side of the above relation simplifies to $n_R\, h_k(\mathcal{F}_R)+\sum_{i\in R} n_i\, g_k(\mathcal{F}_{R\setminus\{i\}})$ (cf. Section 5.5), which in turn suggests the following inequality:

$$(k+1)h_{k+1}(\mathcal{F}_R) + (d+|R|-1-k)h_k(\mathcal{F}_R) \le n_R\, h_k(\mathcal{F}_R) + \sum_{i\in R} n_i\, g_k(\mathcal{F}_{R\setminus\{i\}}) \tag{5.39}$$

that holds true for all $0\le k\le d+|R|-2$. Solving in terms of $h_{k+1}(\mathcal{F}_R)$ results in (5.2). □

6 Upper bounds

Let $S_1,\dots,S_r$ be a partition of a set S into r sets. We say that $A\subseteq \bigcup_{1\le i\le r} S_i$ is a *spanning subset of* S if $A\cap S_i\ne\varnothing$ for all $1\le i\le r$.

Definition 6.1. *Let $P_i, i\in R$, be d-polytopes with vertex sets $V_i, i\in R$. We say that their Cayley polytope $\mathcal{C}_R$ is* R-neighborly *if every spanning subset of $\bigcup_{i\in R} V_i$ of size $|R|\le \ell\le \lfloor\frac{d+|R|-1}{2}\rfloor$ is a face of $\mathcal{C}_R$ (or, equivalently, a face of $\mathcal{F}_R$). We say that the Cayley polytope $\mathcal{C}_R$ is* Minkowski-neighborly *if, for every $\varnothing\subset S\subseteq R$, the Cayley polytope $\mathcal{C}_S$ is S-neighborly.*

The following characterizes R-neighborly Cayley polytopes in terms of the f- and h-vector of $\mathcal{F}_R$.

Lemma 6.2. *The following are equivalent:*

(i) $\mathcal{C}_R$ *is* R*-neighborly,*

(ii) $f_{\ell-1}(\mathcal{F}_R) = \sum_{\varnothing\subset S\subseteq R}(-1)^{|R|-|S|}\binom{n_S}{\ell}$, *for all* $0 \le \ell \le \lfloor\frac{d+|R|-1}{2}\rfloor$,

(iii) $h_\ell(\mathcal{F}_R) = \sum_{\varnothing\subset S\subseteq R}(-1)^{|R|-|S|}\binom{n_S-d-|R|+\ell}{\ell}$, *for all* $0 \le \ell \le \lfloor\frac{d+|R|-1}{2}\rfloor$,

where n_i *is the number of vertices of* P_i *and* $n_S = \sum_{i\in S} n_i$.

Proof. To show the equivalence between (i) and (ii), notice, from the definition of spanning subsets, that every spanning subset of $V_R = \bigcup_{i\in R} V_i$ of size $\ell \ge |R|$ has:

$$\sum_{\substack{\sum_{i\in R} k_i = \ell \\ 1\le k_i\le n_i}} \prod_{i\in R}\binom{n_i}{k_i}$$

elements. Using induction on the size of R, one can check that the above sum of products is equal to the expression on the right-hand side of (ii). Moreover, in the case where $\ell < |R|$, the expression on the right-hand side of (ii) is 0. This, agrees with the fact that there do not exist any spanning subsets of $\bigcup\limits_{i\in R} V_i$ of size $\ell < |R|$.

We next show the equivalence between (ii) and (iii). Taking the $(d-k)$-th derivative of relation (2.4) for $\mathcal{F}_R$, it suffices to show that the values for $f_{\ell-1}(\mathcal{F}_R)$ and $h_\ell(\mathcal{F}_R)$, $0 \le \ell \le k$, in the statement of the theorem satisfy

$$\sum_{i=0}^{k} f_{i-1}(\mathcal{F}_R)\frac{(d-i)!}{(k-i)!}t^{k-i} = \sum_{i=0}^{k}\frac{(d-i)!}{(k-i)!}h_i(\mathcal{F}_R)(t+1)^{k-i}. \tag{6.1}$$

Indeed, we have:

$$\begin{aligned}
&\sum_{i=0}^{k} h_i(\mathcal{F}_R)\tfrac{(d+|R|-1-i)!}{(k-i)!}(t+1)^{k-i}\\
&\qquad= \sum_{i=0}^{k}\sum_{\varnothing\subset S\subseteq R}(-1)^{|R|-|S|}\tbinom{n_S-d-|R|+i}{i}\tfrac{(d+|R|-1-i)!}{(k-i)!}(t+1)^{k-i}\\
&\qquad= \sum_{\varnothing\subset S\subseteq R}(-1)^{|R|-|S|}\sum_{i=0}^{k}\tbinom{n_S-d-|R|+i}{i}\tfrac{(d+|R|-1-i)!}{(k-i)!}\sum_{j=0}^{k-i}\tbinom{k-i}{j}t^j\\
&\qquad= \sum_{\varnothing\subset S\subseteq R}(-1)^{|R|-|S|}\sum_{i=0}^{k}\sum_{j=0}^{k}\tfrac{(d+|R|-1-k+j)!}{j!}\tbinom{n_S-d-|R|+i}{i}t\tbinom{d+|R'|-1-i}{d+|R|-1-k+j}t^j\\
&\qquad= \sum_{\varnothing\subset S\subseteq R}(-1)^{|R|-|S|}\sum_{j=0}^{k}\tfrac{(d+|R|-1-k+j)!}{j!}\sum_{i=0}^{k}\tbinom{n_S-d-|R|+i}{n_S-d-|R|}\tbinom{d+|R|-1-i}{k-i-j}t^j && (6.2)\\
&\qquad= \sum_{\varnothing\subset S\subseteq R}(-1)^{|R|-|S|}\sum_{j=0}^{k}\tfrac{(d+|R|-1-k+j)!}{j!}\tbinom{n_S}{n_S-k+j}t^j && (6.3)\\
&\qquad= \sum_{\varnothing\subset S\subseteq R}(-1)^{|R|-|S|}\sum_{j=0}^{k}\tfrac{(d+|R|-1-j)!}{(k-j)!}\tbinom{n_S}{j}t^{k-j}\\
&\qquad= \sum_{j=0}^{k}\sum_{\varnothing\subset S\subseteq R}(-1)^{|R|-|S|}\tbinom{n_S}{j}\tfrac{(d+|R|-1-j)!}{(k-j)!}t^{k-j},
\end{aligned}$$

where to go from (6.2) to (6.3) we used Relation 5.26 from [10]:

$$\sum_{0\le k\le l}\binom{l-k}{m}\binom{q+k}{n} = \binom{l+q+1}{m+n+1},$$

holding for all non negative integers $l, m, n \ge q$. ☐

6.1 Upper bounds for the lower half of $\boldsymbol{h}(\mathcal{F}_R)$

From the recurrence relation in Theorem 5.1 we arrive at the following theorem.

Theorem 6.3. *For any $\varnothing \subset R \subseteq [r]$ and $0 \le k \le d + |R| - 1$, we have:*

$$g_k(\mathcal{F}_R) \le \sum_{\varnothing\subset S\subseteq R} (-1)^{|R|-|S|}\binom{n_S-d-|R|-1+k}{k}, \quad \textit{and} \tag{6.4}$$

$$h_k(\mathcal{F}_R) \le \sum_{\varnothing\subset S\subseteq R} (-1)^{|R|-|S|}\binom{n_S-d-|R|+k}{k}, \tag{6.5}$$

where $n_S = \sum_{i\in S} n_i$. Equalities hold for all $0 \le k \le \lfloor\frac{d+|R|-1}{2}\rfloor$ if and only if the Cayley polytope $\mathcal{C}_R$ is R-neighborly.

Proof. We are going to show the wanted bounds by induction on $|R|$ and k. Clearly the bounds hold for $|R| = 1$ and for any $0 \le k \le d$ (this is the case of one d-polytope and the bounds of the lemma refer to the well-known bounds on the elements of the h- and g-vector of a polytope).

Suppose now that the bounds for $g_k(\mathcal{F}_R)$ and $h_k(\mathcal{F}_R)$ hold for all $|R| < m$ and for all $0 \le k \le d + |R| - 1$. Consider an R with $|R| = m$. Then, for $k = 0$ we have:

$$\begin{aligned}h_0(\mathcal{F}_R) &= f_{-1}(\mathcal{F}_R) = (-1)^{|R|-1} = -(-1)^{|R|} = \sum_{i=1}^{|R|}(-1)^{|R|-i}\binom{|R|}{i} = \sum_{i=1}^{|R|}(-1)^{|R|-i}\sum_{\substack{\varnothing\subset S\subseteq R\\|S|=i}} 1\\ &= \sum_{i=1}^{|R|}\sum_{\substack{\varnothing\subset S\subseteq R\\|S|=i}}(-1)^{|R|-i} = \sum_{\varnothing\subset S\subseteq R}(-1)^{|R|-|S|} = \sum_{\varnothing\subset S\subseteq R}(-1)^{|R|-|S|}\binom{n_S-d-|R|}{0},\end{aligned}$$

and

$$g_0(\mathcal{F}_R) = h_0(\mathcal{F}_R) - h_{-1}(\mathcal{F}_R) = \sum_{\varnothing\subset S\subseteq R}(-1)^{|R|-|S|}\binom{n_S-d-|R|}{0} = \sum_{\varnothing\subset S\subseteq R}(-1)^{|R|-|S|}\binom{n_S-d-|R|-1}{0}.$$

For $k \ge 1$ we have:

$$\begin{aligned}g_k(\mathcal{F}_R) &= h_k(\mathcal{F}_R) - h_{k-1}(\mathcal{F}_R)\\ &\le \tfrac{n_R-d-|R|+k}{k}h_{k-1}(\mathcal{F}_R) + \sum_{i\in R}\tfrac{n_i}{k}g_{k-1}(\mathcal{F}_{R\smallsetminus\{i\}}) - h_{k-1}(\mathcal{F}_R)\\ &= \tfrac{n_R-d-|R|}{k}h_{k-1}(\mathcal{F}_R) + \sum_{i\in R}\tfrac{n_i}{k}g_{k-1}(\mathcal{F}_{R\smallsetminus\{i\}}).\end{aligned} \tag{6.6}$$

By our inductive hypotheses, we have:

$$h_{k-1}(\mathcal{F}_R) \le \sum_{\varnothing\subset S\subseteq R}(-1)^{|R|-|S|}\binom{n_S-d-|R|+k-1}{k-1} \tag{6.7}$$

and also, for all $i \in R$:

$$\begin{aligned}g_{k-1}(\mathcal{F}_{R\smallsetminus\{i\}}) &\le \sum_{\varnothing\subset S\subseteq R\smallsetminus\{i\}}(-1)^{|R\smallsetminus\{i\}|-|S|}\binom{n_S-d-|R\smallsetminus\{i\}|-1+k-1}{k-1}\\ &= -\sum_{\varnothing\subset S\subseteq R\smallsetminus\{i\}}(-1)^{|R|-|S|}\binom{n_S-d-|R|-1+k}{k-1}.\end{aligned} \tag{6.8}$$

Substituting (6.7) and (6.8) in (6.6) we get:

$$g_k(\mathcal{F}_R) \le \tfrac{n_R-d-|R|}{k}\sum_{\varnothing\subset S\subseteq R}(-1)^{|R|-|S|}\binom{n_S-d-|R|+k-1}{k-1} - \sum_{i\in R}\tfrac{n_i}{k}\sum_{\varnothing\subset S\subseteq R\smallsetminus\{i\}}(-1)^{|R|-|S|}\binom{n_S-d-|R|-1+k}{k-1}. \tag{6.9}$$

Consider the sum $\sum_{i\in R}\frac{n_i}{k}\sum_{\varnothing\subset S\subseteq R\smallsetminus\{i\}}(-1)^{|R|-|S|}\binom{n_S-d-|R|-1+k}{k-1}$; observe that for any given $\varnothing\subset S\subset R$ we get a contribution of $\frac{n_i}{k}$ for $\binom{n_S-d-|R|-1+k}{k-1}$, for any $i\notin S$. In other words, we have the equality:

$$\sum_{i\in R}\frac{n_i}{k}\sum_{\varnothing\subset S\subseteq R\smallsetminus\{i\}}(-1)^{|R|-|S|}\binom{n_S-d-|R|-1+k}{k-1}=\sum_{\varnothing\subset S\subset R}(-1)^{|R|-|S|}\frac{n_{R\smallsetminus S}}{k}\binom{n_S-d-|R|-1+k}{k-1}. \tag{6.10}$$

