THE ABUNDANCY INDEX OF DIVISORS OF SPOOF ODD PERFECT NUMBERS

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Abstract

We call n a spoof odd perfect number if n is odd and $n = km$ for two integers $k, m > 1$ such that $\sigma(k)(m+1) = 2n$, where σ is the sum-of-divisors function. In this paper, we show how results analogous to those of odd perfect numbers could be established for spoof odd perfect numbers (otherwise known in the literature as Descartes numbers). In particular, we predict that an analogue of the Descartes-Frenicle-Sorli conjecture for odd perfect numbers holds for spoof odd perfect numbers. Furthermore, we conjecture that the quasi-Euler prime of a spoof odd perfect number is also its largest quasi-prime factor.

1. Introduction

If N is a positive integer, then we write $\sigma(N)$ for the sum of the divisors of N. A number N is *perfect* if $\sigma(N) = 2N$. A number M is *almost perfect* if $\sigma(M) = 2M-1$. It is currently unknown whether there are infinitely many even perfect numbers, or whether any odd perfect numbers (OPNs) exist. On the other hand, it is known that powers of two (including 1) are almost perfect, although it is still an open problem to rule out other possible forms of even almost perfect numbers, and also, if 1 will be the only odd almost perfect number to be "discovered".

Ochem and Rao recently proved [14] that, if N is an odd perfect number, then $N > 10^{1500}$ and that the largest component (i.e., divisor p^a with p prime) of N is bigger than 10^{62}. This improves on previous results by Brent, Cohen and te Riele [4] in 1991 ($N > 10^{300}$) and Cohen [6] in 1987 (largest component $p^a > 10^{20}$).

An odd perfect number $N = q^r t^2$ is said to be given in Eulerian form if q is prime with $q \equiv r \equiv 1 \pmod 4$ and $\gcd(q, t) = 1$. (The number q is called the *Euler prime*, while the component q^r is referred to as the *Euler factor*. Note that, since q is prime and $q \equiv 1 \pmod 4$, then $q \geq 5$.)

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We denote the *abundancy index* I of the positive integer x as

$$I(x) = \frac{\sigma(x)}{x}.$$

If $w = z$ is the *only* solution to the equation $I(w) = I(z)$, then z is said to be *solitary*. Greening showed that the condition $\gcd(z, \sigma(z)) = 1$ is sufficient (but not necessary) to show that z is solitary. In this paper, we will refer to this result as *Greening's Theorem*. If the equation $I(x) = y$ has no integral solution x for a given rational number y, then the rational y is called an *abundancy outlaw* [13].

In his Ph. D. thesis, Sorli [17] conjectured that $r = 1$.

In the M. Sc. thesis [11], it was conjectured that the divisors q^r and t are related by the inequality $q^r < t$. This conjecture was made on the basis of the result $I(q^r) < I(t)$. In a recent preprint [5], Brown shows that the inequality $q < t$ is true, and that $q^r < t$ holds "in many cases".

In the paper [10], it is shown that $t < q$ is sufficient for the Descartes-Frenicle-Sorli conjecture that $r = 1$ [2]. If proved, the inequality $t < q$ would also imply that q is the largest prime factor of the odd perfect number $N = qt^2$.

In this paper, we will focus on spoof odd perfect numbers (otherwise known as *Descartes numbers* in the literature). We will discuss the analogue of the Descartes-Frenicle-Sorli conjecture for odd perfect numbers (i.e., $r = 1$), for the case of spoof odd perfect numbers. We will also consider the possibility that the quasi-Euler prime of a spoof odd perfect number is also its largest quasi-prime factor.

For recent papers on Descartes/spoof odd perfect numbers, we refer the interested reader to [2] and [7].

2. Preliminaries

A spoof odd perfect number (hereinafter abbreviated as "spoof") is an odd integer $n = km > 1$ such that $\sigma(k)(m + 1) = 2n$.

Notice that the following statements follow from this definition.

Lemma 1. *Let $n = km > 1$ be a spoof.*

- *k must be an odd square.*
- *m satisfies the congruence $m \equiv 1 \pmod 4$.*
- *$m \mid \sigma(k)$.*
- *$(m + 1) \mid 2k$.*
- *$\frac{\sigma(k)}{m} = 2k - \sigma(k) = D(k)$ (This is from [2].)*

Remark 1. We will call m the *quasi-Euler prime* of the spoof n. In this paper, we will restrict m to be composite, since otherwise if m is prime, then the equation

$$\sigma(k)(m+1) = 2n = 2k \cdot m^1$$

will imply that $n = km$ is in fact an *odd perfect number*, whose *Euler prime* m has exponent 1. Thus, since $m \equiv 1 \pmod 4$ by Lemma 1, a lower bound for m is given by $m \geq 9$.

The following result follows directly from Lemma 1.

