Bijection: Parking-like structures and Tree-like structures

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(enumeration, bijection, parking functions, trees, species, labeled structures, Lagrange inversion)

The *parking problem* is introduced in [KW66, §6]:

> A car occupied by a man and his dozing wife enters the street on the left and moves towards the right. The wife awakens at a capricious moment and orders her husband to park immediately! He dutifully parks at his present location, if it is empty, and if not, continues to the right and parks at the next available space.

This *occupancy problem* introduce the *parking function* combinatorial structures: Let U be a finite set. A parking function $\hbar : U \to \mathbb{N}_{>0}$ is a function such that,

$$\#\{\, \hbar(i) \leqslant k \mid i \in U \,\} \geqslant k\,, \qquad \text{for any } k \leqslant u\,. \tag{1}$$

As a hash problem a "car" is a key $u \in U$, the "dozing wives" act as the *hash map* $\hbar$: they compute a "park location" (an index) into the "street" (an array of buckets) $\hbar(u)$. The "husbands" act as an effective *open addressing hash table* which solve the *hash collisions* if several wives have same "caprice". In the sequel we keep the wives activities/hash map and we replace the husbands/open addressing implementation with some other "implementation" used to solve hash collisions. The goal is to enumerate all configurations associated to a fixed implementation; for example how many *chained hash tables with linked lists* it is possible to obtain using *parking functions* as a hash map?

In [PV] the authors redefine (generalized) parking functions species recursively: Let $\chi : \mathbb{N}_{>0} \to \mathbb{N}$ be an non-decreasing function,

$$\mathcal{P}_\chi := \left(\mathcal{E}^{\chi(1)}\right)_0 + \sum_{n\geqslant 1} \left(\mathcal{E}^{\chi(1)}\right)_n \cdot \mathcal{P}_{\rho_n}\,, \qquad \text{with} \qquad \rho_n : m \longmapsto \chi(n+m) - \chi(1)\,, \tag{2}$$

with $\mathcal{E}$ the set species; $\mathcal{E}^k$ denotes the exponentiation of set species: $\mathcal{E}^k := \mathcal{E} \cdot \mathcal{E}^{k-1}$ with $\cdot$ the product operator of species. In other terms, $\mathcal{E}^k$ is the species of k-sequences of sets ($\mathcal{E}^k[U]$ is the sequences of length k of disjoint sets labeled by U). The notation $+$ is the sum of species and $(\mathcal{E}^k)_n$ is the restriction of the species $\mathcal{E}^k$ to sets of cardinality n (the reader may refer to [BLL98] about Species Theory or similary [FS09, chapter II] about Labeled Structures). The parking functions $\mathcal{P}$ (1) are generalized parking functions $\mathcal{P}_\chi$, where we take χ to be the identity map Id (see [SP02, KY03] about generalized parking functions).

Despite the fact that the recursive definition (2) gives an efficient way to generate (generalized) parking functions, the underlying exponential generating series is an inefficient enumerating formula, since the coefficients $p(n) = (n+1)^{n-1}$ are defined as a sum over compositions of n [PV, Theorem 3.4]. An elegant way to enumerate *parking functions* is to use bijection with *forest of rooted trees* $\mathcal{F}$. The species of forest rooted trees $\mathcal{F}$ is defined by the following recursive definition:

$$\mathcal{F} := \mathcal{E}(\mathcal{X} \cdot \mathcal{F})\,, \tag{3}$$

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with $\mathcal{X}$ the singleton species ($\mathcal{X}[U] = \{U\}$ *iff* $U = \varnothing$) and $F(G)$ denoting the (partitional) *composite* of the species G in the species F. This bijection of structures gives the generating series of parking functions:

$$\mathcal{P}(t) = \sum_{n\geqslant 0} p(n)\frac{t^n}{n!}\,, \qquad \text{which satisfies the equation:} \qquad \mathcal{P}(t) = \exp(t\,\mathcal{P}(t))\,.$$

We further generalize the parking functions by replacing the set species $\mathcal{E}$, in (2), with other species $\mathcal{G}$ whose structures model implementations of the hash table (using *parking functions* as hash maps): the *parking-like structures*. In the same way we replace $\mathcal{E}$ by $\mathcal{G}$ in the forest definition (3) and consider the *tree-like structures*. We show and make explicit a bijection between the *parking-like structures* and the *tree-like structures* (Theorem 1). From this isomorphism we obtain generating series defined by functional equation (Corollary 1) computable by *Lagrange inversion*.