In view of (6.10) the inequality in (6.9) becomes:

$$\begin{aligned}
g_k(\mathcal{F}_R) &\le \frac{n_R-d-|R|}{k}\sum_{\varnothing\subset S\subseteq R}(-1)^{|R|-|S|}\binom{n_S-d-|R|+k-1}{k-1}-\sum_{\varnothing\subset S\subset R}(-1)^{|R|-|S|}\frac{n_{R\smallsetminus S}}{k}\binom{n_S-d-|R|-1+k}{k-1}\\
&=\frac{n_R-d-|R|}{k}\binom{n_R-d-|R|-1+k}{k-1}+\frac{n_R-d-|R|}{k}\sum_{\varnothing\subset S\subset R}(-1)^{|R|-|S|}\binom{n_S-d-|R|-1+k}{k-1}\\
&\quad-\sum_{\varnothing\subset S\subset R}(-1)^{|R|-|S|}\frac{n_{R\smallsetminus S}}{k}\binom{n_S-d-|R|-1+k}{k-1}\\
&=\frac{n_R-d-|R|}{k}\binom{n_R-d-|R|-1+k}{k-1}+\sum_{\varnothing\subset S\subset R}(-1)^{|R|-|S|}\left(\frac{n_R-d-|R|}{k}-\frac{n_{R\smallsetminus S}}{k}\right)\binom{n_S-d-|R|-1+k}{k-1}\\
&=\frac{n_R-d-|R|}{k}\binom{n_R-d-|R|-1+k}{k-1}+\sum_{\varnothing\subset S\subset R}(-1)^{|R|-|S|}\frac{n_S-d-|R|}{k}\binom{n_S-d-|R|-1+k}{k-1}\\
&=\frac{n_R-d-|R|+k}{k}\binom{n_R-d-|R|-1+k}{k-1}-\binom{n_R-d-|R|-1+k}{k-1}\\
&\quad+\sum_{\varnothing\subset S\subset R}(-1)^{|R|-|S|}\left(\frac{n_S-d-|R|+k}{k}\binom{n_S-d-|R|-1+k}{k-1}-\binom{n_S-d-|R|-1+k}{k-1}\right)\\
&=\binom{n_R-d-|R|+k}{k}-\binom{n_R-d-|R|-1+k}{k-1}+\sum_{\varnothing\subset S\subset R}(-1)^{|R|-|S|}\left(\binom{n_S-d-|R|+k}{k}-\binom{n_S-d-|R|-1+k}{k-1}\right)\\
&=\binom{n_R-d-|R|+k-1}{k}+\sum_{\varnothing\subset S\subset R}(-1)^{|R|-|S|}\binom{n_S-d-|R|+k-1}{k}\\
&=\sum_{\varnothing\subset S\subseteq R}(-1)^{|R|-|S|}\binom{n_S-d-|R|-1+k}{k}.
\end{aligned}$$

We can now turn our attention to proving the bound for $h_k(\mathcal{F}_R)$. Using the recursive relation (5.2) and the upper bound for $g_k(\mathcal{F}_R)$ that we just proved, we get:

$$\begin{aligned}
h_k(\mathcal{F}_R) &\le \frac{n_R-d-|R|+k}{k}h_{k-1}(\mathcal{F}_R)+\sum_{i\in R}\frac{n_i}{k}g_{k-1}(\mathcal{F}_{R\smallsetminus\{i\}})\\
&\le \frac{n_R-d-|R|+k}{k}\sum_{\varnothing\subset S\subseteq R}(-1)^{|R|-|S|}\binom{n_S-d-|R|+k-1}{k-1}\\
&\quad+\sum_{i\in R}\frac{n_i}{k}\sum_{\varnothing\subset S\subseteq R\smallsetminus\{i\}}(-1)^{|R\smallsetminus\{i\}|-|S|}\binom{n_S-d-|R\smallsetminus\{i\}|-1+k-1}{k-1}\\
&=\frac{n_R-d-|R|+k}{k}\binom{n_R-d-|R|+k-1}{k-1}+\frac{n_R-d-|R|+k}{k}\sum_{\varnothing\subset S\subset R}(-1)^{|R|-|S|}\binom{n_S-d-|R|+k-1}{k-1}\\
&\quad+\sum_{i\in R}\frac{n_i}{k}\sum_{\varnothing\subset S\subseteq R\smallsetminus\{i\}}(-1)^{|R|-1-|S|}\binom{n_S-d-|R|+k-1}{k-1}\\
&=\binom{n_R-d-|R|+k}{k}+\sum_{\varnothing\subset S\subset R}(-1)^{|R|-|S|}\frac{n_R-d-|R|+k}{k}\binom{n_S-d-|R|+k-1}{k-1}\\
&\quad+\sum_{S\subset R}(-1)^{|R|-|S|-1}\frac{n_{R\smallsetminus S}}{k}\binom{n_S-d-|R|+k-1}{k-1}\\
&=\binom{n_R-d-|R|+k}{k}+\sum_{\varnothing\subset S\subset R}(-1)^{|R|-|S|}\left(\frac{n_R-d-|R|+k}{k}-\frac{n_{R\smallsetminus S}}{k}\right)\binom{n_S-d-|R|+k-1}{k-1}\\
&=\binom{n_R-d-|R|+k}{k}+\sum_{\varnothing\subset S\subset R}(-1)^{|R|-|S|}\frac{n_S-d-|R|+k}{k}\binom{n_S-d-|R|+k-1}{k-1}
\end{aligned}$$

$$= \binom{n_R-d-|R|+k}{k} + \sum_{\varnothing\subset S\subset R} (-1)^{|R|-|S|}\binom{n_S-d-|R|+k}{k}$$
$$= \sum_{\varnothing\subset S\subseteq R} (-1)^{|R|-|S|}\binom{n_S-d-|R|+k}{k}. \qquad \square$$

Finally, the equality claim is immediate from Lemma 6.2 .

6.2 Upper bounds for $h_k(\mathcal{F}_R)$ and $h_k(\mathcal{K}_R)$ for all k

Before proceeding with proving upper bounds for the h-vectors of $\mathcal{F}_R$ and $\mathcal{K}_R$ we need to define the following functions.

Definition 6.4. *Let $d \geq 2$, $\varnothing \subset R \subseteq [r]$, $m \geq 0$, $0 \leq k \leq d + |R| - 1$, and $n_i \in \mathbb{N}$, $i \in R$, with $n_i \geq d+1$. We define the functions $\Phi^{(m)}_{k,d}(\boldsymbol{n}_R)$ and $\Psi_{k,d}(\boldsymbol{n}_R)$ by the following conditions:*

1. $\Phi^{(0)}_{k,d}(\boldsymbol{n}_R) = \sum_{\varnothing\subset S\subseteq R}(-1)^{|R|-|S|}\binom{n_S-d-|R|+k}{k}$, $0 \leq k \leq \lfloor\frac{d+|R|-1}{2}\rfloor$,
2. $\Phi^{(m)}_{k,d}(\boldsymbol{n}_R) = \Phi^{(m-1)}_{k,d}(\boldsymbol{n}_R) - \Phi^{(m-1)}_{k-1,d}(\boldsymbol{n}_R)$, $m > 0$,
3. $\Psi_{k,d}(\boldsymbol{n}_R) = \sum_{\varnothing\subset S\subseteq R}\Phi^{(|R|-|S|)}_{k,d}(\boldsymbol{n}_R)$,
4. $\Phi^{(0)}_{k,d}(\boldsymbol{n}_R) = \Psi_{d+|R|-1-k,d}(\boldsymbol{n}_R)$,

where $\boldsymbol{n}_R$ stands for the $|R|$-dimensional vector whose elements are the values n_i, $i \in R$.

Notice that $\Phi^{(0)}_{k,d}(\boldsymbol{n}_R)$ and $\Psi_{k,d}(\boldsymbol{n}_R)$ are well defined, though in a recursive manner (in the size of R), since for any $k > \lfloor\frac{d+|R|-1}{2}\rfloor$, we have:

$$\begin{aligned}
\Phi^{(0)}_{k,d}(\boldsymbol{n}_R) = \Psi_{d+|R|-1-k,d}(\boldsymbol{n}_R) &= \sum_{\varnothing\subset S\subseteq R}\Phi^{(|R|-|S|)}_{d+|R|-1-k,d}(\boldsymbol{n}_S)\\
&= \Phi^{(0)}_{d+|R|-1-k,d}(\boldsymbol{n}_R) + \sum_{\varnothing\subset S\subset R}\Phi^{(|R|-|S|)}_{d+|R|-1-k,d}(\boldsymbol{n}_S)\\
&= \sum_{\varnothing\subset S\subseteq R}(-1)^{|R|-|S|}\binom{n_S-k-1}{d+|R|-1-k} + \sum_{\varnothing\subset S\subset R}\Phi^{(|R|-|S|)}_{d+|R|-1-k,d}(\boldsymbol{n}_S),
\end{aligned} \tag{6.11}$$

where the second sum in (6.11) is to be understood as 0 when $|R| = 1$. In other words, $\Phi^{(0)}_{k,d}(\boldsymbol{n}_R)$, and, thus, also $\Phi^{(m)}_{k,d}(\boldsymbol{n}_R)$ for any $m > 0$, is fully defined for some R and any k, once we know the values $\Phi^{(\ell)}_{k,d}(\boldsymbol{n}_S)$ for all $\varnothing \subset S \subset R$, for all $0 \leq k \leq d + |S| - 1$, and for all $1 \leq \ell \leq |R| - 1$. Moreover, it is easy to verify that $\Phi^{(0)}_{k,d}(\boldsymbol{n}_R)$ satisfies the following recurrence relation:

$$\Phi^{(0)}_{k+1,d}(\boldsymbol{n}_R) = \frac{n_R - d - |R| + k + 1}{k+1}\Phi^{(0)}_{k,d}(\boldsymbol{n}_R) + \sum_{i\in R}\frac{n_i}{k+1}\Phi^{(1)}_{k,d}(\boldsymbol{n}_{R\setminus\{i\}}), \quad 0 \leq k \leq \lfloor\tfrac{d+|R|-2}{2}\rfloor. \tag{6.12}$$

Lemma 6.5. *For any $\varnothing \subset R \subseteq [r]$, any k with $0 \leq k \leq \lfloor\frac{d+|R|-1}{2}\rfloor$, and any α with $0 \leq \alpha \leq \frac{d+1}{d-1}$, we have:*

$$h_k(\mathcal{F}_R) - \alpha\sum_{i\in R}h_{k-1}(\mathcal{F}_{R\setminus\{i\}}) \leq \Phi^{(0)}_{k,d}(\boldsymbol{n}_R) - \alpha\sum_{i\in R}\Phi^{(0)}_{k-1,d}(\boldsymbol{n}_{R\setminus\{i\}}). \tag{6.13}$$

To prove Lemma 6.5 we need the following intermediate result.

