Sieved Enumeration of Interval Orders and Other Fishburn Structures

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ABSTRACT. Following a result of Eriksen and Sjöstrand (2014) we detail a technique to construct structures following the Fishburn distribution from appropriate Mahonian structures.

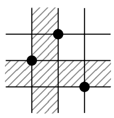
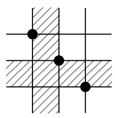
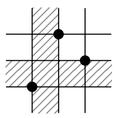
This technique is introduced on a bivincular pattern of Bousquet-Mélou et al. (2010) and then used to introduce a previously unconsidered class of matchings; explicitly, zero alignment matchings according to the number of arcs which are both right-crossed and left nesting.

We then define a statistic on the factorial posets of Claesson and Linusson (2011) counting the number of features which we refer to as mislabelings and demonstrate that according to the number of mislabelings that factorial posets follow the Fishburn distribution.

As a consequence of our approach we find an identity for the Fishburn numbers in terms of the Mahonian numbers.

INTRODUCTION

Eriksen and Sjöstrand [7] provide a remarkable refinement of Zagier's formula for certain Fishburn structures. It follows from their results that the following patterns are equidistributed in permutations:

 ,  and  .

Furthermore they prove that the above patterns are equidistributed with the number of right nestings in non-left nesting matchings. They show the distribution is given by the coefficients $f_{n,k}$ of the following ordinary generating function,

$$\sum_{n=0}\sum_{\pi\in\mathfrak{S}_n} x^n y^{\sigma(\pi)} = \sum_{n=0}\sum_{k=0} f_{n,k} x^n y^k$$
$$= \sum_{m=0} (-1)^m \prod_{i=1}^{m} \frac{(1+(y-1)x)^i - 1}{1-y}.$$

We refer to this as the *Fishburn distribution.*

Observing that the above equation can be written in terms of a substitution of the q-factorial $\boldsymbol{(n)_q!}$ as

$$\sum_{n\geq 0} \boldsymbol{(n)_{x(y-1)+1}!} x^n,$$

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allows for a new approach for the enumeration of Fishburn structures. This makes explicit a link between Fishburn and Mahonian structures. In this paper we demonstrate this technique on three structures. We first recreate Eriksen and Sjöstrand's result on occurrences in permtuations of the first of the above patterns. We then introduce a new Fishburn structure showing that the number of arcs which are both left-nesting and right-crossed in matchings with no alignments follow the Fishburn distribution. Then we identify a new feature on the factorial posets of Claesson and Linusson [4] which we name mislabelings. A factorial poset with zero mislabelings satisfies a condition of Claesson and Linusson giving that the poset is a canonically labeled interval order, thus allowing us to recreate a result of Bousquet-Mélou et al. [2].

As a consequence of the relationship to the q-factorial we provide a new identity for coefficients $f_{n,k}$ of the Fishburn distribution with respect to the Mahonian numbers $m_{n,k}$ (A008302), that

$$f_{n,k} = \sum_{i=k}^{n-2}(-1)^{i+k}\binom{i}{k}\sum_{j=i}^{\binom{n-i}{2}}\binom{j}{i}m_{n-i,j}.$$

Of particular interest is when $k = 0$, which gives an identity for the nth Fishburn number (A022493)

$$\sum_{i=0}^{n-2}(-1)^{i}\sum_{j=i}^{\binom{n-i}{2}}\binom{j}{i}m_{n-i,j}.$$

Terminology and background

For $a, b \in \mathbb{Z}$ with $a < b$ let $[b]$ denote the set $\{1, \dots, b\}$ and $[a, b]$ the set $\{a, \dots, b\}$.

For U a linearly ordered set with $x \in U$ where x is *not* the maximal element of U then, where there is no ambiguity, we shall abuse notation using $x + 1$ to refer to the immediate successor of x in U.

Mahonian numbers. For $n \in \mathbb{N}$ let $\boldsymbol{(n)_q!}$ denote the q-factorial, defined as

$$\boldsymbol{(n)_q!} = \prod_{i=1}^{n}\sum_{j=0}^{i-1}q^j = \prod_{i=1}^{n}\frac{1-q^i}{1-q}.$$

The coefficients of the q-factorial are known as the Mahonian numbers (A008302). The first few terms are shown in Figure 1. We shall use $m_{n,k}$ to denote the kth entry of row n.

Mahonian numbers derive their name from seminal work identifying permutation statistics by Major MacMahon [11]. As a result, and particularly in the case of permutations, structures counted by the q-factorial are often referred to as Mahonian structures.

Permutation patterns. A permutation is a bijection on a finite set U. The results in this paper assume that there is a total order on U. We shall therefore assume throughout that for $n \in \mathbb{N}$ permutations as elements of $\mathfrak{S}_n$ are bijections on the set $[n]$.

For $n, k \in \mathbb{N}$ with $n > k$ take permutations $\pi \in \mathfrak{S}_n$ and $\tau \in \mathfrak{S}_k$. An occurrence of τ as a *classical permutation pattern* in π is a subsequence of π whose entries are in

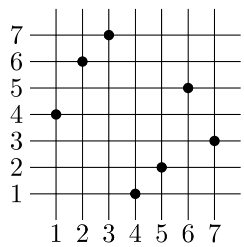
1
1, 1
1, 2, 2, 1
1, 3, 5, 6, 5, 3, 1
1, 4, 9, 15, 20, 22, 20, 15, 9, 4, 1
1, 5, 14, 29, 49, 71, 90, 101, 101, 90, 71, 49, 29, 14, 5, 1
...

