p-Ascent Sequences

Sergey Kitaev
University of Strathclyde
Livingstone Tower, 26 Richmond Street
Glasgow G1 1XH, United Kingdom
sergey.kitaev@cis.strath.ac.uk

Jeffrey B. Remmel*
Department of Mathematics
University of California, San Diego
La Jolla, CA 92093-0112. USA
remmel@math.ucsd.edu

Submitted: Date 1; Accepted: Date 2; Published: Date 3.
MR Subject Classifications: 05A15

Abstract

A sequence $(a_1, \ldots, a_n)$ of nonnegative integers is an *ascent sequence* if $a_0 = 0$ and for all $i \geq 2$, a_i is at most 1 plus the number of ascents in $(a_1, \ldots, a_{i-1})$. Ascent sequences were introduced by Bousquet-Mélou, Claesson, Dukes, and Kitaev in [1], who showed that these sequences of length n are in 1-to-1 correspondence with $(\mathbf{2}+\mathbf{2})$-free posets of size n, which, in turn, are in 1-to-1 correspondence with interval orders of size n. Ascent sequences are also in bijection with several other classes of combinatorial objects including the set of upper triangular matrices with nonnegative integer entries such that no row or column contains all zeros, permutations that avoid a certain mesh pattern, and the set of Stoimenow matchings.

In this paper, we introduce a generalization of ascent sequences, which we call *p-ascent sequences*, where $p \geq 1$. A sequence $(a_1, \ldots, a_n)$ of nonnegative integers is a p-ascent sequence if $a_0 = 0$ and for all $i \geq 2$, a_i is at most p plus the number of ascents in $(a_1, \ldots, a_{i-1})$. Thus, in our terminology, ascent sequences are 1-ascent sequences. We generalize a result of the authors in [15] by enumerating p-ascent sequences with respect to the number of 0s. We also generalize a result of Dukes, Kitaev, Remmel, and Steingrímsson in [4] by finding the generating function for the number of p-ascent sequences which have no consecutive repeated elements. Finally, we initiate the study of pattern-avoiding p-ascent sequences.

1 Introduction

1.1 Ascent sequences

Ascent sequences were introduced by Bousquet-Mélou, Claesson, Dukes, and Kitaev in [1], who showed that these sequences of length n are in 1-to-1 correspondence with $(\mathbf{2}+\mathbf{2})$-*free posets* of size n.

Let $\mathbb{N} = \{0, 1, \ldots, \}$ denote the natural numbers and $\mathbb{N}^*$ denote the set of all words over $\mathbb{N}$. A sequence $(a_1, \ldots, a_n) \in \mathbb{N}^n$ is an *ascent sequence of length* n if and only if it satisfies $a_1 = 0$ and $a_i \in [0, 1 + \mathrm{asc}(a_1, \ldots, a_{i-1})]$ for all $2 \leq i \leq n$. Here, for any integer sequence $(a_1, \ldots, a_i)$, the number of *ascents* of this sequence is

$$\mathrm{asc}(a_1, \ldots, a_i) = |\{j : a_j < a_{j+1}; 1 \leq j < i\}|.$$

*Partially supported by NSF grant DMS 0654060.

For instance, (0, 1, 0, 2, 3, 1, 0, 0, 2) is an ascent sequence which has 4 ascents. We let Asc denote the set of all ascent sequences, where we assume that the empty word is also an ascent sequence. For any $n \geq 1$, we let Asc_n denote the set of all ascent sequences of length n. If $a = (a_1, \ldots, a_n) \in Asc_n$, we let $|a| = n$ be the length of a, $\sum a = a_1 + \cdots + a_n$ equal the sum of the values of a, $|a|_0$ denote the number of occurrences of 0 in a, and $\mathrm{last}(a) = a_n$ denote the last letter of a. We say that $a = (a_1, \ldots, a_n) \in Asc_n$ is an *up-down* ascent sequence if $a_1 < a_2 > a_3 < a_4 > \cdots$. That is, $a = (a_1, \ldots, a_n) \in Asc_n$ is an up-down ascent sequence if $a_i < a_{i+1}$ whenever i is odd, and $a_i > a_{i+1}$ whenever i is even. Throughout this paper, we will often identify a sequence $(a_1, \ldots, a_n)$ in $\mathbb{N}^n$ with the word $a_1 \ldots a_n$. Thus, instead of writing, say, $(0, 0, 0)$, we will simply write 000, or 0^3.

We note that there has been considerable work on ascent sequences in recent years, see, for example, [1, 4, 6, 15]. Ascent sequences are important because they are in bijection with several other interesting combinatorial objects. To be more precise, it follows from the work of [1, 5, 3] that there are natural bijections between Asc_n and the following four classes of combinatorial objects.

- The set of $(\mathbf{2}+\mathbf{2})$-free posets of size n. Here we consider two posets to be equal if they are isomorphic, and an unlabeled poset is said to be $(\mathbf{2}+\mathbf{2})$-free if it does not contain an induced subposet that is isomorphic to $(\mathbf{2}+\mathbf{2})$, the union of two disjoint 2-element chains. $(\mathbf{2}+\mathbf{2})$-free posets are known to be in 1-to-1 correspondence with celebrated *interval orders*.

- The set M_n of upper triangular matrices of nonnegative integers such that no row or column contains all zero entries, and the sum of the entries is n.

- The set R_n of permutations of $[n] = \{1, \ldots, n\}$, where in each occurrence of the pattern 231, either the letters corresponding to the 2 and the 3 are nonadjacent, or else the letters corresponding to the 2 and the 1 are nonadjacent in value. Here, a word contains an occurrence of the pattern 231 if it contains a subsequence of length 3 that is order-isomorphic to 231; see [14] for a comprehensive introduction to the theory of patterns in permutations and words.

- The set Mch_n of Stoimenow matchings on $[2n]$. A *matching* of the set $[2n] = \{1, 2, \ldots, 2n\}$ is a partition of $[2n]$ into subsets of size 2, each of which is called an *arc*. The smaller number in an arc is its *opener*, and the larger one is its *closer*. A matching is said to be *Stoimenow* if it has no pair of arcs $\{a < b\}$ and $\{c < d\}$ that satisfy one (or both) of the following conditions: (a) $a = c + 1$ and $b < d$ and (b) $a < c$ and $b = d + 1$. In other words, a Stoimenow matching has no pair of arcs such that one is nested within the other and either the openers or the closers of the two arcs differ by 1.

Also, Remmel [17] showed that there is an interesting connection between the *Genocchi numbers* G_{2n} and the *median Genocchi numbers* H_{2n-1} and up-down ascent sequences. In particular, Remmel showed that G_{2n} is the number of up-down ascent sequences of length $2n-1$, H_{2n-1} is the number of up-down ascent sequences of length $2n-2$, and that up-down ascent sequences can be used to give a natural combinatorial interpretation of the q-Genocchi numbers of Zeng and Zhou [25].

Let p_n be the number of $(\mathbf{2}+\mathbf{2})$-free posets on n elements or, equivalently, the number of ascent sequences of length n. Bousquet-Mélou et al. [1] showed that the generating function for the number p_n of $(\mathbf{2}+\mathbf{2})$-free posets on n elements is

$$P(t) = \sum_{n\geq 0} p_n t^n = \sum_{n\geq 0} \prod_{i=1}^{n} \left(1-(1-t)^i\right). \tag{1}$$

In fact, Bousquet-Mélou et al. [1] studied a more general generating function

$$F(t,u,v) = \sum_{w\in Asc} t^{|w|} u^{\mathrm{asc}(w)} v^{\mathrm{last}(w)}$$

and found an explicit form for such a generating function. Kitaev and Remmel [15] studied a refined version of this generating function. That is, they found an explicit formula for the generating function

$$G(t,u,v,z,x) := \sum_{w\in Asc} t^{|w|} u^{\mathrm{asc}(w)} v^{\mathrm{last}(w)} z^{|w|_0} x^{\mathrm{run}(w)},$$

where for any ascent sequence w, $\mathrm{run}(w) = 0$ if $w = 0^n$ for some n, and $\mathrm{run}(w) = r$ if $w = 0^r xv$, where x is a positive integer and v is a word. Thus $\mathrm{run}(w)$ keeps track of the initial sequences of 0s that start out w if w does not consist of all zeros. Kitaev and Remmel were able to use their formula for $G(t,u,v,z,x)$ to prove that

$$A(t,z) := \sum_{w\in Asc} t^{|w|} z^{|w|_0} = 1 + \sum_{n\geq 0} \frac{zt}{(1-zt)^{n+1}} \prod_{i=1}^{n} (1-(1-t)^i). \tag{2}$$

1.2 p-ascent sequences

In this paper, we introduce a generalization of ascent sequences, which we call *p-ascent sequences*, where $p \geq 1$. A sequence $(a_1,\ldots,a_n)$ of nonnegative integers is a *p-ascent sequence* if $a_0 = 0$ and for all $i \geq 2$, a_i is at most p plus the number of ascents in $(a_1,\ldots,a_{i-1})$. Thus, in our terminology, ascent sequences are 1-ascent sequences.

We note that p-ascent sequences of length n can be encoded in terms of (usual) ascent sequences of length $n+2p-2$. Indeed, it is easy to see that $(a_1,a_2,\ldots,a_n)$ is a p-ascent sequence if and only if $(0,1,0,1,\ldots,0,1,a_1,a_2,\ldots,a_n)$ is an ascent sequence, where there are $p-1$ 0s and $p-1$ 1s preceding the $a_1 = 0$. Thus, p-ascent sequences can be thought of as a subset of ascent sequences of special type, namely, those ascents sequences that start out with $(01)^{p-1}0$.

