ROOT GEOMETRY OF POLYNOMIAL SEQUENCES II: TYPE $(1,0)$

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ABSTRACT. We consider the sequence of polynomials $W_n(x)$ defined by the recursion $W_n(x) = (ax + b)W_{n-1}(x) + dW_{n-2}(x)$, with initial values $W_0(x) = 1$ and $W_1(x) = t(x - r)$, where a, b, d, t, r are real numbers, $a, t > 0$, and $d < 0$. We show that every polynomial $W_n(x)$ is distinct-real-rooted, and that the roots of the polynomial $W_n(x)$ interlace the roots of the polynomial $W_{n-1}(x)$. We find that, as $n \to \infty$, the sequence of smallest roots of the polynomials $W_n(x)$ converges decreasingly to a real number, and that the sequence of largest roots converges increasingly to a real number. Moreover, by using the Dirichlet approximation theorem, we prove that there is a number to which, for every positive integer $i \geq 2$, the sequence of ith smallest roots of the polynomials $W_n(x)$ converges. Similarly, there is a number to which, for every positive integer $i \geq 2$, the sequence of ith largest roots of the polynomials $W_n(x)$ converges. It turns out that these two convergence points are independent of the numbers t and r, as well as i. We derive explicit expressions for these four limit points, and we determine completely when some of these limit points coincide.

CONTENTS

1. Introduction

In [14], we initiated study of the root geometry of a recursively defined sequence $\{W_n(x)\}_{n\geq 0}$ of polynomials. Originally motivated by the LCGD conjecture from topological graph theory, which says that the genus polynomial of every graph is log-concave, we are led to study the log-concavity of the polynomials in $\{W_n(x)\}_{n\geq 0}$; see [10, 12, 13, 15]. Extending our perspective from the arithmetic property to algebraic structures, and following the spirit of Gian-Carlo Rota, we study the distribution of the zero-sets of the polynomials $\{W_n(x)\}_{n\geq 0}$. Such a distribution is called the *root geometry* of the sequence. General information for the root geometry of collections of polynomials can be found in Marden [21], Rahman and Schmeisser [27], and Prasolov [26].

As encountered naturally in biology, economics, computer sciences, combinatorics, and other branches of mathematics, recurrence relations are among the most familiar objects that mathematicians work on. Proceeding systematically, we [14] introduced the concept of a *polynomial sequence of type* (k, l), which satisfies a recursion of the form

$$W_n(x) \ = \ A(x)W_{n-1}(x) + B(x)W_{n-2}(x)$$

for $n \geq 2$, where $A(x)$ and $B(x)$ are polynomials of degrees k and l, respectively, and where $W_0(x)$ is a constant and $W_1(x)$ a linear polynomial. For the first non-trivial case, namely, sequences of type $(0, 1)$, we showed that subject to some general conditions that hold for all graph genus polynomials, every polynomial is distinct-real-rooted and that the zero-sets of the polynomials $W_{n+1}(x)$ and $W_n(x)$ are interlacing. We also found a sharp bound for the union of all zero-sets over n.

This paper continues the study of root geometry of recursive polynomials, this time for the second non-trivial case, namely, those of type $(1, 0)$. We will confirm the distinct-real-rootedness of such polynomials subject to similar general conditions, and we determine the best bound for the union of all zero-sets. By using Dirichlet's approximation theorem, we calculate some limit points of the union. Classical examples for this kind of recursive polynomials include Chebyshev polynomials of the first and second kinds.

This paper is organized as follows. Section 2 contains the root geometry of polynomial sequences of type $(1, 0)$ as our main result, as well as some applications. We show a particular case for Theorem 2.4 in Section 3, namely, Theorem 2.6, using which we complete the proof of Theorem 2.4 in Section 4.

2. Main Theorem

In this section we describe some information for the root geometry of a polynomial sequence of type $(1, 0)$ as the main result, and we present some applications for illustration.

2.1. **Main result.** We begin with definitions and notation needed or the statement of Theorem 2.4, which is our main result.

Definition 2.1. A polynomial is said to be *distinct-real-rooted* if all its roots are distinct and real. The set of all its roots is called the *zero-set* of a polynomial.

Definition 2.2. Let $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_{n-1}\}$ be ordered sets of real numbers. We say that X *interlaces* Y, denoted $X \bowtie Y$, if

$$x_1 < y_1 < x_2 < y_2 < \cdots < x_{n-1} < y_{n-1} < x_n.$$

A special case is that a singleton set $\{x\}$ interlaces the empty set.

Notation 2.3. Let $\{x_n\}_{n \geq 0}$ be a sequence of real numbers. We write $x_n \searrow x$ if the sequence converges to the number x decreasingly as $n \to \infty$, and we write $x_n \nearrow x$ if it converges to x increasingly.

Our main result is as follows.

Theorem 2.4. *Let $\{W_n(x)\}_{n \geq 0}$ be the polynomial sequence defined recursively by*

$$(2.1) \qquad W_n(x) = (ax + b)W_{n-1}(x) + d \cdot W_{n-2}(x),$$

with $W_0(x) = 1$ and $W_1(x) = t(x - r)$, where $a, t > 0$, $d < 0$, and $b, r \in \mathbb{R}$. Then the polynomial $W_n(x)$ has degree n and is distinct-real-rooted. Also, we define

$$(2.2) \qquad r^{\pm} = -\frac{b}{a} \pm \frac{|a - 2t|\sqrt{-d}}{at},$$

$$(2.3) \qquad x_{\Delta}^{\pm} = \frac{-b \pm 2\sqrt{-d}}{a}, \qquad and$$

$$(2.4) \qquad \xi^{\pm} = \begin{cases} \dfrac{t(ar - b - 2rt) \pm \sqrt{t^2(ar + b)^2 - 4dt(a - t)}}{2t(a - t)}, & \text{if } a \neq t; \\ r - \dfrac{d}{a(ar + b)}, & \text{if } a = t \text{ and } ar + b \neq 0. \end{cases}$$

Denote the zero-set of the polynomial $W_n(x)$ by $R_n = \{\xi_{n,1}, \xi_{n,2}, \ldots, \xi_{n,n}\}$ in increasing order. Then we have the interlacing property $R_{n+1} \bowtie R_n$, and the limits

$$(2.5) \qquad \xi_{n,j} \searrow x_{\Delta}^{-} \quad and \quad \xi_{n,n+1-j} \nearrow x_{\Delta}^{+}, \qquad for \text{ all } j \geq 2.$$

Moreover, we have the following cases of convergence results.

   (i) *If $r \in [r^{-}, r^{+}]$ and $a \leq 2t$, then $\xi_{n,1} \searrow x_{\Delta}^{-}$ and $\xi_{n,n} \nearrow x_{\Delta}^{+}$.*

(ii) *If $r \in (r^-, r^+)$ and $a > 2t$, then $\xi_{n,1} \searrow \xi^-$ and $\xi_{n,n} \nearrow \xi^+$.*

(iii) *If $r = r^-$ and $a > 2t$, or $r < r^-$, then $\xi_{n,1} \searrow \xi^-$ and $\xi_{n,n} \nearrow x_\Delta^+$.*

(iv) *If $r = r^+$ and $a > 2t$, or $r > r^+$, then $\xi_{n,1} \searrow x_\Delta^-$ and $\xi_{n,n} \nearrow \xi^+$.*

Recall from [14] that the sequence of largest roots of a polynomial sequence of type $(0, 1)$ converges to a real number in an oscillating manner. In contrast, for any polynomial sequence of type $(1, 0)$, the sequence of ith smallest roots converges decreasingly as $n \to \infty$, and the sequence of ith largest roots converges increasingly.

Remark. The numbers $\xi^\pm$ are not defined when $a = t$ and $ar + b = 0$. This is an instance of Case (i) of Theorem 2.4.

To give a proof of Theorem 2.4, we state its normalized version as Theorem 2.6. The following notion of $(1, 0)$-sequence of polynomials is the key object we will study; see Section 3. As will be seen, Theorem 2.6 implies Theorem 2.4.

Definition 2.5. Let $\{W_n(x)\}_{n \geq 0}$ be the polynomial sequence defined recursively by
$$W_n(x) = (ax + b)W_{n-1}(x) + dW_{n-2}(x),$$
with $W_0(x) = 1$ and $W_1(x) = x$, where $a > 0$, $b \geq 0$ and $d < 0$. In this context, we say $\{W_n(x)\}_{n \geq 0}$ is a *normalized $(1, 0)$-sequence of polynomials*, or a $(1, 0)$-*sequence* for short. The following theorem concerns the particular case of Theorem 2.4 for which $t = 1$, $r = 0$, and $b \geq 0$.