FIGURE 1. Mahonian Triangle (A008302), $m_{n,k}$

the same relative order as in τ. For example taking $\tau = 132$ and $\pi = 4671253$ then the following subsequences of π correspond to occurrences of τ,

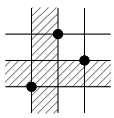
$$465 \quad 475 \quad 153 \quad 253.$$

Permutations may be represented on a grid by dots placed at line intersects such that each line is intersected by exactly one dot. The permutation maps the value indicated by each vertical line to the value of the corresponding horizontal line indicated by the dot placement. For example, the grid below corresponds to the permutation 4671253.



A *mesh pattern*, introduced by Brändén and Claesson [3], consists of a classical permutation pattern and a (potentially empty) set of shaded boxes on the grid representation of that pattern. An occurrence of a mesh pattern consists of an occurrence of the underlying classical permutation such that there are no entries of π contained within the shaded boxes.

For example, there are two occurrences of the following mesh pattern in π,

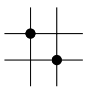
 .

namely 465 and 253. Whereas, although an occurrence of the underlying classical pattern, 475 is not an occurrence the above mesh pattern as 5 occurs between the 4 and 7.

A *vincular pattern* is a mesh pattern where only entire columns may be shaded out. A *bivincular pattern* is a mesh pattern where any shaded boxes must contribute to an entire row or column of shaded boxes. The above mesh pattern is also a bivincular pattern.

A permutation with no occurrences of a pattern is said to *avoid* that pattern.

Occurrences of the pattern

 ,

are known as *inversions* and are counted by the q-factorial (see MacMahon [11]).

Posets. A poset P is defined as a set and an associated binary relation $<_P$ satisfying reflexivity, antisymmetry, and transitivity. A poset constructed on some linearly ordered set U is said to be *naturally labeled* if $i <_P j \implies i <_U j$.

An *interval order* is a poset P where each $z \in P$ can be assigned a closed interval $[l_z, r_z] \in \mathbb{R}$ such that $x <_P y$ if and only if $r_x < l_y$. Equivalent conditions are that an interval order is a poset whose predecessor sets can be assigned a total order by inclusion [1] or that a poset is an interval order if it has no induced subposet isomorphic to the pair of disjoint two element chains, i.e. the poset is $(2+2)$-free [8].

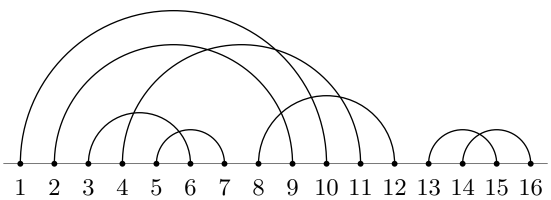
For $i \in P$, let the following notation be used for the predecessor and successor sets of i:

$$\operatorname{Pred} i = \{j \in P : j <_P i\}, \qquad \operatorname{pred} i = |\operatorname{Pred} i|,$$
$$\operatorname{Succ} i = \{\ell \in P : i <_P \ell\}, \qquad \operatorname{succ} i = |\operatorname{Succ} i|.$$

Matchings. A perfect matching of size n is a fixed point free involution of semi-length n. Matchings are typically represented as a set of ordered pairs (i, j) such that $i < j$ or diagrammatically as arcs on the numberline $[2n]$. For example, the matching of size 10

$$\{(1, 10), (2, 9), (3, 6), (4, 11), (5, 7), (8, 12), (13, 15), (14, 16)\},$$

is diagrammatically represented as



A *nesting* arc in a matching is an arc which is entirely encloses another arc when seen diagrammatically, i.e. an (i, j) such that there exists (k, ℓ) with $i < k < \ell < j$. The arc which is enclosed is known as a *nested* arc. If $k = i + 1$ then the arcs are called *left-nesting* and *left-nested* respectively. If $\ell + 1 = j$ then the arcs are *right-nesting* and *right-nested* respectively.

A *crossing* arc in a matching is the leftmost of two intersecting arcs when seen diagrammatically, i.e. an (i, j) such that there exists (k, ℓ) with $i < k < j < \ell$. The same approach as for nestings is taken to define *crossed, left-crossing, left-crossed, right-crossing* and *right-crossed* arcs.

An alignment in a matching is two arcs (i, j) and (k, ℓ) such that $i < j < k < \ell$.

For two arcs (i, j) and (k, ℓ), we say that k is an *embraced nested opener* if k is the opener for an arc nested by (i, j).

Statistics and features. Given some set of structures X a *statistic* ψ is defined as a function taking a structure to a positive integer, i.e. $\psi : X \to \mathbb{Z}_+$.

A *feature* of a structure is a property, aspect or substructure of a combinatorial structure. For example, an inversion in a permutation, or a nesting in a matching.

Background. Studying interval orders and permutations avoiding σ Bousquet-Mélou et al. [2] demonstrate that their ordinary generating function is given by

$$\sum_{n\geq 0}\prod_{k=1}^{n}(1-(1-x)^k) = \boldsymbol{(n)_{-x+1}!}x^n,$$

a function originally considered by Zagier [13] in enumerating a restricted class matchings (non-neighbor nesting matchings).

As an intermediate structure Bousquet-Mélou et al. introduce *ascent sequences*, sequences of the form $b_1 b_2 \dots b_n$ defined recursively with $b_1 = 0$ and $b_{i+1} \in [0, \operatorname{asc}(b_1 b_2 \dots b_i) + 1]$ where asc counts the total number of ascents contained within the sequence. Ascent sequences are used to encode the construction of interval orders via insertions of new maximal elements. Bousquet-Mélou et al. present a case analysis to determine the predecessor set of the ith inserted element which is both dependent on the value b_i and elements previously inserted. Ascent sequences are additionally used to construct non-neighbor nesting matchings and permutaions avoiding σ, thus giving bijective correspondences between the structures.