The last observation allows to obtain a characterization of elements counted by p-ascent sequences in $(\mathbf{2}+\mathbf{2})$-free posets, the set of restricted permutations R_n, the set of upper triangular matrices M_n, and the set of Stoimenow matchings Mch_n whenever we can characterize the images of ascent sequences whose corresponding words start with $(01)^{p-1}0$. We do not get into much detail here providing just one example; we leave the other cases to the interested reader to explore. The $(\mathbf{2}+\mathbf{2})$-free posets corresponding to p-ascent sequences are $(\mathbf{2}+\mathbf{2})$-free posets on $n+2p-2$ elements with the following property. Right before the last $2p-1$ steps in decomposition of such posets (the decomposition is described in [1]; we do not provide its details here due to space concerns), one obtains the poset with p minimum elements and the other $p-1$ elements forming the pattern of the poset in Figure 1 corresponding to the case

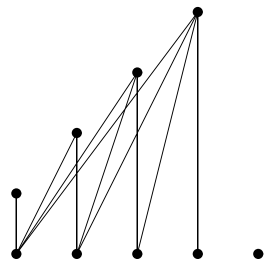

Figure 1: Type of poset obtained right before the last $2p-1$ steps in decomposition of the $(\mathbf{2}+\mathbf{2})$-free poset corresponding to a p-ascent sequence.

$p=5$. Of course, it would be interesting to give a direct characterization of such posets (e.g., in terms of forbidden sub-posets) but we were not able to succeed with that.

The main goal of this paper is to generalize the results of [15] to p-ascent sequences. That is, let $Asc(p)$ denote the set of p-ascent sequences, where, again, we consider the empty word to be a p-ascent sequence for any $p \geq 1$. Thus, the set of ascent sequences Asc is $Asc(1)$ in our terminology. First, we shall study the generating functions

$$G^{(p)}(t,u,v,z,x) := \sum_{w \in Asc(p)} t^{|w|} u^{\mathrm{asc}(w)} v^{\mathrm{last}(w)} z^{|w|_0} x^{\mathrm{run}(w)}. \tag{3}$$

We shall find an explict formula for $G^{(p)}(t,u,v,z,x)$ for any $p \geq 1$ (see Section 2) and then we shall use that formula to prove that

$$A^{(p)}(t,z) := \sum_{w \in Asc(p)} t^{|w|} z^{|w|_0} = 1 + \sum_{n \geq 0} \binom{p+n-1}{n} \frac{zt}{(1-zt)^{n+1}} \prod_{i=1}^{n} (1-(1-t)^i). \tag{4}$$

Duncan and Steingrímsson [6] introduced the study of pattern avoidance in ascent sequences. We initiate a similar study for p-ascent sequences. Given a word $w = w_1 \ldots w_n \in \mathbb{N}^*$, we let $\mathrm{red}(w)$ denote the word that is obtained from w by replacing each copy of the i-th smallest element in w by $i-1$. For example, $\mathrm{red}(238543623) = 015321401$. Then we say that a word $u = u_1 \ldots u_j$ *occurs* in w if there exist $1 \leq i_1 < \cdots < i_j \leq n$ such that $\mathrm{red}(w_{i_1} w_{i_2} \ldots w_{i_j}) = u$. We say that w *avoids* u if u does not occur in w.

For any word $u \in \mathbb{N}^*$ such that $\mathrm{red}(u) = u$, we let $a_{n,p,u}$ denote the number of p-ascent sequences a of length n avoiding u, and $r_{n,p,u}$ denote the number of sequences counted by $a_{n,p,u}$ with no equal consecutive letters, that is, $r_{n,p,u}$ is the number of *primitive sequences* counted by $a_{n,p,u}$. We prove a number of results about $a_{n,p,u}$ and $r_{n,p,u}$. For example, we will show that for all $p \geq 1$,

$$\begin{aligned} r_{n,p,10} &= \binom{p+n-2}{n-1} \text{ and} \\ a_{n,p,10} &= \sum_{s=0}^{n-1} \binom{n-1}{s} \binom{p+s-1}{s}. \end{aligned}$$

This paper is organized as follows. In Section 2, we shall find an explicit formula for $G^{(p)}(t,u,v,z,x)$. Unfortunately, we can not directly set $u=1$ in that formula so that in

Section 3, we shall find a formula for $G^{(p)}(t, 1, 1, 1, x)$ via an alternative proof. This formula will also allow us to find an explicit formula for the generating function for the number of primitive p-ascent sequences. Finally, in Section 4, we shall study $a_{n,p,u}$ and $r_{n,p,u}$ for certain patterns u of lengths 2 and 3.

2 Main results

For $r \geq 1$, let $G_r^{(p)}(t, u, v, z)$ denote the coefficient of x^r in $G^{(p)}(t, u, v, z, x)$. Thus $G_r^{(p)}(t, u, v, z)$ is the generating function of those p-ascent sequences that begin with $r \geq 1$ 0s followed by some element between 1 and p. We let $G_{a,\ell,m,n}^{(p,r)}$ denote the number of p-ascent sequences of length n, which begin with r 0s followed by some element between 1 and p, have a ascents, last letter ℓ, and a total of m zeros. We then let

$$G_r^{(p)}(t, u, v, z) = \sum_{a,\ell,m\geq 0,\ n\geq r+1} G_{a,\ell,m,n}^{(p,r)} t^n u^a v^\ell z^m. \tag{5}$$

Clearly, since the sequences of the form 0^n for some n have no ascents and no initial run of 0s (by definition), we have that the generating function for such sequences is

$$1 + tz + (tz)^2 + \cdots = \frac{1}{1 - tz},$$

where 1 corresponds to the empty word. Thus, we have the following relation between $G^{(p)}$ and $G_r^{(p)}$:

$$G^{(p)}(t, u, v, x, z) = \frac{1}{1 - tz} + \sum_{r\geq 1} x^r G_r^{(p)}(t, u, v, z). \tag{6}$$

Lemma 1. *For $r \geq 1$, the generating function $G_r^{(p)}(t, u, v, z)$ satisfies*

$$(v - 1 - tv(1 - u))G_r^{(p)}(t, u, v, z) = \\ t^{r+1} z^r uv(v^p - 1) + t((v - 1)z - v)G_r^{(p)}(t, u, 1, z) + tuv^{p+1} G_r(t, uv, 1, z). \tag{7}$$

Proof. Our proof follows the same steps as the proof of the $p = 1$ case of the result that was provided in [15]. Fix $r \geq 1$. Let $x' = (x_1, \ldots, x_{n-1})$ be an ascent sequence beginning with r 0s followed by a nonzero element, with a ascents and m zeros, where $x_{n-1} = \ell$. Then $x = (x_1, \ldots, x_{n-1}, i)$ is an ascent sequence if and only if $i \in [0, a + p]$. Clearly, x also begins with r 0s followed by a nonzero element. Now, if $i = 0$, the sequence x has a ascents and $m + 1$ zeros. If $1 \leq i \leq \ell$, x has a ascents and m zeros. Finally if $i \in [\ell + 1, a + p]$, then x has $a + 1$ ascents and m zeros. Counting the sequences $0 \ldots 0q$ with r 0s and $1 \leq q \leq p$ separately, we have

$$
\begin{aligned}
G_r^{(p)}(t,u,v,z) &= t^{r+1}uvz^r\frac{v^p-1}{v-1}+\\
&\sum_{\substack{a,\ell,m\geq 0\\ n\geq r+1}} G_{a,\ell,m,n}^{(p,r)}t^{n+1}\left(u^av^0z^{m+1}+\sum_{i=1}^{\ell}u^av^iz^m+\sum_{i=\ell+1}^{a+p}u^{a+1}v^iz^m\right)\\
&= t^{r+1}uvz^r\frac{v^p-1}{v-1}+t\sum_{\substack{a,\ell,m\geq 0\\ n\geq r+1}} G_{a,\ell,m,n}^{(p,r)}t^nu^az^m\left(z+\frac{v^{\ell+1}-v}{v-1}+u\frac{v^{a+p+1}-v^{\ell+1}}{v-1}\right)\\
&= t^{r+1}uvz^r\frac{v^p-1}{v-1}+tzG_r^{(p)}(t,u,1,z)+\\
&\quad tv\frac{G_r^{(p)}(t,u,v,z)-G_r(t,u,1,z)}{v-1}+tuv\frac{v^pG_r(t,uv,1,z)-G_r(t,u,v,z)}{v-1}.
\end{aligned}
$$

The result follows. □

Next, just like in the proof of the $p=1$ case in [15], we use the kernel method to proceed. Setting $(v-1-tv(1-u))=0$ and solving for v, we obtain that the substitution $v=1/(1+t(u-1))$ will eliminate the left-hand side of (7). We can then solve for $G_r^{(p)}(t,u,1,z)$ to obtain that

$$(1+zt(u-1))G_r^{(p)}(t,u,1,z)=\frac{t^rz^ru(1-\delta_1^p)}{\delta_1^p}+\frac{u}{\delta_1^p}G_r^{(p)}(t,\frac{u}{\delta_1},1,z). \tag{8}$$

Setting $\gamma_1=1+zt(u-1)$, we see that

$$G_r^{(p)}(t,u,1,z)=\frac{t^rz^ru}{\gamma_1\delta_1^p}(1-\delta_1^p)+\frac{u}{\gamma_1\delta_1^p}G_r^{(p)}(t,\frac{u}{\delta_1},1,z) \tag{9}$$

where $\delta_1=1+t(u-1)$.

