Small random instances of the stable roommates problem

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Abstract

Let p_n denote the probability that a random instance of the stable roommates problem of size n admits a solution. We derive an explicit formula for p_n and compute exact values of p_n for $n \leq 12$.

1 Introduction

Matching under preferences is a topic of great practical importance, deep mathematical structure, and elegant algorithmics [1, 2]. A paradigmatic example is the stable roommates problem [3]. Consider an even number n of participants. Each of the participants ranks all the others in strict order of preference. A matching is a set of $n/2$ disjoint pairs of participants. A matching is stable if there is no pair of unmatched participants who both prefer each other to their partner in the matching. Such a pair is said to block the matching. The stable roommates problem is to find a stable matching. The name originates from the problem to assign students to the double bedroomes of a dormitory. Another application is the formation of cockpit crews from a pool of pilots.

An instance of the stable roommates problem is defined by a preference table, in which each participant ranks all other $n-1$ participants, most preferred first. For technical reasons we will assume that each participant puts himself at the very end of his preference list. Here are two examples for $n = 4$:

$$\text{(A)}\quad \begin{array}{lcccc} 1: & 4 & \mathbf{2} & 3 & 1 \\ 2: & 3 & 4 & \mathbf{1} & 2 \\ 3: & 1 & \mathbf{4} & 2 & 3 \\ 4: & \mathbf{3} & 2 & 1 & 4 \end{array} \qquad\qquad \text{(B)}\quad \begin{array}{lcccc} 1: & 3 & 2 & 4 & 1 \\ 2: & 1 & 3 & 4 & 2 \\ 3: & 2 & 1 & 4 & 3 \\ 4: & 1 & 2 & 3 & 4 \end{array} \tag{1}$$

In (A), the marked matching $(1,2)(3,4)$ is stable. In (B), there is no stable matching: whoever is matched with 4 can always form a blocking pair with someone else. Example (B) illustrates the fact that not all instances of the stable roommates problem have a solution. Let p_n denote the probabilty that a random instance, chosen uniformely from all possible instances of size n, admits a solution. Our examples shows that $0 < p_4 < 1$. The exact value is $p_4 = 26/27$. It has been computed by Pittel [4] more than 20 years ago. No other values of p_n are known exactly. Numerical simulations [5] suggest that p_n is a monotonically decreasing function of n that asymptotically decays like $n^{-1/4}$.

In this paper we derive an explicit formula for p_n that we use to compute exact values of p_n for $n \leq 12$. And we discuss a generalization of this approach for odd values of n.

2 Stable Permutations

A matching of size n can be interpreted as a permutation π of $\{1,\ldots,n\}$ that is completely composed of 2-cyles. An obvious generalization is to allow arbitrary permutations π, but for that one needs to extend the definition of stability. A permutation π is called stable if it satisfies the two following conditions:

$$\forall i : i \text{ does not prefer } \pi(i) \text{ to } \pi^{-1}(i) \tag{2a}$$

$$i \text{ prefers } j \text{ to } \pi(i) \Rightarrow j \text{ prefers } \pi(j) \text{ to } i \tag{2b}$$

This definition includes permutations with fixed points. This is the reason why we've added each participant to the very end of his own preference list. But note that (2b) rules out that a stable permutation can have more than one fixed point.

For permutations composed of 2-cycles (matchings) condition (2a) is trivially satisfied and condition (2b) reduces to the usual "no blocking pairs" condition. Condition (2a) enforces each cycle of length ≥ 3 to have a monotonic rank ordering: every member i prefers his predecessor $\pi^{-1}(i)$ to his successor $\pi(i)$, and condition (2b) prevents any member of the cycle to leave the cycle.

The significance of stable permutations for the stable roommates problem arises from the following facts, proven by Tan [6]:

1. Each instance of the stable roommates problem admits at least one stable permutation.

2. If π is a stable permutation for a roommates instance that contains a cycle $C = (v_1, v_2, \ldots, v_{2m})$ of even length, we can get two different stable permutations by replacing C by the 2-cycles $(v_1, v_2), \ldots, (v_{2m-1}, v_{2m})$ or by $(v_2, v_3), \ldots, (v_{2m}, v_1)$.

3. If C is an odd-length cycle in *one* stable permutation for a given roommates instance, then C is a cycle in *all* stable permutations for that instance.

These facts establish the *cycle type* of stable permutations as certificate for the existence of a stable matching: An instance of the stable roommates problem is solvable if and only if the instance admits a stable permutation with no odd cycles.

Consider again the two examples from the previous section. One can easily check that the permutation $(1,2,3)(4)$ is a stable permutation for (B). Since it contains the odd cycle $(1,2,3)$, (B) admits no stable matching. The permutation $(1,3,4,2)$ is stable for (A). According to fact 2, its 4-cycle can be replaced by $(1,3)\,(4,2)$ or by $(3,4)\,(1,2)$, which are in fact both stable matchings.

3 A Formula for $\mathbf{p_n}$

The facts proven by Tan allow us to derive an explicit formula for the probability p_n. The underlying ideas have already been discussed more or less in [4], but the formulas (8) and (13) haven't been published before. We start with an integral representation for $P(\pi)$, the probability that a permutation π is stable.