In studying non-neighbor nesting matchings Zagier gave an asymptoic forumla for their number f_n.

$$f_n \sim n!\frac{12\sqrt{3}}{\pi^{\frac{5}{2}}}e^{\frac{\pi^2}{12}}\left(\frac{6}{\pi^2}\right)^n\sqrt{n}.$$

We refer to f_n as the *Fishburn numbers*.

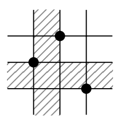
Taking advantage of the equivalent definition for interval orders, that predecessor sets can be given a total order under inclusion, Dukes, Jelínek and Kubitzke [5] provide an intuitive relation between interval orders and their generating function. They show a simple bijection between interval orders and the integer matrices of Dukes and Parviainen [6]. An integer matrix of size n is defined as an $m \times m$, upper triangular, non-row and non-column empty matrix with integer entries summing to n. Dukes, Jelínek and Kubitzke show that two elements are related in an interval order if they share a hook under the diagonal of the integer matrix.

Claesson and Linusson [4] introduce the $n!$ matchings, a subset of matchings with no left-nestings enumerated by $n!$ according to semi-length. Levande [10], solving a conjecture of Claesson and Linusson, finds an additional subset of non left-nesting matchings counted by the Fishburn numbers.

Eriksen and Sjöstrand [7] provide bijections between various Fishburn structures enumerated by $n!$—including non-left matchings and permutations—and a class of filled partition shapes. In doing so they find the full Fishburn distribution for these structures, differing from previous work which had focused solely on avoidance.

Original Fishburn permutation

We lead with a previously studied example. Recall the mesh pattern

$\sigma =$  ,

with avoidance originally given by Bousquet-Mélou et al. [2] and the full distribution given by Eriksen and Sjöstrand [7].

In their paper Eriksen and Sjöstrand show a bijection between permutations and filled partition shapes by using the filled entries in the partition shapes to encode the insertion of elements into an ordered list of blocks. Upon completion the block structure is dropped and the elements read left-to-right return the permutation. Their bijection allows that multiple statistics are equidistributed between the two structures and through this they provide the non-commutative generating function with respect to those statistics.

In this section we shall focus on a small part of their work by considering the distribution of occurrences of σ in isolation from other statistics. This differs from the work of Eriksen and Sjöstrand in that the proof is based on insertion of entries into a permutation rather than encoding the construction. Our application of the sieve principle is the same.

We begin with the fact that the number of inversions in permutations follow the Mahonian distribution. To construct a permutation of size n with i marked occurrences of σ take a permutation of size $n-i$ with i marked inversions. Each marked inversion will be used to insert a new entry which is the first entry of an occurrence of σ. The sieve principle will be then be applied to return those permutations strictly satisfying that *all* occurrences of σ are marked.

Define an order on inversions based on the position in the permutation of the first entry in the tuple and value of the second entry in the tuple. For $a_1a_2\ldots a_n$ a permutation let (a_i, a_j) and $(a_{i'}, a_{j'})$ be two, non-equal inversions. If $i = i'$ it follows $a_i = a_{i'}$ and without loss of generality we can assume $a_j < a_{j'}$. We then define

$$(a_i, a_j) < (a_{i'}, a_{j'}).$$

Otherwise $i \neq i'$ then without loss of generality assume $i < i'$. Then we define

$$(a_i, a_j) < (a_{i'}, a_{j'}).$$

As an example, in the permutation 246531 the following inversions are sorted

$$(4, 1) < (6, 1) < (6, 5).$$

According to the above order each inversion (a_j, a_k) is used to insert a new entry into the permutation. Taking the position before the leftmost entry to be position 0, increment all $a_i > a_k$ by one and insert $a_k + 1$ at position $j - 1$. Thus an occurrence of σ is created.

Example 1. Take the permutation 246531 where we consider the following inversions to be marked

$$(4, 1) < (6, 1) < (6, 5).$$

As the values in the inversions change, at each step the next inversion to be used will be colored in red. Inserted entries will be marked blue.

The inversion $(4, 1)$ is the first inversion under our defined order. Increase all entries greater than 1 by 1

$$357641,$$

and insert 2 at position 1

$$3257641.$$

The next inversion is now labeled $(7, 1)$ with the 7 at position 4. Increase all entries greater than 1 and insert 2 at position 3

$$43628751.$$

Applying the process to the final inversion, now labeled $(8,7)$, leads to the permutation

$$436289751.$$

Note that the inserted entries (marked blue) are all the first entries in an occurrence of σ.

Proposition 2. *The above procedure describes a bijection between permutations with marked inversions marked and permutations with the first entry in an occurrence of σ marked.*

Proof. To show that this mapping is well defined we need to demonstrate that at each step the insertion of an entry does not remove an occurrence of σ previously inserted by this process.

This is enforced by the ordering defined on inversions. Let (a_i, a_j) and $(a_{i'}, a_{j'})$ be inversions.

(1) If $i = i'$ then $a_j < a_{j'}$ and therefore $(a_i, a_j) < (a_{i'}, a_{j'})$. Our insertion process gives that $a_{j'} + 1$ is inserted in the position immediately following that of a_j and thus forming an ascent with a_j. Furthermore as $a_{j'} > a_j$, the minimal entry in the occurrence of σ that a_j is contained in is not incremented. Therefore the occurrence is preserved with the newly inserted entry $a_{j'}$ taking the role of the largest entry in the occurrence.
(2) If $i < i'$ then $(a_i, a_j) < (a_{i'}, a_{j'})$. As $a_{j'}+1$ is inserted further to the right in the permutation the ascent that a_i involved cannot be broken. If $a_j < a_{j'}$ the minimal entry in the occurrence of σ remains unchanged. If $a_j > a_{j'}$ then all entries in the occurrence of σ containing a_j are incremented. If $a_j = a_{j'}$ then $a_{j'}$ replaces the minimal entry of the occurrence of σ containing a_j.

