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Verification of the Firoozbakht Conjecture for Primes up to Four Quintillion

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Abstract

If p_k is the kth prime, the Firoozbakht conjecture states that the sequence $(p_k)^{1/k}$ is strictly decreasing. We use the table of first-occurrence prime gaps in combination with known bounds for the prime-counting function to verify the Firoozbakht conjecture for primes up to four quintillion (4×10^{18}).

Mathematics Subject Classification: 11N05

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1 Introduction

We will examine a conjecture that was first stated in 1982 by the Iranian mathematician Farideh Firoozbakht from the University of Isfahan [8]. It appeared in print in *The Little Book of Bigger Primes* by Paulo Ribenboim [7, p. 185]. The statement is as follows:

Firoozbakht's Conjecture. If p_k is the kth prime, then the sequence $(p_k)^{1/k}$ is strictly decreasing. Equivalently, for all $k \geq 1$ we have

$$p_{k+1}^k < p_k^{k+1}. \tag{1}$$

The Firoozbakht conjecture is one of the strongest upper bounds for prime gaps. As we will see from Table 1 below, it is somewhat stronger than *Cramér's* *conjecture* proposed about half a century earlier by the Swedish mathematician Harald Cramér [1]:

Cramér's Conjecture. If p_k and p_{k+1} are consecutive primes, then we have $p_{k+1} - p_k = O(\log^2 p_k)$ or, more specifically,

$$\limsup_{k\to\infty} \frac{p_{k+1} - p_k}{\log^2 p_k} \;=\; 1.$$

For the sake of numerical comparison with (1), let us use a modified form of Cramér's conjecture stated below.

Modified Cramér Conjecture. If p_k and p_{k+1} are consecutive primes, then

$$p_{k+1} - p_k \;<\; \log^2 p_{k+1}. \tag{2}$$

This modified form allows us to make predictions of an upper bound for any given prime gap; Table 1 lists a few examples of such upper bounds.

TABLE 1
Prime gap bounds predicted by the modified Cramér and Firoozbakht conjectures

k	Consecutive primes p_k	p_{k+1}	Upper bounds for p_{k+1}, as predicted by: Modified Cramér conjecture (solution of $x = p_k + \log^2 x$)	Firoozbakht conjecture (solution of $x^k = p_k^{k+1}$)
5	11	13	19.964	17.769
26	101	103	124.225	120.618
169	1009	1013	1057.493	1051.152
1230	10007	10009	10091.999	10082.220
9593	100003	100019	100135.579	100123.090
78499	1000003	1000033	1000193.874	1000179.012
664580	10000019	10000079	10000278.794	10000261.534

Table 1 shows that, given k and p_k, the Firoozbakht conjecture (1) yields a tighter bound for p_{k+1} than the modified Cramér conjecture (2). Indeed, the Firoozbakht upper bound (last column) is below the Cramér upper bound by approximately $\log p_k$. In *Cramér's probabilistic model of primes* [1, 3] the parameters of the distribution of maximal prime gaps suggest that inequalities (1) and (2) are both true with probability 1; that is, almost all[1] maximal prime gaps in Cramér's model satisfy (1) and (2). One may take this as an indication that any violations of (1) and (2) occur exceedingly rarely (if at all).

[1] In Cramér's model with n urns, the limiting distribution of maximal "prime gaps" is the Gumbel extreme value distribution with scale $a_n \sim n/\operatorname{li} n = O(\log n)$ and mode $\mu_n = n\log(\operatorname{li} n)/\operatorname{li} n + O(\log n) = \log^2 n - \log n \log\log n + O(\log n)$ [3, 9, OEIS A235402]; here $\operatorname{li} n$ denotes the logarithmic integral of n. Hence, for large n, all maximal gap sizes are below $\log n(\log n - 1)$, except for a vanishing proportion of maximal gaps.

2 Computational verification for small primes

When primes p_k are not too large, one can directly verify inequality (1) by computation. A simple program that takes a few seconds to perform the verification for $p_k < 10^6$ is available on the author's website. The program outputs the numeric values of k, p_k, $p_k^{1/k}$, and an `OK` if the value of $p_k^{1/k}$ decreases from one prime to the next. It will output `FAILURE` if the conjectured decrease does not occur. Result: all `OK`s, no `FAILURE`s. Here is a sample of the output:

k	p	p^(1/k)	OK/fail
1	2	2.000000000	OK
2	3	1.732050808	OK
3	5	1.709975947	OK
4	7	1.626576562	OK
5	11	1.615394266	OK
6	13	1.533406237	OK
7	17	1.498919872	OK
8	19	1.444921323	OK
9	23	1.416782203	OK
10	29	1.400360331	OK
11	31	1.366401518	OK
12	37	1.351087503	OK
...	...	...	
78494	999953	1.000176022	OK
78495	999959	1.000176020	OK
78496	999961	1.000176018	OK
78497	999979	1.000176016	OK
78498	999983	1.000176014	OK
78499	1000003	1.000176012	OK
78500	1000033	1.000176010	OK

Thus primes $p_k < 10^6$ do not violate (1). What about larger primes?