Theorem 2.6. *Let $\{W_n(x)\}_{n \geq 0}$ be a $(1, 0)$-sequence. Then the polynomial $W_n(x)$ has degree n and is distinct-real-rooted. Let*

(2.6) $$b_0 = |a - 2|\sqrt{-d} \quad and$$

(2.7) $$x_g^\pm = \begin{cases} \dfrac{-b \pm \sqrt{b^2 - 4d(a - 1)}}{2(a - 1)}, & \text{if } a \neq 1; \\ -\dfrac{d}{b}, & \text{if } a = 1 \text{ and } b \neq 0. \end{cases}$$

Denote the zero-set of the polynomial $W_n(x)$ by $R_n = \{\xi_{n,1}, \xi_{n,2}, \ldots, \xi_{n,n}\}$ in increasing order. Then we have $R_{n+1} \bowtie R_n$ and Result (2.5). Moreover, we have the following.

(i) *If $a \leq 2$ and $b \leq b_0$, then $\xi_{n,1} \searrow x_\Delta^-$ and $\xi_{n,n} \nearrow x_\Delta^+$.*

(ii) *If $a > 2$ and $b < b_0$, then $\xi_{n,1} \searrow x_g^-$ and $\xi_{n,n} \nearrow x_g^+$.*

(iii) *Otherwise, we have $\xi_{n,1} \searrow x_\Delta^-$ and $\xi_{n,n} \nearrow x_g^+$.*

We note that the limit points $x_\Delta^\pm$ are independent of the numbers t and r.

2.2. **Some examples.** In this subsection, we present several applications of our results.

Example 2.7. Let $W_n(x)$ be the polynomials defined by the recursion
$$W_n(x) = 2xW_{n-1}(x) - W_{n-2}(x),$$
with $W_0(x) = 1$ and $W_1(x)$ linear. By Theorem 2.4, every polynomial $W_n(x)$ is of degree n and distinct-real-rooted. Moreover, we have
$$\xi_{n,j} \searrow -1 \quad \text{and} \quad \xi_{n,\,n+1-j} \nearrow 1 \qquad \text{for all } j \geq 2.$$
If $W_1(x) = x$, then Theorem 2.4 (i) gives that
$$(2.8) \qquad \qquad \xi_{n,1} \searrow -1 \quad \text{and} \quad \xi_{n,n} \nearrow 1.$$
In fact, the polynomials $W_n(x)$ are *Chebyshev polynomials of the first kind*, whose zero-sets are known to be
$$R_n = \left\{ \cos \frac{(2j-1)\pi}{2n} : j \in [n] \right\}.$$
If $W_1(x) = 2x$, then Theorem 2.4 (i) also yields Result (2.8). In this case, the polynomials $W_n(x)$ are *Chebyshev polynomials of the second kind*, whose zero-sets are known to be
$$R_n = \left\{ \cos \frac{j\pi}{n+1} : j \in [n] \right\}.$$
If $W_1(x) = x/2$, then Theorem 2.4 (ii) implies
$$\xi_{n,1} \searrow -2/\sqrt{3} \quad \text{and} \quad \xi_{n,n} \nearrow 2/\sqrt{3}.$$
If $W_1(x) = x + 1$, then Theorem 2.4 (iii) implies that
$$\xi_{n,1} \searrow -\sqrt{2} \quad \text{and} \quad \xi_{n,n} \nearrow 1.$$
If $W_1(x) = x - 2$, then Theorem 2.4 (iv) implies
$$\xi_{n,1} \searrow -1 \quad \text{and} \quad \xi_{n,n} \nearrow \sqrt{5}.$$

Example 2.8. Suppose that $W_0(x) = 1$, $W_1(x) = x - 1$, and
$$W_n(x) = xW_{n-1}(x) - W_{n-2}(x).$$
2.4 (i) implies that $\xi_{n,j} \searrow -2$ and $\xi_{n,\,n+1-j} \nearrow 2$ for all $j \geq 1$. In fact, the zero-set R_n of the polynomial $W_n(x)$ is known [29, A130777] to be
$$R_n = \left\{ -2 \cos \frac{2j\pi}{2n+1} : j \in [n] \right\}.$$

Example 2.9. Suppose that $W_0(x) = 1$, $W_1(x) = x + 1$, and
$$W_n(x) = (x+1)W_{n-1}(x) - W_{n-2}(x).$$
2.4 (i) implies that $\xi_{n,j} \searrow -3$ and $\xi_{n,\,n+1-j} \nearrow 1$ for all $j \geq 1$; see [29, A101950].

Example 2.10. Suppose we have $W_0(x) = 1$, $W_1(x) = x - 1$, and

$$W_n(x) = (x - 1)W_{n-1}(x) - W_{n-2}(x).$$

From 2.4 (i) we infer that $\xi_{n,j} \searrow -1$ and $\xi_{n,n+1-j} \nearrow 3$ for all $j \geq 1$; see [29, A104562].

We will prove Theorem 2.4 in the next two sections. In Section 3, we show it for a particular kind of polynomial sequence of type $(1,0)$, in which $W_1(x) = x$ and $b \geq 0$. In Section 4, we complete the proof of Theorem 2.4 by translating and scaling the roots, so as to drop these two conditions.

3. PROOF OF THEOREM 2.6

We start by determining the degree and the leading coefficient of every polynomial in a $(1,0)$-sequence.

Lemma 3.1. *Let $\{W_n(x)\}_{n \geq 0}$ be a $(1,0)$-sequence. Then every polynomial $W_n(x)$ has degree n and leading coefficient a^{n-1}.*

Proof. This lemma follows from consideration of the contributions of the summands $(ax + b)W_{n-1}(x)$ and $d\,W_{n-2}(x)$ to the degree and to the leading coefficient of the polynomial $W_n(x)$. □

The next lemma is a cornerstone for studying the root geometry of polynomials defined by recursions of order 2; see [14].

Lemma 3.2. *Let $A, B \in \mathbb{R}$ such that $A \neq 0$. Define $W_n = AW_{n-1} + BW_{n-2}$ recursively, with $W_0 = 1$ and with some number W_1. Writing $\Delta = A^2 + 4B$ and $g^{\pm} = (2W_1 - A \pm \sqrt{\Delta})/2$, we have*

$$W_n = \begin{cases} \left(1 + \dfrac{n(2W_1 - A)}{A}\right)\left(\dfrac{A}{2}\right)^n, & \text{if } \Delta = 0; \\[2ex] \dfrac{g^{+}(A + \sqrt{\Delta})^n - g^{-}(A - \sqrt{\Delta})^n}{2^n \sqrt{\Delta}}, & \text{if } \Delta \neq 0. \end{cases}$$

In particular, if the complex number $A + \sqrt{\Delta}$ has the polar representation $Re^{i\theta}$, then we have

$$W_n = \left(\frac{R}{2}\right)^n \left(\cos n\theta + \frac{(2W_1 - A)\sin n\theta}{\sqrt{-\Delta}}\right), \qquad \text{if } \Delta < 0.$$

3.1. **The distinct-real-rootedness.** In this subsection we prove the distinct-real-rootedness of every polynomial in a $(1,0)$-sequence. and we derive a bound for the union of the zero-sets of these polynomials. Both of them will be shown by applying an interlacing criterion, which we now develop.

Notation 3.3. For any integers $m \leq n$, we denote the set $\{m, m+1, \ldots, n\}$ by $[m, n]$. When $m = 1$, we denote the set $[1, n]$ by $[n]$. All sets mentioned in this paper are ordered sets, whose elements are arranged in increasing order.

Lemma 3.4. *Let $g(x)$ be a polynomial with zero-set Y. Let $X = \{x_1, x_2, \ldots, x_{m+1}\}$ be a set such that $X \bowtie Y$. And let α and β be numbers such that*

$$\alpha < x_1 < x_2 < \cdots < x_{m+1} < \beta.$$

Then for every $i \in [m+1]$, we have

$$(3.1) \qquad\qquad g(\alpha)g(x_i)(-1)^i \;<\; 0, \qquad and$$

$$(3.2) \qquad\qquad g(x_i)g(\beta)(-1)^{m-i} \;<\; 0.$$

Proof. By the premise $X \bowtie Y$, the polynomial $g(x)$ has no roots less than x_1. In particular, no root lies in the interval (α, x_1). Hence, from the intermediate value theorem, we infer that

$$(3.3) \qquad\qquad g(\alpha)g(x_1) \;>\; 0.$$

This confirms Ineq. (3.1) for $i = 1$.

