# Number of Partitions of an $n$-kilogram Stone into Minimum Number of Weights to Weigh All Integral Weights from 1 to $n$ $\mathrm{kg}(\mathrm{s})$ on a Two-pan Balance 

## Md Towhidul Islam

Assistant Professor
Department of Marketing, Comilla University
Courtbari, Comilla
Bangladesh
towhid@cou.ac.bd
towhid81@gmail.com

## Md Shahidul Islam

Deputy Director
Bangladesh Railway, Bangladesh Civil Service
Dhaka, Bangladesh
shahid1514@gmail.com


#### Abstract

We find out the number of different partitions of an n-kilogram stone into the minimum number of parts so that all integral weights from 1 to $n$ kilograms can be weighed in one weighing using the parts of any of the partitions on a two-pan balance. In comparison to the traditional partitions, these partitions have advantage where there is a constraint on total weight of a set and the number of parts in the partition. They may have uses in determining the optimal size and number of weights and denominations of notes and coins.


Key Words: M-partitions, minimum number of parts, denominations of weights and coins, feasible partitions, two-pan balance.

## Introduction

A seller has an $n$ kilogram stone which he wants to break into the minimum possible number of weights using which on a two-pan balance he can sell in whole kilograms up to $n$ kilogram(s) of goods in one weighing. As in tradition, he can place weights on both the pans but goods on only one pan. We call such a partition a 'feasible' partition. Our intension is to find out the number of all feasible partitions of $n$.

Suppose the minimum possible $m$ weights are $w_{1} \leq w_{2} \leq \cdots \leq w_{m-1} \leq w_{m}$ and $n=w_{1}+$ $w_{2}+\cdots+w_{m-1}+w_{m}$. If we also suppose $u_{i}=[-1,0,1]$, from the description of the problem it can be stated that $u_{1} w_{1} \mp u_{2} w_{2} \mp \cdots \mp u_{m-1} w_{m-1} \mp u_{m} w_{m}$ must make all the positive integers from 1 to $n$. Then we will find out $t(n)$, the number of all such partitions of the ordered integral set of weights $w_{1} \leq w_{2} \leq \cdots \leq w_{m-1} \leq w_{m}$ for every positive integer $n$. To that end we at first define some terms used in the discussion. Then we prove a number of theorems which ultimately lead to our main finding, the recursive functions for $t(n)$ in two spans. O'Shea [1] introduced the concept of M-partitions by partitioning a weight into as few parts as possible so as to be able to weigh any integral weight less than $m$ weighed on a one
scale pan. He maintained the subpartition property of MacMahon's [2] perfect partitions but dropped the uniqueness property and also added a new property; the minimality of number of parts in the partitions. We examine the partitioning situation for a two-pan balance maintaining his minimality of parts.

## Definitions

A feasible set/partition of $n$ is an ordered partition of minimum possible $m$ parts $w_{1} \leq w_{2} \leq$ $\cdots \leq w_{m-1} \leq w_{m}$ made from $n$ such that all integral values from 1 to $n$ can be weighed in one weighing using the parts on a traditional two-pan balance.
$\boldsymbol{R}_{\boldsymbol{i}}=w_{1}+w_{2}+\cdots+w_{i}$. From this, it is clear that $R_{1}=w_{1}, R_{i}=R_{i-1}+w_{i}$ and $R_{m}=w_{1}+$ $w_{2}+\cdots+w_{m}=n$. We assume $R_{0}=0$.
$\boldsymbol{t}(\boldsymbol{n})$ is the number of all feasible partitions of $n$. We assume $t(0)=1$.

## Theorems of Feasibility

Theorem 1. For any feasible set, the lightest weight $w_{1}$ equals 1 kg , i.e. $w_{1}=1$

Proof. Suppose $w_{1} \neq 1$. So, $w_{i} \geq 2$ for all $i$. However, using such a partition, $n-1 \mathrm{~kg}$ cannot be weighed. Placing all $w_{i}$ pieces on one of the pans we see, $w_{1}+w_{2}+\cdots+w_{m-1}+$ $w_{m}=n$. Now, if we take out the smallest piece $w_{1}$ from there, we see $w_{2}+w_{3} \ldots+w_{m-1}+$ $w_{m} \leq n-2$. So, such a partition with $w_{1} \neq 1$ can never weigh $n-1$. Therefore, we conclude the lightest weight $w_{1}=1 \mathrm{~kg}$.