Next we define

$$\delta_k := u-(1-t)^k(u-1) \text{ and} \tag{10}$$
$$\gamma_k := u-(1-zt)(1-t)^{k-1}(u-1) \tag{11}$$

for $k\geq 1$. We also set $\delta_0=\gamma_0=1$. Observe that $\delta_1=u-(1-t)(u-1)=1+t(u-1)$ and $\gamma_1=u-(1-zt)(u-1)=1+zt(u-1)$.

For any function of $f(u)$, we shall write $f(u)|_{u=\frac{u}{\delta_k}}$ for $f(u/\delta_k)$. It is then easy to check that

1. $(u-1)|_{u=\frac{u}{\delta_k}}=\dfrac{(1-t)^k(u-1)}{\delta_k}$,
2. $\delta_s|_{u=\frac{u}{\delta_k}}=\dfrac{\delta_{s+k}}{\delta_k}$,
3. $\gamma_s|_{u=\frac{u}{\delta_k}}=\dfrac{\gamma_{s+k}}{\delta_k}$, and
4. $\dfrac{u}{\delta_s}|_{u=\frac{u}{\delta_k}}=\dfrac{u}{\delta_{s+k}}$.

Using these relations, one can iterate the recursion (9). For example,

$$\begin{aligned}
G_r^{(p)}(t,u,1,z) &= \frac{t^r z^r u(1-\delta_1^p)}{\gamma_1 \delta_1^p} + \\
&\quad \frac{u}{\delta_1^p}\left(\frac{t^r z^r \frac{u}{\delta_1}\left(1-\frac{\gamma_2}{\delta_1}\frac{\delta_2^p}{\delta_1^p}\right)}{\frac{\gamma_2}{\delta_1}\frac{\delta_2^p}{\delta_1^p}} + \frac{\frac{u}{\delta_1}}{\frac{\delta_2^p}{\delta_1^p}} G_r^{(p)}(t,\frac{u}{\delta_2},1,z)\right) \\
&= \frac{t^r z^r u(1-\delta_1^p)}{\gamma_1 \delta_1^p)} + \frac{t^r z^r u^2\left(1-\frac{\delta_2^p}{\delta_1^p}\right)}{\gamma_1\gamma_2\delta_2^p} + \frac{u^2}{\gamma_1\gamma_2\delta_2^p} G_r^{(p)}(t,\frac{u}{\delta_2},1,z).
\end{aligned}$$

In general,

$$\begin{aligned}
\frac{u^k}{\gamma_1\cdots\gamma_k\delta_k^p} G_r^{(p)}(t,\frac{u}{\delta_k},1,z) &= \frac{u^k}{\gamma_1\ldots\gamma_k\delta_k^p}\left(\frac{t^r z^r \frac{u}{\delta_k}\left(1-\frac{\delta_{k+1}^p}{\delta_k^p}\right)}{\frac{\gamma_{k+1}}{\delta_k}\frac{\delta_{k+1}^p}{\delta_k^p}} + \frac{\frac{u}{\delta_k}}{\frac{\gamma_{k+1}}{\delta_k}\frac{\delta_{k+1}^p}{\delta_k^p}} G_r^{(p)}(t,\frac{u}{\delta_{k+1}},1,z)\right) \\
&= \frac{t^r z^r u^{k+1}\left(1-\frac{\delta_{k+1}^p}{\delta_k^p}\right)}{\gamma_1\cdots\gamma_{k+1}\delta_{k+1}^p} + \frac{u^{k+1}}{\gamma_1\cdots\gamma_{k+1}\delta_{k+1}^p} G_r^{(p)}(t,\frac{u}{\delta_{k+1}},1,z).
\end{aligned}$$

Thus, by iterating recursion (9), we can derive that

$$G_r^{(p)}(t,u,1,z) = \frac{t^r z^r u(1-\delta_1^p)}{\gamma_1\delta_1^p} + \sum_{k=2}^{\infty} \frac{t^r z^r u^k\left(1-\frac{\delta_k^p}{\delta_{k-1}^p}\right)}{\gamma_1\cdots\gamma_k\delta_k^p}. \tag{12}$$

Note that since $\delta_0 = 1$, we can rewrite $\frac{t^{r+1} z^r u(1-\delta_1^p)}{\gamma_1\delta_1^p}$ as $\frac{t^r z^r u(\delta_0^p-\delta_1^p)}{\gamma_1\delta_0^p\delta_1^p}$ and we can rewrite $\frac{t^r z^r u^k\left(1-\frac{\delta_k^p}{\delta_{k-1}^p}\right)}{\gamma_1\cdots\gamma_k\delta_k^p}$ as $\frac{t^r z^r u(\delta_{k-1}^p-\delta_k^p)}{\gamma_1\cdots\gamma_k\delta_{k-1}^p\delta_k^p}$. Thus we have proved the following theorem.

Theorem 2.

$$G_r^{(p)}(t,u,1,z) = \sum_{k=1}^{\infty} \frac{t^r z^r u^k(\delta_{k-1}^p-\delta_k^p)}{\gamma_1\cdots\gamma_k\delta_{k-1}^p\delta_k^p}. \tag{13}$$

We have used Mathematica to compute that

$$\begin{aligned}
&G_1^{(2)}(t,u,1,z) = 2uzt^2 + \left(3zu^2+(3z+2z^2)u\right)t^3 + \\
&\left(4zu^3+(13z+9z^2)u^2+(4z+3z^2+2z^3)u\right)t^4 + \\
&\left(5zu^4+(39z+28z^2)u^3+(35z+34z^2+15z^3)u^2+(5z+4z^2+3z^3+2z^4)u\right)t^5 + O[t]^6.
\end{aligned}$$

$$\begin{aligned}
&G_1^{(3)}(t,u,1,z) = 3uzt^2 + \left(6zu^2+(6z+3z^2)u\right)t^3 + \\
&\left(10zu^3+(34z+18z^2)u^2+(10z+6z^2+3z^3)u\right)t^4 + \\
&\left(15zu^4+(125z+70z^2)u^3+(115z+88z^2+30z^3)u^2+(15z+10z^2+6z^3+3z^4)u\right)t^5 + \\
&O[t]^6.
\end{aligned}$$

$$G_1^{(4)}(t,u,1,z) = 4uzt^2 + \big(10zu^2 + (10z+4z^2)u\big)\,t^3 +$$
$$\big(20zu^3 + (70z+30z^2)u^2 + (20z+10z^2+4z^3)u\big)\,t^4 +$$
$$\big(35zu^4 + (305z+140z^2)u^3 + (285z+180z^2+50z^3)u^2 + (35z+20z^2+10z^3+4z^4)u\big)\,t^5 +$$
$$O[t]^6.$$

For example, the coefficient of t^3 in $G_1^{(2)}(t,u,1,z)$, which is $3zu^2 + (3z+2z^2)u$, makes sense since there are eight 2-ascent sequences which start with 0 and are followed by a nonzero element, namely,

$$010, 011, 012, 013, 020, 021, 022, \text{ and } 023,$$

of which three have one zero and two ascents, three have one zero and one ascent, and two have two zeros and one ascent. Similarly, there a 19 3-ascent sequences of length 4 which have only one ascent, namely,

$$0111, 0110, 0100, 0222, 0221, 0220, 0211, 0210, 0200.$$
$$0333, 0332, 0331, 0330, 0322, 0321, 0320, 0311, 0310, \text{ and } 0300,$$

three of which have three zeros, six of which have two zeros, and ten of which have one zero, and this is consistent with the term $(10z+6z^2+3z^3)ut^4$.

Note that we can rewrite (7) as

$$G_r^{(p)}(t,u,v,z) = \frac{t^{r+1}z^r uv(v^p-1)}{v\delta_1 - 1} + \frac{t(z(v-1)-v)}{v\delta_1-1}G_r^{(p)}(t,u,1,z) + \frac{uv^{p+1}t}{v\delta_1-1}G_r^{(p)}(t,uv,1,z). \quad (14)$$

For $s \geq 1$, we let

$$\begin{aligned}\bar{\delta}_s &= \delta_s|_{u=uv} = uv - (1-t)^s(uv-1) \text{ and}\\ \bar{\gamma}_s &= \gamma_s|_{u=uv} = uv - (1-zt)(1-t)^{s-1}(uv-1)\end{aligned}$$

and set $\bar{\delta}_0 = \bar{\gamma}_0 = 1$. Then using (14) and (13), we have the following theorem.