Lemma 6.6. *For any $\varnothing \subset R \subseteq [r]$, any k with $0 \leq k \leq \lfloor\frac{d+|R|-1}{2}\rfloor$, and any α with $0 \leq \alpha \leq \frac{d+1}{d-1}$, we have:*

$$g_k(\mathcal{F}_R) - \alpha\sum_{i\in R}g_{k-1}(\mathcal{F}_{R\setminus\{i\}}) \leq \Phi^{(1)}_{k,d}(\boldsymbol{n}_R) - \alpha\sum_{i\in R}\Phi^{(1)}_{k-1,d}(\boldsymbol{n}_{R\setminus\{i\}}).$$

Proof. Let us recall the recurrence relation from Theorem 5.1:

$$h_k(\mathcal{F}_R) \leq \frac{n_R - d - |R| + k}{k} h_{k-1}(\mathcal{F}_R) + \sum_{i \in R} \frac{n_i}{k} g_{k-1}(\mathcal{F}_{R \setminus \{i\}}).$$

Subtracting $h_{k-1}(\mathcal{F}_R) + \alpha \sum_{i \in R} g_{k-1}(\mathcal{F}_{R \setminus \{i\}})$ from both sides of the inequality we get:

$$g_k(\mathcal{F}_R) - \alpha \sum_{i \in R} g_{k-1}(\mathcal{F}_{R \setminus \{i\}}) \leq \frac{n_R - d - |R|}{k} h_{k-1}(\mathcal{F}_R) + \sum_{i \in R} \left(\frac{n_i}{k} - \alpha \right) g_{k-1}(\mathcal{F}_{R \setminus \{i\}}). \tag{6.14}$$

Observe that the coefficient of $h_{k-1}(\mathcal{F}_R)$ in (6.14) is non-negative:

$$n_R - d - |R| \geq |R|(d+1) - d - |R| = d|R| + |R| - d - |R| = d(|R| - 1) \geq 0.$$

The same holds for the coefficient of $g_{k-1}(\mathcal{F}_{R \setminus \{i\}})$ in (6.14), since:

$$\frac{n_i}{k} \geq \frac{d+1}{\lfloor \frac{d+|R|-1}{2} \rfloor} \geq \frac{d+1}{\frac{d+|R|-1}{2}} = \frac{2d+2}{d+|R|-1} \geq \frac{2d+2}{d+(d-1)-1} = \frac{2d+2}{2d-2} \geq \alpha, \tag{6.15}$$

where we used the fact that $|R| \leq r \leq d-1$. Hence, we can bound (6.14) from above by substituting $g_{k-1}(\mathcal{F}_R)$ and $g_{k-1}(\mathcal{F}_{R \setminus \{i\}})$, $i \in R$, by $\Phi^{(1)}_{k-1,d}(\boldsymbol{n}_R)$ and $\Phi^{(1)}_{k-1,d}(\boldsymbol{n}_{R \setminus \{i\}})$, $i \in R$, respectively. This gives:

$$\begin{aligned} g_k(\mathcal{F}_R) - \alpha \sum_{i \in R} g_{k-1}(\mathcal{F}_{R \setminus \{i\}}) &\leq \frac{n_R - d - |R|}{k} \Phi^{(1)}_{k-1,d}(\boldsymbol{n}_R) + \sum_{i \in R} \left(\frac{n_i}{k} - \alpha \right) \Phi^{(1)}_{k-1,d}(\boldsymbol{n}_{R \setminus \{i\}}) \\ &= \frac{n_R - d - |R|}{k} \Phi^{(1)}_{k-1,d}(\boldsymbol{n}_R) + \sum_{i \in R} \frac{n_i}{k} \Phi^{(1)}_{k-1,d}(\boldsymbol{n}_{R \setminus \{i\}}) \\ &\quad - \alpha \sum_{i \in R} \Phi^{(1)}_{k-1,d}(\boldsymbol{n}_{R \setminus \{i\}}) \\ &= \Phi^{(1)}_{k,d}(\boldsymbol{n}_R) - \alpha \sum_{i \in R} \Phi^{(1)}_{k-1,d}(\boldsymbol{n}_{R \setminus \{i\}}). \end{aligned}$$

□

Having established Lemma 6.6, it is now straightforward to prove Lemma 6.5.

Proof of Lemma 6.5. First observe that $h_i(\mathcal{F}_X)$ may be written as a telescopic sum as follows:

$$h_i(\mathcal{F}_X) = h_0(\mathcal{F}_X) + \sum_{\ell=0}^{i-1} g_{i-\ell}(\mathcal{F}_X). \tag{6.16}$$

Since $h_0(\mathcal{F}_X) = g_0(\mathcal{F}_X)$, the above expansion may be written in the more concise form:

$$h_i(\mathcal{F}_X) = \sum_{\ell=0}^{i} g_{i-\ell}(\mathcal{F}_X). \tag{6.17}$$

Using relations (6.16) and (6.17), and applying Lemma 6.6, we get:

$$\begin{aligned} h_k(\mathcal{F}_R) - \alpha \sum_{i \in R} h_{k-1}(\mathcal{F}_{R \setminus \{i\}}) &= h_0(\mathcal{F}_R) + \sum_{\ell=0}^{k-1} g_{k-\ell}(\mathcal{F}_R) - \alpha \sum_{i \in R} \sum_{\ell=0}^{k-1} g_{k-1-\ell}(\mathcal{F}_{R \setminus \{i\}}) \\ &= h_0(\mathcal{F}_R) + \sum_{\ell=0}^{k-1} \left(g_{k-\ell}(\mathcal{F}_R) - \alpha \sum_{i \in R} g_{k-1-\ell}(\mathcal{F}_{R \setminus \{i\}}) \right) \\ &\leq \Phi^{(0)}_{0,d}(\boldsymbol{n}_R) + \sum_{\ell=0}^{k-1} \left(\Phi^{(1)}_{k-\ell,d}(\boldsymbol{n}_R) - \alpha \sum_{i \in R} \Phi^{(1)}_{k-1-\ell,d}(\boldsymbol{n}_{R \setminus \{i\}}) \right) \end{aligned}$$

$$
\begin{aligned}
&= \Phi^{(0)}_{0,d}(\boldsymbol{n}_R) + \sum_{\ell=0}^{k-1} \Phi^{(1)}_{k-\ell,d}(\boldsymbol{n}_R) - \alpha \sum_{i\in R}\sum_{\ell=0}^{k-1} \Phi^{(1)}_{k-1-\ell,d}(\boldsymbol{n}_{R\smallsetminus\{i\}}) \\
&= \Phi^{(0)}_{k,d}(\boldsymbol{n}_R) - \alpha \sum_{i\in R} \Phi^{(0)}_{k-1,d}(\boldsymbol{n}_{R\smallsetminus\{i\}}),
\end{aligned}
$$

where we also used the fact that $h_0(\mathcal{F}_R) = (-1)^{|R|-1} = \Phi^{(0)}_{0,d}(\boldsymbol{n}_R)$. $\square$

The next theorem provides upper bounds for h-vectors of $\mathcal{F}_R$ and $\mathcal{K}_R$, as well as necessary and sufficient conditions for these upper bounds to be attained.

Theorem 6.7. *For all $0 \le k \le d+|R|-1$, we have:*

(i) $h_k(\mathcal{F}_R) \le \Phi^{(0)}_{k,d}(\boldsymbol{n}_R)$,

(ii) $h_k(\mathcal{K}_R) \le \Psi_{k,d}(\boldsymbol{n}_R)$.

Equalities hold for all k if and only if the Cayley polytope $\mathcal{C}_R$ is Minkowski-neighborly.

Proof. To prove the upper bounds use recursion on the size of $|R|$. For $|R|=1$, the result for both $h_k(\mathcal{F}_R)$ and $h_k(\mathcal{K}_R)$ comes from the UBT for d-polytopes. For $|R|>1$, we assume that the bounds hold for all S with $\varnothing \subset S \subset R$, and for all k with $0 \le k \le d+|S|-1$. Furthermore, the upper bound for $h_k(\mathcal{F}_R)$ for $k \le \lfloor\frac{d+|R|-1}{2}\rfloor$ is immediate from Theorem 6.3. To prove the upper bound for $h_k(\mathcal{K}_R)$, $0 \le k \le \lfloor\frac{d+|R|-1}{2}\rfloor$, we use the following expansion for $h_k(\mathcal{K}_R)$ (cf. [2, Lemma 5.14]):

$$
\begin{aligned}
h_k(\mathcal{K}_R) = &\sum_{j=0}^{\lfloor\frac{|R|}{2}\rfloor} \sum_{s=c-2j-1}^{|R|-2j} \sum_{\substack{S\subseteq R\\|S|=s}} \binom{|R|-s}{2j}\Big(h_{k-2j}(\mathcal{F}_S) - \frac{1}{2j+1}\sum_{i\in S} h_{k-2j-1}(\mathcal{F}_{S\smallsetminus\{i\}})\Big) \\
&+ \sum_{j=0}^{\lfloor\frac{|R|}{2}\rfloor} \sum_{\substack{S\subset R\\|S|=c-2j+1}} \binom{|R|-|S|}{2j}\Big(h_{k-2j}(\mathcal{F}_S) - \frac{1}{2j+1}\sum_{i\in S} h_{k-2j-1}(\mathcal{F}_{S\smallsetminus\{i\}})\Big),
\end{aligned}
\tag{6.18}
$$

where c depends on k, d and $|R|$. Under the assumption that $r<d$, it is easy to show that (see Lemma 6.5 in Section 6.2 below):

$$
h_{k-2j}(\mathcal{F}_S) - \frac{1}{2j+1}\sum_{i\in S} h_{k-2j-1}(\mathcal{F}_{S\smallsetminus\{i\}}) \le \Phi^{(0)}_{k-2j,d}(\boldsymbol{n}_S) - \frac{1}{2j+1}\sum_{i\in S}\Phi^{(0)}_{k-2j-1,d}(\boldsymbol{n}_{S\smallsetminus\{i\}}). \tag{6.19}
$$

Substituting the upper bound from (6.19) in (6.18), and reversing the derivation logic for (6.18), we deduce that $h_k(\mathcal{K}_R) \le \Psi_{k,d}(\boldsymbol{n}_R)$.

For $k > \lfloor\frac{d+|R|-1}{2}\rfloor$ we have:

$$
\begin{aligned}
h_k(\mathcal{F}_R) &= h_{d+|R|-1-k}(\mathcal{K}_R) \le \Psi_{d+|R|-1-k,d}(\boldsymbol{n}_R) = \Phi^{(0)}_{k,d}(\boldsymbol{n}_R), \quad \text{and,}\\
h_k(\mathcal{K}_R) &= h_{d+|R|-1-k}(\mathcal{F}_R) \le \Phi^{(0)}_{d+|R|-1-k,d}(\boldsymbol{n}_R) = \Psi_{k,d}(\boldsymbol{n}_R).
\end{aligned}
$$

The necessary and sufficient conditions are easy consequences of the equality claim in Theorem 6.3. $\square$

For any $d\ge 2$, $\varnothing \subset R \subseteq [r]$, $0 \le k \le d+|R|-1$, and $n_i \in \mathbb{N}$, $i \in R$, with $n_i \ge d+1$, let

$$
\Xi_{k,d}(\boldsymbol{n}_R) = \sum_{\varnothing\subset R\subseteq[r]} (-1)^{r-|R|} f_k\big(C_{d+r-1}(n_R)\big) + \sum_{i=0}^{\lfloor\frac{d+r-2}{2}\rfloor} \binom{i}{k-d-r+1+i} \sum_{\varnothing\subset R\subset[r]} \Phi^{(r-|R|)}_{i,d}(\boldsymbol{n}_R),
$$

where $C_\delta(n)$ stands for the cyclic δ-polytope with n vertices. It is straightforward to verify that for $0 \le k \le \lfloor\frac{d+|R|-1}{2}\rfloor$, $\Xi_{k,d}(\boldsymbol{n}_R)$ simplifies to $\sum_{\varnothing\subset S\subseteq R}(-1)^{|R|-|S|}\binom{n_R}{k}$. We are finally ready to state and prove the main result of the paper.

Theorem 6.8. *Let $P_1, \dots, P_r$ be r d-polytopes, $r < d$, with $n_1, \dots, n_r$ vertices respectively. Then, for all $1 \le k \le d$, we have:*

$$f_{k-1}(P_{[r]}) \le \Xi_{k+r,d}(\boldsymbol{n}_{[r]}).$$

Equality holds for all $0 \le k \le d$ if and only if the Cayley polytope $\mathcal{C}_{[r]}$ of $P_1, \dots, P_r$ is Minkowski-neighborly.