From the premise $X \bowtie Y$, we also know that the polynomial $g(x)$ has exactly one root in the interval (x_{i-1}, x_i) for each integer $i \in [2, m+1]$. Here, the intermediate value theorem implies

$$\begin{aligned} -g(x_1)g(x_2) \;&>\; 0 \qquad (i = 2), \\ -g(x_2)g(x_3) \;&>\; 0 \qquad (i = 3), \\ &\;\;\vdots \\ -g(x_m)g(x_{m+1}) \;&>\; 0 \qquad (i = m+1). \end{aligned}$$

Multiplying the first $i - 1$ of these inequalities yields $g(x_1)g(x_i)(-1)^{i-1} > 0$, or equivalently,

$$(3.4) \qquad\qquad g(x_1)g(x_i)(-1)^i \;<\; 0.$$

Multiplying Ineq. (3.4) by Ineq. (3.3) yields Ineq. (3.1) for $i \in [2, m+1]$. Inequality (3.2) can be obtained similarly. $\qquad\square$

The next lemma provides a way to bound the set of roots of the polynomials in a $(1,0)$-sequence. It serves as the induction step of the theorem that follows.

Lemma 3.5. *Let $\{W_n(x)\}_{n\geq 0}$ be a $(1,0)$-sequence. Let $m \geq 0$ and $\alpha, \beta \in \mathbb{R}$. Let R_m denote the zero-set of the polynomial $W_m(x)$. Suppose that*

$$(3.5) \qquad W_m(\alpha)W_{m+2}(\alpha) \; > \; 0 \qquad and$$

$$(3.6) \qquad W_m(\beta)W_{m+2}(\beta) \; > \; 0,$$

and also that

$$(3.7) \qquad |R_{m+1}| = m+1, \qquad R_{m+1} \subset (\alpha, \beta), \qquad and \qquad R_{m+1} \bowtie R_m.$$

Then we have

$$(3.8) \qquad |R_{m+2}| = m+2, \qquad R_{m+2} \subset (\alpha, \beta), \qquad and \qquad R_{m+2} \bowtie R_{m+1}.$$

Proof. Since $|R_{m+1}| = m+1$ and $R_{m+1} \subset (\alpha, \beta)$, we can write

$$R_{m+1} = \{x_1, x_2, \ldots, x_{m+1}\},$$

where $\alpha < x_1 < x_2 < \cdots < x_{m+1} < \beta$. Since $R_{m+1} \bowtie R_m$, we can infer from Lemma 3.4 that

$$(3.9) \qquad W_m(\alpha)W_m(x_i)(-1)^i \; < \; 0, \qquad and$$

$$(3.10) \qquad W_m(x_i)W_m(\beta)(-1)^{m-i} \; < \; 0,$$

for every $i \in [m+1]$. On the other hand, setting $x = x_i$ in Recursion (2.1), we see that

$$W_{m+2}(x_i) \; = \; (ax_i + b)W_{m+1}(x_i) + d \cdot W_m(x_i)$$
$$= \; d \cdot W_m(x_i) \qquad (\text{since } x_i \in R_{m+1}),$$

or equivalently,

$$(3.11) \qquad W_m(x_i) \; = \; W_{m+2}(x_i)/d.$$

Since $d < 0$, substituting Eq. (3.11) into Ineq. (3.9), we see that

$$W_m(\alpha)W_{m+2}(x_i)(-1)^i > 0.$$

Multiplying this by Ineq. (3.5) of the premises and canceling the square (which is positive), we determine that

$$(3.12) \qquad W_{m+2}(\alpha)W_{m+2}(x_i)(-1)^i \; > \; 0 \qquad \text{for all } i \in [m+1].$$

Inequality (3.5) implies that $W_{m+2}(\alpha) \neq 0$.

Now define $x_0 = \alpha$. Then Ineq. (3.12) holds for $i = 0$ trivially. Consequently, we can replace the index i in the above inequality by $i-1$ for each integer $i \in [m+1]$, which gives that

$$(3.13) \qquad W_{m+2}(\alpha)W_{m+2}(x_{i-1})(-1)^{i-1} \; > \; 0.$$

Multiplying Ineqs. (3.12) and (3.13) produces

$$W_{m+2}(x_{i-1})W_{m+2}(x_i) \; < \; 0.$$

Therefore, by the intermediate value theorem, the polynomial $W_{m+2}(x)$ has a root in the interval (x_{i-1}, x_i). Let z_i be such a root for each $i \in [m+1]$. Then we have the iterated inequality

$$(3.14) \qquad \alpha \; < \; z_1 \; < \; x_1 \; < \; z_2 \; < \; x_2 \; < \; \cdots \; < \; z_m \; < \; x_m \; < \; z_{m+1} \; < \; x_{m+1}.$$

On the other hand, substituting Eq. (3.11) into Ineq. (3.10) gives

$$W_{m+2}(x_i) W_m(\beta)(-1)^{m-i} > 0.$$

Multiplying this by Ineq. (3.6) (and canceling the square), we find that

$$W_{m+2}(x_i) W_{m+2}(\beta)(-1)^{m-i} \; > \; 0.$$

When $i = m + 1$, this latter inequality becomes

$$W_{m+2}(x_{m+1}) W_{m+2}(\beta) \; < \; 0.$$

Again by the intermediate value theorem, the polynomial $W_{m+2}(x)$ has a root z_{m+2} in the interval (x_{m+1}, β). Combining with Relation (3.14), we obtain

$$\alpha \; < \; z_1 \; < \; x_1 \; < \; z_2 \; < \; x_2 \; < \; \cdots \; < \; z_{m+1} \; < \; x_{m+1} \; < \; z_{m+2} \; < \; \beta.$$

We now define $T = \{z_1, z_2, \ldots, z_{m+2}\}$. The ordering immediately above implies that

$$|T| = m + 2, \qquad T \subset (\alpha, \beta), \qquad \text{and} \qquad T \bowtie R_{m+1}.$$

Thus, to complete the proof, it suffices to show that $T = R_{m+2}$. In fact, by the choice of the numbers z_j, we have $T \subseteq R_{m+2}$. By Lemma 3.1, the polynomial $W_{m+2}(x)$ has degree $m + 2$. Thus, the zero-set R_{m+2} has cardinality at most $m + 2$. Since it contains the subset T with cardinality $m + 2$, we conclude that $R_{m+2} = T$. $\qquad\square$

Using Lemma 3.5 as the induction step, we now establish a criterion for a bound on the union $\cup_{n \geq 1} R_n$ of the zero-sets, which will be used in the proof of Theorem 3.12.

Corollary 3.6. *Let $\{W_n(x)\}_{n \geq 0}$ be a $(1, 0)$-sequence, and let R_n denote the zero-set of the polynomial $W_n(x)$. Suppose that there are numbers α and β, with $\alpha < 0 < \beta$, such that Ineqs. (3.5) and (3.6) hold for all $m \geq 0$. Then we have the three relations $|R_n| = n$, $R_n \subset (\alpha, \beta)$, and $R_n \bowtie R_{n-1}$, for all $n \geq 1$.*

Proof. Since $W_1(x) = x$, we have $R_1 = \{0\}$. Thus the inequality $\alpha < 0 < \beta$ is equivalent to the relation $R_1 \subset (\alpha, \beta)$. Since $W_0(x) = 1$, we have $R_0 = \emptyset$. As per Definition 2.2, the relation $R_1 \bowtie R_0$ holds trivially. This proves the desired relations for $n = 1$.

Now suppose that they are true for some index $n = m + 1$, where $m \geq 0$. By induction, all the three relations in (3.7) hold true. Since Ineqs. (3.5) and (3.6) hold by the premise, by Lemma 3.5, we obtain the three relations in (3.8). In other words, the desired relations hold for the index $n = m + 2$. This completes the proof. $\qquad\square$

In order to establish the real-rootedness of every polynomial in a $(1,0)$-sequence, we will construct two real numbers α and β satisfying the premises of Corollary 3.6. Inspired by Lemma 3.2, we employ the following notations.

Notation 3.7. Let $\{W_n(x)\}_{n\geq 0}$ be a $(1,0)$-sequence. We define

$$A(x) = ax + b,$$

$$(3.15) \qquad \Delta(x) = A^2(x) + 4d,$$

$$(3.16) \qquad f(x) = 2W_1(x) - A(x) = (2-a)x - b,$$

$$(3.17) \qquad g^\pm(x) = \big(f(x) \pm \sqrt{\Delta(x)}\,\big)/2, \quad \text{and}$$

$$(3.18) \qquad g(x) = g^-(x)g^+(x) = (1-a)x^2 - bx - d.$$

The roots $x_\Delta^\pm$ of the function $\Delta(x)$ are given in Definition (2.3), and the roots $x_g^\pm$ of the function $g(x)$ are given in Definition (2.7). 3.8 collects some information for the numbers $x_\Delta^\pm$.