Thus the mapping is well defined. To show that the mapping is a bijection we demonstrate that it is both injective and surjective.

Injectivity is enforced by the total order on inversions and that an inversion pair uniquely determines the entry which is inserted.

For surjectivity note that the process we have defined inserts the first entries of marked occurrences of σ in a left-to-right order within the permutation. We can consider the reverse of the insertion operation taking a permutation with marked occurrences of σ to a permutation with marked inversions.

Given a permutation of size n with marked occurrences of σ, take the rightmost marked occurrence. Removing the first entry contained in the occurrence and standardizing the permutation leaves a permutation of size $n-1$ with a marked inversion. Surjectivity follows from repeated application. □

Corollary 3. *Permutations with marked occurrences of σ are given by the ordinary generating function*

$$u(x,z) = \sum_{n\geq 0} \boldsymbol{(n)_{xz+1}!} x^n.$$

Proof. Permutations with respect to inversions are enumerated by the q-factorial. Under the above process an inversion is either marked, in which case a new entry uniquely specifying a marked occurrence of σ is inserted, or it is unmarked. This is equivalent to the substitution $(xz+1)$ in place of q in the q-factorial with the marking of the occurrence of σ denoted by z. □

Recreating Eriksen and Sjöstrand's result we now apply the sieve principle to permutations with *subsets* of σ marked returning those with *all* σ marked. For more details see Wilf [12, Chapter 4, Section 2].

Corollary 4. *Permutations with respect to occurrences of σ are given by the ordinary generating function*

$$\sum_{n\geq 0} \boldsymbol{(n)_{x(y-1)+1}!} x^n.$$

Proof. The previous corollary gives that permutations with respect to marked occurrences of σ are given by the ordinary generating function

$$u(x,z) = \sum_{n\geq 0} \boldsymbol{(n)_{xz+1}!} x^n.$$

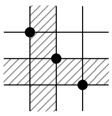
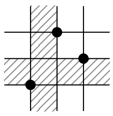
In this set a permutation with k marked occurrences of σ occurs a total of $\binom{k}{i}$ times with i occurrences of σ marked.

Let $f(x,y)$ be the ordinary generating function for permutations with *all* occurrences of σ marked. Consider the substitution of y by $z+1$. This corresponds to remarking occurrences of σ with a z, or unmarking them them with the 1. As such each permutation will occur $\binom{k}{i}$ times with i occurrences of σ now marked by z. We then have that

$$u(x,z) = f(x, z+1).$$

The result then follows through the reverse substitution of z by $y-1$ into $u(x,z)$. $\square$

Remark 5. The distributions of

 and 

given by Eriksen and Sjöstrand [7] can be shown in a near identical manner. Again the key is to note that in occurrences of these patterns each has a point whose value and position are uniquely determined by the other points and that together these other two points form an inversion.

Technique

We can generalize the previous two corollaries to explicitly state a new technique for constructing Fishburn structures. We present it as the following theorem.

Theorem 6. *Let $\mathcal{F}$ be a Mahonian stucture according to the distribution of some q-feature.*

$\mathcal{F}$ follows the Fishburn distribution with respect to some feature p if we can show there is a bijection between $\mathcal{F}$ structures of size n with i marked q-features and $\mathcal{F}$ structures of size $n+i$ with i marked p-features.

Proof. By definition, the distribution of q-features in $\mathcal{F}$ follows the ordinary generating function

$$\sum_{n\geq 0} \boldsymbol{(n)_q!} x^n.$$

Take $\mathcal{F}$ with subsets of q-features marked by some variable w. As a q-feature is either marked or it is not then the generating function for such structures is given by the substitution of q by $w+1$ into the previous equation. We therefore have

$$\sum_{n\geq 0} \boldsymbol{(n)_{w+1}!} x^n.$$

We now use that there exists a bijection between $\mathcal{F}$ structures of size n with i marked q-features and $\mathcal{F}$ structures of size $n+i$ with i marked p-features. This allows that subsets of q-features marked with w can be taken to subsets of p-features marked by z with the inclusion of an additional element. In terms of generating function this corresponds to the substitution of w by xz.

Therefore the ordinary generating function of $\mathcal{F}$ structures with subsets of marked p-features is

$$\sum_{n\geq 0} \boldsymbol{(n)_{xz+1}!} x^n.$$

If subsets of p-features are marked, then each $\mathcal{F}$ structure occurs $\binom{i}{j}$ times with j marked p-features. By the sieve principle (see Wilf [12, Chapter 4, Section 2]), as in the previous corollary, through the substitution of z by $y-1$ it then follows that $\mathcal{F}$ structures with *all* p-features marked are given by the ordinary generating function for the Fishburn distribution

$$\sum_{n\geq 0} \boldsymbol{(n)_{x(y-1)+1}!} x^n.$$

□

Zero alignment matchings

In this section we apply Theorem 6 to identify a new Fishburn statistic on a subset of matchings. Explicitly, matchings with zero alignments follow the Fishburn distribution according to the number of arcs which are both left-nesting and right-crossed.

Recall that two arcs (i,j) and (k,ℓ) are an alignment if $i<j<k<\ell$.

A matching with no alignments (a zero alignment matching) is equivalently characterized as one where *all* the openers in the diagrammatic representation occur before *all* the closers.

Proposition 7. *There are $n!$ zero alignment matchings of semi-length n.*

Furthermore they are enumerated by $\boldsymbol{(n)!}$ when refined according to the number of nestings.

Proof. This is easiest seen via recursion with a bijection between matchings with no alignments and inversion tables. Take the empty matching and the empty inversion table to be in bijection.

Let $b_1b_2\dots b_n$ be an inversion table with each $b_i \in [0, i-1]$ and M the matching constructed from $b_1b_2\dots b_{n-1}$. Label the position to the left of the first closer as 0 and label the positions to the left of an opener right-to-left from 1 to $n-1$. Insert a new arc into M with opener at position b_n and closer at the rightmost position in the matching.