Theorem 2. For any feasible partition, $w_{i} \leq 2 R_{i-1}+1$.

Proof. Two things are clear-
i) The highest value possible to be weighed from $u_{1} w_{1} \mp u_{2} w_{2} \mp \cdots \mp u_{i-1} w_{i-1}$ is $w_{1}+$ $w_{2}+\cdots+w_{i-1}=R_{i-1}$.
ii) So, the lowest possible value made from $w_{i}-\left(u_{1} w_{1} \mp u_{2} w_{2} \mp \cdots \mp u_{i-1} w_{i-1}\right)$ is $w_{i}-R_{i-1}$.
Therefore, for a feasible set, there should not be any integer $n$ in the range $R_{i-1}<n<w_{i}-$ $R_{i-1}$.

That is, $R_{i-1}+1 \geq w_{i}-R_{i-1}$.
$\Rightarrow w_{i} \leq 2 R_{i-1}+1$.

Corollaries of Theorem 2. Interrelationships among $w_{i}, R_{i}$ and $R_{i-1}$
From theorem 2, we get $\quad R_{i-1} \geq \frac{w_{i}-1}{2}$
.... Corollary (1)

$$
\begin{array}{lr}
\Rightarrow R_{i}-w_{i} \geq \frac{w_{i}-1}{2} & \text { (From definition, } \left.R_{i-1}=R_{i}-w_{i}\right) \\
\Rightarrow R_{i} \geq \frac{3 w_{i}-1}{2} & \ldots . \text { Corollary (2) } \\
\Rightarrow w_{i} \leq \frac{2 R_{i}+1}{3} & \ldots . \text { Corollary (3) } \\
\Rightarrow R_{i}-R_{i-1} \leq \frac{2 R_{i}+1}{3} & \text { (From definition, } \left.R_{i}-R_{i-1}=w_{i}\right) \\
\Rightarrow R_{i} \leq 3 R_{i-1}+1 & \ldots . \text { Corollary (4) } \\
\Rightarrow R_{i-1} \geq \frac{R_{i}-1}{3} & \ldots . \text { Corollary (5) }
\end{array}
$$

Theorem 3. The highest ever possible value of part $w_{i}$ is $3^{i-1}$ and the highest ever possible value of $R_{i}$ feasibly partitionable in i parts is $\frac{3^{i}-1}{2}$.

Proof. From Theorem 2, we know $w_{i} \leq 2 R_{i-1}+1$ for $2 \leq i \leq m$; from definition we know $R_{1}=w_{1}$ and from Theorem 1 we know, $w_{1}=1$.

$$
\text { So, } \begin{aligned}
w_{2} & \leq 2 R_{1}+1 \\
& \Rightarrow w_{2} \leq 3
\end{aligned}
$$

So, the highest possible value of $R_{2}$ is $w_{1}+w_{2}=1+3=4$
Going on with $w_{i} \leq 2 R_{i-1}+1$, we see the highest possible values of weights $w_{1}, w_{2}, w_{3} \ldots$, $w_{m}$ are $3^{0}, 3^{1}, 3^{2}, \ldots, 3^{m-1}$. Clearly, $w_{i}=3^{i-1}$.
And the highest possible $R_{i}$ feasibly partitionable in $i$ pieces is $R_{i}=3^{0}+3^{1}+\cdots+3^{i-2}+$ $3^{i-1}=\frac{3^{i}-1}{2}$.

Theorem 4. At least $m$ weights are needed for $\frac{3^{m-1}+1}{2} \leq n \leq \frac{3^{m}-1}{2}$ where $m=\left\lceil\log _{3}(2 n)\right\rceil$.

Proof. From Theorem 3, it is clear that he highest possible value of $R_{m}$ feasibly partitionable in $m$ pieces is $R_{m}=\frac{3^{m}-1}{2}$ and the highest possible value of $R_{m-1}$ feasibly partitionable in
$m-1$ pieces is $R_{m-1}=\frac{3^{m-1}-1}{2}$. So, the next integer $\frac{3^{m-1}-1}{2}+1=\frac{3^{m-1}+1}{2}$ is the lowest value of $n$ feasibly partitionable in $m$ parts.