Lemma 3.8. *Let* $\{W_n(x)\}_{n\geq 0}$ *be a* $(1,0)$-*sequence. Let* $n \geq 0$. *Then we have*

$$(3.19) \qquad A(x_\Delta^\pm) = ax_\Delta^\pm + b = \pm 2\sqrt{-d},$$

$$(3.20) \qquad f(x_\Delta^-) = \frac{2}{a}\Big((a-2)\sqrt{-d} - b\Big),$$

$$(3.21) \qquad f(x_\Delta^+) = \frac{2}{a}\Big((2-a)\sqrt{-d} - b\Big), \quad \text{and}$$

$$(3.22) \qquad W_n(x_\Delta^\pm) = \left(1 + n\cdot\frac{f(x_\Delta^\pm)}{A(x_\Delta^\pm)}\right)\left(\frac{A(x_\Delta^\pm)}{2}\right)^n.$$

Consequently, we have the following.

$$(3.23) \qquad A(x_\Delta^-) < 0 < A(x_\Delta^+),$$

$$(3.24) \qquad f(x_\Delta^-) > 0 \quad \Longleftrightarrow \quad b < (a-2)\sqrt{-d}, \quad \text{and}$$

$$(3.25) \qquad f(x_\Delta^+) < 0 \quad \Longleftrightarrow \quad b > (2-a)\sqrt{-d}.$$

We also have the implication

$$(3.26) \qquad f(x_\Delta^-) > 0 \quad \Longrightarrow \quad a > 2 \quad \Longrightarrow \quad f(x_\Delta^+) < 0.$$

Proof. From Notation 3.7, it is straightforward to compute Eqs. (3.19) — (3.21). Since $\Delta(x_\Delta^\pm) = 0$, 3.2 implies Eq. (3.22). Inequality (3.23) and Relations (3.24) and (3.25) follow from Eqs. (3.19), (3.20) and (3.21), respectively. From Relation (3.24), we see that the inequality $f(x_\Delta^-) > 0$ implies that $a > 2$, and consequently, $b \geq 0 > (2-a)\sqrt{-d}$. Then, from Relation (3.25), we infer Relation (3.26). $\qquad\square$

Our direction is toward showing in Section 3.4 that the numbers $x_\Delta^\pm$ are limit points of the union $\cup_{n\geq1}R_n$. As will be seen, Relation (3.26) plays a crucial role in splitting cases in the statement of our main theorem, Theorem 2.4. In Lemma 3.9 and Lemma 3.10 we are collecting some information regarding the numbers $x_g^\pm$ that we defined in the statement of Theorem 2.6.

Lemma 3.9. *Let $\{W_n(x)\}_{n\geq0}$ be a $(1,0)$-sequence, and suppose that the numbers x_g^- and x_g^+ are both real. Then we have $W_n(x_g^\pm) = (x_g^\pm)^n$. Moreover, we have the following.*

  (i) *If $a \leq 1$, then $0 < x_g^+ \leq x_g^-$, and $W_n(x_g^\pm) > 0$.*
  (ii) *If $a > 1$, then $x_g^- < 0 < x_g^+$, $W_n(x_g^-)(-1)^n > 0$, and $W_n(x_g^+) > 0$.*

Proof. See Appendix A. $\qquad\qquad\square$

We will show in Section 3.3 that, in some cases, the numbers $x_g^\pm$ are also limit points of the union $\cup_{n\geq1}R_n$. In order to give a bound for the union $\cup_{n\geq1}R_n$, we need to be clear about the ordering among the numbers $x_\Delta^\pm$ and $x_g^\pm$. 3.10 collects necessary information for this purpose.

Lemma 3.10. *Suppose that $f(x_\Delta^+) < 0$, where the function $f(x)$ is given by Definition (3.16). Then the numbers $x_g^\pm$ are well-defined and real. Moreover, we have $x_g^+ > x_\Delta^+$. If additionally we have $f(x_\Delta^-) > 0$, then $x_g^- < x_\Delta^-$.*

Proof. See Appendix B. $\qquad\qquad\square$

We are now ready to define a suitable interval (α, β) to be used in applying Corollary 3.6.

Notation 3.11. Let $J_\Delta = (x_\Delta^-, x_\Delta^+)$. With the aid of Relation (3.26), we can define the interval

$$(3.27) \qquad J_0 = \begin{cases} (x_\Delta^-, x_\Delta^+), & \text{if } f(x_\Delta^+) \geq 0; \\ (x_g^-, x_g^+), & \text{if } f(x_\Delta^-) > 0; \\ (x_\Delta^-, x_g^+), & \text{otherwise.} \end{cases}$$

Note that $x_\Delta^- < x_\Delta^+$. We see from Lemma 3.10 that J_0 is a well-defined interval, and that it contains J_Δ as a non-empty subinterval, namely,

$$(3.28) \qquad\qquad J_\Delta \subseteq J_0.$$

This has prepared us to establish the distinct-real-rootedness of the polynomials $W_n(x)$.

Theorem 3.12. *Let $\{W_n(x)\}_{n\geq0}$ be a $(1,0)$-sequence. Then each of the polynomials $W_n(x)$ is distinct-real-rooted. Moreover, we have $R_n \subset J_0$ and $R_{n+1} \bowtie R_n$.*

Proof. Let (α, β) denote the interval J_0. By Definition (3.27), we have

$$\alpha = \begin{cases} x_g^-, & \text{if } f(x_\Delta^-) > 0, \\ x_\Delta^-, & \text{if } f(x_\Delta^-) \leq 0; \end{cases} \quad \text{and} \quad \beta = \begin{cases} x_g^+, & \text{if } f(x_\Delta^+) < 0, \\ x_\Delta^+, & \text{if } f(x_\Delta^+) \geq 0. \end{cases}$$

To apply Corollary 3.6, we will first show that $\alpha < 0$ and $\beta > 0$ and then that Ineqs. (3.5) and (3.6) hold for all $m \geq 0$.

First, we show that $\alpha < 0$. If $f(x_\Delta^-) > 0$, then we have $a > 2$ from Relation (3.26), which implies that $x_g^- < 0$, by Lemma 3.9 (ii). Alternatively, if $f(x_\Delta^-) \leq 0$, then it follows from Definition (2.3) of the number x_Δ^- as $(-b-2\sqrt{-d})/a$ that $x_\Delta^- < 0$, since $a > 0$ and $b \geq 0$. This proves $\alpha < 0$.

Second, we show that $\beta > 0$. If $f(x_\Delta^+) < 0$, then we have $x_g^+ > 0$, by Lemma 3.9. Alternatively, if $f(x_\Delta^+) \geq 0$, we deduce from Relation (3.25) that $b \leq (2-a)\sqrt{-d} < 2\sqrt{-d}$. Thus, from Definition (2.3) of the number x_Δ^+ as $(-b+2\sqrt{-d})/a$, if follows that $x_\Delta^+ > 0$. This proves $\beta > 0$.

Now, for Ineqs. (3.5) and (3.6), we need to show that

(3.29) $W_m(x)W_{m+2}(x) > 0$,   for all $m \geq 0$, and for both $x \in \{\alpha, \beta\}$.

Since $\{\alpha, \beta\} \subseteq \{x_g^\pm, x_\Delta^\pm\}$, we have $x \in \{x_g^\pm, x_\Delta^\pm\}$.

When $x \in \{x_g^\pm\}$, we have from Lemma 3.9 that $W_m(x)W_{m+2}(x) = x^{2m+2} \geq 0$. 3.9 also tells us $x_g^\pm \neq 0$. Thus we have $W_m(x_g^\pm)W_{m+2}(x_g^\pm) > 0$.

When $x \in \{x_\Delta^\pm\}$, Eq. (3.22) implies that

$$W_m(x)W_{m+2}(x) = \left(1 + m \cdot \frac{f(x)}{A(x)}\right)\left(1 + (m+2)\cdot \frac{f(x)}{A(x)}\right)\left(\frac{A(x)}{2}\right)^{2m+2}.$$

To show Ineq. (3.29), it suffices (since $m \geq 0$) to show that $f(x)/A(x) \geq 0$. For $x = x_\Delta^-$, we have $x = \alpha$, with the additional condition $f(x_\Delta^-) \leq 0$. From Ineq. (3.23), we infer that $f(x_\Delta^-)/A(x_\Delta^-) \geq 0$. If $x = x_\Delta^+$, we have $x = \beta$, with the additional condition $f(x_\Delta^+) \geq 0$. Again from Ineq. (3.23), we infer that $f(x_\Delta^+)/A(x_\Delta^+) \geq 0$. $\square$

We see from Theorem 3.12 that the union $\cup_{n \geq 1} R_n$ is contained within the interval J_0. We will show that the interval J_0 is sharp as a bound of that union, by establishing in the next three subsections that each of its endpoints is a limit point of that union; see Theorem 3.16, Theorem 3.18, Theorem 3.22, and Theorem 3.26. Before this, we highlight the important role of the subinterval J_Δ, by proving that it contains almost all members of the union $\cup_{n \geq 1} R_n$.