Proposition 3.1. *Let π be a permutation of $\{1,\ldots,n\}$ and let $F_\pi = \{i : i = \pi(i)\}$ denote the fixed points and $M_\pi = \{i : \pi(i) = \pi^{-1}(i) \neq i\}$ the elements in two cycles of π. The probability that π is a stable permutation for a random instance of the stable roommates problem is given by*

$$P(\pi) = \int_0^1 \mathrm{d}^n x \prod_{(i,j>i)\notin D_\pi} (1 - x_j x_i) \prod_{i \notin M_\pi \cup F_\pi} x_i \prod_{i \in F_\pi} \delta(x_i - 1)\,, \tag{3}$$

where integration is over the n-dimensional unit cube and

$$D_\pi = \{(i,j) : i \neq j\ ,\ i = \pi(j) \vee j = \pi(i)\} \tag{4}$$

is the set of pairs of elements that are cyclic neighbors in π.

Proof. A random instance of the stable roommates problem can be generated as follows: Introduce an $n \times (n-1)$ array of independent random variable X_{ij} $(1 \leq i \neq j \leq n)$, each uniformly distributed in $[0,1]$. Each agent i ranks the agents $j \neq i$ on his preference list in increasing order of the variables X_{ij}. Obviously, such an ordering is uniform for every i, and the orderings by different members are independent. The fact that each agent is at the very end of his hown preference list is taken into account by adding variables $X_{ii} = 1$ to the set of random variables.

Let $P(\pi|x,y)$ denote the conditional probability that the permutation π is stable given $X_{i\pi(i)} = x_i$ and $X_{i\pi^{-1}(i)} = y_i$, and let $F_\pi = \{i : i = \pi(i)\}$ and $M_\pi = \{i : \pi(i) = \pi^{-1}(i) \neq i\}$ denote the fixed points and two cycles of π. Then (2a) tells us

$$P(\pi|x,y) \propto \prod_{i \notin \mathcal{M}_\pi \cup \mathcal{F}_\pi} \Theta(x_i - y_i) \prod_{i \in M_\pi \cup F_\pi} \delta(x_i - y_i) \tag{5}$$

where Θ is the step function

$$\Theta(z) = \begin{cases} 1 & z \geq 0 \\ 0 & z < 0 \end{cases}$$

and $\delta(z)$ is the Dirac delta function.

The second condition (2b) is violated if $X_{ij} < x_i$ and $X_{ji} < x_j$ for some $(i,j) \notin D_\pi$. This happens with probability $x_i x_j$, hence

$$P(\pi|x,y) \propto \prod_{(i,j>i) \notin D_\pi} (1 - x_j x_i)\,, \tag{6}$$

which does not depend on y.

Integrating (5) over y_i gives a factor x_i if i is an element of cycle of length three or more, a factor 1 otherwise. Adding the product $\prod_{i \in F_\pi} \delta(x_i - 1)$ to ensure the constraints $X_{ii} = 1$ finally allows us to integrate over the x_i's to obtain (3). □

Note that (3) differs slightly from the integral representation in [4]: Our integral is valid for any permutation π. If π contains more than one fixed point, the integrand vanishes since the δ-function forces at least one of the factors in the product $\prod(1 - x_i x_j)$ to be zero and $P(\pi) = 0$ as it should.

Obviously $P(\pi)$ depends on π only through the cycle type of π. Let a_k denote the number of cycles of length k in π. We use the notation $\mathbf{a} = [1^{a_1}, 2^{a_2}, \ldots]$ to denote the cycle type, including only those terms with $a_k > 0$. For $n = 4$, the only non-zero integrals are

$$P([2^2]) = \int_0^1 \mathrm{d}^4x\,(1 - x_1 x_3)\,(1 - x_1 x_4)\,(1 - x_2 x_3)\,(1 - x_2 x_4) = \frac{233}{648} \tag{7a}$$

$$P([4^1]) = \int_0^1 \mathrm{d}^4x\,(1 - x_1 x_3)\,(1 - x_2 x_4)\,x_1 x_2 x_3 x_4 = \frac{25}{1296} \tag{7b}$$

$$P([1^1\,3^1]) = \int_0^1 \mathrm{d}^3x\,(1 - x_1)(1 - x_2)(1 - x_3) x_1 x_2 x_3 = \frac{1}{216}\,. \tag{7c}$$

Note that in the last integral, we have already done the trivial integration over $\delta(x_4 - 1)$.

Proposition 3.2. *Let p_n (n even) be the probability that a random instance of the stable roommates problem has a solution. Then*

$$p_n = \sum_{\mathbf{a}\in\mathcal{E}_n} (-1)^{e(\mathbf{a})} c(\mathbf{a}) P(\mathbf{a}) \,, \tag{8}$$

where $\mathcal{E}_n$ is the set of all cycle types of size n with even cycles only. The exponent $e(\mathbf{a})$ is the number of even cycles of length ≥ 4 in $\mathbf{a}$, $e(\mathbf{a}) = \sum_{k=4,6,\ldots} a_k$. The factor $c(\mathbf{a})$ is the number of permutations with cycle type $\mathbf{a}$,

$$c(\mathbf{a}) = \frac{n!}{\prod_k a_k!\, k^{a_k}} \,. \tag{9}$$

Proof. A matching of size n has cycle structure $\mathbf{a} = [2^{n/2}]$, and there are $(n-1)!!$ matchings of size n. Boole's inequality (aka union bound) then tells us that