The generic bijection gives interesting correspondance between *parking-like structures* and trees-like structures: (labeled) *binary trees*, *k-ary trees*, *hypertrees*, *etc.*. Particularly the labeled binary trees are isomorphic to the structures of possible *chained hash tables with linked lists*.

1 Parking like species

Let $\mathcal{G}$ be a species. Without loss of generality we can suppose it is possible to fully recover the underlying set U of any $\mathcal{G}$-structures on U. Indeed the species $\bar{\mathcal{G}} := \mathcal{G} \times \mathcal{E}$ (with $\times$ the cartesian product of species or equivalently $\bar{\mathcal{G}}[U] := \{(g, U)$, for any $g \in \mathcal{G}[U]\}$) is isomorphic to $\mathcal{G}$. We also suppose a total order on U, noted $<_U$:

$$u_1 <_U u_2 <_U \cdots <_U u_n\,, \qquad \text{with } n := \#U\,.$$

1.1 Generalization of the generalization

The definition (2) is a constructive definition of the generalized parking functions [PV, SP02, KY03]. This equation resumes parking functions on U as a sequence of sets (Q_i) which satisfies:

$$\sum_{i=1}^{k} \#Q_i \geqslant k\,, \qquad \text{for any } k \leqslant n := \#U\,. \tag{4}$$

This condition (4) is a translation of the parking condition (1). We generalize this species by replacing $\mathcal{E}$ by other species $\mathcal{G}$:

$$\blacktriangleright^{\mathcal{G}}_{\chi} := \left(\mathcal{G}^{\chi(1)}\right)_0 + \sum_{n\geqslant 1}\left(\mathcal{G}^{\chi(1)}\right)_n \cdot \blacktriangleright^{\mathcal{G}}_{\rho_n}\,, \tag{5}$$

with ρ_n defined as in (2).

Notation : We denote $\mathcal{G}^{\blacktriangleright} := \blacktriangleright^{\mathcal{G}}_{Id}$ the parking-like species over $\mathcal{G}$ associated to the identity map (with $\chi = Id$).

We call $\blacktriangleright^{\mathcal{G}}_{\chi}$ the *parking like species* over the species $\mathcal{G}$ associated to the non-decreasing map χ. (So $\mathcal{P}_\chi = \blacktriangleright^{\mathcal{E}}_{\chi}$ and $\mathcal{P} = \mathcal{E}^{\blacktriangleright}$.) In the sequel we are focusing on this species and we are looking to enumerated $\mathcal{G}^{\blacktriangleright}$-structures *via* a bijection with *tree-like structures*.

Proposition 1: *Let U be a finite set,*

$$\blacktriangleright^{\mathcal{G}}_{\chi}[U] = \left\{(g_i)_{i\in[\chi(u+1)]} \;\middle|\; g_i \in \mathcal{G}[V_i] \text{ such that } (V_i) \in \mathcal{P}_\chi[U]\right\}.$$

Remark 1: The $\blacktriangleright^{\mathcal{G}}_{\chi}$-structures on U are $\chi(u+1)$-sequences of $\mathcal{G}$-structures such that the $\mathcal{G}$-structures appearing at indices between $\chi(u)+1$ and $\chi(u+1)$ are defined on $\varnothing$. If $\chi = Id$ that means the last structure of the sequence is a structure on the emptyset. This point will be important in the bijection Theorem 1.