So, we have proved that at least $m$ weights are needed for $\frac{3^{m-1}+1}{2} \leq n \leq \frac{3^{m}-1}{2}$.

$$
\begin{aligned}
& \Rightarrow 3^{m-1}+1 \leq 2 n \leq 3^{m}-1 \\
& \Rightarrow 3^{m-1}<2 n<3^{m} \\
& \Rightarrow m-1<\log _{3}(2 n)<m
\end{aligned}
$$

However, $m$ is never a fraction. The ceiling function $m=\left\lceil\log _{3}(2 n)\right\rceil$ always gives the $m$ we know to be the correct number of parts for the range $\frac{3^{m-1}+1}{2} \leq n \leq \frac{3^{m}-1}{2}$.

Theorem 5. For $\frac{3^{m-1}+1}{2} \leq n \leq \frac{3^{m-1}+1}{2}+3^{m-2}$, the range of $R_{m-1}$ is $\left\lceil\frac{n-1}{3}\right\rceil \leq R_{m-1} \leq$ $\left\lfloor\frac{2 n+3^{m-2}-1}{4}\right\rfloor$ and for $\frac{3^{m-1}+1}{2}+3^{m-2}+1 \leq n \leq \frac{3^{m}-1}{2}$, the range of $R_{m-1}$ is $\left\lceil\frac{n-1}{3}\right\rceil \leq R_{m-1} \leq$ $\frac{3^{m-1}-1}{2}$.

## Proof.

## The smallest feasible $\mathbf{R}_{\boldsymbol{m}-1}$

Putting $i=m$ in Corollary 5 of Theorem 2 we get, $R_{m-1} \geq \frac{R_{m}-1}{3}$
As $R_{m}=n$, the smallest feasible $R_{m-1}$ is $\left\lceil\frac{n-1}{3}\right\rceil$ for all $n$.
For example, for $n=16$, the smallest feasible $R_{m-1}=\left\lceil\frac{n-1}{3}\right\rceil=5$ and for $n=26$, the smallest feasible $R_{m-1}=\left\lceil\frac{n-1}{3}\right\rceil=9$.

## The largest feasible $\mathbf{R}_{\boldsymbol{m}-1}$

With a little modification of Theorem 3, it is clear that for any $n$ the $m$-part partition $3^{0}, 3^{1}, \ldots, 3^{m-3},\left\lfloor\frac{n-R_{m-2}}{2}\right],\left\lceil\frac{n-R_{m-2}}{2}\right\rceil$ would ensure the highest possible $R_{m-1}$ given the condition that $\left[\frac{n-R_{m-2}}{2}\right] \leq 3^{m-2}$.
Based on whether $\left\lfloor\frac{n-R_{m-2}}{2}\right\rfloor \leq 3^{m-2}$ or not, we can split the range $\frac{3^{m-1}+1}{2} \leq n \leq \frac{3^{m}-1}{2}$ found in Theorem 4 into two mutually exclusive and collectively exhaustive spans: a) $\frac{3^{m-1}+1}{2} \leq n \leq$ $\frac{3^{m-1}+1}{2}+3^{m-2}$ and b) $\frac{3^{m-1}+1}{2}+3^{m-2}+1 \leq n \leq \frac{3^{m}-1}{2}$.
a) For $\frac{3^{m-1}+1}{2} \leq n \leq \frac{3^{m-1}+1}{2}+3^{m-2}$, we see $\left\lfloor\frac{n-R_{m-2}}{2}\right\rfloor \leq 3^{m-2}$. So, the largest feasible $R_{m-1}$ is $3^{0}+3^{1}+\cdots+3^{m-3}+\left\lfloor\frac{n-R_{m-2}}{2}\right\rfloor=\left\lfloor\frac{2 n+3^{m-2}-1}{4}\right\rfloor$.
For example, if $n=16$, the largest feasible $R_{m-1}$ is $\left\lfloor\frac{2 n+3^{m-2}-1}{4}\right\rfloor=10$.
b) However, for $\frac{3^{m-1}+1}{2}+3^{m-2}+1 \leq n \leq \frac{3^{m}-1}{2}$, the condition is not met. From Theorem 3 we know $w_{m-1}$ should never exceed $3^{m-2}$. So the $m$-part partition should be $3^{0}, 3^{1}, \ldots, 3^{m-3}, 3^{m-2}, n-\frac{3^{m-2}-1}{2}$. And the largest feasible $R_{m-1}$ is $3^{0}+3^{1}+\cdots+$ $3^{m-3}+3^{m-2}=\frac{3^{m-1}-1}{2}$.
For example, if $n=26$, the largest feasible $R_{m-1}$ is $\frac{3^{m-1}-1}{2}=13$.
So, we have found the range of $R_{m-1}$ for the two segments as described in the statement of this theorem.