By construction inserted openers occur to the left of all the closers and it is easy to see that entries in the inversion table correspond to the number of nested arcs. □

Recall that a left-nesting arc is an arc (i, j) such that there exists an arc $(i + 1, \ell)$ with $\ell < j$. Recall also that (i, j) is right-crossed if there exists an arc $(k, j-1)$ with $k < i$. We shall call an arc which is both left-nesting and right-crossed a *confused* arc.

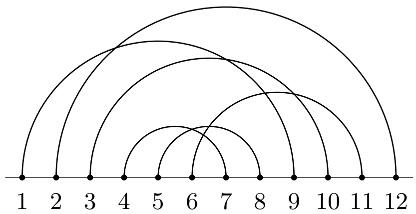
Define an order on embraced nested openers. We shall write embraced nested openers as ordered pairs. Take $((i, j), k)$ and $((i', j'), k')$ where k and k' are closers with (i, j) an arc embracing k and (i', j') an arc embracing k'. If $k = k'$ then, without loss of generality, assume $j < j'$ and define

$$((i', j'), k') < ((i, j), k).$$

Otherwise, without loss of generality, assume $k < k'$ then define

$$((i', j'), k') < ((i, j), k).$$

For example, take the matching $\{(1, 9), (2, 12), (3, 10), (4, 7), (5, 8), (6, 11)\}$.

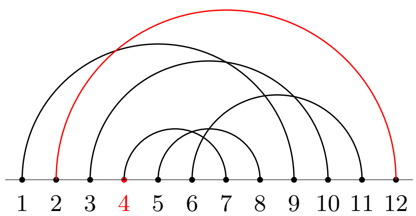


The following subset of embraced nested openers are sorted:

$$((2, 12), 4) \quad < \quad ((1, 9), 4) \quad < \quad ((2, 12), 3).$$

Given a matching with a subset of embraced openers marked, using the above order, for each embraced nested opener $((i, j), k)$ insert a new arc opening immediately to the left of the embraced nested opener k and closing immediately to the right of arc closer j. As $i < k$ the new arc is therefore right-crossed, furthermore as the arc with opener k is nested by (i, j) it follows that the newly inserted arc left nests the arc with opener k. As both right-crossed and left-nesting the inserted arc is confused.

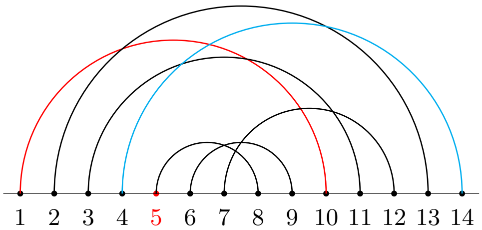
Example 8. We demonstrate on our example matching.



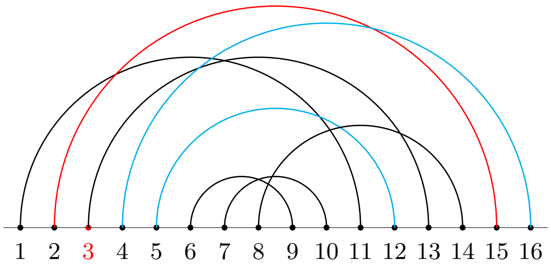
Consider the following nested openers marked.

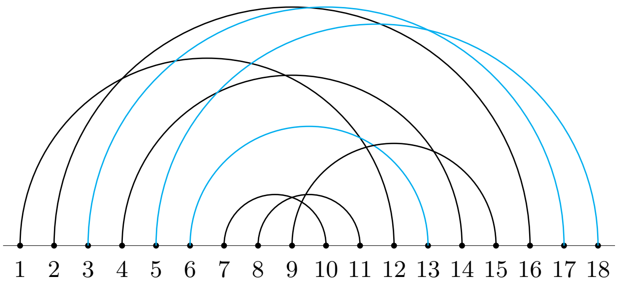
$$((2, 12), 4) \quad ((1, 9), 4) \quad ((2, 12), 3)$$

As in the example for permutations the next nested opener to be considered will be colored red and inserted arcs blue. Inserting a confused arc from the first embraced nested opener results in the following matching.



The next two steps are as follows.





It is easily checked that the inserted arcs are confused and their removal returns the original matching.

Proposition 9. *The above is a bijection between zero alignment matchings with marked embraced openers and zero alignment matchings with marked confused arcs.*

Proof. We are required to show that at each stage the process is well defined: that no alignments are introduced and and that no previously inserted confused arc has its left nesting or right crossed attributes removed.

That no alignment is introduced can be seen by contradiction. As the inserted arc has its opener to the left of an existing opener and its closer to the right of an existing closer no new alignment can be introduced if the original matching was a zero alignment matching.

That each step of the process does not break the right-crossed or left-nesting property of a previously inserted arc is given by the order on nested openers. If two inserted arcs share the same opener as part of their nested opener, then that both arcs are still left nesting is given by the order on nesting arc closers. If two inserted arcs share the same nesting arc closer as part of their nested opener, then that both arcs are right nesting is given by the order of the opener.

Each inserted arc has its opener and closer uniquely determined by the nested opener. Furthermore it is clear that removing inserted arcs returns the original matching. Injectivity and surjectivy are thus simple. □

The following corollary then results from Theorem 6 and the above Proposition.

Corollary 10. *Zero alignment matchings with respect to confused arcs follow the Fishburn distribution.*

Factorial posets

Identifying appropriate statistics and applying the technique given by Theorem 6 to the factorial posets of Claesson and Linusson [4] allows for a new method for the enumeration of interval orders which differs from both the recursive construction of Bousquet-Mélou et al. [2] and the matrix hook bijection of Dukes, Jelínek and Kubitzke [5].