Proof. We start by recalling that:

$$f_{k-1}(\mathcal{F}_{[r]}) = \sum_{i=0}^{d+r-1} \binom{d+r-1-i}{k-i} h_i(\mathcal{F}_{[r]}).$$

In view of Theorem 6.7, the above expression is bounded from above by:

$$\sum_{i=0}^{\lfloor \frac{d+r-1}{2} \rfloor} \binom{d+r-1-i}{k-i} \Phi_{i,d}^{(0)}(\boldsymbol{n}_{[r]}) + \sum_{i=\lfloor \frac{d+r-1}{2} \rfloor+1}^{d+r-1} \binom{d+r-1-i}{k-i} \Phi_{i,d}^{(0)}(\boldsymbol{n}_{[r]}) \tag{6.20}$$

$$= \sum_{i=0}^{\lfloor \frac{d+r-1}{2} \rfloor} \binom{d+r-1-i}{k-i} \Phi_{i,d}^{(0)}(\boldsymbol{n}_{[r]}) + \sum_{i=0}^{\lfloor \frac{d+r-2}{2} \rfloor} \binom{i}{k-d-r+1+i} \Phi_{d+r-1-i,d}^{(0)}(\boldsymbol{n}_{[r]}) \tag{6.21}$$

$$\begin{aligned} &= \sum_{i=0}^{\frac{d+r-1}{2}}{}^{*} \left(\binom{d+r-1-i}{k-i} + \binom{i}{k-d-r+1+i} \right) \sum_{\varnothing \subset R \subseteq [r]} (-1)^{r-|R|} \binom{n_R - d - r + i}{i} \\ &\quad + \sum_{i=0}^{\lfloor \frac{d+r-2}{2} \rfloor} \binom{i}{k-d-r+1+i} \sum_{\varnothing \subset R \subset [r]} \Phi_{i,d}^{(r-|R|)}(\boldsymbol{n}_R) \end{aligned} \tag{6.22}$$

$$= \sum_{\varnothing \subset R \subseteq [r]} (-1)^{r-|R|} f_k(C_{d+r-1}(n_R)) + \sum_{i=0}^{\lfloor \frac{d+r-2}{2} \rfloor} \binom{j}{k-d-r+1+i} \sum_{\varnothing \subset R \subset [r]} \Phi_{i,d}^{(r-|R|)}(\boldsymbol{n}_R) \tag{6.23}$$

where to go:

- from (6.20) to (6.21) we changed the variable of the second sum from i to $d+r-1-i$,
- from (6.21) to (6.22) we wrote the explicit expression of $\Phi_{k,d}^{(0)}(\boldsymbol{n}_R)$ from relation (6.11),
- from (6.22) to (6.23) we used that the number of $(k-1)$-faces of a cyclic δ-polytope with n vertices is $\sum^{*}{}_{i=0}^{\frac{\delta}{2}} \left(\binom{\delta-i}{k-i} + \binom{i}{k-\delta+i} \right) \binom{n-\delta-1+i}{i}$, where $\sum_{i=0}^{\frac{\delta}{2}}{}^{*} T_i$ denotes the sum of the elements $T_0, T_1, \dots, T_{\lfloor \frac{\delta}{2} \rfloor}$ where the last term is halved if δ is even.

Finally, observing that the expression in (6.23) is nothing but $\Xi_{k,d}(\boldsymbol{n}_R)$, and recalling that $f_{k-1}(\mathcal{F}_{[r]}) = f_{k-r}(P_{[r]})$, we arrive at the upper bound in the statement of the theorem. The equality claim is immediate from Theorem 6.7. $\square$

7 Tight bound construction

In this section we show that the bounds in Theorem 6.8 are tight. Before getting into the technical details, we outline our approach. We start by considering the $(d-r+1)$-dimensional moment curve, which we embed in r distinct subspaces of $\mathbb{R}^d$. We consider the r copies of the $(d-r+1)$-dimensional moment curve as different curves, and we perturb them appropriately, so that they become d-dimensional moment-like curves. The perturbation is controlled via a non-negative parameter ζ, which will be chosen appropriately. We then choose points on these r moment-like curves, all parameterized by a positive parameter τ, which will again be chosen appropriately. These points are the vertices of r d-polytopes $P_1, P_2, \ldots, P_r$, and we show that, for all $\varnothing \subset R \subseteq [r]$, the number of $(k-1)$-faces of $\mathcal{F}_R$, where $|R| \le k \le \lfloor \frac{d+|R|-1}{2} \rfloor$, becomes equal to $\Xi_{k,d}(\boldsymbol{n}_R)$ for small enough positive values of ζ and τ. Our construction produces *projected prod-simplicial neighborly* polytopes (cf. [17]). For $\zeta = 0$ our polytopes are essentially the same as those in [17, Theorem 2.6], while for $\zeta > 0$ we get *deformed* versions of those polytopes. The positivity of ζ allows us to ensure the tightness of the upper bound on $f_k(P_{[r]})$, not only for small, but also for large values of k.

At a more technical level (cf. Section C), the proof that $f_{k-1}(\mathcal{F}_R) = \Xi_{k,d}(\boldsymbol{n}_R)$, for all $|R| \le k \le \lfloor \frac{d+|R|-1}{2} \rfloor$, is performed in two steps. We first consider the cyclic $(d-r+1)$-polytopes $\hat{P}_1, \ldots, \hat{P}_r$, embedded in appropriate subspaces of $\mathbb{R}^d$. The $\hat{P}_i$'s are the *unperturbed*, with respect to ζ, versions of the d-polytopes $P_1, P_2, \ldots, P_r$ (i.e., the polytope $\hat{P}_i$ is the polytope we get from P_i, when we set ζ equal to zero). For each $\varnothing \subset R \subseteq [r]$ we denote by $\hat{\mathcal{C}}_R$ the Cayley polytope of $\hat{P}_i, i \in R$, seen as a polytope in $\mathbb{R}^d$, and we focus on the set $\hat{\mathcal{F}}_R$ of its mixed faces. Recall that the polytopes $\hat{P}_i, i \in R$, are parameterized by the parameter τ; we show that there exists a sufficiently small positive value $\tau^\star$ for τ, for which the number of $(k-1)$-faces of $\hat{\mathcal{F}}_R$ is equal to $\Xi_{k,d}(\boldsymbol{n}_R)$ for all $|R| \le k \le \lfloor \frac{d+|R|-1}{2} \rfloor$. For τ equal to $\tau^\star$, we consider the polytopes $P_1, P_2, \ldots, P_r$ (with τ set to $\tau^\star$), and show that for sufficiently small ζ (denoted by $\zeta^\Diamond$), $f_{k-1}(\mathcal{F}_R)$ is equal to $\Xi_{k,d}(\boldsymbol{n}_R)$.

In the remainder of this section we describe our construction in detail. For each $1 \le i \le r$, we define the d-dimensional moment-like curve[2]:

$$\boldsymbol{\gamma}_i(t;\zeta) = (\zeta t^{d-r+2}, \ldots, \zeta t^{d-r+i}, \overset{i\text{–}th\text{ coordinate}}{t}, \zeta t^{d-r+i+2}, \ldots, \zeta t^{d+1}, t^2, \ldots, t^{d-r+1}),$$

and the d-polytope

$$P_i := CH\{\boldsymbol{\gamma}_i(y_{i,1};\zeta), \ldots, \boldsymbol{\gamma}_i(y_{i,n_i};\zeta)\}, \tag{7.1}$$

where the parameters $y_{i,j}$ belong to the sets $Y_i = \{y_{i,1}, \ldots, y_{i,n_i}\}$, $1 \le i \le r$, whose elements are determined as follows. Choose

- $n_{[r]} + d + r$ arbitrary real numbers $x_{i,j}$ and M_s, such that:
  - $0 < x_{i,1} < x_{i,1} + \epsilon < x_{i,2} < x_{i,2} + \epsilon < \cdots < x_{i,n_i} + \epsilon$, for $1 \le i \le r-1$,
  - $0 < x_{r,1} < x_{r,1} + \epsilon < x_{r,2} < x_{r,2} + \epsilon < \cdots < x_{r,n_r} + \epsilon < M_1' < \cdots < M_{d+r}'$,

  where $\epsilon > 0$ is sufficiently small and $x_{i,n_i} < x_{i+1,1}$ for all i, and
- r non-negative integers $\beta_1, \beta_2, \ldots, \beta_r$, such that $\beta_1 > \beta_2 > \cdots > \beta_{r-1} > \beta_r \ge 0$.

We then set $y_{i,j} := x_{i,j}\tau^{\beta_i}$, $\tilde{y}_{i,j} := (x_{i,j} + \epsilon)\tau^{\beta_i}$ and $M_i := M_i'\tau^{\beta_r}$, where τ is a positive parameter. The $y_{i,j}$'s and $\tilde{y}_{i,j}$'s are used to define determinants whose value is positive for a small enough value of τ (see also Lemma C.2 in the Appendix). The positivity of these determinants is crucial in defining supporting hyperplanes for the Cayley polytopes $\hat{\mathcal{C}}_R$ and $\mathcal{C}_R$ in Lemmas 7.1 and 7.2 below.

Next, for each $1 \le i \le r$, we define $\hat{P}_i := \lim_{\zeta \to 0^+} P_i$. Clearly, each $\hat{P}_i$ is a cyclic $(d-r+1)$-polytope embedded in the $(d-r+1)$-flat F_i of $\mathbb{R}^d$, where $F_i = \{x_j = 0 \mid 1 \le j \le r \text{ and } j \ne i\}$. The following lemma establishes the first step towards our construction.

Lemma 7.1. *There exists a sufficiently small positive value $\tau^\star$ for τ, such that, for any $\varnothing \subset R \subseteq [r]$, the set of mixed faces $\hat{\mathcal{F}}_R$ of the Cayley polytope of the polytopes $\hat{P}_1, \ldots, \hat{P}_r$ constructed above, has*

$$f_{k-1}(\hat{\mathcal{F}}_R) = \Xi_{k,d}(\boldsymbol{n}_R), \quad |R| \le k \le \lfloor \tfrac{d+|R|-1}{2} \rfloor.$$

[2]The curve $\gamma_i(t;\zeta)$, $\zeta > 0$, is the image under an invertible linear transformation, of the curve $\hat{\gamma}_i(t) = (t, t^2, \ldots, t^{d-r+i}, t^{d-r+i+2}, \ldots, t^{d+1})$. Polytopes whose vertices are n distinct points on this curve are combinatorially equivalent to the cyclic d-polytope with n vertices.

Proof. Let $\mathscr{U}_i$ be the set of vertices of $\hat{P}_i$ for $1 \le i \le r$ and set $\mathscr{U} := \cup_{i=1}^{r} \mathscr{U}_i$. The objective in the proof is, for each $\varnothing \subset R \subseteq [r]$ and each spanning subset U of the partition $\mathsf{U} = \cup_{i \in R} \mathscr{U}_i$, to exhibit a supporting hyperplane of the $(d+|R|-1)$-dimensional Cayley polytope $\hat{\mathcal{C}}_R$, containing exactly the vertices in U. In that respect, our approach is similar in spirit, albeit much more technically involved, to the proof showing, by defining supporting hyperplanes constructed from Vandermonde determinants, that the cyclic n-vertex d-polytope $C_d(n)$ is neighborly (see, e.g. [22, Corollary 0.8]).

In our proof we need to involve the parameter ζ before taking the limit $\zeta \to 0^+$. This is due to the fact that, when $\varnothing \subset R \subset [r]$, the information of the relative position of the polytopes $\hat{P}_i$, $i \in R$, is lost if we set $\zeta = 0$ from the very first step. To describe our construction, we write each spanning subset U of $\mathsf{U} = \cup_{i \in R} \mathscr{U}_i$ as the disjoint union of non-empty sets U_i, $i \in R$, where $|U_i| = \kappa_i \le n_i$, $U_i = \lim_{\zeta \to 0}\{\boldsymbol{\gamma}_i(y;\zeta) : y \in Y_i'\} = \{\boldsymbol{\gamma}_i(y;0) : y \in Y_i'\}$ and $Y_i' = \{y \in Y_i \mid \boldsymbol{\gamma}_i(y;0) \in U_i\}$. For this particular U, we define the linear equation:

$$H_U(\boldsymbol{x}) = \lim_{\zeta \to 0^+} (-1)^{\frac{|R|(|R|-1)}{2} + \sigma(R)} \zeta^{|R|-r} \mathsf{D}_U(\boldsymbol{x};\zeta), \tag{7.2}$$

where $\boldsymbol{x} = (x_1, x_2, \dots, x_{d+|R|-1})$, and $\mathsf{D}_U(\boldsymbol{x};\zeta)$ is the $(d+|R|) \times (d+|R|)$ determinant[3]:

- whose first column is $(1, \boldsymbol{x})^\top$,
- the next κ_i, $i \in R$, pairs of columns are $(1, \boldsymbol{e}_{i-1}, \gamma_i(y_{i,j};\zeta))^\top$ and $(1, \boldsymbol{e}_{i-1}, \gamma_i(\widetilde{y}_{i,j};\zeta))^\top$ where $\boldsymbol{e}_0, \dots, \boldsymbol{e}_{|R|-1}$ is the standard affine basis of $\mathbb{R}^{|R|-1}$ and $j \in Y_i'$, and
- the last $s := d + |R| - 1 - \sum_{i \in R} \kappa_i$ columns are $(1, \boldsymbol{e}_{|R|-1}, \gamma_{|R|-1}(M_i;\zeta))^\top$, $1 \le i \le s$; these columns exist only if $s > 0$.