3.2. **The quasi-bound J_Δ.** By Theorem 3.12, each of the polynomials $W_n(x)$ is distinct-real-rooted. In order to describe the position of the smallest and the largest roots clearly, we introduce the following notation.

Definition 3.13. A set $\{x_1, x_2, \ldots, x_n\}$ of real numbers is said to be *ordered* or *in increasing order*, if $x_1 < x_2 < \cdots < x_n$.

The following theorem shows that the subinterval J_Δ contains all except at most two roots of each polynomial $W_n(x)$.

Theorem 3.14. *Let $\{W_n(x)\}_{n\geq 0}$ be a $(1,0)$-sequence, in which each polynomial $W_n(x)$ has the ordered zero-set $R_n = \{\xi_{n,1}, \xi_{n,2}, \ldots, \xi_{n,n}\}$.*
(i) When $f(x_\Delta^-) > 0$, we have the following.

- *If $n < -A(x_\Delta^-)/f(x_\Delta^-)$, then $\xi_{n,1} > x_\Delta^-$.*
- *If $n = -A(x_\Delta^-)/f(x_\Delta^-)$, then $\xi_{n,1} = x_\Delta^-$.*
- *If $n > -A(x_\Delta^-)/f(x_\Delta^-)$, then $\xi_{n,1} < x_\Delta^-$. In this case, we have $x_\Delta^- < \xi_{n,2}$ for all $n \geq 2$.*

(ii) When $f(x_\Delta^+) < 0$, we have the following.

- *If $n < -A(x_\Delta^+)/f(x_\Delta^+)$, then $\xi_{n,n} < x_\Delta^+$.*
- *If $n = -A(x_\Delta^+)/f(x_\Delta^+)$, then $\xi_{n,n} = x_\Delta^+$.*
- *If $n > -A(x_\Delta^+)/f(x_\Delta^+)$, then $\xi_{n,n} > x_\Delta^+$. In this case, we have $\xi_{n,n-1} < x_\Delta^+$ for all $n \geq 2$.*

Proof. We define

$$(3.30) \qquad n^\pm = -\frac{A(x_\Delta^\pm)}{f(x_\Delta^\pm)}.$$

Then Eq. (3.22) can be rewritten as

$$(3.31) \qquad W_n(x_\Delta^\pm) = \left(1 - \frac{n}{n^\pm}\right)\left(\frac{A(x_\Delta^\pm)}{2}\right)^n.$$

We shall show (i) and (ii) individually.

(i) Suppose that

$$(3.32) \qquad f(x_\Delta^-) > 0.$$

From Ineq. (3.23), we see that

$$(3.33) \qquad A(x_\Delta^-) < 0.$$

Together with Ineq. (3.32), we can see from Definition (3.30) that

$$(3.34) \qquad n^- > 0.$$

Now, in view of Eq. (3.31) and Ineqs. (3.33) and (3.34), we infer that

$$(3.35) \qquad (-1)^n(n - n^-)W_n(x_\Delta^-) = -\frac{(n - n^-)^2}{n^-}\left(\frac{-A(x_\Delta^-)}{2}\right)^n \leq 0,$$

where the equality holds if and only if $n = n^-$. On the other hand, Relation (3.26) with Ineq. (3.32) tells us $a > 2$. Thus, we infer from Lemma 3.9 (ii) that

$$(3.36) \qquad W_n(x_g^-)(-1)^n > 0.$$

Multiplying Ineqs. (3.35) and (3.36), we find

$$(3.37) \qquad W_n(x_g^-)W_n(x_\Delta^-)(n - n^-) \ \leq \ 0,$$

where the equality holds if and only if $n = n^-$.

According to Definition (3.27), we have $J_0 = (x_g^-, x_g^+)$. From Relation (3.28), we infer that

$$(3.38) \qquad x_g^- < x_\Delta^- < x_\Delta^+ < x_g^+.$$

Thus we can define the interval

$$J^- = (x_g^-, x_\Delta^-).$$

By Theorem 3.12, the roots of the polynomials $W_{n+1}(x)$ and $W_n(x)$ are interlacing in the interval J_0. Since the interval J^- is "a left part" of the interval J_0, we infer that in J^-, the polynomial $W_{n+1}(x)$ has the same number of roots or one more root than the polynomial $W_n(x)$. In notation, we let

$$(3.39) \qquad N_n \ = \ |R_n \cap J^-|.$$

Then the above argument can be expressed as

$$(3.40) \qquad N_n - N_{n-1} \in \{0, 1\} \qquad \text{for all } n \geq 1.$$

Below we show Result (i) by bootstrapping.

Case $n < n^-$. Since the zero-set R_0 is empty, we infer from Definition (3.39) that $N_0 = 0$. We claim that

$$N_n = 0, \qquad \text{for all } n < n^-.$$

If $n^- \leq 1$, then the claim holds true trivially. Below we suppose that $n^- > 1$. We proceed by induction on n. Suppose that $N_{m-1} = 0$ for some $1 \leq m < n^-$. From Relation (3.40), we infer that

$$(3.41) \qquad N_m \in \{0, 1\}.$$

Since $n < n^-$, Ineq. (3.37) implies that $W_n(x_g^-)W_n(x_\Delta^-) > 0$. In other words, at the ends of the interval J^-, the continuous function $W_n(x)$ admits the same sign. Therefore, from the intermediate value theorem, we infer that $W_n(x)$ has an even number of roots in the interval J^-, namely, the integer N_n is even as if $n < n^-$. In view of Eq. (3.41), we find that $N_m = 0$. This proves the claim.

The claim, in fact, states that the polynomial $W_n(x)$ has no roots in the interval $J^- = (x_g^-, x_\Delta^-)$. Recall from Theorem 3.12 that all roots of $W_n(x)$ lie in the interval $J_0 = (x_g^-, x_g^+)$. Therefore, the bound of the roots of $W_n(x)$ can be improved to the interval $[x_\Delta^-, x_g^+)$. Moreover, it is easy to see that $W_n(x_\Delta^-) \neq 0$ from Eq. (3.31). Hence, we obtain that $R_n \subset (x_\Delta^-, x_g^+)$. In particular, we have

$$(3.42) \qquad \xi_{n,1} \ > \ x_\Delta^- \qquad \text{for all } n < n^-.$$

Case $n = n^-$**.** From Eq. (3.31), we see that

$$W_n(x_\Delta^-) \; = \; \left(1 - \frac{n}{n^-}\right)\left(\frac{A(x_\Delta^-)}{2}\right)^n = 0.$$

In other words, the number x_Δ^- is a root of the polynomial $W_n(x)$. We shall show that x_Δ^- is the smallest root of $W_n(x)$.

If $n = 1$, then the polynomial $W_n(x) = W_1(x) = x$ has only one root. Therefore, the root x_Δ^- has to be the smallest root. When $n \geq 2$, the interlacing property $R_n \bowtie R_{n-1}$ implies that $\xi_{n,2} > \xi_{n-1,1}$. From Result (3.42), we infer that $\xi_{n-1,1} > x_\Delta^-$. It follows that $\xi_{n,2} > x_\Delta^-$. In other words, the second smallest root of the polynomial $W_n(x)$ is larger than the root x_Δ^-. It turns out immediately that the root x_Δ^- is the smallest one, namely,

(3.43) $$\xi_{n,1} \; = \; x_\Delta^-, \qquad \text{if } n = n^-.$$

Case $n > n^-$**.** In this case, Ineq. (3.37) reads that $W_n(x_g^-)W_n(x_\Delta^-) < 0$. As in the case $n < n^-$, we can deduce by using the intermediate value theorem that the integer N_n is odd. In particular, we have $N_n \geq 1$.

We claim that

(3.44) $$N_n \; = \; 1 \qquad \text{for all } n > n^-.$$

Combining Results (3.42) and (3.43), we see that for any $m \leq n^-$, the smallest root of the polynomial $W_m(x)$ is at least x_Δ^-. It follows that $N_m = 0$. In particular, we have $N_{\lfloor n^- \rfloor} = 0$. Thus, by Relation (3.40), we have

$$N_{\lfloor n^- \rfloor + 1} \in \{0, 1\}.$$

Since $\lfloor n^- \rfloor + 1 > n^-$, the integer $N_{\lfloor n^- \rfloor + 1}$ is odd. Thus $N_{\lfloor n^- \rfloor + 1} = 1$. Proceeding by induction, we can suppose that $N_n = 1$ for some $n > n^-$. By Relation (3.40), we infer that $N_{n+1} \in \{1, 2\}$. Since the integer N_{n+1} is odd, we deduce that $N_{n+1} = 1$. This confirms the claim.