Lemma 2. *If $n = km > 1$ is a spoof, then*

$$\gcd(k, \sigma(k)) = 2k - \sigma(k).$$

Proof. Let $n = km > 1$ be a spoof. By the definition of spoofs, we have

$$\sigma(k)\,(m+1) = 2n = 2km$$

$$\sigma(k) = m \cdot (2k - \sigma(k))\,.$$

Note that we can also rewrite this equation as

$$2k = \left(\frac{m+1}{m}\right) \cdot \sigma(k) = (m+1) \cdot (2k - \sigma(k))\,,$$

where we have used the equation $\sigma(k)/m = 2k - \sigma(k)$ from Lemma 1.

Consequently, we obtain

$$\gcd(k, \sigma(k)) = \gcd\left(\left(\frac{m+1}{2}\right) \cdot (2k - \sigma(k))\,, m \cdot (2k - \sigma(k))\right).$$

But we know that $\gcd((m+1)/2, m) = \gcd(m+1, m) = 1$ since m and $m+1$ are consecutive (positive) integers. Thus, we get

$$\gcd(k, \sigma(k)) = (2k - \sigma(k)) \cdot \gcd((m+1)/2, m) = (2k - \sigma(k)) \cdot 1 = 2k - \sigma(k),$$

and we are done. $\square$

Remark 2. Since the equation

$$\gcd(k, \sigma(k)) = 2k - \sigma(k)$$

holds if $n = km > 1$ is a spoof (by Lemma 2), and since

$$\frac{\sigma(k)}{m} = 2k - \sigma(k)$$

by Lemma 1, then we have

$$\gcd(k, \sigma(k)) = \frac{\sigma(k)}{m} = 2k - \sigma(k).$$

In particular, if we can show that $\sigma(k) = m$, then this would imply that

$$\gcd(k, \sigma(k)) = \frac{\sigma(k)}{m} = 2k - \sigma(k) = 1,$$

which would further imply that the odd integer $k > 1$ is an almost perfect number and also a solitary number (by Greening's Theorem [1]). Note that $\sigma(k) = m$ implies that, since $k > 1$, we obtain

$$1 < \frac{\sigma(k)}{k} = \frac{m}{k},$$

from which it would follow that $k < m$.

We pattern our approach to a study of spoofs via an analogous method for odd perfect numbers described in the M. Sc. thesis [11], the main results of which were published in [10].

Remark 3. To begin with, note that we do not have the divisibility constraint $\gcd(m, k) = 1$ for spoofs (as compared to odd perfects). (Wikipedia's definition for Descartes numbers in `http://en.wikipedia.org/wiki/Descartes_number` does specify $\gcd(m, k) = 1$. However, this assumption was not used in the papers [2] and [7]. A reference request has been posted online via `http://math.stackexchange.com/q/1178386` to aid in clarifying this. Lastly, in `http://oeis.org/wiki/Descartes_number`, Descartes numbers are defined as odd numbers $n = km$ with $k > 1$, $m > 1$ and $m \nmid k$ such that $\sigma(k) \cdot (m + 1) = 2n = 2km$.)

Consequently, we will be needing the following lemma, whose proof may be found in any standard textbook on (elementary) number theory, and which we will omit in this paper.

Lemma 3. *Let a and b be any two positive integers. Then the following chain of inequalities hold:*

$$a\sigma(b) \leq \sigma(ab) \leq \sigma(a)\sigma(b).$$

We now give our first formal result in this section.

Theorem 4. *Let $n = km > 1$ be a spoof. The following chain of inequalities hold:*

- $1 < \frac{2k}{\sigma(k)} = \frac{m+1}{m} \leq \frac{10}{9} < \frac{9}{5} \leq \frac{\sigma(k)}{k} < 2.$
- $\frac{3}{\sqrt{5}} < \frac{\sigma(\sqrt{k})}{\sqrt{k}} < 2.$

Proof. The first inequality
$$1 < \frac{m+1}{m}$$
is trivial. The inequality
$$\frac{m+1}{m} = 1 + \frac{1}{m} \leq \frac{10}{9}$$
as $m \geq 9$ follows from Remark 1. Similarly, the inequality
$$\frac{9}{5} \leq \frac{\sigma(k)}{k}$$
follows from
$$\frac{\sigma(k)}{k} = \frac{2m}{m+1} = 2\left(\frac{m}{m+1}\right) = 2\left(1 - \frac{1}{m+1}\right) \geq 2\left(1 - \frac{1}{10}\right) = \frac{9}{5}.$$
The inequality
$$\frac{\sigma(k)}{k} < 2$$
follows from
$$\frac{\sigma(k)}{2k} = \frac{m}{m+1} < 1.$$
(In other words, k is deficient.) The inequality
$$\frac{3}{\sqrt{5}} < \frac{\sigma(\sqrt{k})}{\sqrt{k}}$$
follows from Lemma 3, since it implies that
$$\sqrt{\frac{\sigma(k)}{k}} < \frac{\sigma(\sqrt{k})}{\sqrt{k}}.$$
(Note that, by Lemma 1, k is an odd square.) Lastly, the inequality
$$\frac{\sigma(\sqrt{k})}{\sqrt{k}} < \frac{\sigma(k)}{k}$$
follows from Lemma 1 (since $k > 1$ is an odd square), and because then $\sqrt{k} \mid k$ and $\sqrt{k} < k$. $\square$