$$p_n \leq (n-1)!!\, P([2^{n/2}]) \,, \tag{10}$$

where equality holds if and only if the stability of different matchings were independent. This is not true in our case. Fact 2 from above tells us that stable matchings may come in pairs. Every stable permutation that consists of exactly one even length cycle of size $z \geq 4$ and $(n-z)/2$ cycles of size 2 corresponds to two stable matchings. These pairs have been counted twice in the sum in (10). The number of permutations of cycle type $[2^{(n-z)/2}\, z^1]$ is $n!\,\big((n-z)!!\,z\big)^{-1}$ and we get

$$p_n \geq (n-1)!!\, P([2^{n/2}]) - \sum_{z=4,6,\ldots}^{n} \frac{n!}{(n-z)!!\, z} P([2^{(n-z)/2}\, z^1]) \,. \tag{11}$$

The $\geq$ is again a consequence of Boole's inequality. Equality in (11) would only hold if the stability of pairs of permutations were independent events, but we know from fact 2 that stable pairs again may come in pairs: we have a quartet of stable permutation for each permutation that is composed of precisely two cycles of length ≥ 4 and 2-cycles. Again we can express the corrections by $P([])$ and a combinatorial prefactor. Iterating this reasoning (which is of course the well known inclusion-exclusion principle) yields (8).

The formula (9) for the number of permutations of a given cycle type is well known. Yet we will give a short proof for completeness. Write down the cycle structure in terms of a_k pairs of parentheses enclosing k dots, like

$$(\cdots)(\cdots)(\cdot\cdot)(\cdot) \tag{12}$$

for $n = 9$ and $\mathbf{a} = [1^1,\, 2^1,\, 3^2]$. Now imagine that the n dots are replaced left to right with a permutation of $\{1, \ldots, n\}$. Then the parentheses induces the desired cycle structure on this permutation. There are $n!$ permutations, but some of them result in the same "cycled" permutations. First, a cycle of length k can have k different leftmost values in $(\cdots)$, which gives a factor k^{a_k} of overcounting. And pairs of parentheses that hold the same number of dots can be arranged in any order, which gives a factor $a_k!$ of overcounting. This yields (9). $\square$

Corollary 3.3. *Let $\mathcal{O}_n$ denote the set of all cycle types of size n that contain at most one fixed point and at least one odd cycle. Then*

$$1 - p_n = \sum_{\mathbf{a}\in\mathcal{O}_n} (-1)^{e(\mathbf{a})} c(\mathbf{a}) P(\mathbf{a}) \,, \tag{13}$$

Proof. Since $P(\mathbf{a}) = 0$ if $\mathbf{a}$ has more than one fixed point, we can extend the sum to run over all cycle types with at least one odd cycle. Then the right hand side of (13) is the probability that a random instance of the stable roommates problem has a stable permutation with at least one odd cycle. But this equals the probability that a random instance of the stable roommates problem has no solution. $\square$

4 Evaluation of p_n

We already know the values of the integrals $P(\mathbf{a})$ for $n = 4$, see (7). When we insert these values into (8) or (13) we get

$$\begin{aligned} p_4 &= 3\,P([2^2]) - 6\,P([4^1]) = \frac{26}{27} = 0.962962\ldots \\ 1 - p_4 &= 8\,P([1^1, 3^1]) = \frac{1}{27}\,, \end{aligned} \tag{14}$$

the value computed by Pittel in 1993 [4]. It seems straightforward to compute p_n for larger values of n, since all we need to do is to evaluate and sum the corresponding integrals $P(\mathbf{a})$. This is not easy, however. Pittel wrote "For $n = 6$, the computations by hand become considerably lengthier and we gave up after a couple of half-hearted attempts."

The computations become "lengthier" for two reasons: the number of integrals in (8) and (13) increase with n, and the evaluation of each individual integral gets harder. Let us first look at the number of integrals:

Lemma 4.1. *Let $p(n)$ denote the number of unordered partitions of n, and let n be even. Then*

$$|\mathcal{E}_n| = p\left(\frac{n}{2}\right) \tag{15a}$$

$$|\mathcal{O}_n| = p(n) - p(n-2) - p\left(\frac{n}{2}\right)\,. \tag{15b}$$

Proof. From $\sum_k k a_k = n$, or from glancing at (12), it is obvious that there is a one-to-one correspondence between the set of cycle types of size n and the set of integer partitions of n.

Every cycle type $\mathbf{a} \in \mathcal{E}_n$ corresponds to a partition of n into even numbers and vice versa. Every partition of n into even numbers corresponds to a unique partition of $n/2$ and vice versa—simply divide or mutiply all parts of the partition by two. This proves (15a).

The number of all cycle types is $p(n)$, and the number of all cycle types that contain at least two fixed points is $p(n-2)$. Hence the number of cycle types that contain at most one fixed point is $p(n) - p(n-2)$. For $|\mathcal{O}_n|$ we also need to subtract the number of cycle types with even cycles only, which is $p(n/2)$. This proves (15b). $\square$

There is no closed formula for the partition numbers $p(n)$, but they are known for all $n \leq 10\,000$ [7]. And we need $p(n)$ only for small values of n to get

n	4	6	8	10	12	14	16	18
$\lvert\mathcal{E}_n\rvert$	2	3	5	7	11	15	22	30
$\lvert\mathcal{O}_n\rvert$	1	3	6	13	24	43	74	124

In this regime of n, the number of integrals is no problem. So let us turn our attention to the individual integrals.