Theorem 3. *For all $r \geq 1$,*

$$G_r^{(p)}(t,u,v,z) =$$
$$t^r z^r \left(\frac{tuv(v^p-1)}{v\delta_1-1} + \frac{t(z(v-1)-v)}{v\delta_1-1}\sum_{k\geq 1}\frac{(\delta_{k-1}^p - \delta_k^p)}{\gamma_1\cdots\gamma_k\delta_{k-1}^p\delta_k^p} + \frac{tuv^{p+1}}{v\delta_1-1}\sum_{k\geq 1}\frac{(\bar{\delta}_{k-1}^p - \bar{\delta}_k^p)}{\bar{\gamma}_1\cdots\bar{\gamma}_k\bar{\delta}_{k-1}^p\bar{\delta}_k^p}\right). \quad (15)$$

We have used Mathematica to compute that

$$G_1^{(2)}(t,u,v,z) = (uvz+uv^2z)t^2 + (2uvz+uv^2z+u^2v^2z+2u^2v^3z+2uz^2)t^3 +$$
$$\big(3uvz+3u^2vz+uv^2z+5u^2v^2z+5u^2v^3z+u^3v^3z+3u^3v^4z+3uz^2+3u^2z^2+$$
$$2u^2vz^2+2u^2v^2z^2+2u^2v^3z^2+2uz^3\big)\,t^4 +$$
$$\big(4uvz+13u^2vz+4u^3vz+uv^2z+13u^2v^2z+7u^3v^2z+9u^2v^3z+12u^3v^3z+$$
$$16u^3v^4z+u^4v^4z+4u^4v^5z+4uz^2+13u^2z^2+4u^3z^2+9u^2vz^2+3u^3vz^2+$$
$$7u^2v^2z^2+5u^3v^2z^2+5u^2v^3z^2+7u^3v^3z^2+9u^3v^4z^2+3uz^3+9u^2z^3+$$
$$2u^2vz^3+2u^2v^2z^3+2u^2v^3z^3+2uz^4\big)\,t^5 + O[t]^6.$$

$$
\begin{aligned}
&G_1^{(3)}(t,u,v,z) = (uvz + uv^2z + uv^3z)t^2 + \\
&\left(3uvz + 2uv^2z + u^2v^2z + uv^3z + 2u^2v^3z + 3u^2v^4z + 3uz^2\right)t^3 + \\
&\left(6uvz + 6u^2vz + 3uv^2z + 9u^2v^2z + uv^3z + 10u^2v^3z + u^3v^3z + 9u^2v^4z + 3u^3v^4z + \right.\\
&\quad \left. 6u^3v^5z + 6uz^2 + 6u^2z^2 + 3u^2vz^2 + 3u^2v^2z^2 + 3u^2v^3z^2 + 3u^2v^4z^2 + 3uz^3\right)t^4 + \\
&\left(10uvz + 34u^2vz + 10u^3vz + 4uv^2z + 34u^2v^2z + 16u^3v^2z + uv^3z + 28u^2v^3z + 25u^3v^3z + \right.\\
&\quad 19u^2v^4z + 34u^3v^4z + u^4v^4z + 40u^3v^5z + 4u^4v^5z + 10u^4v^6z + 10uz^2 + 34u^2z^2 + 10u^3z^2 + \\
&\quad 18u^2vz^2 + 6u^3vz^2 + 15u^2v^2z^2 + 9u^3v^2z^2 + 12u^2v^3z^2 + 12u^3v^3z^2 + 9u^2v^4z^2 + 15u^3v^4z^2 + \\
&\quad \left. 18u^3v^5z^2 + 6uz^3 + 18u^2z^3 + 3u^2vz^3 + 3u^2v^2z^3 + 3u^2v^3z^3 + 3u^2v^4z^3 + 3uz^4\right)t^5 + O[t]^6.
\end{aligned}
$$

For example, the term $3uv^2zt^4$ that appears in $G_1^{(3)}(t,u,v,z)$ corresponds to the sequences 0222, 0322, and 0332.

It is easy to see from Theorem 3 that

$$G_r^{(p)}(t,u,v,z) = t^{r-1}z^{r-1}G_1^{(p)}(t,u,v,z). \tag{16}$$

The relation (16) is also easy to see combinatorially since every ascent sequence counted by $G_r^{(p)}(t,u,v,z)$ is of the form $0^{r-1}a$, where a is a p-ascent sequence counted by $G_1^{(p)}(t,u,v,z)$.

Note that

$$
\begin{aligned}
G^{(p)}(t,u,v,z,x) &= \frac{1}{1-tz} + \sum_{r\geq 1} G_r^{(p)}(t,u,v,z)x^r \\
&= \frac{1}{1-tz} + \sum_{r\geq 1} t^{r-1}z^{r-1}G_1^{(p)}(t,u,v,z)x^r \\
&= \frac{1}{1-tz} + \frac{x}{1-tzx}G_1^{(p)}(t,u,v,z).
\end{aligned}
$$

Thus we have the following theorem.

Theorem 4.

$$G^{(p)}(t,u,v,z,x) = \frac{1}{1-tz} + \frac{x}{1-tzx}G_1^{(p)}(t,u,v,z). \tag{17}$$

3 Specializations of our general results

In this section, we shall compute the generating function for p-ascent sequences by length and the number of zeros.

For $n \geq 1$, let $H_{a,b,\ell,n}^{(p)}$ denote the number of p-ascent sequences of length n with a ascents and b zeros which have last letter ℓ. Then we first wish to compute

$$H^{(p)}(t,u,v,z) = \sum_{n\geq 1,\ a,b,\ell\geq 0} H_{a,b,\ell,n}^{(p)} u^a z^b v^\ell t^n. \tag{18}$$

Using the same reasoning as in the previous section, we see that

$$\begin{aligned}
H^{(p)}(t,u,v,z) &= tz + \sum_{\substack{a,b,\ell\geq 0\\ n\geq 1}} H^{(p)}_{a,b,\ell,n} t^{n+1}\left(u^a v^0 z^{b+1} + \sum_{i=1}^{\ell} u^a v^i z^b + \sum_{i=\ell+1}^{a+p} u^{a+1} v^i z^b\right)\\
&= tz + t\sum_{\substack{a,b,\ell\geq 0\\ n\geq r+1}} H_{a,b,\ell,n} t^n u^a z^b\left(z + \frac{v^{\ell+1}-v}{v-1} + u\frac{v^{a+p+1}-v^{\ell+1}}{v-1}\right)\\
&= tz + tzH^{(p)}(t,u,1,z) + \frac{tv}{v-1}\left(H^{(p)}(t,u,v,z) - H^{(p)}(t,u,1,z)\right) + \\
&\quad \frac{tuv}{v-1}\left(H^{(p)}(t,uv,1,z) - H^{(p)}(t,u,v,z)\right).
\end{aligned}$$

Solving for $H^{(p)}(t,u,v,z)$, we see that we have the following lemma.

Lemma 5.

$$\begin{aligned}
&(v\delta_1 - 1)H^{(p)}(t,u,v,z) = \\
&\qquad (v-1)tz + t(z(v-1)-v)H^{(p)}(t,u,1,z) + tuv^{p+1}H^{(p)}(t,uv,1,z). \quad (19)
\end{aligned}$$

Again, the substitution $v = \frac{1}{\delta_1}$ eliminates the left-hand side of (19). We can then solve for $H^{(p)}(u,1,z,t)$ to obtain the recursion

$$H^{(p)}(t,u,1,z) = \frac{(1-\delta_1)z}{\gamma_1} + \frac{u}{\gamma_1\delta_1^p}H^{(p)}(t,\frac{u}{\delta_1},1,z). \quad (20)$$

We can iterate the recursion (20) in the same manner as we iterated the recursion (9) in the previous section to prove that

$$H^{(p)}(t,u,1,z) = \sum_{n\geq 0}\frac{(\delta_n - \delta_{n+1})zu^n}{\gamma_1\cdots\gamma_{n+1}\delta_n^p}. \quad (21)$$

Notice that for all $n \geq 0$,

$$\begin{aligned}
\delta_n - \delta_{n+1} &= (u-(1-t)^n(u-1)) - (u-(1-t)^{n+1}(u-1))\\
&= -(1-t)^n(u-1)(1-(1-t)\\
&= (1-u)t(1-t)^n.
\end{aligned}$$

Thus, as a power series in u, we can conclude the following.

Theorem 6.

$$H^{(p)}(t,u,1,z) = \sum_{n=0}^{\infty}\frac{zt(1-u)u^n(1-t)^n}{\delta_n^p\prod_{i=1}^{n+1}\gamma_i}. \quad (22)$$

We would like to set $u=1$ in the power series $\sum_{s=0}^{\infty}\frac{zt(1-u)u^s(1-t)^s}{\delta_s\prod_{i=1}^{s+1}\gamma_i}$, but the factor $(1-u)$ in the series does not allow us to do that in this form. Thus our next step is to rewrite the series in a form where it is obvious that we can set $u = 1$ in the series. To that end, observe that for $k \geq 1$,

$$\delta_k = u - (1-t)^k(u-1) = 1 + u - 1 - (1-t)^k(u-1) = 1 - ((1-t)^k - 1)(u-1),$$

so that by Newton's binomial theorem,

$$\begin{aligned}
\frac{1}{\delta_k^p} &= \left(\frac{1}{1-(u-1)((1-t)^k-1)}\right)^p \\
&= \sum_{n=0}^{\infty} \binom{p-1+n}{n} ((u-1)((1-t)^k-1))^n \\
&= \sum_{n=0}^{\infty} \binom{p-1+n}{n} (u-1)^n \left(\sum_{m=0}^{n} (-1)^{n-m} \binom{n}{m} (1-t)^{km}\right). \qquad (23)
\end{aligned}$$

Substituting (23) into (22), we see that

$$\begin{aligned}
&H^{(p)}(t,u,1,z) = \\
&\frac{zt(1-u)}{\gamma_1} + \sum_{k\geq 1} \frac{zt(1-u)u^k(1-t)^k}{\prod_{i=1}^{k+1}\gamma_i} \sum_{n\geq 0} \binom{p-1+n}{n} (u-1)^n \sum_{m=0}^{n} (-1)^{n-m} \binom{n}{m} (1-t)^{km} = \\
&\frac{zt(1-u)}{\gamma_1} + \sum_{n\geq 0}\sum_{m=0}^{n} (-1)^{n-m-1} \binom{n}{m} (u-1)^{n-m} zt \sum_{k\geq 1} \frac{(u-1)^{m+1}u^k(1-t)^{k(m+1)}}{\prod_{i=1}^{k+1}\gamma_i} = \\
&\frac{zt(1-u)}{\gamma_1} + \sum_{n\geq 0} \binom{p-1+n}{n} \sum_{m=0}^{n} (-1)^{n-m-1} \binom{n}{m} (u-1)^{n-m} \frac{zt}{(1-zt)^{m+1}} \times \\
&\sum_{k\geq 1} \frac{(u-1)^{m+1}(1-zt)^{m+1}u^k(1-t)^{k(m+1)}}{\prod_{i=1}^{k+1}\gamma_i}.
\end{aligned}$$

In [15], we have proved the following lemma.