The quantity $\sigma(R)$ above is a non-negative integer counting the total number of row swaps required to shift, for all $j \in [r] \setminus R$, the $(|R|+j)$-th row of $\mathsf{D}_U(\boldsymbol{x};\zeta)$ to the bottom of the determinant, so that the powers of $y_{i,j}$ in each column are in increasing order (notice that if $R \equiv [r]$ no such row swaps are required). Moreover, $\sigma(R)$ depends only on R and not on the choice of the spanning subset U of U.

The equation $H_U(\boldsymbol{x}) = 0$ is the equation of a hyperplane in $\mathbb{R}^{d+|R|-1}$ that passes through the points in U. We claim that, for any choice of U, and for all vertices $\boldsymbol{u}$ in $\mathscr{U} \setminus U$, we have $H_U(\boldsymbol{u}) > 0$. To prove our claim, notice first that, for each $j \in [r] \setminus R$, the $(|R|+j)$-th row of the determinant $\mathsf{D}_U(\boldsymbol{u};\zeta)$ will contain the parameters $y_{i,j}^{d-r+1+j}, \widetilde{y}_{i,j}^{d-r+1+j}$ multiplied by ζ. After extracting ζ from each of these rows and shifting it to its *proper* position (i.e., the position where the powers along each column increase), we will have a term $\zeta^{r-|R|}$ and a sign $(-1)^{\sigma(R)}$ (induced from the $\sigma(R)$ row swaps required altogether). These terms cancel out with the term $(-1)^{\sigma(R)}\zeta^{|R|-r}$ in (7.2). We can, therefore, transform $H_U(\boldsymbol{u})$ in the form of the determinant $D_{\kappa}(\mathsf{Y};\mu_1,\dots,\mu_m)$ shown below:

$$D_{\kappa}(\mathsf{Y};\mu_1,\dots,\mu_m) := (-1)^{\frac{n(n-1)}{2}} \begin{vmatrix} y_{1,1}^{\mu_1} & \cdots & y_{1,\kappa_1}^{\mu_1} & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & y_{2,1}^{\mu_1} & \cdots & y_{2,\kappa_2}^{\mu_1} & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & y_{n,1}^{\mu_1} & \cdots & y_{n,\kappa_1}^{\mu_1} \\ y_{1,1}^{\mu_2} & \cdots & y_{1,\kappa_1}^{\mu_2} & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & y_{2,1}^{\mu_2} & \cdots & y_{2,\kappa_2}^{\mu_2} & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & y_{n,1}^{\mu_2} & \cdots & y_{n,\kappa_n}^{\mu_2} \\ y_{1,1}^{\mu_3} & \cdots & y_{1,\kappa_1}^{\mu_3} & y_{2,1}^{\mu_3} & \cdots & y_{2,\kappa_2}^{\mu_3} & \cdots & y_{n,1}^{\mu_3} & \cdots & y_{n,\kappa_n}^{\mu_3} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ y_{1,1}^{\mu_m} & \cdots & y_{1,\kappa_1}^{\mu_m} & y_{2,1}^{\mu_m} & \cdots & y_{2,\kappa_2}^{\mu_m} & \cdots & y_{n,1}^{\mu_m} & \cdots & y_{n,\kappa_n}^{\mu_m} \end{vmatrix},$$

[3] We refer the reader to Figs. 1 and 2 in the Appendix for an example of $\mathsf{D}_U(\boldsymbol{x};\zeta)$, $\zeta > 0$, and $\mathsf{D}_U(\boldsymbol{x};0)$.

by means of the following determinant transformations:

(i) By subtracting rows 2 to $|R|$ of $H_U(\boldsymbol{u})$ from its first row.

(ii) By shifting the first column of $H_U(\boldsymbol{u})$ to the right, so that all columns of $H_U(\boldsymbol{u})$ are arranged in increasing order according to their parameter. Clearly, this can be done with an *even* number of column swaps.

The determinant $D_{\mathrm{K}}(\mathsf{Y};\mu_1,\ldots,\mu_m)$ is strictly positive for all τ between 0 and some value $\hat{\tau}(R,U,\boldsymbol{u})$, that, depends (only) on the choice of R, U and $\boldsymbol{u}$. Since there is a finite number of possible such determinants, the value $\hat{\tau}^\star := \min_{R,U,\boldsymbol{u}} \hat{\tau}(R,U,\boldsymbol{u})$ is necessarily positive. Choosing some $\tau^\star \in (0,\hat{\tau}^\star)$ makes all these determinants simultaneously positive; this completes our proof. □

The following lemma establishes the second (and last) step of our construction.

Lemma 7.2. *There exists a sufficiently small positive value $\zeta^\lozenge$ for ζ, such that, for any $\varnothing \subset R \subseteq [r]$, the set $\mathcal{F}_R$ of mixed faces of the Cayley polytope $\mathcal{C}_R$ of the polytopes $P_1,\ldots,P_r$ in (7.1) has*

$$f_{k-1}(\mathcal{F}_R) = \Xi_{k,d}(\boldsymbol{n}_R), \qquad \textit{for all} \quad |R| \le k \le \lfloor \tfrac{d+|R|-1}{2} \rfloor.$$

Proof. Briefly speaking, the value $\zeta^\lozenge$ is determined by replacing the limit $\zeta \to 0^+$ in the previous proof, by a specific value of ζ for which the determinants we consider are positive.

More precisely, let $\mathscr{U}_i$ be the set of vertices of P_i, $1 \le i \le r$, and set $\mathscr{U} := \cup_{i=1}^{r} \mathscr{U}_i$. Our goal is, for each $\varnothing \subset R \subseteq [r]$ and each spanning subset U of the partition $\mathsf{U} = \cup_{i\in R} \mathscr{U}_i$, to exhibit a supporting hyperplane of the Cayley polytope $\mathcal{C}_R$, containing exactly the vertices in U. To this end, we define the linear equation $\widetilde{H}_U(\boldsymbol{x};\zeta) = 0$, $\boldsymbol{x} = (x_1, x_2, \ldots, x_{d+|R|-1})$, with

$$\widetilde{H}_U(\boldsymbol{x};\zeta) = (-1)^{\frac{|R|(|R|-1)}{2}+\sigma(R)} \zeta^{|R|-r} \mathsf{D}_U(\boldsymbol{x};\zeta), \quad \zeta > 0, \tag{7.3}$$

where $\mathsf{D}_U(\boldsymbol{x};\zeta)$ is the determinant in the proof of Lemma 7.1, where we have set τ to $\tau^\star$. Clearly, for each $\boldsymbol{u} \in \mathscr{U} \setminus U$, we have $\lim_{\zeta\to 0^+} \widetilde{H}_U(\boldsymbol{u};\zeta) = H_U(\boldsymbol{u}) > 0$. This immediately implies that for each combination of U and $\boldsymbol{u}$ there exists a value $\hat{\zeta}(U,\boldsymbol{u})$ such that, for all $\zeta \in (0,\hat{\zeta}(U,\boldsymbol{u}))$, $\widetilde{H}_U(\boldsymbol{u};\zeta) > 0$, which, due to the positivity of ζ, yields that $\zeta^{r-|R|}\widetilde{H}_U(\boldsymbol{u};\zeta) > 0$. Since the number of possible combinations for U and $\boldsymbol{u}$ is finite, the minimum $\hat{\zeta}^\lozenge := \min_{U,\boldsymbol{u}}\{\hat{\zeta}(U,\boldsymbol{u})\}$ is well defined and positive. Taking $\zeta^\lozenge$ to be any value in $(0,\hat{\zeta}^\lozenge)$, satisfies our demands. □

7.1 Examples of determinants appearing in the tightness construction

The determinant in Fig. 1 is the determinant $\mathsf{D}_U(\boldsymbol{x};\zeta)$ that corresponds to the linear equation $H_U(\boldsymbol{x})$ defined in the proof of Lemma 7.1, in the case where $R = [r]$ and $Y_i = \{y_{i,1},\ldots,y_{i,\kappa_i}\}$, for all $1 \le i \le r$. The determinant in Fig. 2 is the same as in Fig. 1 after having taken the limit $\zeta \to 0^+$.

$$
\left|\begin{array}{c:ccccc:cccccc:ccccccc}
1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\
x_1 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \ddots & \vdots \\
x_{r-1} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\
\hdashline
x_r & y_{1,1} & \widetilde{y}_{1,1} & \cdots & y_{1,\kappa_1} & \widetilde{y}_{1,\kappa_1} & \zeta y_{2,1}^{d-r+2} & \zeta \widetilde{y}_{2,1}^{d-r+2} & \cdots & \zeta y_{2,\kappa_2}^{d-r+2} & \zeta \widetilde{y}_{2,\kappa_2}^{d-r+2} & \cdots & \zeta y_{r,1}^{d-r+2} & \zeta \widetilde{y}_{r,1}^{d-r+2} & \cdots & \zeta y_{r,\kappa_r}^{d-r+2} & \zeta \widetilde{y}_{r,\kappa_r}^{d-r+2} & \cdots & \zeta M_s^{d-r+2} \\
x_{r+1} & \zeta y_{1,1}^{d-r+3} & \zeta \widetilde{y}_{1,1}^{d-r+3} & \cdots & \zeta y_{1,\kappa_1}^{d-r+3} & \zeta y_{{}_{\epsilon}1,\kappa_1}^{d-r+3} & y_{2,1} & \widetilde{y}_{2,1} & \cdots & y_{2,\kappa_2} & \widetilde{y}_{2,\kappa_2} & \cdots & \zeta y_{r,1}^{d-r+3} & \zeta \widetilde{y}_{r,1}^{d-r+3} & \cdots & \zeta y_{r,\kappa_r}^{d-r+3} & \zeta \widetilde{y}_{r,\kappa_r}^{d-r+3} & \cdots & \zeta M_s^{d-r+3} \\
x_{r+2} & \zeta y_{1,1}^{d-r+4} & \zeta \widetilde{y}_{1,1}^{d-r+4} & \cdots & \zeta y_{1,\kappa_1}^{d-r+4} & \zeta \widetilde{y}_{1,\kappa_1}^{d-r+4} & \zeta y_{2,1}^{d-r+4} & \zeta \widetilde{y}_{2,1}^{d-r+4} & \cdots & \zeta y_{2,\kappa_2}^{d-r+4} & \zeta \widetilde{y}_{2,\kappa_2}^{d-r+4} & \cdots & \zeta y_{r,1}^{d-r+4} & \zeta \widetilde{y}_{r,1}^{d-r+4} & \cdots & \zeta y_{r,\kappa_r}^{d-r+4} & \zeta \widetilde{y}_{r,\kappa_r}^{d-r+4} & \cdots & \zeta M_s^{d-r+4} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\
x_{2r-1} & \zeta y_{1,1}^{d+1} & \zeta \widetilde{y}_{1,1}^{d+1} & \cdots & \zeta y_{1,\kappa_1}^{d+1} & \zeta \widetilde{y}_{1,\kappa_1}^{d+1} & \zeta y_{2,1}^{d+1} & \zeta \widetilde{y}_{2,1}^{d+1} & \cdots & \zeta y_{2,\kappa_2}^{d+1} & \zeta \widetilde{y}_{2,\kappa_2}^{d+1} & \cdots & y_{r,1}^{d+1} & \widetilde{y}_{r,1}^{d+1} & \cdots & y_{r,\kappa_r}^{d+1} & \widetilde{y}_{r,\kappa_r}^{d+1} & \cdots & M_s \\
\hdashline
x_{2r} & y_{1,1}^2 & \widetilde{y}_{1,1}^2 & \cdots & y_{1,\kappa_1}^2 & \widetilde{y}_{1,\kappa_1}^2 & y_{2,1}^2 & \widetilde{y}_{2,1}^2 & \cdots & y_{2,\kappa_2}^2 & \widetilde{y}_{2,\kappa_2}^2 & \cdots & y_{r,1}^2 & \widetilde{y}_{r,1}^2 & \cdots & y_{r,\kappa_r}^2 & \widetilde{y}_{r,\kappa_r}^2 & \cdots & M_s^2 \\
x_{2r+1} & y_{1,1}^3 & \widetilde{y}_{1,1}^3 & \cdots & y_{1,\kappa_1}^3 & \widetilde{y}_{1,\kappa_1}^3 & y_{2,1}^3 & \widetilde{y}_{2,1}^3 & \cdots & y_{2,\kappa_2}^3 & \widetilde{y}_{2,\kappa_2}^3 & \cdots & y_{r,1}^3 & \widetilde{y}_{r,1}^3 & \cdots & y_{r,\kappa_r}^3 & \widetilde{y}_{r,\kappa_r}^3 & \cdots & M_s^3 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
x_{d+r-1} & y_{1,1}^{d-r+1} & \widetilde{y}_{1,1}^{d-r+1} & \cdots & y_{1,\kappa_1}^{d-r+1} & \widetilde{y}_{1,\kappa_1}^{d-r+1} & y_{2,1}^{d-r+1} & \widetilde{y}_{2,1}^{d-r+1} & \cdots & y_{2,\kappa_2}^{d-r+1} & \widetilde{y}_{2,\kappa_2}^{d-r+1} & \cdots & y_{r,1}^{d-r+1} & \widetilde{y}_{r,1}^{d-r+1} & \cdots & y_{r,\kappa_r}^{d-r+1} & \widetilde{y}_{r,\kappa_r}^{d-r+1} & \cdots & M_s^{d-r+1}
\end{array}\right|
$$