3 What if we do not know $k = \pi(p_k)$?

For large primes p_k, the exact values of $k = \pi(p_k)$ are not readily available. Nevertheless, the Firoozbakht conjecture can often be verified in such cases too. When we do not know $\pi(p_k)$ exactly, we can use these bounds for the prime-counting function $\pi(x)$:

$$\pi(x) < \frac{x}{\log x - 1.1} \quad \text{for } x \geq 60184 \qquad \text{[2, p. 9, Theorem 6.9]} \tag{3}$$

$$\pi(x) < \frac{x}{\log x - 1.2} \quad \text{for } x \geq 4 \quad \text{(from (3) + computer check for } x < 10^5\text{)}. \tag{4}$$

Taking the log of both sides of (1) and rearranging, we find that the Firoozbakht conjecture (1) is equivalent to

$$\pi(p_k) < \frac{\log p_k}{\log p_{k+1} - \log p_k}. \tag{5}$$

If we know p_k and p_{k+1} (where $p_k > 60184$) but do not know $\pi(p_k)$, then instead of (5) we may check the stronger condition

$$\frac{p_k}{\log p_k - 1.1} < \frac{\log p_k}{\log p_{k+1} - \log p_k}. \tag{6}$$

For a larger range of applicability ($p_k > 4$)[2] we may check another (still stronger) condition:

$$\frac{p_k}{\log p_k - 1.2} < \frac{\log p_k}{\log p_{k+1} - \log p_k}. \tag{7}$$

If (6) or (7) is true, so is (5); and the exact value of $\pi(p_k)$ is not needed to check (6), (7).

4 Verification for all gaps of a given size g

We will take (6) and (7) one step further and make p_k a variable (x); then $p_{k+1} = x + g$, where g is the gap size. We can now solve the resulting simultaneous inequalities

$$0 < \frac{x}{\log x - 1.1} < \frac{\log x}{\log(x+g) - \log x} \quad \text{with} \quad x > 60184, \tag{8}$$

or, if we are interested in a larger range of applicability,

$$0 < \frac{x}{\log x - 1.2} < \frac{\log x}{\log(x+g) - \log x} \quad \text{with} \quad x > 4. \tag{9}$$

Here we use the gap size g as a parameter. In combination with a table of first-occurrence prime gaps [5], the solution of (8) and/or (9) will tell us whether a prime gap of size g may violate the Firoozbakht conjecture for primes $p_k \approx x$. Consider the following examples.

Example 1. *Can a prime gap of size* 150 *violate the Firoozbakht conjecture?* To answer this question, we substitute $g = 150$ into (8),

$$0 < \frac{x}{\log x - 1.1} < \frac{\log x}{\log(x+150) - \log x} \quad \text{with} \quad x > 60184,$$

[2] regardless of the computation of Sect. 2 which already proves (1) for all $p_k < 10^6 \ldots$

solve for x and find the "safe bound"

$$x \geq 365323 \quad \text{(or, more precisely, } x > 365322.7038\text{);}$$

that is, a gap of 150 does not violate (1) if such a gap occurs between primes above 365323. But there are no prime gaps of size 150 below 365323; in fact, the table of first-occurrence prime gaps [5] indicates that the first such gap follows the prime 13626257. Therefore, a prime gap of size 150 can never violate the Firoozbakht conjecture (1).

Example 2. *Can a prime gap of size* 2 *(twin primes) violate the Firoozbakht conjecture?* We substitute $g = 2$ into (9),

$$0 < \frac{x}{\log x - 1.2} < \frac{\log x}{\log(x+2) - \log x} \quad \text{with} \quad x > 4,$$

solve for x and find the "safe bound"

$$x \geq 8 \quad \text{(or, more precisely, } x > 7.8745\text{).}$$

So a prime gap of size 2 does not violate (1) if this gap occurs between primes above 8. But we already know that gaps between primes below 8 do not violate the conjecture either (Section 2). Therefore, a prime gap of size 2 (twin primes) can never violate (1).

We have repeated the computation of the above examples for all even values of the gap size $g \in [2, 1476]$ and found that *none of these gap sizes could possibly violate* (1). A tabulation of "safe bounds" by gap size is available on the author's website [4]. For $g = 2$ and 4, we had to manually check (1) for a couple of gaps of size g between primes below the respective safe bounds. For $g \in [6, 1476]$, the actual first occurrence of prime gap g is already safe. ("Close calls" occur for record prime gaps in OEIS sequence A005250 [9].) From the prime gaps table [5] we also know that gaps larger than 1476 do not occur below 4×10^{18}. Thus the validity of Firoozbakht conjecture (1) has been verified for primes up to 4×10^{18}. We have obtained the following theorem:

Theorem *Inequality (1) is true for all primes* $p_k < 4 \times 10^{18}$.

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References

[1] H. Cramér, On the order of magnitude of the difference between consecutive prime numbers. *Acta Arith.* **2** (1936), 23-46.

[2] P. Dusart, Estimates of some functions over primes without R.H., preprint arXiv:1002.0442 (2010). http://arxiv.org/abs/1002.0442

[3] A. Kourbatov, The distribution of maximal prime gaps in Cramér's probabilistic model of primes. *Int. Journal of Statistics and Probability* **3** (2014), No. 2, 18-29. arXiv:1401.6959. http://arxiv.org/abs/1401.6959

[4] A. Kourbatov, Checking the Firoozbakht conjecture: safe bounds, 2015. http://www.javascripter.net/math/primes/firoozbakhtconjecturebounds.htm

[5] T. R. Nicely, First occurrence prime gaps, preprint, 2014. http://www.trnicely.net/gaps/gaplist.html.

[6] T. Oliveira e Silva, S. Herzog, and S. Pardi, Empirical verification of the even Goldbach conjecture and computation of prime gaps up to $4 \cdot 10^{18}$, *Math. Comp.* **83** (2014), 2033-2060. http://dx.doi.org/10.1090/s0025-5718-2013-02787-1

[7] P. Ribenboim, *The Little Book of Bigger Primes*, Springer, New York, 2004. http://dx.doi.org/10.1007/b97621

[8] C. Rivera (ed.), Conjecture 30. The Firoozbakht Conjecture, 2002. http://www.primepuzzles.net/conjectures/

[9] N. J. A. Sloane (ed.), *The On-Line Encyclopedia of Integer Sequences*, 2014. Published electronically at http://oeis.org/A235402.