Claesson and Linusson [4] define the *factorial posets*, a set of labeled interval orders counted by $n!$, as follows. A factorial poset P on some linearly ordered underlying set U is a naturally labeled poset with the additional condition that, for $i, j, k \in U$,

$$i <_U j <_P k \implies i <_P k.$$

This is referred to as the *factorial condition*.

Easily seen to be equivalent, a poset is factorial if and only if for all $k \in P$ there exists $j \in [0, k-1]$ such that $\operatorname{Pred} k = [1, j]$. As the predecessor sets can be linearly ordered by inclusion it follows that factorial posets are interval orders.

Claesson and Linusson take advantage of this by using entries of an inversion table to encode the construction of a factorial poset, thus giving that the two structures are in bijection. We include their result for completeness.

Theorem 11 (Claesson and Linusson [4])**.** *Factorial posets on* $[n]$ *are in bijection with inversion tables of length* n*.*

Proof. As a poset is factorial if and only if for all $k \in P$ there exists $j \in [0, k-1]$ such that $\operatorname{Pred} k = [1, j]$. An inversion table $b_1 b_2 \ldots b_n$ is given by setting b_k to the value $j \in [0, k-1]$. ☐

Claesson and Linusson identify numerous statistics preserved by their bijection. In particular that the number of *incomparable pairs* in factorial posets, defined as

$$|\{(i, j) \in P \times P : i \not<_P j, i <_U j\}|,$$

are enumerated by the q-factorial.

Taking two factorial posets to be equivalent if they are structurally isomorphic, Claesson and Linusson demonstrate that posets satisfying that for all $i \in [n-1]$

$$\operatorname{pred} i \leq \operatorname{pred}(i+1) \quad \text{or} \quad \operatorname{succ} i > \operatorname{succ}(i+1)$$

are unique representatives of their class.

Again we include their result.

Proposition 12 (Claesson and Linusson [4])**.** *There is exactly one way to label a* $(2+2)$*-free poset such that it satisfies*

$$\operatorname{pred} i \leq \operatorname{pred}(i+1) \quad \text{or} \quad \operatorname{succ} i > \operatorname{succ}(i+1).$$

Proof. A poset satisfying the above condition has that for all $i \in [n]$ the pairs

$$(\operatorname{succ} i, \operatorname{pred} i)$$

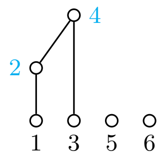
are weakly decreasing on the first coordinate and weakly increasing on the second. The factorial condition gives that for $i, j \in [n]$ the pairs $(\operatorname{succ} i, \operatorname{pred} i)$ and (succ j, pred j) are equal if an only if the pairs are indistinguishable within the poset, thus giving a canonical labeling. □

We extend this notion to consider a new feature on factorial posets, explicitly elements which *fail* to satisfy this property.

Definition 13 (Mislabeling)**.** Define a mislabeling in a factorial poset on $[n]$ to be an $i \in [n-1]$ such that

$$\operatorname{pred} i > \operatorname{pred}(i+1) \quad \text{and} \quad \operatorname{succ} i \leq \operatorname{succ}(i+1).$$

Example 14. The poset

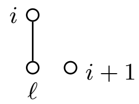


Has set of mislabelings $\{2, 4\}$.

By definition a factorial poset with zero mislabelings satisfies the condition from Proposition 12 and is thus a unique representative of its isomorphism class.

A consequence of the factorial condition is that if $\operatorname{pred} i > \operatorname{pred}(i+1)$ then i and $i+1$ are incomparable as if $i <_P i+1$ then the factorial condition requires that for all $\ell <_U i \implies \ell <_P i+1$. Furthermore there must exist ℓ such that $\ell <_P i$ but that $\ell \not<_P i+1$.

Therefore an equivalent condition to $\operatorname{pred} i > \operatorname{pred}(i+1)$ is that there exists an induced subposet isomorphic to $(2+1)$ with the following labeling



Sieved enumeration of interval orders. Recall that we write incomparable pairs as (i, j) with $i <_U j$ and let U be some linearly ordered set with $|U| = n$. For some $k \in [0, n-1]$ take a poset P built on the first $n-k$ elements of U with k marked incomparable pairs.

Define an order on incomparable pairs. Let (i, j) and (i', j') be two pairs. If $j = j'$ without loss of generality assume $i <_U i'$. Then we define

$$(i', j') < (i, j).$$

Otherwise $j \neq j'$ then without loss of generality assume $j <_U j'$. Then we define

$$(i, j) < (i', j').$$

To illustrate, the following pairs are sorted according to the above order.

$$(2, 3) < (1, 3) < (4, 6) < (3, 6).$$

In this order, each pair (i, j) is then used to insert a new element into the poset. This new element has predecessor set

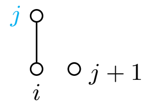
$$\{h \in P : h \leq_U i\},$$

and successor set

$$\operatorname{Succ} j.$$

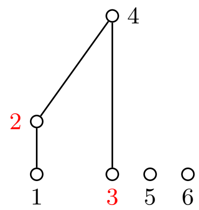
Increment all $k \in P$ with $k \geq_U j$ to its immediate successor in U, giving the newly inserted element the value j.

By definition this introduces an occurrence of



into the new poset with the inserted element marked in blue. Furthermore as the successor set of the inserted element j is equal to that of the successor set of the element now labeled $j+1$ it follows that the newly inserted element with label j is a mislabeling.

Example 15. Consider our earlier factorial poset built on [6].