Remark 5. Note the rational approximations
$$\frac{10}{9} \approx 1.11111$$
and
$$\frac{3}{\sqrt{5}} \approx 1.34164,$$

so that we obtain

$$\frac{m+1}{m} \leq \frac{10}{9} < \frac{3}{\sqrt{5}} < \frac{\sigma(\sqrt{k})}{\sqrt{k}}.$$

This inequality is an analogue of a similar result for odd perfect numbers, as detailed in [11].

Following in the footsteps of [11], we now claim the following statement.

Theorem 6. *Let $n = km > 1$ be a spoof. Then $k \neq m$.*

Proof. We adapt the proof from the paper [8].

To this end, assume that $n = km > 1$ is a spoof, and suppose to the contrary that $k = m$.

It follows that

$$\sigma(k)(m+1) = 2km \Longrightarrow \frac{\sigma(k)}{k} = \frac{2k}{k+1}.$$

This last equation implies

$$\sigma(k) = \frac{2k^2}{k+1} = \frac{2k^2-2}{k+1} + \frac{2}{k+1} = \frac{2(k^2-1)}{k+1} + \frac{2}{k+1} = 2(k-1) + \frac{2}{k+1}.$$

Since $\sigma(k)$ and $2(k-1)$ are both integers, the immediately preceding equation implies that

$$(k+1) \mid 2$$

from which we conclude that $k + 1 \leq 2$. This implies $m = k \leq 1$, which contradicts our initial assumption $n = km > 1$.

This finishes the proof. $\square$

Remark 7. Theorem 6 just says that if $n = km > 1$ is a spoof, then n is not a square. This is an analogue of a similar result for odd perfect numbers.

By Theorem 6 and trichotomy, we now know that either $k < m$ or $m < k$ is true (but *not both*). We again consult [11] (and also, [10]) for guidance as to the correct inequality between these two.

Note that, by Remark 1, m is the quasi-Euler prime of n. Since k is an odd square, from the results in [11] and [10], we conjecture that $m < k$ must be the right inequality to pursue.

We embark on a proof for this conjecture in the succeeding theorem, subject to a reasonable assumption.

Theorem 8. *If $n = km > 1$ is a spoof, then $k > 1$ is an odd almost perfect number if and only if $k < m$.*

Proof. Suppose that $n = km > 1$ is a spoof. Notice that, by using the equation

$$\gcd(k, \sigma(k)) = 2k - \sigma(k) = D(k)$$

from Lemma 1, we get the implication **"if** $\gcd(k, \sigma(k)) = 1$**, then** $k > 1$ **is odd almost perfect (and also solitary)",** because of Greening's Theorem.

Thus, we first show that if $k > 1$ is an odd almost perfect number, then the inequality $k < m$ holds. An easy proof of this fact is given by the considerations in Remark 2. Here is an alternative that adds a different perspective to the proof, and which uses the criterion in [8] . Since $n = km > 1$ is a spoof, we have

$$\frac{\sigma(k)}{k} = I(k) = \frac{2m}{m+1}.$$

By the criterion in [8], since $k > 1$ is an odd almost perfect number, we get

$$\frac{2k}{k+1} < I(k) = \frac{2m}{m+1}$$

from which we obtain $k < m$.

Next, we prove the other direction of the theorem. Suppose that $n = km > 1$ is a spoof with $k < m$. We want to show that $k > 1$ is an odd almost perfect number. To this end, since $k < m$ we have

$$\frac{\sigma(k)}{m} < \frac{\sigma(k)}{k} = I(k) = \frac{2m}{m+1} < 2$$

by Theorem 4. Since k is a square and m is odd, then $\sigma(k)/m$ is odd, and consequently we know that $\sigma(k)/m \geq 1$. Together with $\sigma(k)/m < 2$, this implies that

$$\frac{\sigma(k)}{m} = 1.$$

Now by the considerations in Remark 2, we get

$$\gcd(k, \sigma(k)) = 2k - \sigma(k) = \frac{\sigma(k)}{m} = 1$$

which implies that the odd number $k > 1$ is almost perfect and solitary.

This concludes the proof. □

Remark 9. The result in Theorem 8 is consistent with the prediction in Remark 7 and that of the conjectured nonexistence of odd almost perfect numbers other than 1.