Example 1: Let $\mathcal{L}$ be the linear order species (defined by $\mathcal{L} = 1 + \mathcal{X} \cdot \mathcal{L}$ with 1 neutral species (such that $1[U] = \{U\}$ *iff* $U = \varnothing$)). The $\mathcal{L}$-structures are isomorphic to the permutations. The $\mathcal{L}^{\blacktriangleright}$-structures on $[0], [1], [2]$ and $[3]$ are:

$$\mathcal{L}^{\blacktriangleright}[0] = \{(\cdot)\}\ ; \quad \mathcal{L}^{\blacktriangleright}[1] = \{(1 \mid \cdot)\}\ ; \quad \mathcal{L}^{\blacktriangleright}[2] = \{(12 \mid \cdot \mid \cdot),\ (21 \mid \cdot \mid \cdot),\ (1 \mid 2 \mid \cdot),\ (2 \mid 1 \mid \cdot)\}\ ;$$

$$\mathcal{L}^{\blacktriangleright}[3] = \left\{ \begin{array}{c} (123 \mid \cdot \mid \cdot \mid \cdot),\ (132 \mid \cdot \mid \cdot \mid \cdot),\ (213 \mid \cdot \mid \cdot \mid \cdot),\ (231 \mid \cdot \mid \cdot \mid \cdot),\ (312 \mid \cdot \mid \cdot \mid \cdot),\ (321 \mid \cdot \mid \cdot \mid \cdot), \\ (12 \mid 3 \mid \cdot \mid \cdot),\ (21 \mid 3 \mid \cdot \mid \cdot),\ (12 \mid \cdot \mid 3 \mid \cdot),\ (21 \mid \cdot \mid 3 \mid \cdot),\ (13 \mid 2 \mid \cdot \mid \cdot),\ (31 \mid 2 \mid \cdot \mid \cdot), \\ (13 \mid \cdot \mid 2 \mid \cdot),\ (31 \mid \cdot \mid 2 \mid \cdot),\ (23 \mid 1 \mid \cdot \mid \cdot),\ (32 \mid 1 \mid \cdot \mid \cdot),\ (23 \mid \cdot \mid 1 \mid \cdot),\ (32 \mid \cdot \mid 1 \mid \cdot), \\ (1 \mid 23 \mid \cdot \mid \cdot),\ (1 \mid 32 \mid \cdot \mid \cdot),\ (2 \mid 13 \mid \cdot \mid \cdot),\ (2 \mid 31 \mid \cdot \mid \cdot),\ (3 \mid 12 \mid \cdot \mid \cdot),\ (3 \mid 21 \mid \cdot \mid \cdot), \\ (1 \mid 2 \mid 3 \mid \cdot),\ (2 \mid 1 \mid 3 \mid \cdot),\ (1 \mid 3 \mid 2 \mid \cdot),\ (3 \mid 1 \mid 2 \mid \cdot),\ (2 \mid 3 \mid 1 \mid \cdot),\ (3 \mid 2 \mid 1 \mid \cdot) \end{array} \right\}.$$

1.2 Tree-like species and bijection

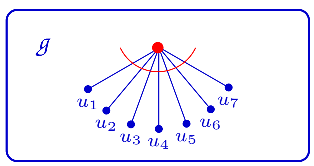
Similary to the replacement on the generalized parking functions species, we replace the set species with another species in the definition of forest species (3):

$$\mathcal{T}_{\mathcal{G}} := \mathcal{G}(\mathcal{X} \cdot \mathcal{T}_{\mathcal{G}})\ .$$

We call $\mathcal{T}_{\mathcal{G}}$ the *tree-like species* over the species $\mathcal{G}$. Focusing on the identity map, we obtain the following isomorphism with $\mathcal{G}^{\blacktriangleright}$:

Theorem 1: *There is a bijection between $\mathcal{G}^{\blacktriangleright}$-structures and $\mathcal{T}_{\mathcal{G}}$-structures ($\mathcal{G}^{\blacktriangleright}[U] \simeq \mathcal{T}_{\mathcal{G}}[U]$, for any finite set U).*

Proof (by drawing): Let U be a finite set of cardinality n. Let g be a $\mathcal{G}$-structure on U represented (as in [BLL98]) by:



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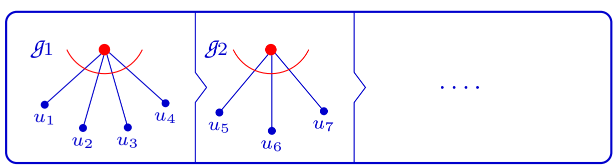
Let (g_i) be a $\mathcal{G}^{\blacktriangleright}$-structure on U represented by:

$(g_i) =$ 

,

This structure is a $n+1$-sequence of $\mathcal{G}$-structures g_i which satisfies:

$$\sum_{i=1}^{k} \#V_i \geqslant k \qquad \text{for any} \quad k \leqslant n\,; \qquad \text{(Proposition 1)}$$

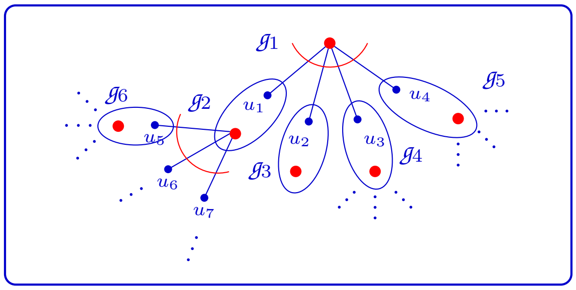
with V_i the underlying set of g_i, for any i (analogous to (1)). Using the order $<_U$ on U we associate to the parking like structure (g_i) the total order $<_q$ defined by:

$$u <_q u' \quad \Longleftrightarrow \quad \begin{cases} u <_U u' & \text{with } u, u' \text{ elements of the underlying set of } g_i, \\ j < k & \text{with } u \ (\textit{resp. } u') \text{ an element of the underlying set of } g_j \ (\textit{resp. } g_k), \end{cases}$$

Following the idea of the construction between parking function and forest of rooted trees we associate to the $\mathcal{G}^{\blacktriangleright}$-structure (g_i) a $\mathcal{T}_{\mathcal{G}}$-structure f defined by:

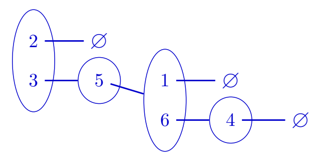
- set g_1 be the root,
- set vertices $u_i \to g_{i+1}$, for any $u_i \in U$.

This construction is summarized by the following schema (where vertices are represented by circles):



Furthermore this construction is reversible: g_1 is the root of f without the vertices, using the order on U one obtains a deterministic way of ordering each underlying $\mathcal{G}$-structures from vertices. Finally by induction on the height of f we show that the resulting sequence of $\mathcal{G}$-structures satisfies (1). ∎

Example 2: Let $(23 \mid \cdot \mid 5 \mid 16 \mid \cdot \mid 4 \mid \cdot)$ be a (usual) parking function on $[6]$. We associate the order $2 <_q 3 <_q 5 <_q 1 <_q 6 <_q 4$ from the parking function (using the natural order on $[6]$). The bijection described in the proof of the Theorem is a generalization of the FOATA-RIORDAN bijection [FR74]. The forest of $\mathcal{F}[6]$ associated to this construction is:



From the species theory this theorem immediately yields the following equalities of exponential generating series computable by *Lagrange inversion*:

Corollary 1: *Let $\mathcal{G}^{\blacktriangleright}(t)$ and $\mathcal{T}_{\mathcal{G}}(t)$ be respectively the exponential generating series of $\mathcal{G}^{\blacktriangleright}$ and $\mathcal{T}_{\mathcal{G}}$,*

$$\mathcal{G}^{\blacktriangleright}(t) = \mathcal{T}_{\mathcal{G}}(t) \qquad \textit{and} \qquad \mathcal{T}_{\mathcal{G}}(t) = \mathcal{G}(t\mathcal{T}_{\mathcal{G}}(t))\,.$$

2 Applications on some species

2.1 Linear order species and chained hash tables with linked lists

We recall the exponential generating series of the linear order: $\mathcal{L}(t) = (1-t)^{-1}$. So thanks to the Corollary 1, the exponential generating series of $\mathcal{T}_{\mathcal{L}} \simeq \mathcal{L}^{\blacktriangleright}$ is:

$$\mathcal{T}_{\mathcal{L}}(t) = \frac{1-\sqrt{1-4t}}{2t}\,. \qquad \text{[A001761]}$$

In Example 1 we listed all possible *hash tables with linked lists* (using parking functions as hash map) on a set of key $U = [0], [1], [2]$ and $[3]$. This formula suggests an isomorphism with labeled binary trees; considering a parking function as a *staircase walk* where *tread*/horizontal steps are decorated by linear orders of same length (see [PV, §2]), the classical bijection between Dyck paths and binary trees gives a bijection between $\mathcal{L}^{\blacktriangleright}$-structures and labeled binary trees (when we create a node from a tread, we associate *via* the bijection the label to decorate it).