Lemma 7.

$$\begin{aligned}
\psi_{m+1}(u) &= \sum_{k\geq 0} \frac{(u-1)^{m+1}(1-zt)^{m+1}u^k(1-t)^{k(m+1)}}{\prod_{i=1}^{k+1}\gamma_i} \\
&= -\sum_{j=0}^{m} (u-1)^j (1-zt)^j u^{m-j} \prod_{i=j+1}^{m} (1-((1-t)^i). \qquad (24)
\end{aligned}$$

It thus follows that

$$\begin{aligned}
H^{(p)}(t,u,1,z) &= \frac{zt(1-u)}{\gamma_1} + \sum_{n\geq 0} \binom{p-1+n}{n} \sum_{m=0}^{n} (-1)^{n-m-1} \binom{n}{m} (u-1)^{n-m} \frac{zt}{(1-zt)^{m+1}} \times \\
&\left(-\frac{(u-1)^{m+1}(1-zt)^{m+1}}{\gamma_1} - \sum_{j=0}^{m} (u-1)^j (1-zt)^j u^{m-j} \prod_{i=j+1}^{m} (1-(1-t)^i)\right).
\end{aligned}$$

There is no problem in setting $u = 1$ in this expression to obtain that

$$H^{(p)}(t,1,1,z) = \sum_{n\geq 0} \binom{p-1+n}{n} \frac{zt}{(1-zt)^{n+1}} \prod_{i=1}^{n} (1-(1-t)^i). \tag{25}$$

Clearly, our definitions ensure that $1 + H(t,1,1,z) = A^{(p)}(t,z)$ as defined in the introduction so that we have the following theorem.

Theorem 8. *For all $p \geq 1$,*

$$A^{(p)}(t,z) = \sum_{w\in Asc(p)} t^{|w|} z^{|w|_0} = 1 + \sum_{n\geq 0} \binom{p-1+n}{n} \frac{zt}{(1-zt)^{n+1}} \prod_{i=1}^{n} (1-(1-t)^i). \tag{26}$$

The case $p = 1$ in Theorem 8 gives exactly the same formula for $A^{(1)}(t,z)$ as that derived in [15], which should be the case. We also note that the authors conjectured in [15] that

$$1 + \sum_{k=0}^{\infty} \frac{zt}{(1-zt)^{k+1}} \prod_{i=1}^{k} (1-((1-t)^i) = 1 + \sum_{m=1}^{\infty} \prod_{i=1}^{m} (1-(1-t)^{i-1}(1-zt)). \tag{27}$$

This was proved independently by Jelínek [12], Levande [16], and Yan [23]. It would be interesting to find an analogue of this relation for $p > 1$.

We have used Mathematica to compute the first few terms of $A^{(p)}(t,z)$ for $p = 2, 3, 4$:

$$A^{(2)}(t,z) = 1 + zt + \left(2z + z^2\right) t^2 + \left(6z + 4z^2 + z^3\right) t^3 + \left(21z + 18z^2 + 6z^3 + z^4\right) t^4 + \left(84z + 87z^2 + 36z^3 + 8z^4 + z^5\right) t^5 + \left(380z + 456z^2 + 222z^3 + 60z^4 + 10z^5 + z^6\right) t^6 + O[t]^7.$$

$$A^{(3)}(t,z) = 1 + zt + \left(3z + z^2\right) t^2 + \left(12z + 6z^2 + z^3\right) t^3 + \left(54z + 36z^2 + 9z^3 + z^4\right) t^4 + \left(270z + 222z^2 + 72z^3 + 12z^4 + z^5\right) t^5 + \left(1490z + 140z^2 + 564z^3 + 120z^4 + 15z^5 + z^6\right) t^6 + O[t]^7.$$

$$A^{(4)}(t,z) = 1 + zt + \left(4z + z^2\right) t^2 + \left(20z + 8z^2 + z^3\right) t^3 + \left(110z + 60z^2 + 12z^3 + z^4\right) t^4 + \left(660z + 450z^2 + 90z^3 + 16z^4 + z^5\right) t^5 + \left(4300z + 3480z^2 + 1140z^3 + 200z^4 + 20z^5 + z^6\right) t^6 + O[t]^7.$$

Next we can use the same techniques as in [4] to find the generating function for the number of *primitive p-ascent sequences*. That is, let $r_{n,p}$ denote the number of p-ascent sequences a of length n such that a has no consecutive repeated letters and $a_{n,p}$ denote the number of p-ascent sequences a of length n.

If

$$\begin{aligned} R^{(p)}(t) &= 1 + \sum_{n\geq 1} r_{n,p} t^n \text{ and} \\ A^{(p)}(t) &= 1 + \sum_{n\geq 1} a_{n,p} t^n, \end{aligned}$$

then it is easy to see that

$$A^{(p)}(t) = A^{(p)}(t,1) = R^{(p)}\left(\frac{t}{1-t}\right) = R^{(p)}(t + t^2 + \cdots), \tag{28}$$

since each element in a primitive p-ascent sequence can be repeated any specified number of times.

Setting $x = \frac{t}{1-t}$ so that $t = \frac{x}{1+x}$, we see that (28) implies that

$$R^{(p)}(x) = A^{(p)}\left(\frac{x}{1+x}\right). \tag{29}$$

But by (26), we know that

$$A^{(p)}(t) = 1 + \sum_{n=0}^{\infty} \binom{p-1+n}{n} \frac{t}{(1-t)^{n+1}} \prod_{i=1}^{n} (1-(1-t)^i).$$

Hence,

$$\begin{aligned} R^{(p)}(x) &= 1 + \sum_{n=0}^{\infty} \binom{p-1+n}{n} \frac{\frac{x}{1+x}}{(1-\frac{x}{1+x})^{n+1}} \prod_{i=1}^{n} \left(1 - \left(1 - \frac{x}{1+x}\right)^i\right) \\ &= 1 + x\sum_{n=0}^{\infty} \binom{p-1+n}{n} (1+x)^n \prod_{i=1}^{n} \left(1 - \left(\frac{1}{1+x}\right)^i\right). \end{aligned}$$

Thus, we have the following theorem.

Theorem 9. *For all $p \geq 1$,*

$$R^{(p)}(t) = 1 + t\sum_{n=0}^{\infty} \binom{p-1+n}{n} (1+t)^n \prod_{i=1}^{n} \left(1 - \left(\frac{1}{1+t}\right)^i\right). \tag{30}$$

For example, we have computed that

$R^{(2)}(t) = 1 + t + 2t^2 + 6t^3 + 21t^4 + 87t^5 + 413t^6 + 2213t^7 + 13205t^8 + 86828t^9 + O[t]^{10}$,
$R^{(3)}(t) = 1 + t + 3t^2 + 12t^3 + 54t^4 + 276t^5 + 1574t^6 + 9916t^7 + 68394t^8 + 512671t^9 + O[t]^{10}$,
$R^{(4)}(t) = 1 + t + 4t^2 + 20t^3 + 110t^4 + 670t^5 + 4470t^6 + 32440t^7 + 254490t^8 + 2146525t^9 + O[t]^{10}$.

We note that by (27), in the case of $z = 1$, we have that

$$A^{(1)}(t) = 1 + \sum_{m=1}^{\infty} \prod_{i=1}^{m} (1-(1-t)^i) = 1 + \sum_{n\geq 0} \frac{t}{(1-t)^{n+1}} \prod_{i=1}^{n} (1-(1-t)^i).$$

This leads to two forms for $R^{(1)}(t) = A^{(1)}\left(\frac{t}{1+t}\right)$. That is, it follows that

$$R^{(1)}(t) = 1 + t\sum_{n=0}^{\infty} (1+t)^n \prod_{i=1}^{n} \left(1 - \left(\frac{1}{1+t}\right)^i\right) = 1 + \sum_{m=1}^{\infty} \prod_{i=1}^{m} \left(1 - \frac{1}{(1+t)^i}\right),$$

where the second formula was derived in [4]. It would be interesting to find an analogue of this equality for $p > 1$.