By Definition (3.39) of the number N_n as $|R_n \cap (x_g^-, x_\Delta^-)|$, Result (3.44) implies that $\xi_{n,1} < x_\Delta^-$ if $n > n^-$. From Eq. (3.31), we see that $W_n(x_\Delta^-) \neq 0$. Together with Result (3.44), we obtain that $\xi_{n,2} > x_\Delta^-$ as if $n > n^-$ and $n \geq 2$.

This completes the proof of (i).

(ii) Suppose that $f(x_\Delta^+) < 0$. Analogous to Ineq. (3.37), one may show that

(3.45) $$W_n(x_\Delta^+)W_n(x_g^+)(n - n^+) \leq 0,$$

where the equality holds if and only if $n = n^+$.

According to Definition (3.27), we have $J_0 = (x_g^-, x_g^+)$ or $J_0 = (x_\Delta^-, x_g^+)$. From Relation (3.28), we can define

$$N_n^+ \; = \; |R_n \cap (x_\Delta^+, x_g^+)|.$$

Since the interval (x_Δ^+, x_g^+) is "a right part" of the interval J_0, the interlacing property implies that

$$(3.46) \qquad N_n^+ - N_{n-1}^+ \in \{0, 1\} \qquad \text{for all } n \geq 1.$$

Similar to (i), by using the intermediate value theorem, the parity arguments, with the aids of Ineq. (3.45) and Relation (3.46), we can show the desired results in (ii).

This completes the proof of Theorem 3.14. $\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

3.3. The numbers $x_g^\pm$ can be limit points.

From Theorem 3.14, we see that when $f(x_\Delta^-) > 0$, the smallest roots are eventually less than the number x_Δ^-. 3.16 shows that they converge to the number x_g^-. Similarly, when $f(x_\Delta^+) < 0$, the largest roots are eventually larger than the number x_Δ^+. 3.18 shows that they converge to the number x_g^+ in that case.

In the proofs of these two theorems, we will use Formula (3.47), which is obtained from Lemma 3.2 straightforwardly.

Proposition 3.15. *If $\Delta(x_0) > 0$, then*

$$(3.47) \quad W_n(x_0) \; = \; \frac{\left(A(x_0) + \sqrt{\Delta(x_0)}\right)^n}{2^n \sqrt{\Delta(x_0)}} \left[g^+(x_0) - g^-(x_0)\left(\frac{A(x_0) - \sqrt{\Delta(x_0)}}{A(x_0) + \sqrt{\Delta(x_0)}}\right)^n \right].$$

Theorem 3.16. *Let $\{W_n(x)\}_{n \geq 0}$ be a $(1,0)$-sequence. If $f(x_\Delta^-) > 0$, then $\xi_{n,1} \searrow x_g^-$.*

Proof. Suppose that $f(x_\Delta^-) > 0$. Then $a > 2$ by Relation (3.26).

By Theorem 3.14 (i), we have $\xi_{n,1} \in (x_g^-, x_\Delta^-)$ for large n. The property $R_{n+1} \bowtie R_n$ obtained in Theorem 3.12 implies that the sequence $\xi_{n,1}$ decreases. Therefore, there exists a number $\xi^- \in [x_g^-, x_\Delta^-)$ such that $\lim_{n \to \infty} \xi_{n,1} = \xi^-$. Suppose, to the contrary, that $\xi^- \neq x_g^-$. Let $x_0 \in (x_g^-, \xi^-)$. Then

$$(3.48) \qquad x_g^- \; < \; x_0 \; < \; \xi^- \; < \; x_\Delta^- \; < \; -b/a.$$

We claim that

$$(3.49) \qquad g^-(x_0) \; \leq \; 0.$$

Since the interval (x_g^-, ξ^-) contains no roots of any polynomial $W_n(x)$, by the intermediate value theorem, we infer that

$$W_n(x_g^-)W_n(x_0) \; > \; 0.$$

Since $a > 2$, by Lemma 3.9 (ii), we have $W_n(x_g^-)(-1)^n > 0$. Together with the above inequality, we derive that $W_n(x_0)(-1)^n > 0$. In particular, we have

$$(3.50) \qquad W_n(x_0) \; > \; 0 \qquad \text{for all positive even integers } n.$$

On the other hand, since $a > 0$, the function $A(x) = ax + b$ is increasing. From Relation (3.48), we see that $x_0 < -b/a$, and thus, $A(x_0) < A(-b/a) = 0$. Since $d < 0$, from Definition (3.15), we deduce that

$$A(x_0) - \sqrt{\Delta(x_0)} \; < \; A(x_0) + \sqrt{\Delta(x_0)} \; = \; A(x_0) + \sqrt{A^2(x_0) + 4d} \; < \; 0.$$

Therefore, we have

$$(3.51) \qquad \frac{A(x_0) - \sqrt{\Delta(x_0)}}{A(x_0) + \sqrt{\Delta(x_0)}} \; > \; 1.$$

Set n to be a large even integer. By Definition (2.3), the function $\Delta(x) = (ax + b)^2 + 4d$ is a quadratic polynomial with a positive leading coefficient. From Relation (3.48), we see that $x_0 < x_\Delta^-$, and thus, $\Delta(x_0) > 0$. By Ineq. (3.50), the right hand side of Formula (3.47) is positive. In view of Ineq. (3.51), we deduce that $g^-(x_0) \leq 0$. This confirms the claim.

Since $a > 2$, by Definition (3.16), the function $f(x) = (2-a)x - b$ is decreasing. Since $x_0 < x_\Delta^-$, we infer that $f(x_0) > f(x_\Delta^-) > 0$. Thus, by Definition (3.17), we obtain that $g^+(x_0) = (f(x_0) + \sqrt{\Delta(x_0)})/2 > 0$. Together with Ineq. (3.49), we derive that

$$(3.52) \qquad g(x_0) \; = \; g^-(x_0)g^+(x_0) \; \leq \; 0.$$

From Definition (3.18), we see that $g(x) = (1-a)x^2 - bx - d$ is a quadratic polynomial with negative leading coefficient. Thus we have $g(x) > 0$ for all $x \in (x_g^-,\, x_g^+)$. By Lemma 3.10, we have $x_g^- < x_\Delta^- < x_\Delta^+ < x_g^+$. Together with Relation (3.48), we deduce that $x_0 \in (x_g^-,\, x_g^+)$, and thus, $g(x_0) > 0$, contradicting Ineq. (3.52). This completes the proof. $\square$

To show that the number x_g^+ can be a limit point, we need the following lemma.

Lemma 3.17. *If $f(x_\Delta^+) < 0$, then $f(x) < 0$ for all $x \in (x_\Delta^+,\, x_g^+)$.*

Proof. See Appendix C. $\square$

The proof of the limit point x_g^+ is similar to the proof of the limit point x_g^-.

Theorem 3.18. *Let $\{W_n(x)\}_{n \geq 0}$ be a $(1,0)$-sequence. If $f(x_\Delta^+) < 0$, then $\xi_{n,n} \nearrow x_g^+$.*

Proof. Suppose that $f(x_\Delta^+) < 0$. Suppose, by way of contradiction, that the convergent point of the largest roots is not the number x_g^+. Then, there exists a number $x_0 \in (x_\Delta^+,\, x_g^+)$ such that $W_n(x_0)W_n(x_g^+) > 0$, that is,

$$(3.53) \qquad W_n(x_0) > 0.$$

Similar to the proof of Theorem 3.16, we can show that $A(x_0) > 0$, $\Delta(x_0) > 0$, and

$$A(x_0) + \sqrt{\Delta(x_0)} \; > \; A(x_0) - \sqrt{\Delta(x_0)} \; = \; A(x_0) - \sqrt{A^2(x_0) + 4d} \; > \; 0.$$

Therefore, we have

$$(3.54) \qquad \frac{A(x_0) - \sqrt{\Delta(x_0)}}{A(x_0) + \sqrt{\Delta(x_0)}} \in (0, 1).$$

Set n to be a large even integer. By Ineq. (3.53), the right hand side of Formula (3.47) is positive. In view of Relation (3.54), we infer that

$$(3.55) \qquad g^+(x_0) \geq 0.$$

By Lemma 3.17, we have $f(x_0) < 0$. Therefore, $g^-(x_0) = f(x_0) - \sqrt{\Delta(x_0)} < 0$, and thus,

$$g(x_0) = g^-(x_0)g^+(x_0) \leq 0.$$

Below, we show that $g(x_0) > 0$, which implies an immediate contradiction. Recall from Definition (3.18) that $g(x) = (1 - a)x^2 - bx - d$.