## The Main Result: The Recursive Functions for $\boldsymbol{t}(\boldsymbol{n})$

## Theorem 6.

$$
\begin{aligned}
& t\left(\frac{3^{m-1}+1}{2} \leq n \leq \frac{3^{m-1}+1}{2}+3^{m-2}\right) \\
&=\sum_{R_{m-1}=\left\lceil\frac{n-1}{3}\right\rceil}^{\left\lfloor\frac{2 n+3^{m-2}-1}{4}\right\rceil} t\left(R_{m-1}\right)-\sum_{R_{m-1}=\left\lceil\left.\frac{3 n+2}{5} \right\rvert\,\right.}^{\left\lfloor\frac{2 n+3^{m-2}-1}{4}\right\rfloor} \sum_{R_{m-2}=\left\lceil\frac{R_{m-1}-1}{3}\right\rceil}^{2 R_{m-1}-n-1} t\left(R_{m-2}\right)
\end{aligned}
$$

and

$$
t\left(\frac{3^{m-1}+1}{2}+3^{m-2}+1 \leq n \leq \frac{3^{m}-1}{2}\right)=\sum_{R_{m-1}=\left\lceil\frac{n-1}{3}\right\rceil}^{\frac{3^{m-1}-1}{2}} t\left(R_{m-1}\right)
$$

Proof. In order to determine the number of feasible partitions of $n$ we will derive two different recursive functions for the two spans from Theorem 5.
a) Determining $t(n)$ for the range $\frac{3^{m-1}+1}{2} \leq n \leq \frac{3^{m-1}+1}{2}+3^{m-2}$

To count the total number of feasible partitions of $n$, we have to count all feasible partitions of all $R_{m-1}$ possible to be broken from $n$ because adding an additional last part $w_{m}$ to these $m-1$ part feasible partitions of $R_{m-1}$ will turn them into $m$ part feasible partitions of $n$; the
condition is that no $w_{m-1}$ possible to be made from each of these $R_{m-1}$ values exceeds the corresponding $w_{m}$.
From Theorem 5 we know, for $\frac{3^{m-1}+1}{2} \leq n \leq \frac{3^{m-1}+1}{2}+3^{m-2}$ the range of $R_{m-1}$ is $\left\lceil\frac{n-1}{3}\right\rceil \leq$ $R_{m-1} \leq\left\lfloor\frac{2 n+3^{m-2}-1}{4}\right\rfloor$ and $w_{m}=\left\lceil\frac{n-R_{m-2}}{2}\right\rceil$. So, the formula at first seems to be

$$
t\left(\frac{3^{m-1}+1}{2} \leq n \leq \frac{3^{m-1}+1}{2}+3^{m-2}\right)=\sum_{R_{m-1}=\left\lceil\frac{n-1}{3}\right\rceil}^{\left\lfloor\frac{2 n+3^{m-2}-1}{4}\right\rfloor} t\left(R_{m-1}\right)
$$

However this would count some partitions in duplication as for some values of $n$ in this range, some of the $w_{m-1}$ values feasibly broken from each of these $R_{m-1}$ values are larger than the corresponding $w_{m}=\left\lceil\frac{n-R_{m-2}}{2}\right\rceil$. If arranged in ascending order, it would be clear that the partitions are counted in duplication for some other $n$. We have to exclude those duplications from the count by finding out such partitions of these $R_{m-1}$ values.

To do that, we will at first find the range of such problematic $R_{m-1}$ values and then we will set the range of incompatible $R_{m-2}$ values for each of these $R_{m-1}$ values.