Remark 10. The author e-mailed Dean Hickerson on March 12, 2011, and here's what he got to say regarding the problem of determining the status of squares with respect to solitude or friendliness:

AUTHOR: *"[I] [n]otice[d] that all of the squares from* 1 *to* 121 *are solitary (since they satisfy* $\gcd(n, \sigma(n)) = 1$*). (See OEIS sequence A014567 [14].) Does my observation hold true in general?"*

HICKERSON: "No. The smallest square for which $\gcd(n, \sigma(n))$ is not equal to 1 is 196: $\gcd(196, \sigma(196)) = 7$. (But it's easy to show that 196 is solitary.)"

In 1995 I found a square that isn't solitary; I don't know if there are any smaller ones: $26334^2 = 693479556 = 2^2 3^4 7^2 11^2 19^2$. There are at least 5 other numbers with the same abundancy index:

$$8640 = 2^6 \cdot 3^3 \cdot 5$$

$$52416 = 2^6 \cdot 3^2 \cdot 7 \cdot 13$$

$$71814642425856 = 2^{13} \cdot 3^4 \cdot 11^3 \cdot 31 \cdot 43 \cdot 61$$

$$2168446760665473024 = 2^{13} \cdot 3^{10} \cdot 11 \cdot 23 \cdot 43 \cdot 107 \cdot 3851$$

$$5321505362711814144 = 2^{13} \cdot 3^6 \cdot 11 \cdot 23 \cdot 43 \cdot 137 \cdot 547 \cdot 1093$$

AUTHOR: *"I had expected my conjecture to fail for even squares. Notwithstanding, have you also found any odd squares which are not solitary?"*

HICKERSON: "No, I haven't. It's easy to find odd squares for which $\gcd(n, \sigma(n))$ is not equal to 1; e.g. if $n = 21^2 = 441$ then $\gcd(n, \sigma(n)) = 3$. But 441 is solitary."

Hence, it appears that the friendly or solitary status of odd squares is still an open problem. [12]

In a letter to Mersenne dated November 15, 1638, Descartes showed that

$$d = 3^2 7^2 11^2 13^2 22021 = 3^2 7^2 11^2 13^2 19^2 61 = 198585576189$$

would be an odd perfect number if 22021 were prime. Hence, the depth of Descartes' understanding of the subject should not be underestimated. In fact, to this day, it has recently been shown that Descartes' example is the only spoof with less than seven distinct quasi-prime factors [7].

Let us try to compute the abundancy index of d and thereafter make suitable conjectures regarding spoofs in general.

$$I(d) = \frac{13}{9} \cdot \frac{57}{49} \cdot \frac{133}{121} \cdot \frac{183}{169} \cdot \frac{381}{361} \cdot \frac{62}{61}$$

$$= \frac{426027470778}{198585576189} = \frac{23622}{11011} \approx 2.14531$$

So Descartes' spoof is in fact abundant! A natural question to ask at this point would be: *Are all Descartes numbers necessarily abundant?*

Additionally, observe that the quasi-Euler prime factor of d (i.e., 22021) is also its largest quasi-prime factor. Lastly, take note that the exponent for the quasi-Euler prime is 1.

In the next section, we delve deeper into these considerations, still following in the footsteps of [11].

3. Main Results

Recall from the previous section that a spoof is an odd integer

$$n = km > 1$$

such that

$$\sigma(k)(m+1) = 2n.$$

First, we want to show that $(m+1)/k < \sigma(k)/m$, if $m < k$.

In fact, under the assumption that $m < k$ (or equivalently, that $k > 1$ is not an odd almost perfect number), we have

$$3 \leq D(k) = \frac{\sigma(k)}{m} = \frac{2k}{m+1},$$

so that we obtain

$$\frac{m+1}{k} \leq \frac{2}{3} < 3 \leq \frac{\sigma(k)}{m}.$$

We state this result as our first lemma for this section.

Lemma 4. *Let $n = km > 1$ be a spoof. If $m < k$, then we have the chain of inequalities*

$$\frac{m+1}{k} \leq \frac{2}{3} < 3 \leq \frac{\sigma(k)}{m}.$$

Following exactly the same method in [11], we can prove the following result.

Lemma 5. *Let $n = km > 1$ be a spoof. If $m < k$ is not an odd almost perfect number, then we have the chain of inequalities*

$$\frac{131}{45} \leq \frac{m+1}{m} + \frac{\sigma(k)}{k} < 3 < \frac{11}{3} \leq \frac{m+1}{k} + \frac{\sigma(k)}{m}.$$

Remark 11. The inequalities

$$\frac{131}{45} \leq \frac{m+1}{m} + \frac{\sigma(k)}{k} < 3 < \frac{11}{3} \leq \frac{m+1}{k} + \frac{\sigma(k)}{m}$$

in Lemma 5 are also an analogue of similar results for odd perfect numbers. (See the M. Sc. thesis [11] for more details.)