Figure 1: The determinant $\mathsf{D}_U(\boldsymbol{x};\zeta)$, for $R = [r]$.

$$
\begin{array}{c|ccccc|ccccc|c|ccccc|ccc|}
1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\
x_1 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots & \vdots & 0 & 0 & 0 \\
x_{r-1} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & 1 \\
x_r & y_{1,1} & \widetilde{y}_{1,1} & \cdots & y_{1,\kappa_1} & \widetilde{y}_{1,\kappa_1} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
x_{r+1} & 0 & 0 & \cdots & 0 & 0 & y_{2,1} & \widetilde{y}_{2,1} & \cdots & y_{2,\kappa_2} & \widetilde{y}_{2,\kappa_2} & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
x_{2r-1} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & y_{r,1} & \widetilde{y}_{r,1} & \cdots & y_{r,\kappa_r} & \widetilde{y}_{r,\kappa_r} & M_1 & \cdots & M_s \\
x_{2r} & y_{1,1}^2 & \widetilde{y}_{1,1}^2 & \cdots & y_{1,\kappa_1}^2 & \widetilde{y}_{1,\kappa_1}^2 & y_{2,1}^2 & \widetilde{y}_{2,1}^2 & \cdots & y_{2,\kappa_2}^2 & \widetilde{y}_{2,\kappa_2}^2 & \cdots & y_{r,1}^2 & \widetilde{y}_{r,1}^2 & \cdots & y_{r,\kappa_r}^2 & \widetilde{y}_{r,\kappa_r}^2 & M_1^2 & \cdots & M_s^2 \\
x_{2r+1} & y_{1,1}^3 & \widetilde{y}_{1,1}^3 & \cdots & y_{1,\kappa_1}^3 & \widetilde{y}_{1,\kappa_1}^3 & y_{2,1}^3 & \widetilde{y}_{2,1}^3 & \cdots & y_{2,\kappa_2}^3 & \widetilde{y}_{2,\kappa_2}^3 & \cdots & y_{r,1}^3 & \widetilde{y}_{r,1}^3 & \cdots & y_{r,\kappa_r}^3 & \widetilde{y}_{r,\kappa_r}^3 & M_1^3 & \cdots & M_s^3 \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
x_{d+r-1} & y_{1,1}^{d-r+1} & \widetilde{y}_{1,1}^{d-r+1} & \cdots & y_{1,\kappa_1}^{d-r+1} & \widetilde{y}_{1,\kappa_1}^{d-r+1} & y_{2,1}^{d-r+1} & \widetilde{y}_{2,1}^{d-r+1} & \cdots & y_{2,\kappa_2}^{d-r+1} & \widetilde{y}_{2,\kappa_2}^{d-r+1} & \cdots & y_{r,1}^{d-r+1} & \widetilde{y}_{r,1}^{d-r+1} & \cdots & y_{r,\kappa_r}^{d-r+1} & \widetilde{y}_{r,\kappa_r}^{d-r+1} & M_1^{d-r+1} & \cdots & M_s^{d-r+1} \\
\multicolumn{1}{c}{} & \multicolumn{5}{c}{\kappa_1 \text{ pairs of points from } (\boldsymbol{e}_0, \gamma_1(\cdot))} & \multicolumn{5}{c}{\kappa_2} & \multicolumn{1}{c}{} & \multicolumn{5}{c}{\kappa_r} & \multicolumn{3}{c}{\text{auxiliary columns if necessary}}
\end{array}
$$

Figure 2: The determinant $\mathsf{D}_U(\boldsymbol{x};0) = \lim_{\zeta\to 0^+}\mathsf{D}(\boldsymbol{u};\zeta)$, for $R=[r]$.

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A Special sets related to the derivation of the Dehn-Sommerville equations

To prove Lemma 3.3, we introduce a couple of sets that appear in the face counting of $\partial \mathcal{Q}_R$. For any $m \in \mathbb{N}$, and $S \subseteq [m]$ we define:

$$\mathcal{A}_m(S,k) \coloneqq \{(S_1, S_2, \dots, S_k) \mid S \subseteq S_1 \subset S_2 \subset \dots \subset S_k \subset [m]\}, \tag{A.1}$$

$$\mathcal{B}_m(S,k) \coloneqq \{(S_0, S_1, \dots, S_{k-1}) \mid S = S_0 \subset S_1 \subset \dots \subset S_{k-1} \subset [m]\}. \tag{A.2}$$

Furthermore, we denote by $\mathsf{A}_m(S,k)$ and $\mathsf{B}_m(S,k)$ the cardinalities of $\mathcal{A}_m(S,k)$ and $\mathcal{B}_m(S,k)$ respectively. It is immediate to see that:

$$\mathsf{A}_m(S,k) = \mathsf{A}_{m-|S|}(\varnothing,k), \quad \text{and}$$
$$\mathsf{B}_m(S,k) = \mathsf{B}_{m-|S|}(\varnothing,k).$$

Lemma A.1. *For any $k, m \in \mathbb{N}$, with $k \le m$, we have:*

(i) $\mathsf{B}_m(\varnothing,k) = k!\, S_m^k$,

(ii) $\mathsf{A}_m(\varnothing,k) = k!\, S_{m+1}^{k+1}$.

Proof. Recall that the Stirling number S_m^k counts the number of elements of the set $\binom{[m]}{k}$ of all partitions of $[m] = \{1, 2, \dots, m\}$ into k subsets.

In order to prove (i) let $\sigma : [k] \to [k]$ be a permutation of the integers in $[k]$ and $T = (T_1, \dots, T_k)$ be a partition of $[m]$ into k subsets. We claim that the map φ which sends each pair (σ, T) to the chain

$$\varnothing \subset T_{\sigma(1)} \subset (T_{\sigma(1)} \cup T_{\sigma(2)}) \subset \dots \subset \bigcup_{i=1}^{k-1} T_{\sigma(i)} \subset \bigcup_{i=1}^{k} T_{\sigma(i)} = [m]$$

is a bijection between $[k] \times \binom{[m]}{k}$ and $\mathcal{B}_m(\varnothing,k)$.

To prove our claim, notice first that, since the sets $T_1, \dots, T_k$ are non-empty, the inclusions in the chain $\varphi(\sigma,T)$ are strict and thus φ is well defined. To prove that φ is injective, let σ, τ be two permutations of $[k]$, and $T = (T_1, \dots, T_k)$, $T' = (T'_1, \dots, T'_k)$ be two partitions of $[m]$ into k subsets. We assume that $\varphi(\sigma,T) = \varphi(\tau,T')$ and we will prove that $\sigma = \tau$ and $\{T_1, \dots, T_k\} = \{T'_1, \dots, T'_k\}$. We use induction on the size of $[m]$, the case $m = 1$ being trivial. We next assume that our assumption holds true for any proper subset of $[m]$ and any $k < m$ and we prove it for $[m]$. To this end, since $\varphi(\sigma,T) = \varphi(\tau,T')$, we have that the chains

$$\varnothing \subset T_{\sigma(1)} \subset (T_{\sigma(1)} \cup T_{\sigma(2)}) \subset \dots \subset \bigcup_{i=1}^{k-2} T_{\sigma(i)},$$
$$\varnothing \subset T'_{\tau(1)} \subset (T'_{\tau(1)} \cup T'_{\tau(2)}) \subset \dots \subset \bigcup_{i=1}^{k-2} T'_{\tau(i)}$$

are identical. Thus, using the induction hypothesis, we deduce that $T_{\sigma(i)} = T'_{\tau(i)}$ and $\sigma(i) = \tau(i)$ for all $1 \le i \le k-1$. Clearly, $\sigma(k), \tau(k) \in K = [k] \setminus \{\sigma(i) : 1 \le i \le k-1\}$. But since $|K| = 1$, we have that $\sigma(k) = \tau(k)$. Moreover, since $[m] = \bigcup_{i=1}^{k-1} T_{\sigma(i)} = \bigcup_{i=1}^{k} T'_{\tau(i)}$ we deduce that $T_{\sigma(k)} = T'_{\tau(k)}$. This completes our induction. Finally, to prove that φ is onto, we consider a chain $\varnothing \subset S_1 \subset S_2 \subset \dots \subset S_{k-1} \subset [m]$ in $\mathcal{B}_m(\varnothing,k)$ and we set $T_k \coloneqq [m] \setminus S_{k-1}, T_{k-1} \coloneqq S_{k-1} \setminus S_{k-2}, \dots, T_2 \coloneqq S_2 \setminus S_1$ and $T_1 \coloneqq S_1$. It is immediate to see that $T_1, \dots, T_k$ is a partition of $[m]$ into k non-empty sets and that $\varphi(A, \mathtt{id}) = \varnothing \subset S_1 \subset S_2 \subset \dots \subset S_k$, where $\mathtt{id}$ is the identity permutation in $[k]$.

To prove (ii), notice that

$$\mathcal{A}_m(\varnothing,k) = \{(S_1, \dots, S_k) \mid \varnothing \subseteq S_1 \subset \dots \subset S_k \subset [m]\}$$

$$= \{(S_1,\dots,S_k) \mid \varnothing \subset S_1 \subset \cdots \subset S_k \subset [m]\} \bigcup \{(S_1,\dots,S_k) \mid \varnothing = S_1 \subset \cdots \subset S_k \subset [m]\}$$
$$= \{(S_1,\dots,S_k) \mid \varnothing \subset S_1 \subset \cdots \subset S_k \subset [m]\} \bigcup \{(S_2,\dots,S_k) \mid \varnothing \subset S_2 \subset \cdots \subset S_k \subset [m]\}.$$

Using (i), we have:

$$\begin{aligned}
\mathsf{A}_m(\varnothing,k) &= \mathsf{B}_m(\varnothing,k) + \mathsf{B}_m(\varnothing,k-1)\\
&= \sum_{j=0}^{k+1}(-1)^{k+1-j}\tbinom{k+1}{j}j^m + \sum_{j=0}^{k}(-1)^{k-j}\tbinom{k}{j}j^m\\
&= (k+1)^m + \sum_{j=0}^{k}(-1)^{k+1-j}\left(\tbinom{k+1}{j} - \tbinom{k}{j}\right)j^m\\
&= (k+1)^m + \sum_{j=0}^{k}(-1)^{k+1-j}\tbinom{k}{j-1}j^m\\
&= (k+1)^m + \sum_{i=0}^{k-1}(-1)^{k-i}\tbinom{k}{i}(i+1)^m\\
&= \sum_{i=0}^{k}(-1)^{k-i}\tbinom{k}{i}(i+1)^m\\
&= k!\, S_{m+1}^{k+1}.
\end{aligned}$$

□

The following combinatorial identities are used in the proof of Lemma 3.5.

