SUPERCONTINUANTS

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Abstract. Morier-Genoud, Ovsienko and Tabachnikov introduced supersymmetric frieze patterns (see ArXiv 1501.07476). This note gives a solution to Problem 1 from that article: determine the formula for the entries of a superfrieze.

This note gives a solution to Problem 1 from the article [3]: determine the formula for the entries of a superfrieze.

Let $\mathcal{R} = \mathcal{R}_0 \oplus \mathcal{R}_1$ be an arbitrary supercommutative ring, and the sequences $\{v_i\}$, $\{w_i\}$, with $v_i \in \mathcal{R}_0$, $w_i \in \mathcal{R}_1$, be defined by the initial conditions $v_{-1} = 0$, $v_0 = 1$, $w_0 = 0$ and the recurrence relation

$$v_i = a_i v_{i-1} - v_{i-2} - \beta_i w_{i-1}, \quad w_i = w_{i-1} + \beta_i v_{i-1} \quad (i \in \mathbb{Z}). \tag{1}$$

In particular,

$$v_1 = a_1, \quad v_2 = a_1 a_2 - 1 + \beta_1\beta_2, \quad v_3 = a_1 a_2 a_3 - a_1 - a_3 + a_1\beta_2\beta_3 + a_3\beta_1\beta_2 + \beta_1\beta_3;$$
$$w_1 = \beta_1, \quad w_2 = a_1\beta_2 + \beta_1, \quad w_3 = a_1 a_2 \beta_3 + a_1\beta_2 + \beta_1\beta_2\beta_3 + \beta_1 - \beta_3.$$

The problem is to express v_n, w_n in terms of $a_1, \ldots, a_n$ and $\beta_1, \ldots, \beta_n$. Such expression will be called *supercontinuants*. (See [2] for the properties of the classical continuants.)

We define two sequences of supercontinuants

$$\{\mathcal{K}\big(\begin{smallmatrix} & a_1 & \\ \beta_1 & & \beta_1\end{smallmatrix} \big| \begin{smallmatrix} & a_2 & \\ \beta_2 & & \beta_2\end{smallmatrix} \big| \ldots \big| \begin{smallmatrix} & a_n & \\ \beta_n & & \beta_n\end{smallmatrix}\big)\} \quad \text{and} \quad \{\mathcal{K}\big(\begin{smallmatrix} & a_1 & \\ \beta_1 & & \beta_1\end{smallmatrix} \big| \ldots \big| \begin{smallmatrix} & a_{n-1} & \\ \beta_{n-1} & & \beta_{n-1}\end{smallmatrix} \big| \beta_n\big)\}$$

by the initial conditions $\mathcal{K}() = 1$, $\mathcal{K}\big(\begin{smallmatrix} & a_1 & \\ \beta_1 & & \beta_1\end{smallmatrix}\big) = a_1$, $\mathcal{K}(\beta_1) = \beta_1$ and the recurrence relations

$$\begin{aligned}\mathcal{K}\big(\begin{smallmatrix} & a_1 & \\ \beta_1 & & \beta_1\end{smallmatrix} \big| \ldots \big| \begin{smallmatrix} & a_n & \\ \beta_n & & \beta_n\end{smallmatrix}\big) &= a_n \mathcal{K}\big(\begin{smallmatrix} & a_1 & \\ \beta_1 & & \beta_1\end{smallmatrix} \big| \ldots \big| \begin{smallmatrix} & a_{n-1} & \\ \beta_{n-1} & & \beta_{n-1}\end{smallmatrix}\big) - \mathcal{K}\big(\begin{smallmatrix} & a_1 & \\ \beta_1 & & \beta_1\end{smallmatrix} \big| \ldots \big| \begin{smallmatrix} & a_{n-2} & \\ \beta_{n-2} & & \beta_{n-2}\end{smallmatrix}\big) \\ &\quad - \beta_n \mathcal{K}\big(\begin{smallmatrix} & a_1 & \\ \beta_1 & & \beta_1\end{smallmatrix} \big| \ldots \big| \begin{smallmatrix} & a_{n-2} & \\ \beta_{n-2} & & \beta_{n-2}\end{smallmatrix} \big| \beta_{n-1}\big), \\ \mathcal{K}\big(\begin{smallmatrix} & a_1 & \\ \beta_1 & & \beta_1\end{smallmatrix} \big| \ldots \big| \begin{smallmatrix} & a_{n-1} & \\ \beta_{n-1} & & \beta_{n-1}\end{smallmatrix} \big| \beta_n\big) &= \beta_n \mathcal{K}\big(\begin{smallmatrix} & a_1 & \\ \beta_1 & & \beta_1\end{smallmatrix} \big| \ldots \big| \begin{smallmatrix} & a_{n-1} & \\ \beta_{n-1} & & \beta_{n-1}\end{smallmatrix}\big) + \mathcal{K}\big(\begin{smallmatrix} & a_1 & \\ \beta_1 & & \beta_1\end{smallmatrix} \big| \ldots \big| \begin{smallmatrix} & a_{n-2} & \\ \beta_{n-2} & & \beta_{n-2}\end{smallmatrix} \big| \beta_{n-1}\big).\end{aligned} \tag{2}$$