Finally if we replace t by $t + t^2 + \cdots + t^k = t\frac{(t^k-1)}{t-1}$ in (30), then we can obtain the generating function for the number of p-ascent sequences a such that the maximum length of a consecutive sequence of repeated letters is less than or equal to k:

$$1 + t\frac{t^k-1}{t-1}\sum_{n=0}^{\infty}\binom{p-1+n}{n}\left(\frac{t^{k+1}-1}{t-1}\right)^n\prod_{i=1}^{n}\left(1-\left(\frac{t-1}{t^{k+1}-1}\right)^i\right). \tag{31}$$

4 Pattern avoidance in p-ascent sequences

In this section, we shall prove some simple results about pattern avoidance in p-ascent sequences thus extending the studies initiated in [6] for ascent sequences.

We begin by considering patterns of length 2. There are three such patterns, 00, 01, and 10. Recall that $a_{n,p,u}$ (resp., $r_{n,p,u}$) is the number of (resp., primitive) p-ascent sequences of length n that avoid a pattern u.

4.1 01-avoiding p-ascent sequences

The only p-ascent sequences that avoid 01 are the sequences that consist of all zeros so that $a_{n,p,01} = 1$ for all $n, p \geq 1$ and $r_{n,p,01}$ equals 1 if $n = 1$ and 0 otherwise.

4.2 10-avoiding p-ascent sequences

Let us consider $r_{n,p,10}$. In this case, we are looking for p-ascent sequences which avoid 10 and have no repeated letters. It is clear that any such a sequence a must be of the form $a = a_1 \dots a_n$, where $0 = a_1 < a_2 < \cdots < a_n$. For each $1 \leq i \leq n$, the word $a_1 \dots a_i$ has $i - 1$ ascents so that $a_{i+1} \leq i - 1 + p$. It follows that $r_{n,p,10}$ counts all words $a_1 a_2 \dots a_n$, where $0 = a_1 < a_2 < \cdots < a_n \leq p + n - 2$. Hence

$$r_{n,p,10} = \binom{p+n-2}{n-1}. \tag{32}$$

Note that it follows from Newton's Binomial Theorem that

$$\begin{aligned} R_{10}^{(p)}(t) &= 1 + \sum_{n\geq 1} r_{n,p,10} t^n \\ &= 1 + \sum_{n\geq 1}\binom{p-1+n-1}{n-1} t^n \\ &= 1 + \frac{t}{(1-t)^p}. \end{aligned} \tag{33}$$

It is easy to see that the p-ascent sequences counted by $a_{n,p,10}$ arise by taking a sequence $d_1 \dots d_s$ counted by $r_{s,p,10}$ for some $s \leq n$ and replacing each letter d_i by one or more copies so that the resulting word is of length n. The number of ways to do this for a given $d_1 \dots d_s$ is the number of solutions to $b_1 + \cdots + b_s = n$, where $b_i \geq 1$, which is $\binom{n-1}{s-1}$. Thus

$$a_{n,p,10} = \sum_{s=1}^{n} \binom{n-1}{s} r_{s,p,10} = \sum_{s=1}^{n} \binom{n-1}{s-1}\binom{p+s-2}{s-1} = \sum_{s=0}^{n-1} \binom{n-1}{s}\binom{p+s-1}{s}. \tag{34}$$

It also follows that

$$\begin{aligned} A_{10}^{(p)}(t) &= 1 + \sum_{n\geq 1} a_{n,p,10} t^n \\ &= R_{10}^{(p)}\left(\frac{t}{1-t}\right) \\ &= 1 + \frac{t}{1-t}\frac{1}{(1-\frac{t}{1-t})^p} \\ &= 1 + \frac{t(1-t)^{p-1}}{(1-2t)^p}. \end{aligned} \tag{35}$$

We note that the sequence $(a_{n,2,10})_{n\geq 1}$ starts out $1, 3, 8, 20, 48, 112, 256, \ldots$ and this is the sequence A001792 in the OEIS [19] which has many combinatorial interpretations.

4.3 00-avoiding p-ascent sequences

Next, consider avoiding the pattern 00. If a p-ascent sequence $a = a_1 \ldots a_n$ avoids 00, then all its elements must be distinct. Note that for each $2 \leq i \leq n$, $a_1 \ldots a_{i-1}$ can have at most $i-2$ ascents so that $a_i \leq p + i - 2$. Let $\max(a)$ denote the maximum of $\{a_1, \ldots, a_n\}$. If a avoids 00, then by the pigeon hole principle, it must be the case that $\max(a) \geq n-1$. Thus, if a avoids 00, then $n-1 \leq \max(a) \leq n+p-2$.

Now consider 2-ascent sequences that avoid 00. Suppose that $a = a_1 \ldots a_n$ is a 2-ascent sequence which avoids 00. Then we know that $\max(a) \in \{n-1, n\}$. If $\max(a) = n$, a must be strictly increasing and there must be some smallest $k \geq 1$ such that $a_k = k$, In such a situation, it is easy to see that a must be of the form $0, 1, \ldots, k-2, k, k+1, \ldots n$. Thus there are $n-1$ 2-ascent sequences a of length n such that a avoids 00 and $\max(a) = n$.

Next, suppose that $a = a_1 \ldots a_n$ is a 2-ascent sequence that avoids 00 and $\max(a) = n-1$. Then there are two cases. Namely, it could be that there is no $k \leq n$ such that $a_k = k$. In that case, a is the increasing sequence $a = 012 \ldots (n-1)$. Otherwise, let j equal the smallest i such that $a_i = i$. Then a must be strictly increasing up to a_j so that a starts out $012 \ldots (j-2)j$. Since $\max(a) = n-1$, it follows that $\{a_1, \ldots, a_n\} = \{0, 1, \ldots, n-1\}$ so that there must be some $j < k \leq n$ such that $a_k = j-1$. In that case, $a_{k-1} > a_k$ so that a has at least one descent. However, if $\max(a) = n-1$, a can have at most one descent. Thus, once we have placed $j-1$, the remaining elements must be placed in increasing order. It is then easy to check that no matter where we place $j-1$ after position j, the resulting sequence will be a 2-ascent sequence. It follows that the number of 2-ascent sequences which avoid 00 and have one descent is $\sum_{j=1} n-1(n-j) = \binom{n-1}{2}$.

It is easy to check that wherever we place $j-1$, $j-1$ will cause a descent and there can be at most one descent in a. Thus, we have the following theorem.

Theorem 10. *For all $n \geq 1$,*

$$a_{n,2,00} = n - 1 + 1 + \binom{n-1}{2} = 1 + \binom{n}{2}. \tag{36}$$

We computed that the sequence $(a_{n,3,00})_{n\geq 1}$ starts out

$$1, 3, 9, 24, 57, 122, 239, 435, 745, 1213, 1893, 2850, \ldots.$$

This is the sequence A089830 in the OEIS [19], whose generating function is

$$\frac{1 - 3x + 6x^2 - 5x^3 + 3x^4 - x^5}{(1-x)^6}.$$

In this case, if $a = a_1 \ldots a_n$ is a 3-ascent sequence which avoids 00, then we know that $n - 1 \leq \max(a) \leq n + 1$. We shall prove that

$$\sum_{n\geq 1} a_{n,3,00} x^n = \frac{x(1 - 3x + 6x^2 - 5x^3 + 3x^4 - x^5)}{(1-x)^6}$$

by classifying the 3-ascent sequences a which avoid 00 by the $\max(a)$ and $\mathrm{des}(a)$, where $\mathrm{des}(a)$ is the number of *descents* in a, that is, the number of elements followed by smaller elements.

Case 1. $\mathrm{des}(a) = 0$.
Suppose that $a = a_1 \ldots a_n$ is an increasing 3-ascent sequence that avoids 00. Now, if $\max(a) = n-1$, then $a = 012\ldots(n-1)$. If $\max(a) = n$, then exactly one element from $[n] = \{1, \ldots, n-1\}$ does not appear in a. If i does not appear in a, then $a = 01\ldots(i-1)(i+1)(i+2)\ldots n$, which is a 3-ascent sequence. Thus, there are $n - 1$ increasing 3-ascent sequences whose maximum is n. Finally, if $\max(a) = n + 1$, then two elements from $[n]$ do not appear in a. Again, it is easy to check that no matter which two elements from $[n]$ we leave out, the resulting increasing sequence will be a 3-ascent sequence. Thus, there are $\binom{n}{2}$ increasing 3-ascent sequences whose maximum is $n + 1$. Therefore, the total number of increasing 3-ascents sequences of length n is $1 + (n-1) + \binom{n}{2} = \binom{n+1}{2}$.

Case 2. $\mathrm{des}(a) = 1$.
In this case, if $a = a_1 \ldots a_n$ is a 3-ascent sequence such that $\mathrm{des}(a) = 1$ and a avoids 00, then $\max(a) \in \{n-1, n\}$. Suppose that $a_j > a_{j-1}$. Then we have two subcases.

Subcase 2.1. $a_j = j + 1$.
In this case, there must be two elements $1 \leq u < v \leq j$, which do not appear in $a_1 \ldots a_j$. Clearly, we have $\binom{j}{2}$ ways to pick u and v. We then have three subcases.

Subcase 2.1.1. Both u and v appear in a. In this case, a must start out $a_1 \ldots a_j uv$ so that $a_{j+3} \ldots a_n$ must be an increasing sequence from $[n] - [j+1]$ of length $n - j - 2$. Clearly, there are $n - j - 1$ such subsequences and it is easy to check that we can attach any such subsequence at the end of the sequence $a_1 \ldots a_j uv$ to obtain a 3-ascent sequence avoiding 00.