Note the rational approximation

$$\frac{131}{45} \approx 2.91111.$$

Next, we now try to determine the correct ordering between the quantities

$$\frac{m+1}{\sqrt{k}}$$

and

$$\frac{\sigma(\sqrt{k})}{m}.$$

First, we establish the inequation

$$\frac{m+1}{\sqrt{k}} \neq \frac{\sigma(\sqrt{k})}{m},$$

subject to a reasonable divisibility assumption concerning k and m, which is supported by the observation

$$\gcd(k', m') = 1,$$

as exhibited by the divisors k' and m' of the single Descartes number

$$n' = k'm' = 198585576189$$

that we know of. Suppose to the contrary that

$$\frac{m+1}{\sqrt{k}} = \frac{\sigma(\sqrt{k})}{m}.$$

Then we have

$$m(m+1) = \sqrt{k}\sigma(\sqrt{k}).$$

We now realize that, without assuming a divisibility constraint, we will be unable to obtain anything fruitful from this last equation. (We wish to emphasize the assertions in Remark 3.) So we suppose (for the sake of deriving a contradiction) that $\gcd(m, \sqrt{k}) = 1$. (Note that this implies $\gcd(m, k) = 1$.) This assumption implies the divisibility conditions

$$m \mid \sigma(\sqrt{k})$$

and

$$\sqrt{k} \mid (m+1)\,.$$

In fact, we have that
$$\frac{m+1}{\sqrt{k}} = \frac{\sigma(\sqrt{k})}{m}$$
is an integer. This further implies that
$$\frac{m+1}{\sqrt{k}} \cdot \frac{\sigma(\sqrt{k})}{m} = \frac{m+1}{m} \cdot \frac{\sigma(\sqrt{k})}{\sqrt{k}}$$
must be an integer, which contradicts the constraint
$$1 < \frac{3}{\sqrt{5}} < \frac{m+1}{m}\frac{\sigma(\sqrt{k})}{\sqrt{k}} < \frac{m+1}{m}\frac{\sigma(k)}{k} = 2.$$
We state this result as our next lemma.

Lemma 6. *Let $n = km > 1$ be a spoof. If $\gcd(m, \sqrt{k}) = 1$, then we have the inequation*
$$\frac{m+1}{\sqrt{k}} \neq \frac{\sigma(\sqrt{k})}{m}.$$

Remark 12. If $n = km > 1$ is a spoof and $\gcd(m, \sqrt{k}) = 1$, then it follows that $\gcd(m, k) = 1$.

Note that, in Descartes' example:
$$d = 3^2 7^2 11^2 13^2 22021 = 3^2 7^2 11^2 13^2 19^2 61 = 198585576189$$
so that
$$m = 22021 = 19^2 \cdot 61,$$
$$k = 3^2 \cdot 7^2 \cdot 11^2 \cdot 13^2 = 9018009,$$
and
$$\sqrt{k} = 3 \cdot 7 \cdot 11 \cdot 13 = 3003.$$
Consequently, we have
$$\gcd(m, k) = \gcd(m, \sqrt{k}) = 1$$
and
$$\sqrt{k} < m < k,$$
at least for the Descartes number d.

Lastly, if $\gcd(m, k) = 1$, then since the sum-of-divisors function σ is weakly multiplicative, we obtain
$$\sigma(n) = \sigma(k)\sigma(m).$$
Also, from the definition of Descartes numbers, we have
$$2n = \sigma(k)(m+1).$$

Consequently, we have the biconditional

$$\sigma(n) = 2n \Longleftrightarrow \sigma(m) = m + 1.$$

But $\sigma(m) = m + 1$ is true if and only if m is prime. Since m is the quasi-Euler prime, by Remark 1 we know that $\sigma(m) > m + 1$, so that we get $\sigma(n) > 2n$. This last inequality agrees with our earlier computation of the abundancy index for the Descartes number $d = 198585576189$.

Taking off from the considerations in Remark 12, we now predict the following conjectures.

Conjecture 1. *Let $n = km > 1$ be a spoof. Then the following conditions hold:*

- *The inequality $I(n) > 2$ holds. (i.e., n is abundant.)*
- *If $\gcd(m, k) = 1$, then the inequality $\sqrt{k} < m < k$ holds.*

Remark 13. Conjecture 1 is an analogue of the Descartes-Frenicle-Sorli conjecture on odd perfect numbers, for the case of spoofs.

Note that the inequality $\sqrt{k} < m$ implies that m is the largest quasi-prime factor of the spoof $n = km$.

We now go back to our earlier problem of determining the correct ordering between the quantities

$$\frac{m+1}{\sqrt{k}}$$

and

$$\frac{\sigma(\sqrt{k})}{m}.$$

We closely follow the method in the MSE post cited in [9]. Essentially, we want to show that the biconditionals

$$m < \sqrt{k} \Longleftrightarrow m + 1 < \sigma(\sqrt{k}) \Longleftrightarrow \frac{m+1}{\sqrt{k}} < \frac{\sigma(\sqrt{k})}{m}$$

are true. (However, we will find that an additional assumption (i.e., $m < \sqrt{k}$) is sufficient in order to prove that this biconditional is indeed true.)

