Number of Partitions of an *n*-kilogram Stone into Minimum Number of Weights to Weigh All Integral Weights from 1 to *n* kg(s) on a Two-pan Balance

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Abstract

We find out the number of different partitions of an n-kilogram stone into the minimum number of parts so that all integral weights from 1 to n kilograms can be weighed in one weighing using the parts of any of the partitions on a two-pan balance. In comparison to the traditional partitions, these partitions have advantage where there is a constraint on total weight of a set and the number of parts in the partition. They may have uses in determining the optimal size and number of weights and denominations of notes and coins.

Key Words: M-partitions, minimum number of parts, denominations of weights and coins, feasible partitions, two-pan balance.

Introduction

A seller has an n kilogram stone which he wants to break into the minimum possible number of weights using which on a two-pan balance he can sell in whole kilograms up to nkilogram(s) of goods in one weighing. As in tradition, he can place weights on both the pans but goods on only one pan. We call such a partition a 'feasible' partition. Our intension is to find out the number of all feasible partitions of n.

Suppose the minimum possible *m* weights are $w_1 \le w_2 \le \cdots \le w_{m-1} \le w_m$ and $n = w_1 + w_2 + \cdots + w_{m-1} + w_m$. If we also suppose $u_i = [-1,0,1]$, from the description of the problem it can be stated that $u_1w_1 \mp u_2w_2 \mp \cdots \mp u_{m-1}w_{m-1} \mp u_mw_m$ must make all the positive integers from 1 to *n*. Then we will find out t(n), the number of all such partitions of the ordered integral set of weights $w_1 \le w_2 \le \cdots \le w_{m-1} \le w_m$ for every positive integer *n*. To that end we at first define some terms used in the discussion. Then we prove a number of theorems which ultimately lead to our main finding, the recursive functions for t(n) in two spans. O'Shea [1] introduced the concept of M-partitions by partitioning a weight into as few parts as possible so as to be able to weigh any integral weight less than *m* weighed on a one

scale pan. He maintained the subpartition property of MacMahon's [2] perfect partitions but dropped the uniqueness property and also added a new property; the minimality of number of parts in the partitions. We examine the partitioning situation for a two-pan balance maintaining his minimality of parts.

Definitions

- A *feasible set/partition* of n is an ordered partition of minimum possible m parts $w_1 \le w_2 \le \cdots \le w_{m-1} \le w_m$ made from n such that all integral values from 1 to n can be weighed in one weighing using the parts on a traditional two-pan balance.
- $R_i = w_1 + w_2 + \dots + w_i$. From this, it is clear that $R_1 = w_1$, $R_i = R_{i-1} + w_i$ and $R_m = w_1 + w_2 + \dots + w_m = n$. We assume $R_0 = 0$.
- t(n) is the number of all feasible partitions of n. We assume t(0) = 1.

Theorems of Feasibility

Theorem 1. For any feasible set, the lightest weight w_1 equals 1 kg, i.e. $w_1 = 1$

Proof. Suppose $w_1 \neq 1$. So, $w_i \geq 2$ for all *i*. However, using such a partition, n - 1 kg cannot be weighed. Placing all w_i pieces on one of the pans we see, $w_1 + w_2 + \dots + w_{m-1} + w_m = n$. Now, if we take out the smallest piece w_1 from there, we see $w_2 + w_3 \dots + w_{m-1} + w_m \leq n - 2$. So, such a partition with $w_1 \neq 1$ can never weigh n - 1. Therefore, we conclude the lightest weight $w_1 = 1$ kg.

Theorem 2. For any feasible partition, $w_i \leq 2R_{i-1} + 1$.

Proof. Two things are clear-

- i) The highest value possible to be weighed from $u_1w_1 \mp u_2w_2 \mp \cdots \mp u_{i-1}w_{i-1}$ is $w_1 + w_2 + \cdots + w_{i-1} = R_{i-1}$.
- ii) So, the lowest possible value made from $w_i (u_1w_1 \mp u_2w_2 \mp \cdots \mp u_{i-1}w_{i-1})$ is $w_i R_{i-1}$.

Therefore, for a feasible set, there should not be any integer *n* in the range $R_{i-1} < n < w_i - R_{i-1}$.

That is, $R_{i-1} + 1 \ge w_i - R_{i-1}$. $\Rightarrow w_i \le 2R_{i-1} + 1$.