Subcase 2.1.2. u appears in a, but v does not appear in a.

In this case, a must be of the form $a_1 \dots a_j u\gamma$, where γ is the increasing sequence $(j+2)(j+3)\dots n$.

Subcase 2.1.3. v appears in a, but u does not appear in a.
In this case, a must be of the form $a_1 \dots a_j v\gamma$, where γ is the increasing sequence $(j+2)(j+3)\dots n$.

It follows that the number of 3-ascent sequences counted in Case 2.1 is $\sum_{j=2}^{n-1}\binom{j}{2}(n-j+1)$. One can verify by Mathematica that

$$\sum_{j=2}^{n-1}\binom{j}{2}(n-j+1) = \binom{n}{3} + \binom{n+1}{4}.$$

Case 2.2. $a_j = j$.
In this case, there is one element u in $[j]$ which does not appear in $a_1 \dots a_j$, so that the sequence must start out $a_1 \dots a_j u$. The rest of the sequence must be the increasing rearrangement of $\{j+1,\dots,n\} - \{v\}$ for some $v \in \{j+1,\dots,n\}$. Thus, we have $j-1$ choices for u and $n-j$ choices for v. Hence the number of 3-ascent sequences in Case 2.2 is $\sum_{j=2}^{n-1}(j-1)(n-j)$. One can check by Mathematica that $\sum_{j=2}^{n-1}(j-1)(n-j) = \binom{n}{3}$.

Thus, the number of 3-ascent sequences with one descent, which avoid 00 is $2\binom{n}{3} + \binom{n+1}{4}$.

Case 3 $\mathrm{des}(a) = 2$.
In this case, it must be that $\max(a) = n-1$, so that a must contain all the elements in the sequence $0,1,\dots,n-1$. Now, suppose that the first descent of a occurs at position j. Then we have two cases.

Case 3.1 $a_j = j$.
In this case, there must be u, $1 \le u \le j-1$, which does not appear in $a_1 \dots a_j$ and $a_{j+1} = u$. We have $j-1$ choices for u. The sequence $a_{j+2} \dots a_n$ must be a rearrangement of $(j+1)(j+2)\dots(n-1)$, which has one descent. The bottom element of the descent pair that occurs in $a_{j+2} \dots a_n$ must equal s for some $j+1 \le s \le n-2$ and the top element of the descent must equal t, where $s+1 \le t \le n-1$. It is easy to check that any choice of s and t will yield a 3-ascent sequence, so that the number of choices for the sequence $a_{j+2} \dots a_n$ is

$$\begin{aligned} \sum_{s=(j+1)}^{n-2} n-1-s &= \sum_{r=1}^{n-2-j} n-1-(r+j) \\ &= \sum_{r=1}^{n-2-j} n-1-j-r = \binom{n-1-j}{2}. \end{aligned}$$

It follows that the number of 3-ascent sequences in Case 3.1 is $\sum_{j=2}^{n-2}(j-1)\binom{n-1-j}{2}$, which can be shown by Mathematica to be equal to $\binom{n-1}{4}$.

Case 3.2 $a_j = j+1$.
In this, there must be two elements $1 \le u \le v \le j$ that do note appear in $a_1 \dots a_j$. We have $\binom{j}{2}$

ways to choose u and v. We then have two subcases.

Case 3.2.1 $a_{j+1} = v$.
In this case, our sequences start out $a_1 \dots a_j = (j+1)v$ and where every u occurs in the sequence $a_{j+2} \dots a_n$, it will cause a second descent so that there are $n-j-1$ choices in this case.

Case 3.2.2 $a_{j+1} = u$.
In this case, the sequence $a_{j+2} \dots a_n$ consists of the sequence $v(j+2)(j+3)\dots(n-1)$ and we can argue as we did in Case 3.1 that there are $\binom{n-j-1}{2}$ choices for the sequence $a_{j+2} \dots a_n$.

It follows that the total number of choices for the sequence $a_{j+1} \dots a_n$ in Case 3 is $n-j-1+\binom{n-j-1}{2} = \binom{n-j}{2}$. Thus the total number of choices in Case 3.2 is

$$\sum_{j=1}^{n-2} \binom{j}{2}\binom{n-j}{2} = \binom{n+1}{5}.$$

Note that the last equality can be checked by Mathematica.

Putting all the cases together, we see that the number of 3-ascent sequences of length n, which avoid 00 is equal to

$$\binom{n+1}{2} + 2\binom{n}{3} + \binom{n+1}{4} + \binom{n-1}{4} + \binom{n+1}{5}.$$

Using the fact that $\binom{n+1}{4} + \binom{n+1}{5} = \binom{n+2}{5}$, we see that we have the following theorem.

Theorem 11. *For all* $n \geq 1$,

$$a_{n,3,00} = \binom{n+1}{2} + 2\binom{n}{3} + \binom{n-1}{4} + \binom{n+2}{5}.$$

Note that it follows from Newton's binomial theorem that

$$\begin{aligned}
\sum_{n\geq 1} \binom{n+1}{2} x^n &= \frac{x}{(1-x)^3}, \\
\sum_{n\geq 1} 2\binom{n}{3} x^n &= \frac{2x^3}{(1-x)^4}, \\
\sum_{n\geq 1} \binom{n-1}{4} x^n &= \frac{x^5}{(1-x)^5}, \text{ and} \\
\sum_{n\geq 1} \binom{n+2}{5} x^n &= \frac{x^3}{(1-x)^6}.
\end{aligned}$$

Adding these series together and simplifying, we have the following theorem.

Theorem 12. *The generating function*

$$\sum_{n\geq 1} a_{n,3,00} x^n = \frac{x(1-3x+6x^2-5x^3+3x^4-x^5)}{(1-x)^6}.$$

We note that Burstein and Mansour [2] gave a combinatorial interpretation to the n-th element in sequence A089830 as the number of words $w = w_1 \dots w_{n-1} \in \{1,2,3\}^*$, which avoid the vincular pattern 21-2 (also denoted in the literature $\underline{21}2$; see [14]). That is, there are no subsequences of the form $w_i w_{i+1} w_j$ in w such that $i+1 < j$ and $w_i = w_j > w_{i+1}$. We ask the question whether one can construct a simple bijection between such words and the set of 3-ascent sequences of length n, which avoid 00.

We note that the sequence $(a_{n,4,00})_{n\geq 1}$ starts out $1, 4, 16, 58, 190, 564, 1526, 3794 \dots$. This sequence does appear in the OEIS.

4.4 012-avoiding p-ascent sequences

Now suppose that $a = a_1 \dots a_n$ is a p-ascent sequence such that a avoids 012. The first thing to observe is that if $a_i = 1$ for some i, then since $a_1 = 0$, it must be the case that $a_j \in \{0,1\}$ for all $j \geq i$. The second thing to observe is that $a_i \leq p$ for all i. That is, the only way that a can have an element $a_k > p$ is if $a_1 \dots a_{k-1}$ has at least $a_k - p$ ascents. Since the first ascent in a p-ascent sequence must be of one of the forms $01, 02, \dots, 0p$, such an ascent sequence would not avoid 012.

2-ascent sequences. Now, suppose that $a = a_1 \dots a_n$ is a 2-ascent sequence such that a avoids 012. If a has no 1s, then $a_i \in \{0,2\}$ for all $i \geq 2$, so that there are 2^{n-1} such 2-ascent sequences. If a contains a 1, then let k be the smallest j such that a_j equals 1. It then follows that $a_i \in \{0,2\}$ for $2 \leq i < k$ and $a_j \in \{0,1\}$ for $k < j \leq n$. Thus, there are 2^{n-2} such 2-ascent sequences, so that the number of 2-ascent sequences that avoid 012 and contain a 1 is $(n-1)2^{n-2}$. Hence, for $n \geq 1$,

$$a_{n,2,012} = 2^{n-1} + (n-1)2^{n-2} = (n+1)2^{n-2}. \tag{37}$$

We note that the sequence $(a_{n,2,012})_{n\geq 1}$ starts out $1, 3, 8, 20, 48, 112, 256, \dots$, and this is, again, as in the case of $(a_{n,2,10})_{n\geq 1}$, the sequence A001792 in the OEIS [19]. Next, we will explain this fact combinatorially.