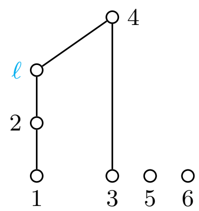
We consider the following incomparable pairs to be marked

$$(2,3) < (1,3) < (4,6) < (3,6)$$

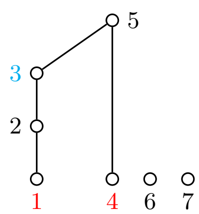
As the values within the pairs change at each stage we will denote the next incomparable pair to be used in red. Inserted elements will be colored blue.

The pair $(2,3)$ specifies the new element ℓ to be inserted defined by

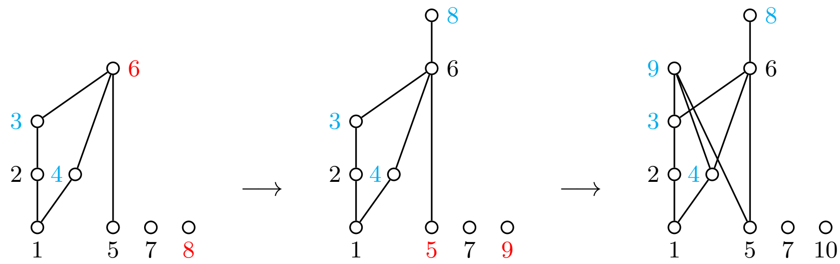
$$\operatorname{Pred} \ell = \{h \in U : h \leq_U 2\} = \{1,2\} \quad \text{and} \quad \operatorname{Succ} \ell = \operatorname{Succ} 3 = \{4\}.$$



All elements with label greater or equal to 3 are incremented by one and the newly inserted element ℓ is given the label 3.



The remaining steps are as follows.



Thus we have returned a poset of size 10 with set of marked mislabelings $\{3, 4, 8, 9\}$.

Proposition 16. *The procedure described above gives a bijection between factorial posets with marked incomparable pairs and factorial posets with marked mislabelings.*

Proof. We shall first show that the process described above is well defined. This is equivalent to showing that at each insertion the following properties are preserved for the resulting poset: it is naturally labeled, it satisfies the factorial condition and it does not remove any mislabelings previously inserted by the process.

For the incomparable pair (i, j) and newly inserted element ℓ we have that ℓ covers all $\{h \in P : h <_U i\}$ and thus by construction it is both naturally labeled and satisfies the factorial condition. Elements smaller than j under U remain unchanged. The newly inserted element is given the same successor set as the element which previously had that label and thus the insertion does not break the factorial condition for any element larger under U than j and also ensures that the naturally labeled property is preserved.

That no previously inserted mislabelings are removed by the process is given by the order on incomparable pairs and the factorial property, thus the process is a mapping between factorial posets with marked incomparable pairs to factorial posets with marked mislabelings.

Next we show that the mapping is bijective. That it is injective follows from the total order defined on incomparable pairs and that the insertion of a new element is uniquely determined by an incomparable pair.

It remains to show surjectivity. The process we have defined inserts mislabelings in order according to U. We can consider the reverse of the insertion operation taking a factorial poset with marked mislabelings to a factorial poset with marked incomparable pairs.

Given a factorial poset of size n with k marked mislabelings take the mislabeling with the largest value j and remove it from the poset. As j is a mislabeling there exists $\ell <_P j$ such that $\ell \not< j+1$. Take the largest such ℓ and mark the incomparable pair consisting of $(\ell, j+1)$. Thus we have returned a poset of size $n-1$ with $i-1$ mislabelings and 1 marked incomparable pair. Surjectivity follows from repeated application.

As it is both surjective and injective the mapping is a bijection. □

The following corollary then results from Theorem 6 and Proposition 16.

Corollary 17. *Factorial posets follow the Fishburn distribution according to the number of mislabelings.*

$$\begin{array}{c} 1 \\ 2 \\ 6,\ 1 \\ 24,\ 9 \\ 120,\ 72,\ 5 \\ 720,\ 600,\ 98,\ 1 \\ 5040,\ 5400,\ 1450,\ 76 \\ 40320,\ 52920,\ 20100,\ 2200,\ 35 \\ 362880,\ 564480,\ 279300,\ 48750,\ 2299,\ 9 \\ \dots \end{array}$$

FIGURE 2. Unsieved Fishburn distribution: number of structures of size n with i marked p-features, $u_{n,i}$

Substitution of $y = 0$ into the ordinary generating function for the Fishburn distribution,

$$\sum_{n \geq 0} \boldsymbol{(n)_{x(y-1)+1}!} x^n|_{y=0},$$

returns the ordinary generating function for factorial posets with no mislabelings. Proposition 12 gives that such posets are unique representatives of their isomorphism class thus yielding, as expected, the result of Bousquet-Mélou et al. [2] that the generating function for unlabeled interval orders is given by

$$\sum_{n \geq 0} \boldsymbol{(n)_{-x+1}!} x^n.$$

FISHBURN DISTRIBUTION

We obtain the following corollaries concerning the Fishburn distribution from Theorem 6 and its proof.

Corollary 18. *For some appropriate structure let p be a feature which follows the Fishburn distribution and q a feature which follows the Mahonian distribution.*

Letting $u_{n,i}$ denote the number of structures of size n with i marked p-features,

$$\sum_{n \geq 0} \sum_{i \geq 0} u_{n,i} x^n z^i = \sum_{n \geq 0} \boldsymbol{(n)_{xz+1}!} x^n,$$

we have that

$$u_{n,i} = \sum_{j=i}^{\binom{n-i}{2}} \binom{j}{i} m_{n-i,j}.$$

The first few terms of $u_{n,i}$ are shown in Figure 2

Proof. Theorem 6 gives that a q-factorial structure of size $n - i$ with i marked q-features can be extended to a structure of size n with i marked p-features.

For a Mahonian structure of size $n - i$ with j q-features then i are selected. The number of q-factorial structures of size $n - i$ with j q-features is given by Mahonian number $m_{n-i,j}$.