Corollaries of Theorem 2. Interrelationships among w_i , R_i and R_{i-1}

From theorem 2, we get $R_{i-1} \ge \frac{w_i - 1}{2}$ Corollary (1) $\Rightarrow R_i - w_i \ge \frac{w_i - 1}{2}$ (From definition, $R_{i-1} = R_i - w_i$) $\Rightarrow R_i \ge \frac{3w_i - 1}{2}$ Corollary (2) $\Rightarrow w_i \le \frac{2R_i + 1}{3}$ Corollary (3) $\Rightarrow R_i - R_{i-1} \le \frac{2R_i + 1}{3}$ (From definition, $R_i - R_{i-1} = w_i$) $\Rightarrow R_i \le 3R_{i-1} + 1$ Corollary (4) $\Rightarrow R_{i-1} \ge \frac{R_i - 1}{3}$ Corollary (5)

Theorem 3. The highest ever possible value of part w_i is 3^{i-1} and the highest ever possible value of R_i feasibly partitionable in *i* parts is $\frac{3^{i-1}}{2}$.

Proof. From Theorem 2, we know $w_i \le 2R_{i-1} + 1$ for $2 \le i \le m$; from definition we know $R_1 = w_1$ and from Theorem 1 we know, $w_1 = 1$. So, $w_2 \le 2R_1 + 1$ $\Rightarrow w_2 \le 3$

So, the highest possible value of R_2 is $w_1 + w_2 = 1 + 3 = 4$

Going on with $w_i \leq 2R_{i-1} + 1$, we see the highest possible values of weights $w_1, w_2, w_3 \dots$, w_m are $3^0, 3^1, 3^2, \dots, 3^{m-1}$. Clearly, $w_i = 3^{i-1}$.

And the highest possible R_i feasibly partitionable in *i* pieces is $R_i = 3^0 + 3^1 + \dots + 3^{i-2} + 3^{i-1} = \frac{3^{i-1}}{2}$.

Theorem 4. At least m weights are needed for $\frac{3^{m-1}+1}{2} \le n \le \frac{3^{m-1}}{2}$ where $m = \lceil log_3(2n) \rceil$.

Proof. From Theorem 3, it is clear that he highest possible value of R_m feasibly partitionable in *m* pieces is $R_m = \frac{3^m - 1}{2}$ and the highest possible value of R_{m-1} feasibly partitionable in m-1 pieces is $R_{m-1} = \frac{3^{m-1}-1}{2}$. So, the next integer $\frac{3^{m-1}-1}{2} + 1 = \frac{3^{m-1}+1}{2}$ is the lowest value of *n* feasibly partitionable in *m* parts.

So, we have proved that at least *m* weights are needed for $\frac{3^{m-1}+1}{2} \le n \le \frac{3^{m-1}}{2}$.

$$\Rightarrow 3^{m-1} + 1 \le 2n \le 3^m - 1$$
$$\Rightarrow 3^{m-1} < 2n < 3^m$$
$$\Rightarrow m - 1 < \log_3(2n) < m$$

However, *m* is never a fraction. The ceiling function $m = \lceil log_3(2n) \rceil$ always gives the *m* we know to be the correct number of parts for the range $\frac{3^{m-1}+1}{2} \le n \le \frac{3^m-1}{2}$.

Theorem 5. For $\frac{3^{m-1}+1}{2} \le n \le \frac{3^{m-1}+1}{2} + 3^{m-2}$, the range of R_{m-1} is $\left[\frac{n-1}{3}\right] \le R_{m-1} \le \left[\frac{2n+3^{m-2}-1}{4}\right]$ and for $\frac{3^{m-1}+1}{2} + 3^{m-2} + 1 \le n \le \frac{3^{m-1}}{2}$, the range of R_{m-1} is $\left[\frac{n-1}{3}\right] \le R_{m-1} \le \frac{3^{m-1}-1}{2}$.

Proof.

The smallest feasible R_{m-1}

Putting i = m in Corollary 5 of Theorem 2 we get, $R_{m-1} \ge \frac{R_m - 1}{3}$ As $R_m = n$, the smallest feasible R_{m-1} is $\left[\frac{n-1}{3}\right]$ for all n. For example, for n = 16, the smallest feasible $R_{m-1} = \left[\frac{n-1}{3}\right] = 5$ and for n = 26, the smallest feasible $R_{m-1} = \left[\frac{n-1}{3}\right] = 9$.