It is easy to see that each sequence counted by $(a_{n,2,012})_{n\geq 1}$ can be obtained by taking a number of 2s (maybe none) followed by a number of 1s, and placing any number of 0s (maybe none) between these 1s and 2s making sure that the total length of the sequence is n, and this sequence begins with a 0. On the other hand, it is also straightforward to see that sequences counted by $(a_{n,2,10})_{n\geq 1}$ are of two types: they are either of the form

$$\underbrace{0\dots0}_{i_0\geq 1}\underbrace{1\dots1}_{i_1\geq 1}\underbrace{2\dots2}_{i_2\geq 1}\dots, \tag{38}$$

or of the form

$$\underbrace{0\dots0}_{i_0\geq 1}\underbrace{1\dots1}_{i_1\geq 1}\underbrace{2\dots2}_{i_2\geq 1}\dots\underbrace{a\dots a}_{i_a\geq 1}\underbrace{(a+2)\dots(a+2)}_{i_{a+2}\geq 1}\underbrace{(a+3)\dots(a+3)}_{i_{a+3}\geq 1}\underbrace{(a+4)\dots(a+4)}_{i_{a+4}\geq 1}\dots, \tag{39}$$

where $a \geq 0$ exists. A bijection between the classes of sequences is given by turning sequences of the form (38) into

$$\underbrace{0\dots0}_{i_0}2\underbrace{0\dots0}_{i_1-1}2\underbrace{0\dots0}_{i_2-1}\dots,$$

and the sequences of the form (39) into

$$\underbrace{0\dots0}_{i_0}2\underbrace{0\dots0}_{i_1-1}2\underbrace{0\dots0}_{i_2-1}\dots2\underbrace{0\dots0}_{i_a-1}1\underbrace{0\dots0}_{i_{a+2}-1}1\underbrace{0\dots0}_{i_{a+3}-1}1\underbrace{0\dots0}_{i_{a+4}-1}\dots.$$

3-ascent sequences. Now, suppose that $a = a_1 \dots a_n$ is a 3-ascent sequence such that a avoids 012. If a has no 1s, then $a_i \in \{0, 2, 3\}$ for all $i \geq 2$. It is then easy to see that if $b_1 \dots b_n$ is the sequence that arises from $a_1 \dots a_n$ by replacing each 2 by a 1 and each 3 by a 2, then b is a 2-ascent sequence that avoids 012. Thus, there are $(n+1)2^{n-2}$ such sequences. Now, suppose that a contains a 1. Then let k be the smallest j such that a_j equals 1. It then follows that $a_i \in \{0, 2, 3\}$ for $2 \leq i < k$ and $a_j \in \{0, 1\}$ for $k < j \leq n$. It is then easy to see that if $b_1 \dots b_{k-1}$ is the sequence that arises from $a_1 \dots a_{k-1}$ by replacing each 2 by a 1 and each 3 by a 2, then $b_1 \dots b_{k-1}$ is a 2-ascent sequence that avoids 012. Thus, from our argument above, it follows that there are $k2^{k-3}$ choices for $a_1 \dots a_{k-1}$ and 2^{n-k} choices for $a_{k+1} \dots a_n$. Therefore, given k, we have $k2^{n-3}$ choices for a. Thus,

$$\begin{aligned}
a_{n,3,012} &= (n+1)2^{n-2} + \sum_{k=2}^{n} k2^{n-3} \\
&= 2^{n-3}\left(2n + 2 + \sum_{k=2}^{n} k\right) \\
&= 2^{n-3}\left(2n + 2 + \binom{n+1}{2} - 1\right) && (40) \\
&= 2^{n-3}\frac{4n + 4 + n^2 + n - 2}{2} = 2^{n-4}(n^2 + 5n + 2). && (41)
\end{aligned}$$

We note that the sequence $(a_{n,3,012})_{n\geq 1}$ starts out $1, 4, 13, 38, 104, 272, 688, \dots$ and this is the sequence A049611 in the OEIS [19] having several combinatorial interpretations.

p-ascent sequences for an arbitrary p. In general, we can obtain a simple recursion for $a_{n,p,012}$. That is, suppose that $a = (a_1, \dots, a_n)$ is a p-ascent sequence such that a avoids 012. Now, if a has no 1s, then $a_i \in \{0, 2, 3, \dots, p\}$ for all $i \geq 2$. It is then easy to see that if $b = (b_1, \dots, b_n)$ is the sequence that arises from a by replacing each $i \geq 2$, by an $i - 1$, then b is a $(p-1)$-ascent sequences that avoids 012. Thus, there are $a_{n,p-1,012}$ such sequences. Now suppose that a contains a 1. Then let k be the smallest j such that a_j equals 1. It then follows that $a_i \in \{0, 2, 3, \dots, p\}$ for $2 \leq i < k$ and $a_j \in \{0, 1\}$ for $k < j \leq n$. It is then easy to see that if $b_1 \dots b_{k-1}$ is the sequence that arises from $a_1 \dots a_{k-1}$ by replacing each $i \geq 2$ by an $i-1$, then $b_1 \dots b_{k-1}$ is a 2-ascent sequences that avoids 012. It follows that there are $a_{k-1,p-1,012}$ choices for $a_1 \dots a_{k-1}$ and 2^{n-k} choices for $a_{k+1} \dots a_n$. Thus, given k, we have $2^{n-k}a_{k-1,p-1,012}$ choices for a. It follows that

$$a_{n,p,012} = a_{n,p-1,012} + \sum_{k=2}^{n} a_{k-1,p-1,012}2^{n-k}. \tag{42}$$

Thus, for example,

$$\begin{aligned}
a_{n,4,012} &= 2^{n-4}(n^2 + 5n + 2) + \sum_{k=2}^{n} 2^{k-5}((k-1)^2 + 5(k-1) + 2)2^{n-k} \\
&= 2^{n-5}\left(2n^2 + 10n + 4 + \sum_{k=1}^{n-1}(k^2 + 5k + 2)\right) \\
&= 2^{n-5}(2n^2 + 10n + 4 + (1/3)(n^3 + 6n^2 - n - 6)) \\
&= \frac{2^{n-5}}{3}(n^3 + 12n^2 + 29n + 6).
\end{aligned}$$

We note that the sequence $(a_{n,4,012})_{n\geq 1}$ starts out

$$1, 5, 19, 63, 192, 552, 1520, 4048, 10496, 26264, \ldots$$

and this is the sequence A049612 in the OEIS [19].

References

[1] M. Bousquet-Mélou, A. Claesson, M. Dukes, S. Kitaev, Unlabeled (2+2)-free posets, ascent sequences and pattern avoiding permutations. *J. Combin. Theory Ser. A*, **117** Issue 7 (2010), 884–909.

[2] A. Burstein and T. Mansour, Words restricted by 3-letter generalized multipermutation patterns, Annals of Combinatorics, **7:1** (2003), 1-14.

[3] A. Claesson, M. Dukes, and S. Kitaev, A direct encoding of Stoimenow's matchings as ascent sequences, *Australalasian J. Combin.* **49** (2011) 47–59.

[4] M. Dukes, S. Kitaev, J. Remmel, and E. Steingrimsson, Enumerating $2+2$-free posets by indistinguishable elements, *J. Combin.*, **2**(1) (2011), 139–163.

[5] M. Dukes and R. Parviainen, Ascent sequences and upper triangular matrices containing non-negative integers, Electronic J. Combin., **17** no.1 (2010), #R53.

[6] P. Duncan and E. Steingrímsson, Pattern avoidance in ascent sequences, Electronic J. Combin., **18** Issue 1 (2011), # P226.

[7] M. H. El-Zahar, Enumeration of ordered sets, in: I. Rival (Ed.), *Algorithms and Order*, Kluwer Academic Publishers, Dordrecht, 1989, 327–352.

[8] P. C. Fishburn, *Interval Graphs and Interval Orders*, Wiley, New York, 1985.

[9] P. C. Fishburn, Intransitive indifference in preference theory: a survey, *Oper. Res.* **18** (1970) 207–208.

[10] P. C. Fishburn, Intransitive indifference with unequal indifference intervals, *J. Math. Psych.* **7** (1970) 144–149.

[11] P. E. Haxell, J. J. McDonald and S. K. Thomasson, Counting interval orders, *Order* **4** (1987) 269–272.

[12] V. Jelínek, Counting general and sefl-dual interval orders, J. Combinatorial Theory (A), **119** (2012), 599-614.

[13] S. M. Khamis, Height counting of unlabeled interval and N-free posets, *Discrete Math.* **275** (2004) 165–175.

[14] S. Kitaev, Patterns in permutations and words, Springer-Verlag, 2011.

[15] S. Kitaev and J. Remmel, Enumerating $(2+2)$-free posets by the number of minimal elements and other statistics, *Discrete Appl. Math.* **159**, Issue 17 (2011) 2098–2108.

[16] P. Levande, Fishburn diagrams, Fishburn numbers, and their refined generating functions, J. of Combinatorial Theory (A), **120** (2013), 194-217.

[17] J. Remmel, Up-down ascent sequences and the q-Genocchi numbers, to appear Pure Mathematics and Applications.

[18] M. Skandera, A characterization of $(3+1)$-free posets, *J. Combin. Theory Ser. A* **93**, no. 2 (2001) 231–241.

[19] N. J. A. Sloane, The on-line encyclopedia of integer sequences, published electronically at `http://oeis.org`.

[20] R. P. Stanley, *Enumerative combinatorics Vol. 1*, volume 49 of *Cambridge Studies in Advanced Mathematics*, Cambridge University Press, Cambridge, 1997.

[21] A. Stoimenow, Enumeration of chord diagrams and an upper bound for Vassiliev invariants, *J. Knot Theory Ramifications* **7** no. 1 (1998) 93–114.

[22] J. Wimp and D. Zeilberger, Resurrecting the asymptotics of linear recurrences, *J. Math. Anal. Appl.*, **111** no. 1 (1985) 162–176.

[23] S.H.F. Yan, On a conjecture about enumerating (2+2)-free posets, Europ. J. Combinatorics, **32** (2011), 282-287.

[24] D. Zagier, Vassiliev invariants and a strange identity related to the Dedeking eta-function, *Topology* **40** (2001) 945–960.

[25] J. Zeng and J. Zhou, A q-analog of the Seidel generation of Genocchi numbers, *European J. Combin.* **27** (2006) 364–381