From (1) and (2) it easily follows that

$$v_n = \mathcal{K}\big(\begin{smallmatrix} & a_1 & \\ \beta_1 & & \beta_1\end{smallmatrix} \big| \ldots \big| \begin{smallmatrix} & a_n & \\ \beta_n & & \beta_n\end{smallmatrix}\big), \qquad w_n = \mathcal{K}\big(\begin{smallmatrix} & a_1 & \\ \beta_1 & & \beta_1\end{smallmatrix} \big| \ldots \big| \begin{smallmatrix} & a_{n-1} & \\ \beta_{n-1} & & \beta_{n-1}\end{smallmatrix} \big| \beta_n\big).$$

The classical continuants $K(a_1, \ldots, a_n)$, corresponding to reduced regular continued fractions

$$a_1 - \cfrac{1}{a_2 - \ddots - \cfrac{1}{a_n}},$$

are defined by

$$K() = 1, \quad K(a_1) = a_1, \quad K(a_1, \ldots, a_n) = a_n K(a_1, \ldots, a_{n-1}) - K(a_1, \ldots, a_{n-2}).$$

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There is Euler's rule which allows one to write down all summands of $K(a_1,\dots,a_n)$: starting with the product $a_1a_2\dots a_n$, we strike out adjacent pairs a_ia_{i+1} in all possible ways. If a pair a_ia_{i+1} is struck out, then it must be replaced by -1. We can represent Eulers rule graphically by constructing all "Morse code" sequences of dots and dashes having length n, where each dot contributes 1 to the length and each dash contributes 2. For example $K(a_1,a_2,a_3,a_4)$ consists of the following summands:

• • • •	$\mapsto a_1a_2a_3a_4$	• •—• •	$\mapsto -a_1a_4$
•—• • •	$\mapsto -a_3a_4$	•—• •—•	$\mapsto 1$
• • •—•	$\mapsto -a_1a_2$		

By analogy with Euler's rule, we can construct a similar rule for calculation of supercontinuants.

Theorem 1. *The summands of* $\mathcal{K}\big(\overset{a_1}{\beta_1\ \beta_1} \mid \overset{a_2}{\beta_2\ \beta_2} \mid \dots\big)$ *can be obtained from the product* $\beta_1\beta_1\beta_2\beta_2\dots$ *by the following rule: we strike out adjacent pairs and adjacent 4-tuples* $\beta_i\beta_i\beta_{i+1}\beta_{i+1}$ *in all possible ways; for deleted pairs and 4-tuples we make the substitutions* $\beta_i\beta_i\to a_i$, $\beta_i\beta_{i+1}\to 1$, $\beta_i\beta_i\beta_{i+1}\beta_{i+1}\to -1$.

This rule can be represented graphically as well. To each monomial there corresponds a sequence of total length $2n$ (or $2n-1$) consisting of dots (of the length one), dashes (of the length two) and long dashes (of the length four). For example, the monomials of $\mathcal{K}\big(\overset{a_1}{\beta_1\ \beta_1} \mid \overset{a_2}{\beta_2\ \beta_2} \mid \beta_3\big)$ can be obtained from the product $\beta_1\beta_1\beta_2\beta_2\beta_3$ as follows:

•—• •—• •	$\mapsto a_1a_2\beta_3$	• •—• •—•	$\mapsto \beta_1$
•—• • •—•	$\mapsto a_1\beta_2$	•—•—•—• •	$\mapsto -\beta_3$
• •—• • •	$\mapsto \beta_1\beta_2\beta_3$		

Let us note that the odd variables anticommute with each other. In particular, $\beta_i^2=0$, and in each pair $\beta_i\beta_i$ at least one variable must be struck out. Supercontinuants become the usual continuants if all odd variables are replaced by zeros.