First, we show that the following claim holds.

Lemma 7. *Let a and b be integers, and let $I(x) = \sigma(x)/x$ be the abundancy index of the (positive) integer x. Then the following statements are true.*

- *If $I(a) + I(b) < \sigma(a)/b + \sigma(b)/a$, then $a < b \Longleftrightarrow \sigma(a) < \sigma(b)$.*
- *If $\sigma(a)/b + \sigma(b)/a < I(a) + I(b)$, then $a < b \Longleftrightarrow \sigma(b) < \sigma(a)$.*

Proof. Here, we only prove the first claim, as the proof for the second claim is very similar.

To this end, suppose that $I(a) + I(b) < \sigma(a)/b + \sigma(b)/a$. Then we have:

$$\frac{\sigma(a)}{a} + \frac{\sigma(b)}{b} < \frac{\sigma(a)}{b} + \frac{\sigma(b)}{a}.$$

This is equivalent to:

$$b\sigma(a) + a\sigma(b) < a\sigma(a) + b\sigma(b),$$

which is in turn equivalent to:

$$(b - a)\sigma(a) - (b - a)\sigma(b) < 0.$$

This last inequality is equivalent to:

$$(a - b)(\sigma(a) - \sigma(b)) > 0.$$

Consequently, $a < b \Longleftrightarrow \sigma(a) < \sigma(b)$ holds, as desired. □

Remark 14. As pointed out to the author by a referee, in Lemma 7, a, b, $\sigma(a)$, and $\sigma(b)$ could be replaced by any 4 arbitrary real numbers. Lemma 7 is *not directly* related to odd perfect numbers (or to spoofs for that matter), nor to properties of the sum-of-divisors function. This does not mean, though, that we will not be able to utilize this lemma to derive results on odd perfect numbers (or more so, on spoofs). (Details on an implementation of this idea for establishing some biconditionals involving the divisors of odd perfect numbers are in `http://math.stackexchange.com/q/548528`.)

Next, we have the following observation.

Remark 15. Let $n = km >$ be a spoof. By Theorem 4, we have the inequality

$$\frac{m+1}{m} < \frac{\sigma(\sqrt{k})}{\sqrt{k}}.$$

We can rewrite this as

$$\frac{m+1}{\sigma(\sqrt{k})} < \frac{m}{\sqrt{k}}.$$

Consequently, we have the implications

$$\sigma(\sqrt{k}) < m + 1 \Longrightarrow \sqrt{k} < m$$

and

$$m < \sqrt{k} \Longrightarrow m + 1 < \sigma(\sqrt{k}).$$

Notice that, an easy way to prove the biconditionals

$$m < \sqrt{k} \Longleftrightarrow m + 1 < \sigma(\sqrt{k}) \Longleftrightarrow \frac{m+1}{\sqrt{k}} < \frac{\sigma(\sqrt{k})}{m}$$

would then be to rule out the "middle case"

$$\frac{m+1}{\sigma(\sqrt{k})} < 1 < \frac{m}{\sqrt{k}}.$$

In general, this approach would be difficult. (See an earlier version of the paper [10] in `http://arxiv.org/pdf/1103.1090v1.pdf` for a similar approach in the case of odd perfect numbers). In order to hurdle this obstacle, we make use of

$$\frac{m+1}{\sigma(\sqrt{k})} < \frac{m}{\sqrt{k}}$$

and Lemma 6.

We are now ready to prove the following statement.

Theorem 16. *Let* $n = km > 1$ *be a spoof. If* $\gcd(m, k) = 1$ *and* $m < \sqrt{k}$*, then the biconditionals*

$$m < \sqrt{k} \Longleftrightarrow m + 1 < \sigma(\sqrt{k}) \Longleftrightarrow \frac{m+1}{\sqrt{k}} < \frac{\sigma(\sqrt{k})}{m}$$

hold.

Proof. As hinted in Remark 15, an easy way to prove the biconditionals

$$m < \sqrt{k} \Longleftrightarrow m + 1 < \sigma(\sqrt{k}) \Longleftrightarrow \frac{m+1}{\sqrt{k}} < \frac{\sigma(\sqrt{k})}{m}$$

is to rule out the "middle case"

$$\frac{m+1}{\sigma(\sqrt{k})} < 1 < \frac{m}{\sqrt{k}}.$$

Indeed, our hypothesis $m < \sqrt{k}$ rules out the inequality $\sqrt{k} < m$ on the right.