From corollary 3 of Theorem 2 it is clear that $w_{m-1} \leq \frac{2 R_{m-1}+1}{3}$ and from definition, $w_{m}=$ $n-R_{m-1}$. It is noticeable that $w_{m-1}$ will be greater than $w_{m}$ if $w_{m} \leq w_{m-1}-1$.

$$
\begin{aligned}
& \Rightarrow n-R_{m-1} \leq \frac{2 R_{m-1}+1}{3}-1 . \\
& \Rightarrow \frac{3 n+2}{5} \leq R_{m-1} .
\end{aligned}
$$

This lower limit of $R_{m-1}$ taken in consideration along with the range of $R_{m-1}$ set in Theorem 5 redefines the range of such problematic $R_{m-1}$ as $\frac{3 n+2}{5} \leq R_{m-1} \leq\left\lfloor\frac{2 n+3^{m-2}-1}{4}\right\rfloor$ where some of the $w_{m-1}$ values broken from $R_{m-1}$ are larger than the corresponding $w_{m}$.
As $w_{m-1}$ must not be larger than $w_{m}$, we have to exclude those partitions of these $R_{m-1}$ in $m-1$ parts where $w_{m}+1 \leq w_{m-1} \leq \frac{2 R_{m-1}+1}{3}$. (As from corollary 3 of Theorem 2 we know, $w_{m-1} \leq \frac{2 R_{m-1}+1}{3}$.)
That is, we have to exclude partitions with $n-R_{m-1}+1 \leq w_{m-1} \leq \frac{2 R_{m-1}+1}{3}$ Or, $R_{m-1}-n+R_{m-1}-1 \geq R_{m-1}-w_{m-1} \geq R_{m-1}-\frac{2 R_{m-1}+1}{3}$ (deducting the terms from $\left.R_{m-1}\right)$.

$$
\text { Or, } \frac{R_{m-1}-1}{3} \leq R_{m-2} \leq 2 R_{m-1}-n-1 \text { for each of these problematic } R_{m-1} \text {. }
$$

Finally the formula stands,

$$
\begin{aligned}
& t\left(\frac{3^{m-1}+1}{2} \leq n \leq \frac{3^{m-1}+1}{2}+3^{m-2}\right) \\
&\left.=\sum_{R_{m-1}=\left\lceil\frac{n-1}{3}\right\rceil} t\left(R_{m-1}\right)-\sum_{R_{m-1}=\left\lceil\frac{2 n+3^{m-2}-1}{5}\right\rceil}^{4}\right\rceil R_{m-2}=\left\lceil\frac{R_{m-1}-1}{3}\right\rceil
\end{aligned}
$$

b) Determining $t(n)$ for the range $\frac{3^{m-1}+1}{2}+3^{m-2}+1 \leq n \leq \frac{3^{m}-1}{2}$

From Theorem 5 we know, for $\frac{3^{m-1}+1}{2}+3^{m-2}+1 \leq n \leq \frac{3^{m}-1}{2}$ the range of $R_{m-1}$ is $\left\lceil\frac{n-1}{3}\right\rceil \leq R_{m-1} \leq \frac{3^{m-1}-1}{2}$. However, unlike for the range of $n$ in part (a), all possible $w_{m-1}$ values broken from $R_{m-1}$ in this range are less than the corresponding $w_{m}$. So, to count the total possible number of feasible partitions of $n$ in this range, we have only to count $t\left(R_{m-1}\right)$ for all possible $R_{m-1}$ values for $n$; no chance of duplication arises. So, the formula stands,

$$
t\left(\frac{3^{m-1}+1}{2}+3^{m-2}+1 \leq n \leq \frac{3^{m}-1}{2}\right)=\sum_{R_{m-1}=\left\lceil\frac{n-1}{3}\right\rceil}^{\frac{3^{m-1}-1}{2}} t\left(R_{m-1}\right)
$$

When we find out the $t\left(\frac{3^{m-1}+1}{2}+3^{m-2}+1 \leq n \leq \frac{3^{m}-1}{2}\right)$ values starting from $\frac{3^{m}-1}{2}$ backwards, we see the terms of sequence A005704 of the OEIS [3] come in triplicates.

## References

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