The largest feasible R_{m-1}

With a little modification of Theorem 3, it is clear that for any *n* the *m*-part partition $3^{0}, 3^{1}, ..., 3^{m-3}, \left\lfloor \frac{n-R_{m-2}}{2} \right\rfloor, \left\lceil \frac{n-R_{m-2}}{2} \right\rceil$ would ensure the highest possible R_{m-1} given the condition that $\left\lfloor \frac{n-R_{m-2}}{2} \right\rfloor \leq 3^{m-2}$. Based on whether $\left\lfloor \frac{n-R_{m-2}}{2} \right\rfloor \leq 3^{m-2}$ or not, we can split the range $\frac{3^{m-1}+1}{2} \leq n \leq \frac{3^{m-1}+1}{2}$ found in Theorem 4 into two mutually exclusive and collectively exhaustive spans: a) $\frac{3^{m-1}+1}{2} \leq n \leq \frac{3^{m-1}+1}{2} \leq \frac$ For example, if n = 16, the largest feasible R_{m-1} is $\left\lfloor \frac{2n+3^{m-2}-1}{4} \right\rfloor = 10$.

b) However, for $\frac{3^{m-1}+1}{2} + 3^{m-2} + 1 \le n \le \frac{3^{m}-1}{2}$, the condition is not met. From Theorem 3 we know w_{m-1} should never exceed 3^{m-2} . So the *m*-part partition should be $3^{0}, 3^{1}, \dots, 3^{m-3}, 3^{m-2}, n - \frac{3^{m-2}-1}{2}$. And the largest feasible R_{m-1} is $3^{0} + 3^{1} + \dots + 3^{m-3} + 3^{m-2} = \frac{3^{m-1}-1}{2}$.

For example, if n = 26, the largest feasible R_{m-1} is $\frac{3^{m-1}-1}{2} = 13$.

So, we have found the range of R_{m-1} for the two segments as described in the statement of this theorem.

The Main Result: The Recursive Functions for t(n)

Theorem 6.

$$t\left(\frac{3^{m-1}+1}{2} \le n \le \frac{3^{m-1}+1}{2} + 3^{m-2}\right)$$

$$= \sum_{R_{m-1}=\left\lceil \frac{n-1}{3} \right\rceil}^{\left\lfloor \frac{2n+3^{m-2}-1}{4} \right\rfloor} t(R_{m-1}) - \sum_{R_{m-1}=\left\lceil \frac{3n+2}{5} \right\rceil}^{2R_{m-1}-n-1} \sum_{R_{m-2}=\left\lceil \frac{R_{m-1}-1}{3} \right\rceil}^{2R_{m-1}-n-1} t(R_{m-2})$$

and

$$t\left(\frac{3^{m-1}+1}{2}+3^{m-2}+1\le n\le \frac{3^m-1}{2}\right) = \sum_{R_{m-1}=\left\lfloor\frac{n-1}{3}\right\rfloor}^{\frac{3^m-1-1}{2}} t(R_{m-1})$$

Proof. In order to determine the number of feasible partitions of n we will derive two different recursive functions for the two spans from Theorem 5.

m-1

a) Determining t(n) for the range $\frac{3^{m-1}+1}{2} \le n \le \frac{3^{m-1}+1}{2} + 3^{m-2}$

To count the total number of feasible partitions of n, we have to count all feasible partitions of all R_{m-1} possible to be broken from n because adding an additional last part w_m to these m-1 part feasible partitions of R_{m-1} will turn them into m part feasible partitions of n; the condition is that no w_{m-1} possible to be made from each of these R_{m-1} values exceeds the corresponding w_m .

From Theorem 5 we know, for $\frac{3^{m-1}+1}{2} \le n \le \frac{3^{m-1}+1}{2} + 3^{m-2}$ the range of R_{m-1} is $\left[\frac{n-1}{3}\right] \le R_{m-1} \le \left[\frac{2n+3^{m-2}-1}{4}\right]$ and $w_m = \left[\frac{n-R_{m-2}}{2}\right]$. So, the formula at first seems to be $t\left(\frac{3^{m-1}+1}{2}\le n\le \frac{3^{m-1}+1}{2}+3^{m-2}\right) = \sum_{\substack{R_{m-1}=\left[\frac{n-1}{2}\right]}}^{\left[\frac{2n+3^{m-2}-1}{4}\right]} t(R_{m-1})$

However this would count some partitions in duplication as for some values of n in this range, some of the w_{m-1} values feasibly broken from each of these R_{m-1} values are larger than the corresponding $w_m = \left[\frac{n-R_{m-2}}{2}\right]$. If arranged in ascending order, it would be clear that the partitions are counted in duplication for some other n. We have to exclude those duplications from the count by finding out such partitions of these R_{m-1} values.