Supercontinuants can be expressed as determinants.

Theorem 2.

$$\mathcal{K}\big(\overset{a_1}{\beta_1\ \beta_1} \mid \dots \mid \overset{a_n}{\beta_n\ \beta_n}\big) = \begin{vmatrix} a_1 & -1+\beta_1\beta_2 & \beta_1\beta_3 & \cdots & \beta_1\beta_{n-1} & \beta_1\beta_n \\ -1 & a_2 & -1+\beta_2\beta_3 & \cdots & \beta_2\beta_{n-1} & \beta_2\beta_n \\ 0 & -1 & a_3 & \cdots & \beta_3\beta_{n-1} & \beta_3\beta_n \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & -1 & a_{n-1} & -1+\beta_{n-1}\beta_n \\ 0 & 0 & \cdots & 0 & -1 & a_n \end{vmatrix}, \tag{3}$$

$$\mathcal{K}\big(\overset{a_1}{\beta_1\ \beta_1} \mid \dots \mid \overset{a_{n-1}}{\beta_{n-1}\ \beta_{n-1}} \mid \beta_n\big) = \begin{vmatrix} a_1 & -1+\beta_1\beta_2 & \beta_1\beta_3 & \cdots & \beta_1\beta_{n-1} & \beta_1 \\ -1 & a_2 & -1+\beta_2\beta_3 & \cdots & \beta_2\beta_{n-1} & \beta_2 \\ 0 & -1 & a_3 & \cdots & \beta_3\beta_{n-1} & \beta_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & -1 & a_{n-2} & -1+\beta_{n-2}\beta_{n-1} & \beta_{n-2} \\ 0 & \cdots & 0 & -1 & a_{n-1} & \beta_{n-1} \\ 0 & 0 & \cdots & 0 & -1 & \beta_n \end{vmatrix}.$$

The second determinant in Theorem 2 is well-defined because odd variables occupy only one column. The proofs of Theorems 1 and 2 follow by induction from recurrence relations (2), and we do not dwell on them.

The supercontinuants of the form $\mathcal{K}\big(\beta_1 | \substack{a_2 \\ \beta_2\ \beta_2} | \dots | \substack{a_{n-1} \\ \beta_{n-1}\ \beta_{n-1}} | \beta_n\big)$ also may be defined by the rule from the Theorem 1. For example

$$\mathcal{K}(\beta_1|\beta_2) = \beta_1\beta_2 + 1, \quad \mathcal{K}\big(\beta_1 | \substack{a_2 \\ \beta_2\ \beta_2} | \beta_3\big) = a_2\beta_1\beta_3 + \beta_1\beta_2 + \beta_2\beta_3 + 1.$$

These supercontinuants can be represented in terms of determinants as well (we assume that the determinant is expanded in the first column, and the same rule is applied to all determinants of smaller matrices).

Theorem 3. *The supercontinuants* $\mathcal{K}\big(\beta_1 | \substack{a_2 \\ \beta_2\ \beta_2} | \dots | \substack{a_{n-1} \\ \beta_{n-1}\ \beta_{n-1}} | \beta_n\big)$ *satisfy the recurrence relation*

(4)
$$\mathcal{K}\big(\beta_1 | \substack{a_2 \\ \beta_2\ \beta_2} | \dots | \substack{a_{n-1} \\ \beta_{n-1}\ \beta_{n-1}} | \beta_n\big) = -\beta_n \mathcal{K}\big(\beta_1 | \substack{a_2 \\ \beta_2\ \beta_2} | \dots | \substack{a_{n-1} \\ \beta_{n-1}\ \beta_{n-1}}\big) + \mathcal{K}\big(\beta_1 | \substack{a_2 \\ \beta_2\ \beta_2} | \dots | \beta_{n-1}\big) \qquad (n \geq 2)$$