Lemma A.2. *For any $m \geq 1$, we have:*

$$\sum_{i=0}^{m} i!\, S_{m+1}^{i+1}(t-1)^{m-i} = \sum_{j=0}^{m-1} E_m^j t^{m-j}, \tag{A.3}$$

and

$$\sum_{i=0}^{m-1} (i+1)!\, S_{m}^{i+1}(t-1)^{m-i-1} = \sum_{j=0}^{m-1} E_m^j t^{m-1-j}. \tag{A.4}$$

Proof. Observe that:

$$\begin{aligned}
\sum_{i=0}^{m} i!\, S_{m+1}^{i+1}(t-1)^{m-i} &= \sum_{i=0}^{m}\left(\frac{1}{i+1}\sum_{j=0}^{i+1}(-1)^{i+1-j}\tbinom{i+1}{j}j^{m+1}\right)\left(\sum_{j'=0}^{m-i}(-1)^{m-i-j'}\tbinom{m-i}{j'}t^{j'}\right)\\
&= \sum_{i=0}^{m}\left(\sum_{j=0}^{i+1}(-1)^{i+1-j}\tbinom{i}{j-1}j^{m}\right)\left(\sum_{j'=0}^{m-i}(-1)^{m-i-j'}\tbinom{m-i}{j'}t^{j'}\right)\\
&= \sum_{j=0}^{m+1}\sum_{j'=0}^{m}(-1)^{m+1-j-j'}j^m\sum_{i=0}^{m}\tbinom{m-i}{j'}\tbinom{i}{j-1}t^{j'}\\
&= \sum_{j=1}^{m+1}\sum_{j'=0}^{m}(-1)^{m+1-j-j'}j^m\tbinom{m+1}{j+j'}t^{j'}\\
&\overset{\ell:=m-j'}{=} \sum_{j=1}^{m+1}\sum_{\ell=0}^{m}(-1)^{\ell-j+1}j^m\tbinom{m+1}{m-\ell+j}t^{m-\ell}\\
&\overset{i:=\ell-j+1}{=} \sum_{\ell=0}^{m}\sum_{i=0}^{\ell}(-1)^{i}(\ell-i+1)^m\tbinom{m+1}{i}t^{m-\ell}\\
&= \sum_{\ell=0}^{m} E_m^{\ell}t^{m-\ell} = \sum_{\ell=0}^{m-1} E_m^{\ell}t^{m-\ell},
\end{aligned}$$

where in the last sum we used the fact that $E_m^m = 0$ for all $m \geq 1$.

To prove (A.4), we distinguish between the cases $m = 1$ and $m > 1$. For $m = 1$ we have:

$$\sum_{i=0}^{m-1} (i+1)! S_m^{i+1} (t-1)^{m-i-1} = (t-1)^0 = 1 = t^0 = \sum_{i=0}^{m-1} E_m^i t^{m-1-i},$$

where we used the fact that $S_1^1 = E_1^0 = 1$. For $m > 1$, we set $\mathtt{S}_m(t) := \sum_{i=0}^{m} i!\, S_{m+1}^{i+1} t^{m-i}$ and $\mathtt{E}_m(t) := \sum_{i=0}^{m-1} E_m^i t^{m-i}$, so that relation (A.3) is equivalent to $\mathtt{S}_m(t-1) = \mathtt{E}_m(t)$. We then have:

$$\begin{aligned}
\sum_{i=0}^{m-1} (i+1)! S_m^{i+1} (t-1)^{m-i-1} &= m \sum_{i=0}^{m-1} i!\, S_m^{i+1} (t-1)^{m-i-1} - \sum_{i=0}^{m-1} (m-i-1)\, i!\, S_m^{i+1} (t-1)^{m-i-1} \\
&= m\, \mathtt{S}_{m-1}(t-1) + (t-1)\, \mathtt{S}'_{m-1}(t-1) \\
&= m\, \mathtt{E}_{m-1}(t) - (t-1) \mathtt{E}'_{m-1}(t) \\
&= \sum_{i=0}^{m-1} \Big(m E_{m-1}^i - (m-1-i) E_{m-1}^i + (m-i) E_{m-1}^{i-1} \Big) t^{m-1-i} \\
&= \sum_{i=0}^{m-1} E_m^i t^{m-1-i},
\end{aligned}$$

where in the last equality we used the recurrence relation of Eulerian numbers:

$$E_m^i = (m-i) E_{m-1}^{i-1} + (i+1) E_{m-1}^i. \qquad \square$$

B Relations appearing in the derivation of the recurrence relation for the h-vector of $\mathcal{F}_R$

B.1 McMullen's relation restated

McMullen [18], in his original proof of the Upper Bound Theorem for polytopes, proved that for any simplicial d-polytope P the following relation holds:

$$(k+1) h_{k+1}(\partial P) + (d-k) h_k(\partial P) = \sum_{v \in \mathrm{vert}(\partial P)} h_k(\partial P/v), \qquad 0 \leq k \leq d-1. \tag{B.1}$$

Below we rewrite these relations in terms of generating functions.

Lemma B.1 (McMullen 1970)**.** *For any simplicial d-polytope P*

$$d\, \mathtt{h}(\partial P; t) + (1-t) \mathtt{h}'(\partial P; t) = \sum_{v \in \mathit{vert}(\partial P)} \mathtt{h}(\partial P/v; t). \tag{B.2}$$

Proof. Multiplying both sides of (B.1) by t^{d-k-1}, and summing over all $0 \leq k \leq d$, we get:

$$\sum_{k=0}^{d} (k+1) h_{k+1}(\partial P) t^{d-k-1} + \sum_{k=0}^{d} (d-k) h_k(\partial P) t^{d-k-1} = \sum_{k=0}^{d} \sum_{v \in \mathrm{vert}(\partial P)} h_k(\partial P/v) t^{d-k-1}. \tag{B.3}$$

For the right-hand side of (B.3) we have:

$$\sum_{k=0}^{d} \sum_{v \in \mathrm{vert}(\partial P)} h_k(\partial P/v) t^{d-k-1} = \sum_{v \in \mathrm{vert}(\partial P)} \sum_{k=0}^{d} h_k(\partial P/v) t^{d-1-k} = \sum_{v \in \mathrm{vert}(\partial P)} \mathtt{h}(\partial P/v; t), \tag{B.4}$$

whereas for the left-hand side of (B.3) we get:

$$\begin{aligned}\sum_{k=0}^{d}(k+1)h_{k+1}(\partial P)t^{d-k-1} &+ \sum_{k=0}^{d}(d-k)h_k(\partial P)t^{d-k-1} \\ &= \sum_{k=0}^{d} kh_k(\partial P)t^{d-k} + \sum_{k=0}^{d}(d-k)h_k(\partial P)t^{d-k-1} \\ &= d\sum_{k=0}^{d} h_k(\partial P)t^{d-k} + (1-t)\sum_{k=0}^{d}(d-k)h_k(\partial P)t^{d-k-1} \\ &= d\,\mathtt{h}(\partial P;t) + (1-t)\,\mathtt{h}'(\partial P;t).\end{aligned} \tag{B.5}$$

Substituting (B.4) and (B.5) in (B.3) we recover the relation in the statement of the lemma. □

B.2 One more auxiliary set

Recall that $\mathsf{D}(R,T,X,\ell)$ denotes the cardinality of the set:

$$\mathcal{D}(R,T,X,\ell) := \{(S_1,\dots,S_\ell) : X \subseteq S_1 \subset S_2 \subset \cdots \subset S_\ell \subset R \text{ and } S_i = T \text{ for some } 1 \le i \le \ell\}.$$

The following lemma expresses the sum of the cardinalities $\mathsf{D}(R,T,X,\ell)$, over all T with $X \subseteq T \subset R$, in terms of the Stirling numbers of the second kind.

Lemma B.2. *For any $\ell \in \mathbb{N}$, and X, R with $\varnothing \subseteq X \subset R$, we have:*

$$\sum_{X\subseteq T\subset R} \mathsf{D}(R,T,X,\ell) = \ell\,\ell!\,S^{\ell+1}_{|R|-|X|+1}. \tag{B.6}$$

Proof. The left-hand side of (B.6) is the cardinality of the set

$$\mathcal{Y} = \{(S_1,\dots,S_\ell) : X \subseteq T \subset R, X \subseteq S_1 \subset S_2 \subset \cdots \subset S_\ell \subset R \text{ and } S_i = T \text{ for some } 1 \le i \le \ell\},$$

which is nothing but ℓ copies of the set

$$\mathcal{Z} = \{(S_1,S_2,\dots,S_\ell) \mid X \subseteq S_1 \subset S_2 \subset \cdots \subset S_\ell \subset R\}.$$

Indeed,

$$\begin{aligned}\mathcal{Y} =&\{(S_1,\dots,S_\ell) : X \subseteq T \subset R, X \subseteq S_1 \subset S_2 \subset \cdots \subset S_\ell \subset R \text{ and } S_i = T \text{ for some } 1 \le i \le \ell\} \\ =&\{i : 1 \le i \le \ell\} \times \{(S_1,\dots,S_\ell) : X \subseteq T \subset R, X \subseteq S_1 \subset S_2 \subset \cdots \subset S_\ell \subset R \text{ and } S_i = T\} \\ =&\{i : 1 \le i \le \ell\} \times \{(S_1,\dots,S_\ell) : X \subseteq S_1 \subset S_2 \subset \cdots \subset S_\ell \subset R\} \\ =&\{i : 1 \le i \le \ell\} \times \mathcal{Z}.\end{aligned}$$

By Lemma A.1(ii), the cardinality of $\mathcal{Z}$ is $\ell!\,S^{\ell+1}_{|R|-|X|+1}$ and this completes our proof. □

C Determinants used in the tightness construction

Definition C.1. *Let $Y_i = \{y_{i,1},\dots,y_{i,\kappa_1}\}$, $1 \le i \le n$, be non-empty disjoint sets of real numbers. Set $K := \kappa_1 + \kappa_2 + \cdots + \kappa_n$, $m := K - 2n - 2$ and let $\mu_1 < \mu_2 < \cdots < \mu_m$ be non-negative integers. We denote by Y the partition $Y_1 \cup \cdots \cup Y_n$ and we define the $K \times K$ matrix $\Delta_K(\mathsf{Y};\mu_1,\dots,\mu_m)$ as follows:*

$$
\Delta_K(\mathsf{Y};\mu_1,\ldots,\mu_m) \coloneqq
\begin{array}{cc}
 & \scriptstyle \textit{row} \\[-4pt]
 & \scriptstyle \textit{index} \\
\left(\begin{matrix}
y^{\mu_1}_{1,1} & \cdots & y^{\mu_1}_{1,\kappa_1} & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & y^{\mu_1}_{2,1} & \cdots & y^{\mu_1}_{2,\kappa_2} & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 & y^{\mu_1}_{3,1} & \cdots & y^{\mu_1}_{3,\kappa_3} & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & y^{\mu_1}_{n,1} & \cdots & y^{\mu_1}_{n,\kappa_1} \\
y^{\mu_2}_{1,1} & \cdots & y^{\mu_2}_{1,\kappa_1} & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & y^{\mu_2}_{2,1} & \cdots & y^{\mu_2}_{2,\kappa_2} & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 & y^{\mu_2}_{3,1} & \cdots & y^{\mu_2}_{3,\kappa_3} & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & y^{\mu_2}_{n,1} & \cdots & y^{\mu_2}_{n,\kappa_n} \\
y^{\mu_3}_{1,1} & \cdots & y^{\mu_3}_{1,\kappa_1} & y^{\mu_3}_{2,1} & \cdots & y^{\mu_3}_{2,\kappa_2} & y^{\mu_3}_{3,1} & \cdots & y^{\mu_3}_{3,\kappa_3} & \cdots & y^{\mu_3}_{n,1} & \cdots & y^{\mu_3}_{n,\kappa_n} \\
y^{\mu_4}_{1,1} & \cdots & y^{\mu_4}_{1,\kappa_1} & y^{\mu_4}_{2,1} & \cdots & y^{\mu_4}_{2,\kappa_2} & y^{\mu_4}_{3,1} & \cdots & y^{\mu_4}_{3,\kappa_3} & \cdots & y^{\mu_4}_{n,1} & \cdots & y^{\mu_4}_{n,\kappa_n} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\
y^{\mu_m}_{1,1} & \cdots & y^{\mu_m}_{1,\kappa_1} & y^{\mu_m}_{2,1} & \cdots & y^{\mu_m}_{2,\kappa_2} & y^{\mu_m}_{3,1} & \cdots & y^{\mu_m}_{3,\kappa_3} & \cdots & y^{\mu_m}_{n,1} & \cdots & y^{\mu_m}_{n,\kappa_n}
\end{matrix}\right)
&
\begin{matrix}
\scriptstyle 1 \\ \scriptstyle 2 \\ \scriptstyle 3 \\ \vdots \\ \scriptstyle n \\ \scriptstyle n+1 \\ \scriptstyle n+2 \\ \scriptstyle n+3 \\ \vdots \\ \scriptstyle 2n \\ \scriptstyle 2n+1 \\ \scriptstyle 2n+2 \\ \vdots \\ \scriptstyle K
\end{matrix}
\end{array}
$$

We denote by $D_{\mathrm{K}}(\mathsf{Y};\mu_1,\ldots,\mu_m)$ *the signed determinant* $(-1)^{\frac{n(n-1)}{2}}\left|\Delta_K(\mathsf{Y};\mu_1,\ldots,\mu_m)\right|$.