To do that, we will at first find the range of such problematic R_{m-1} values and then we will set the range of incompatible R_{m-2} values for each of these R_{m-1} values.

From corollary 3 of Theorem 2 it is clear that $w_{m-1} \le \frac{2R_{m-1}+1}{3}$ and from definition, $w_m = n - R_{m-1}$. It is noticeable that w_{m-1} will be greater than w_m if $w_m \le w_{m-1} - 1$.

$$\Rightarrow n - R_{m-1} \le \frac{2R_{m-1}+1}{3} - 1$$
$$\Rightarrow \frac{3n+2}{5} \le R_{m-1}.$$

This lower limit of R_{m-1} taken in consideration along with the range of R_{m-1} set in Theorem 5 redefines the range of such problematic R_{m-1} as $\frac{3n+2}{5} \le R_{m-1} \le \left\lfloor \frac{2n+3^{m-2}-1}{4} \right\rfloor$ where some of the w_{m-1} values broken from R_{m-1} are larger than the corresponding w_m .

As w_{m-1} must not be larger than w_m , we have to exclude those partitions of these R_{m-1} in m-1 parts where $w_m + 1 \le w_{m-1} \le \frac{2R_{m-1}+1}{3}$. (As from corollary 3 of Theorem 2 we know, $w_{m-1} \le \frac{2R_{m-1}+1}{3}$.)

That is, we have to exclude partitions with $n - R_{m-1} + 1 \le w_{m-1} \le \frac{2R_{m-1} + 1}{3}$

Or,
$$R_{m-1} - n + R_{m-1} - 1 \ge R_{m-1} - w_{m-1} \ge R_{m-1} - \frac{2R_{m-1} + 1}{3}$$
 (deducting the terms from R_{m-1}).

Or,
$$\frac{R_{m-1}-1}{3} \le R_{m-2} \le 2R_{m-1} - n - 1$$
 for each of these problematic R_{m-1}

Finally the formula stands,

$$t\left(\frac{3^{m-1}+1}{2} \le n \le \frac{3^{m-1}+1}{2} + 3^{m-2}\right)$$
$$= \sum_{R_{m-1}=\left\lceil\frac{n-1}{3}\right\rceil}^{\left\lfloor\frac{2n+3^{m-2}-1}{4}\right\rfloor} t(R_{m-1}) - \sum_{R_{m-1}=\left\lceil\frac{3n+2}{5}\right\rceil}^{\left\lfloor\frac{2n+3^{m-2}-1}{4}\right\rfloor} \sum_{R_{m-2}=\left\lceil\frac{R_{m-1}-1}{3}\right\rceil}^{2R_{m-1}-n-1} t(R_{m-2})$$

b) Determining t(n) for the range $\frac{3^{m-1}+1}{2} + 3^{m-2} + 1 \le n \le \frac{3^{m}-1}{2}$ From Theorem 5 we know, for $\frac{3^{m-1}+1}{2} + 3^{m-2} + 1 \le n \le \frac{3^{m}-1}{2}$ the range of R_{m-1} is $\left[\frac{n-1}{3}\right] \le R_{m-1} \le \frac{3^{m-1}-1}{2}$. However, unlike for the range of n in part (a), all possible w_{m-1} values broken from R_{m-1} in this range are less than the corresponding w_m . So, to count the total possible number of feasible partitions of n in this range, we have only to count $t(R_{m-1})$ for all possible R_{m-1} values for n; no chance of duplication arises. So, the formula stands,

$$t\left(\frac{3^{m-1}+1}{2}+3^{m-2}+1\le n\le \frac{3^m-1}{2}\right) = \sum_{R_{m-1}=\left\lceil\frac{n-1}{3}\right\rceil}^{\frac{3^{m-1}-1}{2}} t(R_{m-1})$$

When we find out the $t\left(\frac{3^{m-1}+1}{2}+3^{m-2}+1\le n\le \frac{3^{m}-1}{2}\right)$ values starting from $\frac{3^{m}-1}{2}$ backwards, we see the terms of sequence A005704 of the OEIS [3] come in triplicates.

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