and can be expressed in the following form:

$$\mathcal{K}\big(\beta_1 | \substack{a_2 \\ \beta_2\ \beta_2} | \dots | \substack{a_{n-1} \\ \beta_{n-1}\ \beta_{n-1}} | \beta_n\big) = \begin{vmatrix} \beta_1 & \beta_2 & \beta_3 & \cdots & \beta_{n-1} & 1 \\ -1 & a_2 & -1+\beta_2\beta_3 & \cdots & \beta_2\beta_{n-1} & \beta_2 \\ 0 & -1 & a_3 & \cdots & \beta_3\beta_{n-1} & \beta_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & -1 & a_{n-2} & -1+\beta_{n-2}\beta_{n-1} & \beta_{n-2} \\ 0 & \cdots & 0 & -1 & a_{n-1} & \beta_{n-1} \\ 0 & 0 & \cdots & 0 & -1 & \beta_n \end{vmatrix}.$$

The proof of formula (4) is an application of the rule from Theorem 1. The determinant formula follows by induction from the recurrence relation (4).

Finally, the even supercontinuants $\mathcal{K}\big(\substack{a_1 \\ \beta_1\ \beta_1} | \dots | \substack{a_n \\ \beta_n\ \beta_n} \big)$ can be also expressed as Berezinians. Recall that the Berezinian of the matrix

$$\mathrm{Ber}\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where where A and D have even entries, and B and C have odd entries, is given by the formula

$$\det(A - BD^{-1}C)\det(D)^{-1}, \tag{5}$$

see, e.g., [1].

Theorem 4.

$$\mathcal{K}\big(\substack{a_1 \\ \beta_1\ \beta_1} | \dots | \substack{a_n \\ \beta_n\ \beta_n} \big) = \mathrm{Ber}\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where

$$A=\begin{pmatrix} a_1 & -1 & 0 & \cdots & 0\\ -1 & a_2 & -1 & \ddots & \vdots\\ 0 & -1 & a_3 & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & -1\\ 0 & \cdots & 0 & -1 & a_n \end{pmatrix},\qquad B=\begin{pmatrix} \beta_1 & \beta_2 & \beta_3 & \cdots & \beta_n\\ 0 & \beta_2 & \beta_3 & \ddots & \vdots\\ 0 & 0 & \beta_3 & \ddots & \beta_n\\ \vdots & \ddots & \ddots & \ddots & \beta_n\\ 0 & \cdots & 0 & 0 & \beta_n \end{pmatrix},$$

$$C=\begin{pmatrix} -\beta_1 & 0 & 0 & \cdots & 0\\ 0 & -\beta_2 & 0 & \ddots & \vdots\\ 0 & 0 & -\beta_3 & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & 0\\ 0 & \cdots & 0 & 0 & -\beta_n \end{pmatrix},\qquad D=\begin{pmatrix} 1 & 0 & 0 & \cdots & 0\\ 0 & 1 & 0 & \ddots & \vdots\\ 0 & 0 & 1 & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & 0\\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix}.$$

Theorem 4 is direct corollary of (3) and (5).

It follows from recurrence relations (2) and (4) that the number of terms in supercontinuants $\mathcal{K}\big({}_{\beta_1}{}^{a_1}{}_{\beta_1} | \dots | {}_{\beta_n}{}^{a_n}{}_{\beta_n} \big)$, $\mathcal{K}\big({}_{\beta_1}{}^{a_1}{}_{\beta_1} | \dots | {}_{\beta_{n-1}}{}^{a_{n-1}}{}_{\beta_{n-1}} |\beta_n\big)$ and $\mathcal{K}\big(\beta_1| {}_{\beta_2}{}^{a_2}{}_{\beta_2} | \dots | {}_{\beta_{n-1}}{}^{a_{n-1}}{}_{\beta_{n-1}} |\beta_n\big)$ coincide respectively with the sequences (see [4])

$$A077998 : 1, 3, 6, 14, 31, 70, 157, 353, 793, 1782, 4004, \dots$$
$$A006054 : 1, 2, 5, 11, 25, 56, 126, 283, 636, 1429, 3211, \dots$$
$$A052534 : 1, 2, 4, 9, 20, 45, 101, 227, 510, 1146, 2575, \dots$$

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