- When $a < 1$, the quadratic function $g(x)$ has a positive leading coefficient. On the other hand, by Lemma 3.9 (i), the number x_g^+ is the smaller root of the function $g(x)$. Therefore, we have $g(x) > 0$ for all $x < x_g^+$. In particular, we have $g(x_0) > 0$.
- When $a = 1$, we have $g(x) = -bx - d$. If $b = 0$, then $g(x) = -d > 0$ for all $x \in \mathbb{R}$. Otherwise $b > 0$, then the function $g(x)$ is decreasing. Thus, for any $x < x_g^+$, we have $g(x) > g(x_g^+) = 0$. In particular, we have $g(x_0) > 0$.
- When $a > 1$, the leading coefficient of the quadratic function $g(x)$ is negative. On the other hand, by Lemma 3.9 (ii), the number x_g^+ is the larger root of the function $g(x)$. Therefore, we have $g(x) > 0$ for all $x \in (x_g^-, x_g^+)$. In particular, we have $g(x_0) > 0$, by Lemma 3.10.

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

3.4. The numbers $x_\Delta^\pm$ are limit points.

In this subsection, we will show that the numbers $x_\Delta^\pm$ are limit points of the union $\cup_{n \geq 1} R_n$. For proof convenience, we will adopt the polar coordinate system.

Definition 3.19. Let $\theta \in \mathbb{R}$. We define the *principal value* of the number θ, denote by $\mathrm{pv}(\theta)$, to be the unique number $\theta' \in [0, 2\pi)$ such that the difference $\theta - \theta'$ is an integral multiple of the number 2π. In the polar coordinate system, we adopt the wording

- *the angle θ*, to mean the angle of size θ;
- *the ray θ*, to mean the ray starting from the origin with the incline angle $\mathrm{pv}(\theta)$; and
- *the line θ*, to mean the line on which lies the ray θ.

Let $\psi \in [0, \pi)$. We say that the angle θ lies

- *to the left of the line ψ*, if $\theta \in (\psi + 2k\pi, \psi + (2k+1)\pi)$ for some integer k;

- *to the right of the line ψ, if $\theta \in (\psi + (2k-1)\pi, \ \psi + 2k\pi)$ for some integer k;*
- *on the line ψ, if the line θ coincides with the line ψ.*

For example, when $\psi \in [0, \pi/2)$ (resp., $\psi \in (\pi/2, \pi)$), the angle θ lies to the left of the line ψ if and only if it is above (resp., below) the line ψ, intuitively.

To characterize the sign of the value $W_n(x)$ for the real numbers x such that $\Delta(x) < 0$, we need the following notation.

Notation 3.20. Let $\{W_n(x)\}_{n \geq 0}$ be a $(1, 0)$-sequence. Let $x \in \mathbb{R}$ such that $\Delta(x) < 0$. Define the angle $\theta_x \in (0, \pi)$ by

$$(3.56) \qquad \tan \theta_x \ = \ \frac{\sqrt{-\Delta(x)}}{A(x)}.$$

Define the angle $\psi_x \in (0, \pi)$ by

$$(3.57) \qquad \tan \psi_x \ = \ \frac{-\sqrt{-\Delta(x)}}{(2-a)x - b}.$$

Here is a characterization for the sign of the value $W_n(x_0)$.

Theorem 3.21. *Let $\{W_n(x)\}_{n \geq 0}$ be a $(1, 0)$-sequence. Let $x \in \mathbb{R}$ such that $\Delta(x) < 0$. Then we have*

- $W_n(x) < 0$ *if and only if the angle $n\theta_x$ lies to the left of the line ψ_x;*
- $W_n(x) = 0$ *if and only if the angle $n\theta_x$ lies on the line ψ_x;*
- $W_n(x) > 0$ *if and only if the angle $n\theta_x$ lies to the right of the line ψ_x.*

Proof. Since $\Delta(x) < 0$, the complex number $A(x) + \sqrt{\Delta(x)}$ has the real part $A(x)$ and the imaginary part $\sqrt{-\Delta(x)}$. Therefore, by Definition (3.56) of the angle θ_x, we have

$$A(x) + \sqrt{\Delta(x)} = Re^{i\theta_x},$$

where $R = \sqrt{A^2(x) - \Delta(x)}$. Let

$$h = \big((2-a)x - b\big)/\sqrt{-\Delta(x)}, \qquad \text{and} \qquad F = \cos(n\theta_x) + h \cdot \sin(n\theta_x).$$

Since $\Delta(x) < 0$, by Lemma 3.2, we have

$$W_n(x) \ = \ \left(\frac{R}{2}\right)^n \left(\cos(n\theta_x) + \frac{(2-a)x - b}{\sqrt{-\Delta(x)}} \cdot \sin(n\theta_x)\right) \ = \ \left(\frac{R}{2}\right)^n F.$$

Since $R > 0$, we have $W_n(x) > 0$ if and only if $F > 0$. On the other hand, in view of Definition (3.57), the line ψ_x has slope $-1/h$.

If $h = 0$, we have $\psi_x = \pi/2$. In this case, we have $F = \cos(n\theta_x)$, and thus, the above sign relation reduces to that $W_n(x)$ and $\cos \psi_x$ have the same sign. In other words, we have $W_n(x) > 0$ if and only if the angle $n\theta_x$ lies in the open right half-plane; and $W_n(x) < 0$ if and only if the angle $n\theta_x$ lies in the left open half-plane.

Consequently, we have $W_n(x) = 0$ if and only if the line $n\theta_x$ coincides with the vertical line $\pi/2$. This proves the desired relations. Below we can suppose that $h \neq 0$.

Assume that $h > 0$. From the definition of the function F, it is elementary to deduce the following equivalence relation

$$F > 0 \quad \Longleftrightarrow \quad \sin(n\theta_x) > -\frac{1}{h}\cos(n\theta_x)$$

$$\Longleftrightarrow \quad \begin{cases} \tan(n\theta_x) > -1/h, & \text{if } \cos(n\theta_x) > 0; \\ \sin(n\theta_x) > 0, & \text{if } \cos(n\theta_x) = 0; \\ \tan(n\theta_x) < -1/h, & \text{if } \cos(n\theta_x) < 0. \end{cases}$$

Therefore, we have $W_n(x) > 0$ if and only if the angle $n\theta_x$ belongs to the set $S_1 \cup S_2 \cup S_3$, where

$$S_1 = \{\psi \in \mathbb{R} : \cos\psi > 0 \text{ and } \tan\psi > -1/h\} = \cup_{k\in\mathbb{Z}}(\psi_x - \pi + 2k\pi, \, \pi/2 + 2k\pi),$$

$$S_2 = \{\psi \in \mathbb{R} : \cos\psi = 0 \text{ and } \sin\psi > 0\} = \cup_{k\in\mathbb{Z}}\{\pi/2 + 2k\pi\},$$

$$S_3 = \{\psi \in \mathbb{R} : \cos\psi < 0 \text{ and } \tan\psi < -1/h\} = \cup_{k\in\mathbb{Z}}(\pi/2 + 2k\pi, \, \psi_x + 2k\pi).$$

It is routine to deduce their union, which is

$$S_1 \cup S_2 \cup S_3 = \cup_{k\in\mathbb{Z}}(\psi_x + (2k-1)\pi, \, \psi_x + 2k\pi).$$

From Definition 3.19, we see that $W_n(x) > 0$ if and only if the angle $n\theta_x$ lies to the right of the line ψ_x. By symmetry, we have $W_n(x) < 0$ if and only if the angle $n\theta_x$ lies to the left of the line ψ_x. It follows that $W_n(x) = 0$ if and only if the angle $n\theta_x$ lies on the line ψ_x.

When $h < 0$, we have the following equivalence relation in the same vein:

$$F > 0 \quad \Longleftrightarrow \quad \begin{cases} \tan(n\theta_x) < -1/h, & \text{if } \cos(n\theta_x) > 0; \\ \sin(n\theta_x) < 0, & \text{if } \cos(n\theta_x) = 0; \\ \tan(n\theta_x) > -1/h, & \text{if } \cos(n\theta_x) < 0. \end{cases}$$

In the same way we can find the same desired relations. This completes the proof. □

Now we are ready to show that the number x_Δ^+ is a limit point.

Theorem 3.22. *Let $\{W_n(x)\}_{n\geq 0}$ be a $(1,0)$-sequence, with ordered zero-set*

$$\{\xi_{n,1}, \, \xi_{n,2}, \, \ldots, \, \xi_{n,n}\}.$$

Then we have $\xi_{n,\,n+1-i} \nearrow x_\Delta^+$ as $n \to \infty$, for all $i \geq 2$ if $f(x_\Delta^+) < 0$, and for all $i \geq 1$ otherwise.