When we expand the product in (3), we get a sum of easy-to-integrate terms of the form $x_1^{b_1} \cdot x_n^{b_n}$, but there too many terms to be integrated by hand.

	p_4	p_6	p_8	p_{10}	p_{12}
(8):	0.20 sec.	19.8 sec.	5 min.	20 min.	15.5 days
(13):	0.02 sec.	3.5 sec.	6 min.	25 min.	13.9 days

Table 1: Times to compute p_n according (8) or (13).

Lemma 4.2. *A full expanion of the integrand in* (3) *yields* $2^{f(\mathbf{a})}$ *terms, where*

$$f(\mathbf{a}) = \frac{1}{2}n(n-3) + a_1 + a_2\,. \tag{16}$$

Proof. If we expand the integrand, each factor in the product

$$\prod_{\substack{i<j \\ (i,j)\notin D_\pi}}^{n} (1 - x_i x_j) \tag{17}$$

doubles the number of terms. Hence we need to show that (16) is the number of factors in this product. Think of the n variables x_i as the vertices of a graph G. Each factor $(1 - x_i x_j)$ in (17) corresponds to an edge of G. Without the constraint $(i, j) \notin D_\pi$, G is the complete graph with $\frac{1}{2}n(n-1)$ edges. Each cycle of length $k \geq 3$ in $\mathbf{a}$ corresponds to a cycle in G with k edges that are removed from the complete graph. Each cycle of length 2 corresponds to an edge that is also removed. This gets us

$$f(\mathbf{a}) = \frac{1}{2}n(n-1) - \sum_{k\geq 3} k a_k - a_2 = \frac{1}{2}n(n-1) - \left(\sum_k k a_k - 2a_2 - a_1\right) - a_2$$

and (16) follows from $\sum_k k a_k = n$. $\square$

The maximum number of terms arises for pure matchings, i.e., for $a_2 = n/2$ and $a_1 = 0$. It reads 2^4, 2^{12}, 2^{24}, 2^{40} and 2^{60} for $n = 4, 6, 8, 10$ and 12. Hence it is no surprise that Pittel gave up on the integrals for $n = 6$. The integration is better left to a computer.

We used the computer-algebra system Mathematica [8] for the exact evaluation of the integrals $P(\mathbf{a})$. Figure 1 shows the Mathematica code that sets up the integrand and performs the integration. The full Mathematica code is available online [9].

Using our Mathematica code, we computed the values of p_n for $n \leq 12$ both from (8) and (as a crosscheck) from (13). The results are

$$p_6 = \frac{181431847}{194400000} = 0.93329139403292181070\ldots \tag{18a}$$

$$p_8 = \frac{809419574956627}{889426440000000} = 0.91004667564933981499\ldots \tag{18b}$$

$$p_{10} = \frac{2536546575452094345792177420 7}{2846049012732144844800000000 0} = 0.89125189485653484085\ldots \tag{18c}$$

$$p_{12} = \frac{13544124829485098788469430650439043569062157071}{15469783933925839494793980316271247360000000000} = 0.87552126696367780620\ldots\,. \tag{18d}$$

The values of the corresponding individual integrals are listed in Tables 2 and 3.

We ran our Mathematica code on a computer equipped with 2 Intel® Xeon® CPUs E5-1620 with 3.60 GHz clock rate and 32 GByte of memory. The total computation times are shown in Table 1. Table 3 also shows the times to compute the individual

```
Integrand[a_] := Module[(* computes integrand corresponding to cycle pattern a *)
  {n,i,j,l,result,cycle},
  If[a[[1]]>1,result=0, (* more than one fixed point *)
    n=Sum[k*a[[k]],{k,1,Length[a]}];
    If[a[[1]]>0, (* take care of fixed point *)
      n=n-1;result=Product[(1-x[i]),{i,1,n}],
      result = 1 (* no fixed point *)
    ];
    result=result*Product[(1-x[i]*x[j]),{i,1,n-1},{j,i+1,n}];
    (* remove 2-cycles from product *)
    result=result/Product[(1-x[2*i-1]*x[2*i]),{i,1,a[[2]]}];
    (* cycles larger than 2 *)
    result=result*Product[x[i],{i,2*a[[2]]+1,n}];
    For[cycle=3,cycle<=Length[a],cycle++,
      l=Sum[i*a[[i]],{i,2,cycle-1}]+1;
      For[i=l,i<=l+cycle*(a[[cycle]]-1),i+=cycle,
        For[j=0,j<cycle,j++,result=result/(1-x[i+j]*x[i+Mod[(j+1),cycle]])]
      ]
    ]
  ];
  result
];

P[a_] := Module[
  {y,n,k},
  n=Sum[k*a[[k]],{k,1,Length[a]}];
  If[a[[1]]>0,n=n-1];
  y = Integrand[a];
  For[k=n,k>=1,k--,y=Integrate[y,{x[k],0,1}]];
  y
];
```

Figure 1: Mathematica code to compute the integrals $P(\mathbf{a})$ (3). The procedure `Integrand[a]` returns the integrand as a function of variables `x[1]`,...,`x[n]` (or `x[n-1]` if the cycle type `a` contains a fixed point), the procedure `P[a]` evaluates the integral by exactly integrating variable by variable.