Here we present an alternative for a different perspective on the proof. To this end, suppose that $n = km > 1$ is a spoof with $\gcd(m, k) = 1$ and $m < \sqrt{k}$. (Given our assumption that $m < \sqrt{k}$, and since we know *a priori* that $k > 1$, notice that $m + 1 = \sigma(\sqrt{k})$ cannot happen, as this will imply that $m < \sqrt{k} < m + 1$, contradicting the fact that m and $m + 1$ are consecutive integers.)

The inequality $m < \sqrt{k}$ implies

$$m + 1 < \sigma(\sqrt{k})$$

(by Remark 15).

Now, $m < \sqrt{k}$ and $m+1 < \sigma(\sqrt{k})$ imply that

$$\frac{m+1}{\sqrt{k}} < \frac{\sigma(\sqrt{k})}{m}.$$

Additionally,

$$\frac{m+1}{\sqrt{k}} < \frac{\sigma(\sqrt{k})}{m}$$

together with the inequality

$$\frac{m+1}{m} < \frac{\sigma(\sqrt{k})}{\sqrt{k}}$$

from Theorem 4, implies that

$$m+1 < \sigma(\sqrt{k}).$$

Summarizing, we now have the chain of implications

$$m < \sqrt{k} \Longrightarrow m+1 < \sigma(\sqrt{k}) \Longrightarrow \frac{m+1}{\sqrt{k}} < \frac{\sigma(\sqrt{k})}{m} \Longrightarrow m+1 < \sigma(\sqrt{k}).$$

It remains to show either of the implications

$$\frac{m+1}{\sqrt{k}} < \frac{\sigma(\sqrt{k})}{m} \Longrightarrow m < \sqrt{k},$$

or

$$m+1 < \sigma(\sqrt{k}) \Longrightarrow m < \sqrt{k}.$$

We take an indirect approach. Noting the considerations in Remark 14, we set

$$a = m, b = \sqrt{k}, \sigma(a) = m+1, \sigma(b) = \sigma(\sqrt{k})$$

in Lemma 7, and thereby obtain the following:

- If $\frac{m+1}{m} + \frac{\sigma(\sqrt{k})}{\sqrt{k}} < \frac{m+1}{\sqrt{k}} + \frac{\sigma(\sqrt{k})}{m}$, then $m < \sqrt{k} \Longleftrightarrow m+1 < \sigma(\sqrt{k})$.
- If $\frac{m+1}{m} + \frac{\sigma(\sqrt{k})}{\sqrt{k}} = \frac{m+1}{\sqrt{k}} + \frac{\sigma(\sqrt{k})}{m}$, then either $m = \sqrt{k}$ or $m+1 = \sigma(\sqrt{k})$.
- If $\frac{m+1}{\sqrt{k}} + \frac{\sigma(\sqrt{k})}{m} < \frac{m+1}{m} + \frac{\sigma(\sqrt{k})}{\sqrt{k}}$, then $m < \sqrt{k} \Longleftrightarrow \sigma(\sqrt{k}) < m+1$.

Note that the following implications from Remark 15:

$$\sigma(\sqrt{k}) < m+1 \Longrightarrow \sqrt{k} < m$$

and

$$m < \sqrt{k} \Longrightarrow m+1 < \sigma(\sqrt{k}),$$

show that the implication $m < \sqrt{k} \Longrightarrow \sigma(\sqrt{k}) < m+1$ is false (since $m < \sqrt{k}$ is true by assumption). Consequently, the biconditional

$$m < \sqrt{k} \Longleftrightarrow \sigma(\sqrt{k}) < m+1$$

is not true. Therefore, the inequality

$$\frac{m+1}{m} + \frac{\sigma(\sqrt{k})}{\sqrt{k}} \leq \frac{m+1}{\sqrt{k}} + \frac{\sigma(\sqrt{k})}{m}$$

must hold.

It remains to consider the case

$$\frac{m+1}{m} + \frac{\sigma(\sqrt{k})}{\sqrt{k}} = \frac{m+1}{\sqrt{k}} + \frac{\sigma(\sqrt{k})}{m}.$$

Notice that the equation

$$\frac{m+1}{m} + \frac{\sigma(\sqrt{k})}{\sqrt{k}} = \frac{m+1}{\sqrt{k}} + \frac{\sigma(\sqrt{k})}{m}$$

is true if and only if

$$m = \sqrt{k}$$

or

$$m+1 = \sigma(\sqrt{k}).$$

If both $m = \sqrt{k}$ and $m+1 = \sigma(\sqrt{k})$ hold, then $\sigma(m) = m+1$, so that m is prime. Since m is the quasi-Euler prime, this is a contradiction (by Remark 1). Therefore, either $m = \sqrt{k}$ or $m+1 = \sigma(\sqrt{k})$ is true, but not both.

By assumption, $\gcd(m, k) = 1$. Since this implies $\gcd(m, \sqrt{k}) = 1$, we cannot have $m = \sqrt{k}$, as we know that both $m \geq 9$ (by Remark 1) and $k > m^2 \geq 81$ (since $m < \sqrt{k}$ by assumption).