The previous construction (of tree-like structure in the proof of Theorem 1) completed by the bijection between linear orders and decreasing binary trees provides a bijection between $\mathcal{L}^{\blacktriangleright}$-structures and "*labeled incomplete ternary trees on n vertices in which each left and middle child have a larger label than their parent*" (see [A001761]).

2.2 Partition species

The *species of partitions* is defined by the equation: $\mathcal{P}ar := \mathcal{E}(\mathcal{E}_+)$ with $\mathcal{E}_+$ the restriction of $\mathcal{E}$ to non-empty sets. Thanks to the corollary 1, the generating series of $\mathcal{P}ar^{\blacktriangleright}$ satisfies the equation:

$$\mathcal{P}ar^{\blacktriangleright}(t) = \exp(e^{t\,\mathcal{P}ar^{\blacktriangleright}(t)} - 1) = 1 + t + 4\frac{t^2}{2!} + 29\frac{t^3}{3!} + 311\frac{t^4}{4!} + 4447\frac{t^5}{5!} + \cdots \qquad \text{[A030019]}$$

and the construction (proof of the Theorem 1) yields the bijection between $\mathcal{P}ar^{\blacktriangleright}$-structures and " spanning trees in the complete hypergraph on n vertices" (see [A030019]).

2.3 Composition species

The *species of compositions* (or *ballots*) is defined by the equation: $\mathcal{C} := \mathit{1} + \mathcal{E}_+ \cdot \mathcal{C}$ with $\mathit{1}$ the neutral species ($\mathit{1}[U] = \{U\}$ *iff* $U = \varnothing$ and it is empty in otherwise). The generating series of the $\mathcal{C}^\blacktriangleright$-structures satisfies:

$$\mathcal{C}^\blacktriangleright(t) = \frac{1}{2 - \exp(t\,\mathcal{C}^\blacktriangleright(t))} = 1 + t + 5\frac{t^2}{2!} + 46\frac{t^3}{3!} + 631\frac{t^4}{4!} + 11586\frac{t^5}{5!} + \cdots \qquad \text{[A052894]}$$

2.4 Subsets species

The *subsets species* is $\mathcal{S} := \mathcal{E} \cdot \mathcal{E}$. The construction of the Theorem 1 yields a bijection with *ditrees* (see [A097629]) and the generating series satisfies $s(t) = \exp(2t\,s(t))$.

2.5 k-ary trees

Let $\mathcal{F}_k$ be the species of the k-ary trees ($\mathcal{F}_k := \mathit{1} + \mathcal{X} \cdot (\mathcal{F}_k)^k$ with $k \geqslant 1$). The construction defines a bijection between $\mathcal{F}_k^\blacktriangleright$-structures and $\mathcal{F}_{k+1}$-structures (*Note*: $\mathcal{F}_1^\blacktriangleright = \mathcal{L}^\blacktriangleright$). From [BP71], there also exists an isomorphism with labeled dissections of a k-ball.

3 Perspective

The bijection can be extended between *parking-like structures* over $\mathcal{G}$ associated to the any non-decreasing linear function $\chi : m \mapsto a \cdot m + b$ and *tree-like structures* associated to the species $\mathcal{T}_\mathcal{G}^{a,b}$ defined as $\mathcal{G}(\mathcal{X} \cdot \mathcal{T}_\mathcal{G}^{a,0})^{a+b}$ (with the root have $a + b$ ordered edges and others nodes have a ordered edges). This extension is a generalization of the bijective proof [Yan01, Theorem 1].

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