$\mathbf{a}$	$P(\mathbf{a})$	$\mathbf{a}$	$P(\mathbf{a})$
$[2^3]$	$\frac{448035973}{5832000000}$	$[1^1, 2^1, 3^1]$	$\frac{38077}{86400000}$
$[2^1, 4^1]$	$\frac{307841}{144000000}$	$[1^1, 5^1]$	$\frac{26257}{777600000}$
$[6^1]$	$\frac{2591729}{11664000000}$	$[3^2]$	$\frac{1742111}{7776000000}$
$[2^4]$	$\frac{1245959394495647}{107585022182400000}$	$[1^1, 7^1]$	$\frac{49958102093}{384232222080000000}$
$[2^2, 4^1]$	$\frac{5211637894488503}{268962555456000000000}$	$[1^1, 2^2, 3^1]$	$\frac{441974732789}{12807740736000000}$
$[2^1, 6^1]$	$\frac{914248620325799}{537925110912000000000}$	$[1^1, 3^1, 4^1]$	$\frac{1249592153}{9605805552000000}$
$[4^2]$	$\frac{1493807915753}{1195389135360000000}$	$[1^1, 2^1, 5^1]$	$\frac{58105985423}{25615481472000000}$
$[8^1]$	$\frac{622186155317}{498078806400000000}$	$[2^1, 3^2]$	$\frac{76670733315619}{4482709257600000000}$
		$[3^1, 5^1]$	$\frac{58105985423}{25615481472000000}$
$[2^5]$	$\frac{4338571669164186607574318852 03}{322741958043825225400320000000000}$	$[1^1, 2^3, 3^1]$	$\frac{188269700322702515039071 9}{8196621156668577153024000000000}$
$[2^3, 4^1]$	$\frac{479469348803275157810485993 7}{322741958043825225400320000000000}$	$[1^1, 3^3]$	$\frac{158398327239405983477}{51228882229178607206400000000000}$
$[2^2, 6^1]$	$\frac{72615811763168183011218671 3}{645483916087650450800640000000000}$	$[1^1, 2^1, 3^1, 4^1]$	$\frac{276587867939346662063 3}{4098310578334288576512000000000}$
$[2^1, 4^2,]$	$\frac{940896019692712489785718 31}{1290967832175300901601280000000000}$	$[1^1, 2^2, 5^1]$	$\frac{433660294766995569476 9}{3278648462667430861209600000000}$
$[2^1, 8^1]$	$\frac{188126210428003843609396 21}{258193566435060180320256000000000}$	$[1^1, 4^1, 5^1]$	$\frac{126601947989502609349}{4098310578334288576512000000000}$
$[4^1, 6^1]$	$\frac{106782268656219441751350 83}{2581935664350601803202560000000000}$	$[1^1, 3^1, 6^1]$	$\frac{633196266619396193087}{20491552891671442882560000000000}$
$[10^1]$	$\frac{427088041880355671404433 57}{10327742657402407212810240000000000}$	$[1^1, 2^1, 7^1]$	$\frac{789921304062168675601}{1170945879524082450432000000000}$
		$[1^1, 9^1]$	$\frac{126599549126442677035 3}{40983105783342885765120000000000}$
		$[2^2, 3^2]$	$\frac{2918990176269285877130918549}{2581935664350601803202560000000000}$
		$[2^1, 3^1, 5^1]$	$\frac{188453690890826324796193 57}{258193566435060180320256000000000}$
		$[3^2, 4^1]$	$\frac{214102877135791172223668 71}{5163871328701203606405120000000000}$
		$[3^1, 7^1]$	$\frac{610894828667022260751797}{147539180820034388754432000000000}$
		$[5^2]$	$\frac{854187443629530134228140 3}{20655485314804814425620480000000000}$

Table 2: Probabilities $P(\mathbf{a})$ for $n = 6, 8, 10$ (top to bottom). Cycle types with (right) and without (left) odd cycles.

integrals for $n = 12$. Some of theses integrals (marked with a $\star$) could not be computed by the simple iterative scheme in Figure 1 because Mathematica ran out of memory. In these cases we expanded the integrand in a polynomial in the variable x_n (or x_{n-1} if there is a fixed point) and applied interative integration to each coefficient of this polynomial. This reduces the memory consumption, but it slows down the computation. With a larger memory (like 64 GByte instead of 32 GByte), this could have been avoided and p_{12} could have been computed somewhat faster.