We, now, parameterize all $y_{i,j}$'s as follows: for each $1 \le i \le n$ we choose arbitrary real numbers $0 < x_{i,1} < x_{i,2} < \cdots < x_{i,\kappa_i}$ and non-negative integers $0 \le \beta_n < \beta_{n-1} < \cdots < \beta_1$. Then, we set $y_{i,j} \coloneqq x_{i,j}\tau^{\beta_i}$ where τ is a positive parameter, and consider $D_{\mathrm{K}}(\mathsf{Y};\mu_1,\ldots,\mu_m)$ as a polynomial in τ. In the next lemma we essentially show that, for sufficiently small τ, the determinant $D_{\mathrm{K}}(\mathsf{Y};\mu_1,\ldots,\mu_m)$ is strictly positive.

Lemma C.2. *If the elements of the sets* Y_i, $1 \le i \le r$, *are parameterized as above and* $\mathsf{Y} = Y_1 \cup \cdots \cup Y_n$, *then* $D_{\mathrm{K}}(\mathsf{Y};\mu_1,\ldots,\mu_m) = A\tau^a + O(\tau^{a+1})$, *where* $A > 0$ *and* a *is a positive integer.*

Proof. To prove our claim we use the Binet-Cauchy theorem [3]. More precisely, let J be a subset of $\{1,2,\ldots,n(m+1)\}$ of size K. We denote by $L_{[K],J}$ the $K \times K$ matrix whose columns are the columns of L at indices from J and by $R_{J,[K]}$ the $K \times K$ matrix whose rows are the rows of R at indices from J. The Binet-Cauchy theorem states that:

$$
\det(LR) = \sum_J \det(L_{[K],J})\det(R_{J,[K]}), \tag{C.1}
$$

where the sum is taken over all subsets J of $\{1,2,\ldots,n(m+1)\}$ of size K.

To apply the Binet-Cauchy theorem in our case, notice that the matrix $\Delta_K(\mathsf{Y};\mu_1,\ldots,\mu_m)$ can be factorized into a product of a $K \times n(m+1)$ matrix L and an $n(m+1) \times K$ matrix R as shown below:

$$
L = \begin{array}{cc}
 & \begin{array}{ccccccccccccccccccccccccccc} 1 & 2 & 3 & 4 & \cdots & m+1 & 1+1(m+1) & 2+1(m+1) & 3+1(m+1) & 4+1(m+1) & \cdots & m+1+1(m+1) & 1+2(m+1) & 2+2(m+1) & 3+2(m+1) & 4+2(m+1) & \cdots & m+1+2(m+1) & \cdots & 1+(n-1)(m+1) & 2+(n-1)(m+1) & 3+(n-1)(m+1) & 4+(n-1)(m+1) & \cdots & m+1+(n-1)(m+1) \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ \vdots \\ n \\ n+1 \\ n+2 \\ n+3 \\ \vdots \\ 2n \\ 2n+1 \\ 2n+2 \\ \vdots \\ K \end{array} &
\begin{pmatrix}
1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 1 & \cdots & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & 0 & \cdots & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1
\end{pmatrix}
\end{array},
$$

$$
R = \begin{pmatrix}
y_{1,1}^{\mu_1} & \cdots & y_{1,\kappa_1}^{\mu_1} & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
y_{1,1}^{\mu_2} & \cdots & y_{1,\kappa_1}^{\mu_2} & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
y_{1,1}^{\mu_3} & \cdots & y_{1,\kappa_1}^{\mu_3} & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
y_{1,1}^{\mu_m} & \cdots & y_{1,\kappa_1}^{\mu_m} & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & y_{2,1}^{\mu_1} & \cdots & y_{2,\kappa_2}^{\mu_1} & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & y_{2,1}^{\mu_2} & \cdots & y_{2,\kappa_2}^{\mu_2} & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & y_{2,1}^{\mu_3} & \cdots & y_{2,\kappa_2}^{\mu_3} & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & y_{2,1}^{\mu_m} & \cdots & y_{2,\kappa_2}^{\mu_m} & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 & y_{3,1}^{\mu_1} & \cdots & y_{3,\kappa_3}^{\mu_1} & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 & y_{3,1}^{\mu_2} & \cdots & y_{3,\kappa_3}^{\mu_2} & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 & y_{3,1}^{\mu_3} & \cdots & y_{3,\kappa_3}^{\mu_3} & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & y_{3,1}^{\mu_m} & \cdots & y_{3,\kappa_3}^{\mu_m} & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & y_{n,1}^{\mu_1} & \cdots & y_{n,\kappa_n}^{\mu_1} \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & y_{n,1}^{\mu_2} & \cdots & y_{n,\kappa_n}^{\mu_2} \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & y_{n,1}^{\mu_3} & \cdots & y_{n,\kappa_n}^{\mu_3} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & y_{n,1}^{\mu_m} & \cdots & y_{n,\kappa_n}^{\mu_m}
\end{pmatrix}.
$$

The numbers over and sideways of L indicate the column and row numbers, respectively.

Recall that $y_{i,j} = x_{i,j}\tau^{\beta_i}$. Then it is not hard to see that, for each $J \subseteq \{1, \dots, n(m+1)\}$ with $|J| = K$, the sub-matrix $L_{[K],J}$ is independent of τ while $R_{J,[K]}$ is a block-diagonal matrix whose blocks are generalized Vandermonde determinants (cf. [8]) from which we can extract powers of τ. More precisely, we set $k^{\langle i \rangle} := k + (i-1)(m+1)$ and we write each index set J of the

Binet-Cauchy expansion as $J_1 \cup J_2 \cup \cdots \cup J_n$, where $J_i \subseteq \left\{1^{\langle i \rangle}, \ldots, (m+1)^{\langle i \rangle}\right\}$. We then have:

$$\det(R_{J,[K]}) = \tau^{a(J)} \prod_{i=1}^{n} \mathrm{GVD}(J_i), \tag{C.2}$$

where

$$a(J) = \beta_1 \sum_{j^{\langle 1 \rangle} \in J_1} \mu_j + \beta_2 \sum_{j^{\langle 2 \rangle} \in J_2} \mu_j + \cdots + \beta_n \sum_{j^{\langle n \rangle} \in J_n} \mu_j,$$

and $\mathrm{GVD}(J_i)$ is a positive generalized Vandermonde determinant[4], independent of τ, depending on the $y_{i,j}$'s with $j \in J_i$. Thus, combining (C.1) and (C.2), we deduce that $D_{\mathrm{K}}(\mathsf{Y}; \mu_1, \ldots, \mu_m)$ is a polynomial in τ. To prove our claim it suffices to find the subset J for which $a(J)$ is minimal and, for this J, evaluate the sign of the coefficient of $\tau^{a(J)}$.

Notice that a term $\det(L_{[K],J}) \det(R_{J,[K]})$ in the Cauchy-Binet expansion of $D_{\mathrm{K}}(\mathsf{Y}; \mu_1, \mu_2, \ldots, \mu_m)$ vanishes in the following two cases:

(i) $k^{\langle i \rangle}, k^{\langle j \rangle} \in J$ for some $3 \leq k \leq m+1$; in this case the $k^{\langle i \rangle}$-th and $k^{\langle j \rangle}$-th columns of $L_{[K],J}$ are identical, and thus $\det(L_{[K],J}) = 0$.

(ii) $|J_i| \neq k_i$ for at least some $1 \leq i \leq n$; in this case $R_{J,[K]}$ is a block-diagonal square matrix with non-square non-zero blocks. The determinant of such a matrix is always zero.[5]

Among all possible index sets $J = J_1 \cup \cdots \cup J_n$ for which the product $\det(L_{[K],J}) \det(R_{J,[K]})$ does not vanish, we have to find the one for which the exponent $a(J)$ in (C.2) is the minimum possible. To do this, we combine condition (i) above with the fact that $\beta_1 > \cdots > \beta_n$ and we deduce that the minimum exponent $M(J)$ is attained if, for all $1 \leq i \leq r$:

- $1^{\langle i \rangle}, 2^{\langle i \rangle} \in J_i$, and
- if $\kappa^{\langle i \rangle} \in J_i$ and $\lambda^{\langle i+1 \rangle} \in J_{i+1}$ for some $\kappa, \lambda > 2$, then $\kappa < \lambda$.

Moreover, since from condition (ii) we have $|J_i| = k_i$, we conclude that:

- $J_1 = J_1^\star := \{1^{\langle 1 \rangle}, 2^{\langle 1 \rangle}, 3^{\langle 1 \rangle}, \ldots, k_1^{\langle 1 \rangle}\} = \{1, \ldots, k_1\}$,
- $J_2 = J_2^\star := \{1^{\langle 2 \rangle}, 2^{\langle 2 \rangle}, (k_1+1)^{\langle 2 \rangle}, \ldots, (k_1+k_2-2)^{\langle 2 \rangle}\}$,
- $J_3 = J_3^\star := \{1^{\langle 3 \rangle}, 2^{\langle 3 \rangle}, (k_1+k_2-1)^{\langle 3 \rangle}, \ldots, (k_1+k_2+k_3-4)^{\langle 3 \rangle}\}$

  etc.

For the above choice of $J^\star = J_1^\star \cup \cdots \cup J_n^\star$, the matrix $L_{[K],J}$ is:

[4] It is a well-known fact that, if the parameters in the columns of the generalized Vandermonde determinant are in strictly increasing order, then the Vandermonde determinant is itself strictly positive (see [8] for a proof of this fact).

[5] To see this, consider the Laplace expansion of the matrix with respect to the columns of its top-left block.

$$
L_{[K],J^\star} = \left(\begin{array}{ccc|ccc|ccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\
\hline
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\
\hline
0 & 0 & \mathrm{I}_{k_1-2} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \mathrm{I}_{k_2-2} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{I}_{k_3-2} & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \mathrm{I}_{k_n-2}
\end{array}\right)
\begin{array}{c}
\text{row index} \\
1 \\ 2 \\ 3 \\ \vdots \\ n \\ \hline n+1 \\ n+2 \\ n+3 \\ \vdots \\ 2n \\ \hline \vdots \\ \\ \\ \\ \\
\end{array}
$$

Thus, in order to find the sign of our original determinant, we have to evaluate $\det(L_{[K],J^\star})$. To do this, we perform the appropriate row and column swaps so that $L_{[K],J^\star}$ becomes the identity matrix. More precisely,

- we perform $n-1+(n-2)+(n-3)+\cdots+1 = \frac{n(n-1)}{2}$ row swaps so that, for all $1 \le i \le n$, row $n+i$ is shifted upwards and paired with row i, to become a 2×2 identity matrix,
- we then perform an even number of column swaps to shift each I_{k_i-2} to its "proper" position (i.e., so that we get an identity matrix along with the corresponding 2×2 block of the previous step).

We therefore conclude that the sign of the dominant term of the expansion of the determinant of the matrix $\Delta_K(\mathsf{Y};\mu_1,\ldots,\mu_m)$ as a polynomial in τ, is $(-1)^{\frac{n(n-1)}{2}}$ and this completes our proof. $\square$