The maximum number of q-features a q-factorial structure of size $n - i$ can have is $\binom{n-i}{2}$. Thus j is bounded

$$i \leq j \leq \binom{n-i}{j}.$$

□

Remark 19. The row sums of Figure 2 (A179525), i.e.

$$\sum_{i=0} u_{n,i},$$

have previously been studied by Jelínek [9] as counting *primitive row Fishburn matrices*, upper-triangular, binary non-row empty matrices, according to the sum of the entries. Jelínek considers such matrices as part of his work on counting self-dual interval orders; he demonstrates a relation between the generating functions of self-dual interval orders enumerated by a reduced size function and primitive row Fishburn matrices.

We note that the coefficient of $x^n z^k$ in the refined formula

$$\sum_{n\geq 0} \boldsymbol{(n)_{xz+1}!} x^n$$

can be interpreted as counting the number of primitive row Fishburn matrices such that:

(1) There are a total of k entries in the matrix that are not the first to occur in their row.
(2) The entries in the matrix sum to n.

Corollary 20. *Recalling that $f_{n,k}$ denotes the coefficient in the Fishburn distribution*

$$\sum_{n\geq 0}\sum_{k\geq 0} f_{n,k} x^n y^k = \sum_{n\geq 0} \boldsymbol{(n)_{x(y-1)+1}!} x^n,$$

we have that

$$\begin{aligned} f_{n,k} &= \sum_{i=k}^{n-2} (-1)^{i+k} \binom{i}{k} u_{n,i} \\ &= \sum_{i=k}^{n-2} (-1)^{i+k} \binom{i}{k} \sum_{j=i}^{\binom{n-i}{2}} \binom{j}{i} m_{n-i,j}. \end{aligned}$$

The first few terms are shown in Figure 3

Proof. Again for some appropriate structure let p be a feature which follows the Fishburn distribution.

Recall from the proof of Theorem 6 that to take structures with subsets of p-features marked to structures with all *all* p-features marked corresponds to the substitution of $y-1$ by z.

The result then follows from the previous corollary and binomial expansion. □

Remark 21. In the above corollary the upper bound $n-2$ for the initial summation is justified by noting that an occurrence of σ can be uniquely determined by the first element in the occurrence and that two more entries must follow in the permutation. Therefore there can be no more than $n-2$ occurrences of a Fishburn statistic in a structure of size n.

This is sufficient for our purposes however we note that this is *not* the least upper bound (easily checked empirically). We leave this as an open question.

1
2
5, 1
15, 9
53, 62, 5
217, 407, 95, 1
1014, 2728, 1222, 76
5335, 19180, 13710, 2060, 35
31240, 142979, 146754, 39644, 2254, 9
...

FIGURE 3. Fishburn distribution, $f_{n,k}$

Question 22. *Is there an aesthetically pleasing expression for the least upper bound for the value of a Fishburn statistic?*

REFERENCES

[1] Kenneth P. Bogart. "An obvious proof of Fishburn's interval order theorem". In: *Discrete Mathematics* 118.1–3 (1993), pp. 239 –242 (cit. on p. 4).

[2] Mireille Bousquet-Mélou, Anders Claesson, Mark Dukes, and Sergey Kitaev. "(2+2) free Posets, Ascent Sequences and Pattern Avoiding Permutations". In: *Journal of Combinatorial Theory, Series A* 117.7 (Oct. 2010), pp. 884–909 (cit. on pp. 2, 5, 12, 16).

[3] Petter Brändén and Anders Claesson. "Mesh Patterns and the Expansion of Permutation Statistics as Sums of Permutation Patterns". In: *Electronic Journal of Combinatorics* 18.2 (2011) (cit. on p. 3).

[4] Anders Claesson and Svante Linusson. "n! matchings, n! posets". In: *Proceedings of the American Mathematical Society* 139 (2011), pp. 435–449 (cit. on pp. 2, 5, 12).

[5] Mark Dukes, Vít Jelínek, and Martina Kubitzke. "Composition matrices, (2+2)-free posets and their specializations". In: *The Electronic Journal of Combinatorics* 18.1 (2011) (cit. on pp. 5, 12).

[6] Mark Dukes and Robert Parviainen. "Ascent sequences and upper triangular matrices containing non-negative integers". In: *Electronic Journal of Combinatorics* 17 (2010) (cit. on p. 5).

[7] Niklas Eriksen and Jonas Sjöstrand. "Equidistributed statistics on matchings and permutations". In: *The Electronic Journal of Combinatorics* 21.4 (2014) (cit. on pp. 1, 5, 8).

[8] Peter C. Fishburn. "Intransitive indifference with unequal indifference intervals". In: *Journal of Mathematical Psychology* 7.1 (1970), pp. 144 –149 (cit. on p. 4).

[9] Vít Jelínek. "Counting general and self-dual interval orders". In: *Journal of Combinatorial Theory, Series A* 119.3 (2012), pp. 599 –614 (cit. on p. 17).

[10] Paul Levande. "Fishburn Diagrams, Fishburn Numbers and Their Refined Generating Functions". In: *Journal of Combinatorial Theory, Series A* 120.1 (Jan. 2013), pp. 194–217 (cit. on p. 5).

[11] Percy A. MacMahon. *Combinatory Analysis*. Dover Books on Mathematics Series. Dover Publications, 2004 (cit. on pp. 2, 4).

[12] Herbert S. Wilf. *generatingfunctionology: Third Edition*. Ak Peters Series. Taylor & Francis, 2005 (cit. on pp. 8, 9).

[13] Don Zagier. "Vassiliev invariants and a strange identity related to the Dedekind eta-function". In: *Topology* 40.5 (2001), pp. 945 –960 (cit. on p. 5).

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