5 Odd values of n

For odd values of n there are no stable matchings, of course. But there are still stable permutations: Tan's results listed in Section 2 as well as Proposition 3.1 also hold for odd values of n. This allows us to generalize the stable roommates problem to odd values of n. The most obvious generalization is to accept one fixed point, i.e., to reject

$\mathbf{a}$	$P(\mathbf{a})$	Time [sec.]
$[2^6]$	$\frac{3258999084948836441264401998576021937572114296 27}{2572934463890545624774134806202233860915200000000000}$	265 018
$[2^4, 4^1]$	$\frac{12091159747917346526056815633241221409634 07221}{1225206887566926487987683241048682790912000000000000}$	265 091*
$[2^2, 4^2]$	$\frac{162880721523276100530009504091642251866 50151}{428822410648424270795689134367038976819200000000000000}$	205 089*
$[4^3]$	$\frac{231703173390597300042186053017177445722753}{25729344638905456247741348062022338609152000000000000000}$	206 115*
$[2^3, 6^1]$	$\frac{33782941779419325090531728722989244869237 64591}{514586892778109124954826961240446772183040000000000000}$	49 493
$[2^1, 4^1, 6^1]$	$\frac{417880592074077264470531487240272595070729}{21441120532421213539784456718351948840960000000000000}$	49 220
$[6^2]$	$\frac{185333091274829953004403478473488053746 2353}{20583475711124364998193078449617870887321600000000000000}$	49 293
$[2^2, 8^1]$	$\frac{508893194633666952579907861671385135829521}{1340070033276325846236528544896996802560000000000000}$	47 303
$[4^1, 8^1]$	$\frac{441292524174200578516771544953621967 6971}{4900827550267705951950732964194731163648000000000000}$	47 520
$[2^1, 10^1]$	$\frac{534847302611912533616087474053948145143 57}{274446342814991533309241045994904945164288000000000000}$	53 836
$[12^1]$	$\frac{148266416698941643400769418325578083838 93}{16466780568899491998554462759694296709857280000000000000}$	53 299
$[1^1, 2^4, 3^1]$	$\frac{1225039668944721076022427373084381694 03}{92082047225749044611868930453995520000000000000}$	92 605
$[1^1, 2^1, 3^3]$	$\frac{12225438801691226225608777385 75059}{935436670229831564311049452231065600000000000}$	42 996
$[1^1, 2^2, 3^1, 4^1]$	$\frac{193959334006722457965074605079586629657}{61879135735703357979175921265084989440000000000000}$	7 038
$[1^1, 3^1, 4^2]$	$\frac{28550252007671725137350817321068 63}{5729549605157718331405177894915276800000000000}$	8 354
$[1^1, 2^3, 5^1]$	$\frac{843789529005571058535031791056610124 7317}{123758271471406715958351842530169978880000000000000}$	10 784
$[1^1, 3^2, 5^1]$	$\frac{17623003994516463168472342350120 3937}{3535950613468763313095766929433427968000000000000}$	10 498
$[1^1, 2^1, 4^1, 5^1]$	$\frac{179574886149520118907830203999929 0739}{1375091905237852399537242694779666432000000000000}$	9 092
$[1^1, 2^1, 3^1, 6^1]$	$\frac{269429158480038509775930570738484 0499}{2062637857856778599305864042169499648000000000000}$	8 314
$[1^1, 5^1, 6^1]$	$\frac{217474090499631792087688788933 4183}{436537112773921396678489744374497280000000000000}$	8 376
$[1^1, 2^2, 7^1]$	$\frac{169585272084207649246611891502862 8181}{541216930632973976494250040219402240000000000000}$	8 059
$[1^1, 4^1, 7^1]$	$\frac{880779354382117073759631138574292 59}{1767975306734381656547883464716713984000000000000}$	8 082
$[1^1, 3^1, 8^1]$	$\frac{192709920613409576198802125822368 03}{386744598348145987369849507906781184000000000000}$	8 098
$[1^1, 2^1, 9^1]$	$\frac{119714788881285340316454265465524 6797}{916727936825234933024828463186444288000000000000}$	8 096
$[1^1, 11^1]$	$\frac{369923219620277787348067489855403 3731}{7425496288284402957501110551810198732800000000000000}$	8 001
$[2^3, 3^2]$	$\frac{161499154693883709213457621140881273357670713}{245041377513385297597536648209736558182400000000000000}$	209 450
$[2^2, 3^1, 5^1]$	$\frac{434864354582543378889269461785635127582 8603}{114352642839579805545517102497877060485120000000000000}$	56 207
$[2^1, 3^2, 4^1]$	$\frac{357327697339220191285941924523831564051}{18296422854332768887282736399660329677619200000000}$	212 763*
$[3^4]$	$\frac{983682430884312065518701902481276903 0613}{108907278894837910043349621426549581414400000000000000}$	216 063
$[2^1, 3^1, 7^1]$	$\frac{910543359462850455167213506257224588746 31}{4667454809778767573286412346852124917760000000000000}$	50 230
$[3^2, 6^1]$	$\frac{618748213165565813756023362735829302961763}{68611585703747883327310261498726236291072000000000000000}$	49 614
$[2^1, 5^2]$	$\frac{267426270217559786779749455618801978444 53}{1372231714074957666546205229974524725821440000000000}$	59 920
$[3^1, 4^1, 5^1]$	$\frac{618297201305342041531210311850338468155 23}{6861158570374788332731026149872623629107200000000000}$	59 587
$[3^1, 9^1]$	$\frac{206087452177563327772393466331282655631 23}{22870528567915961109103420499575412097024000000000000}$	53 642
$[5^1, 7^1]$	$\frac{176508223822125299754789495189389335334 41}{19603310201070823807802931856778924654592000000000000}$	53 495