It remains to consider the equation $m+1 = \sigma(\sqrt{k})$.

By Remark 15, we have

$$\frac{m+1}{\sigma(\sqrt{k})} < \frac{m}{\sqrt{k}}.$$

We therefore obtain $\sqrt{k} < m$. This contradicts the hypothesis $m < \sqrt{k}$.

Since the inequality

$$\frac{m+1}{m} + \frac{\sigma(\sqrt{k})}{\sqrt{k}} < \frac{m+1}{\sqrt{k}} + \frac{\sigma(\sqrt{k})}{m}$$

implies that the biconditional

$$m < \sqrt{k} \Longleftrightarrow m+1 < \sigma(\sqrt{k})$$

is true, and because this biconditional further implies that

$$m+1<\sigma(\sqrt{k}) \Longleftrightarrow \frac{m+1}{\sqrt{k}}<\frac{\sigma(\sqrt{k})}{m}$$

is true (still under the hypothesis $m<\sqrt{k}$), then we finally obtain that the biconditionals

$$m<\sqrt{k} \Longleftrightarrow m+1<\sigma(\sqrt{k}) \Longleftrightarrow \frac{m+1}{\sqrt{k}}<\frac{\sigma(\sqrt{k})}{m}$$

hold.

This finishes the proof. □

Remark 17. Let us double-check the findings in Theorem 16 using the only spoof that we know of, as a test case.

In Descartes' example, we have

$$m=22021=19^2\cdot 61$$

and

$$\sqrt{k}=3\cdot 7\cdot 11\cdot 13=3003,$$

so that we obtain

$$m+1=22022=2\cdot 11011,$$

$$\sigma(\sqrt{k})=(3+1)\cdot(7+1)\cdot(11+1)\cdot(13+1)=5376=2^8\cdot 3\cdot 7$$

and

$$\sqrt{k}<m.$$

Notice that

$$\sigma(\sqrt{k})<m+1$$

and that

$$\frac{\sigma(\sqrt{k})}{m}=\frac{5376}{22021}<1<\frac{22022}{3003}=\frac{m+1}{\sqrt{k}},$$

in perfect agreement with the results in Theorem 16.

Remark 18. In general, note that Theorem 16 in fact gives the inequality

$$\frac{m+1}{m}+\frac{\sigma(\sqrt{k})}{\sqrt{k}}<\frac{m+1}{\sqrt{k}}+\frac{\sigma(\sqrt{k})}{m}.$$

The lower bound

$$1+\sqrt{\frac{9}{5}}<\frac{m+1}{m}+\frac{\sigma(\sqrt{k})}{\sqrt{k}}$$

follows from the estimates in Theorem 4, so that we have the lower bound

$$1+\sqrt{\frac{9}{5}} < \frac{m+1}{\sqrt{k}} + \frac{\sigma(\sqrt{k})}{m}.$$

Note the rational approximation

$$1+\sqrt{\frac{9}{5}} \approx 2.34164$$

We end this section with the following conjecture.

Conjecture 2. *Let $n = km > 1$ be a spoof. If $\gcd(m,k) = 1$, then the following inequality holds:*

$$\sigma(\sqrt{k}) < m.$$

Remark 19. We remark that Conjecture 2 is consistent with the corresponding assertion in Conjecture 1.

4. Future Research

If $n = km > 1$ is a spoof, an improvement to the upper bound $I(k) < 2$ would rule out the case $k < m$ in Theorem 8. (Corresponding to this endeavor is an ongoing effort to improve the upper bound $I(t) < 2$ where $t = \sqrt{\frac{N}{q^r}}$ and N is an odd perfect number given in Eulerian form. A referee has pointed out to the author that improving $I(t) < 2$ is equivalent to getting an upper bound on the smallest prime factor of N. This can easily be seen by considering the inequality $2(t-1)/t < I(t)$.)

Lastly, note that we have not utilized computers to either validate or otherwise calibrate the findings in this paper. We leave this as a computational project for other researchers to pursue.

5. Concluding Remarks

In the same way that 1 is the only odd multiperfect number which has been discovered (as well as being the single odd almost perfect number currently known), it would seem that Descartes' spoof would be the only one of its kind. This is evidenced by the following theorems, taken from the respective indicated papers:

Theorem 20. *[*Banks, et. al. (2008), [1]*]. If n is a cube-free spoof which is not divisible by 3, then $n = k\sigma(k)$ for some odd almost perfect number k, and n has more than one million distinct prime divisors.*

Theorem 21. *[*Dittmer (2014), [6]*]. The only spoof with less than seven quasi-prime factors is Descartes' example,* $3^2 \cdot 7^2 \cdot 11^2 \cdot 13^2 \cdot 22021^1$*.*

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