Table 3: Probabilities $P(\mathbf{a})$ for $n = 12$ and the times to compute them. Times marked with $\star$ refer to a slower, more memory efficient integration procedure (see text).

one participant from the dormitory (or put him into a single bedroom), and to ask for a stable matching of the remaining $n-1$ participants. Let p_n (for n odd) denote the probability that a random instance admits such a solution. Following the same reasoning as in Proposotion 3.2 and Corollary 3.3, we get

$$p_n = \sum_{\mathbf{a}\in\mathcal{E}_n^1} (-1)^{e(\mathbf{a})} c(\mathbf{a}) P(\mathbf{a})\,, \tag{19}$$

$$1 - p_n = \sum_{\mathbf{a}\in\mathcal{O}_n^3} (-1)^{e(\mathbf{a})} c(\mathbf{a}) P(\mathbf{a})\,, \tag{20}$$

where $\mathcal{E}_n^1$ is the set of all cycle types of size n consisting of one fixed point and even cycles and $\mathcal{O}_n^3$ is the set of all cycle types of size n that contain at least one cycle of odd length ≥ 3. Table 4 lists the values of the corresponding integrals $P(\mathbf{a})$ for odd $n \leq 11$. The resulting values of p_n are

$$p_3 = \frac{3}{4} = 0.75 \tag{21a}$$

$$p_5 = \frac{4075}{6912} = 0.5895543981481481\ldots \tag{21b}$$

$$p_7 = \frac{246462083}{518400000} = 0.4754284008487654\ldots \tag{21c}$$

$$p_9 = \frac{11365049284140796201}{29144725585920000000} = 0.38995218021992023\ldots \tag{21d}$$

$$p_{11} = \frac{176967745750762518431538515329}{546441410444571810201600000000} = 0.3238549318705289\ldots \tag{21e}$$

It seems counterintuitive that $p_{2k-1} < p_{2k}$, but note that the enforced fixed-point for an odd number of participants represents someone who is happy to be matched with anybody else. This high destabilizing potential is a result of the rule that every participant has to put himself at the very end of his preference list.

6 Conclusions and Outlook

We have seen that p_n, the probabilty of a random instance of the stable roommmates problem of size n to admit a solution, can be expressed as a sum over cycle types of permutations of size n. Each term in the sum is an integral with an exponential number of terms. The latter restricts an exact evaluation of p_n to $n \leq 12$. In spite of this limitation, the method is far more efficient than the exhaustive enumeration over the $[(n-1)!]^{n-1}$ different instances of size n. For $n = 12$, this number is 4.1×10^{83}, or 4100 times the number of atoms in the visible universe (which is usually estimated as 10^{80}).

Our results for $n \leq 12$ don't shed new light on the ultimate behavior of p_n as n becomes large, but they suggest that exact evaluation of p_n for any larger values of n is likely to be infeasible without some unexpected new approach.

The approach outlined in this paper can easily be modified to work for the stable matching problem on general graphs, where each participant corresponds to a vertex of a graph G and ranks only those participants adjacent to him in G. If G is the complete graph, we recover the stable roommates problem. In the case of bipartite graphs G (known as stable marriage problem) we have $p_n = 1$. For non-bipartite graphs, p_n seems to be a monotonically decreasing function of n that may or may not approach a non-zero value, depending on the number of short cycles in G [10].

$\mathbf{a}$	$P(\mathbf{a})$	$\mathbf{a}$	$P(\mathbf{a})$
$[1^1, 2^1]$	$\frac{1}{4}$	$[3^1]$	$\frac{1}{8}$
$[1^1, 2^2]$	$\frac{833}{20736}$	$[2^1, 3^1]$	$\frac{491}{27648}$
$[1^1, 4^1]$	$\frac{1}{2304}$	$[5^1]$	$\frac{191}{82944}$
$[1^1, 2^3]$	$\frac{110831617}{23328000000}$	$[2^2, 3^1]$	$\frac{5103637}{2592000000}$
$[1^1, 2^1, 4^1]$	$\frac{797731}{23328000000}$	$[2^1, 5^1]$	$\frac{1945639}{9331200000}$
$[1^1, 6^1]$	$\frac{6541}{2916000000}$	$[3^1, 4^1]$	$\frac{336349}{18662400000}$
$[1^1, 3^2]$	$\frac{2183}{972000000}$	$[7^1]$	$\frac{558779}{31104000000}$
$[1^1, 2^4]$	$\frac{12242957448855683129}{27541765678694400000000}$	$[2^3, 3^1]$	$\frac{39406434169244998649}{22033412542955520000000}$
$[1^1, 2^2, 4^1]$	$\frac{2998148628185909}{13115126513664000000000}$	$[2^2, 5^1]$	$\frac{234360972607515209}{14688941695303680000000}$
$[1^1, 4^2]$	$\frac{184134811312313}{27541765678694400000000}$	$[2^1, 3^1, 4^1]$	$\frac{3502136387768779}{29377883390607360000000}$
$[1^1, 2^1, 6^1]$	$\frac{3617070987119831}{27541765678694400000000}$	$[3^3]$	$\frac{374799675933251}{48963138984345600000000}$
$[1^1, 8^1]$	$\frac{26303739761759}{39345379540992000000000}$	$[2^1, 7^1]$	$\frac{12476274579169301}{10492101210931200000000}$
$[1^1, 2^1, 3^2]$	$\frac{1206877128048157}{91805885595648000000000}$	$[3^1, 6^1]$	$\frac{16811008475015879}{22033412542955520000000}$
$[1^1, 3^1, 5^1]$	$\frac{2455964944171}{367223542382592000000}$	$[4^1, 5^1]$	$\frac{671436255551711}{88133650171822080000000}$
		$[9^1]$	$\frac{1864835590786319}{24481569492172800000000}$
$[1^1, 2^5]$	$\frac{8885348647878412034499217035 1}{2581935664350601803202560000000000}$	$[2^4, 3^1]$	$\frac{71010742456357082830658884073 9}{516387132870120360640512000000000000}$
$[1^1, 2^3, 4^1]$	$\frac{4572509990406797552502341}{34425808858008024042700800000000000}$	$[2^3, 5^1]$	$\frac{109197089334060411167876570117}{1032774265740240721281024000000000000}$
$[1^1, 2^1, 4^2]$	$\frac{268054718171660435931803}{8606452214502006010675200000000000}$	$[2^2, 3^1, 4^1]$	$\frac{143523730213219992257056584 71}{206554853148048144256204800000000000}$
$[1^1, 2^2, 6^1]$	$\frac{116883178613702023566706 7}{1721290442900401202135040000000000}$	$[2^1, 3^3]$	$\frac{183002200715285406357445301}{459010784773440320569344000000000000}$
$[1^1, 4^1, 6^1]$	$\frac{743715115011041403407}{5737634809668004007116800000000000}$	$[2^2, 7^1]$	$\frac{2046277321274807954881575 91}{295078361640068777508864000000000000}$
$[1^1, 2^1, 8^1]$	$\frac{1005168225457531674538 91}{3227419580438252254003200000000000}$	$[2^1, 3^1, 6^1]$	$\frac{547557978971950371021494551}{1377032354320320961708032000000000000}$
$[1^1, 10^1]$	$\frac{669334190408902252822 03}{51638713287012036064051200000000000}$	$[2^1, 4^1, 5^1]$	$\frac{246101769371735646036211342 7}{6196645594441444327686144000000000000}$
$[1^1, 2^2, 3^2]$	$\frac{4386643900008678909343237}{6454839160876504508006400000000000}$	$[3^2, 5^1]$	$\frac{418665267598213008160712 11}{206554853148048144256204800000000000}$
$[1^1, 3^2, 4^1]$	$\frac{5314998536465979483 83}{4098310578334288576512000000000000}$	$[3^1, 4^2]$	$\frac{3734906146624600673780 83}{184423976025042985943040000000000000}$
$[1^1, 2^1, 3^1, 5^1]$	$\frac{80439233844576137718876 7}{2581935664350601803202560000000000}$	$[2^1, 9^1]$	$\frac{182277891278802756936253723}{459010784773440320569344000000000000}$
$[1^1, 5^2]$	$\frac{16733378688533122315949}{12909678321753009016012800000000000}$	$[3^1, 8^1]$	$\frac{627375711619366876512268 13}{309832279722072216384307200000000000}$
$[1^1, 3^1, 7^1]$	$\frac{19128779897378689455131}{14753918082003438875443200000000000}$	$[4^1, 7^1]$	$\frac{179085961953969171115519 79}{885235084920206332526592000000000000}$
$[11^1]$	$\frac{62675660640300931114214381}{309832279722072216384307200000000000}$	$[5^1, 6^1]$	$\frac{696399653569180926527422 1}{34425808858008024042700800000000000}$

Table 4: Probabilities $P(\mathbf{a})$ for $n = 3, 5, 7, 9, 11$ (top to bottom).

References

[1] David F. Manlove. *Algorithmics of Matching under Preferences*, volume 2 of *Series on Theoretical Computer Science*. World Scientific, Singapore, 2013.

[2] Dan Gusfield and Robert W. Irving. *The Stable Marriage Problem: Structure and Algorithms*. The MIT Press, Cambridge, Massachusetts, 1989.

[3] David Gale and Lloyd S. Shapley. College admissions and the stability of marriage. *American Mathematical Monthly*, 69:9–15, 1962.

[4] Boris Pittel. On a random instance of a 'stable roommates' problem: Likely behavior of the proposal algorithm. *Combinatorics, Probability and Computing*, 2:53–92, 1993.

[5] Stephan Mertens. Stable roommates problem with random preferences. *Journal of Statistical Mechanics: Theory and Experiment*, 2015(1):P01020, 2015.

[6] Jimmy J.M. Tan. A necessary and sufficient condition for the existence of a complete stable matching. *J. Algorithms*, 12:154–178, 1991.

[7] N. J. A. Sloane. The On-Line Encyclopedia of Integer Sequences. A000041. $a(n)$ = number of partitions of n (the partition numbers).

[8] Wolfram Research Inc. Mathematica. `http://www.wolfram.com/mathematica`.

[9] Stephan Mertens. Stable matchings. `http://www.ovgu.de/mertens/research/matchings`.

[10] Stephan Mertens. Random stable matchings. *J. Stat. Mech.*, 2005(10):P10008, 2005.

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