Proof. Let $i \geq 1$. From the property $R_{n+1} \bowtie R_n$ obtained in Theorem 3.12, we see that the sequence $\xi_{n,n+1-i}$ $(n \geq i)$ increases as $n \to \infty$. Since all the roots are bounded by the interval J_0, the sequence $\xi_{n,n+1-i}$ converges. Suppose that $\lim_{n \to \infty} \xi_{n,n+1-i} = \ell_i$.

Suppose that $f(x_\Delta^+) < 0$ and $i \geq 2$. From Theorem 3.14 (ii), we see that $\xi_{n,n+1-i} < x_\Delta^+$ for large n, which implies that $\ell_i \leq x_\Delta^+$. Suppose, to the contrary, that $\ell_i < x_\Delta^+$. When n is large, the polynomial $W_n(x)$ has exactly $i - 2$ distinct roots in the interval (ℓ_i, x_Δ^+), that is, the roots $\xi_{n,n+2-i}, \xi_{n,n+3-i}, \ldots, \xi_{n,n-1}$. Thus, by the intermediate value theorem, we infer that

$$W_n(\ell_i)W_n(x_\Delta^+)(-1)^{i-2} > 0 \qquad \text{for large } n.$$

On the other hand, we have $A(x_\Delta^+) > 0$ by Ineq. (3.23). Since $f(x_\Delta^+) < 0$, by Eq. (3.22), we infer that $W_n(x_\Delta^+) < 0$ for large n. Multiplying it with the above inequality, we obtain that

$$(3.58) \qquad\qquad W_n(\ell_i)(-1)^i < 0 \qquad \text{for large } n.$$

Since $\ell_i \in J_\Delta$, we have $\Delta(\ell_i) < 0$. By Theorem 3.21, there is an integer M such that for all integers $n > M$, the angle $n\theta_{\ell_i}$ lies on the same side of the line ψ_{ℓ_i}. This is impossible, because $\theta_{\ell_i} \in (0, \pi)$. Hence, we have $\ell_i = x_\Delta^+$.

Now suppose that $f(x_\Delta^+) \geq 0$ and $i \geq 1$. From Theorem 3.12, we see that $R_n \subset J_0 = J_\Delta$. Thus we have $\xi_{n,n+1-i} < x_\Delta^+$ for large n, which implies that $\ell_i \leq x_\Delta^+$. Suppose, to the contrary, that $\ell_i < x_\Delta^+$. The polynomial $W_n(x)$ has exactly $i - 1$ roots in the interval (ℓ_i, x_Δ^+) for large n, that is, the roots $\xi_{n,n+2-i}$, $\xi_{n,n+3-i}, \ldots, \xi_{n,n}$. Thus, we get

$$W_n(\ell_i)W_n(x_\Delta^+)(-1)^{i-1} > 0 \qquad \text{for large } n.$$

On the other hand, since $A(x_\Delta^+) > 0$ and $f(x_\Delta^+) \geq 0$, we have $W_n(x_\Delta^+) > 0$ for all n. Multiplying it with the above inequality, we obtain Eq. (3.58) again, which is absurd for the same reason. This completes the proof. $\qquad \square$

Applying the same idea to show that the number x_Δ^- is also a limit point, we find that the angles $n\theta_{\ell_i}$ reside on both sides of the line ψ_{ℓ_i}, but alternatively. This leads us to show that the alternation is impossible. The next two lemmas serves for this aim, depending on the rationality of the number θ_x. Let $\pi\mathbb{Q} = \{q\pi \colon q \in \mathbb{Q}\}$.

Lemma 3.23. *Let $\theta = q\pi/p$, where p is a positive integer, q is an integer, and $(p, q) = 1$. Then the sequence $\{\mathrm{pv}(n\theta)\}_{n \geq 1}$ is periodic, with the minimum period*

$$(3.59) \qquad\qquad p_0 = \begin{cases} p, & \text{if } q \text{ is even;} \\ 2p, & \text{if } q \text{ is odd.} \end{cases}$$

Moreover, we have

$$(3.60) \qquad \{\mathrm{pv}(n\theta) \colon n \in [p_0]\} = \{2j\pi/p_0 \colon j \in [0, p_0 - 1]\}.$$

Proof. See Appendix D. $\qquad \square$

Lemma 3.24. *Let $\theta \in \mathbb{R}\backslash\pi\mathbb{Q}$. Then for any nonempty open interval $I \subset (0, 2\pi)$, there exists an arbitrarily large integer m such that $\mathrm{pv}(m\theta) \in I$.*

Proof. See Appendix E for a proof by using Dirichlet's approximation theorem. $\square$

By using the above two lemmas, we can show the impossibility of the aforementioned alternation.

Lemma 3.25. *Let $x \in \mathbb{R}$ such that $\Delta(x) < 0$. Let $M > 0$. Suppose that the lines $n\theta_x$ and ψ_x do not coincide with each other for all $n > M$. Then there exists an arbitrarily large integer n such that the angles $n\theta_x$ and $(n+1)\theta_x$ lie on the same side of the line ψ_x.*

Proof. Recall from Notation 3.20 that

$$(3.61) \qquad\qquad \theta_x \in (0, \pi) \quad \text{and} \quad \psi_x \in (0, \pi).$$

Assume that $\theta_x \notin \pi\mathbb{Q}$. Let $I = (\psi_x, \psi_x + \epsilon)$, where

$$\epsilon = \min((\pi - \theta_x)/2, \, \pi - \psi_x/2) \in (0, \, \pi/2).$$

Using $\epsilon \le \pi - \psi_x/2$, we deduce that $I \subset (0, 2\pi)$. By Lemma 3.24 and using $\epsilon \le (\pi - \theta_x)/2$, we infer that there is an arbitrarily large integer n such that

$$\mathrm{pv}(n\theta_x) \in I \subseteq (\psi_x, \, \psi_x + (\pi - \theta_x)/2).$$

Together with Relation (3.61), we deduce that

$$\mathrm{pv}(n\theta_x) + \theta_x \in (\psi_x, \, \psi_x + (\pi - \theta_x)/2 + \theta_x) \subset (\psi_x, \, \psi_x + \pi),$$

which implies that the angle $(n+1)\theta_x$ lies to the left of the line ψ_x. On the other hand, by using $\epsilon \le \pi - \psi_x/2$, we derive that $\mathrm{pv}(n\theta_x) \in I \subset (\psi_x, \, \psi_x + \pi)$, which implies that the angle $n\theta_x$ also lies to the left of the line ψ_x.

Now suppose that $\theta_x \in \pi\mathbb{Q}$. By Relation (3.61), we can denote $\theta_x = q\pi/p$, where $p \in \mathbb{Z}^+$, $q \in [p-1]$, and $(p, q) = 1$. It follows that

$$(3.62) \qquad \pi/p_0 + \theta_x \; = \; \pi/p_0 + q\pi/p \; \le \; \pi/p_0 + (p-1)\pi/p \; \le \; \pi.$$

We will use the above inequality in the sequel.

In the polar coordinate system, the p_0 rays $2j\pi/p_0$ ($j \in [0, p_0 - 1]$) partition the full circle equally into p_0 angles of size $2\pi/p_0$. Note that the lines $n\theta_x$ and ψ_x do not coincide for $n > M$. By Eq. (3.60), the set $\{\mathrm{pv}(n\theta_x) \colon n \in \mathbb{Z}\}$ of lines is finite, which does not contain the line ψ_x. Therefore, the minimum angle among the angles formed by the line ϕ_x and one of the above rays is of size at most a half of the size $2\pi/p_0$. In other words, there is an integer $j_0 \in [0, p_0 - 1)$ such that

$$|2j_0\pi/p_0 - \psi_x| \in (0, \, \pi/p_0].$$

By Eq. (3.60), we can suppose that $\mathrm{pv}(n_0\theta_x) = 2j_0\pi/p_0$, where $n_0 \in [p_0]$. Then the above range relation can be rewritten as

$$(3.63) \qquad\qquad |\mathrm{pv}(n_0\theta_x) - \psi_x| \in (0, \, \pi/p_0].$$

If $\mathrm{pv}(n_0\theta_x) - \psi_x > 0$, then the above range relation gives that

$$\mathrm{pv}(n_0\theta_x) \in (\psi_x,\ \psi_x + \pi/p_0] \subset (\psi_x,\ \psi_x + \pi),$$

which implies that the angle $n_0\theta_x$ lies to the left of the line ψ_x. By the above relation, and using Relation (3.61) and Ineq. (3.62), we infer that

$$\mathrm{pv}(n_0\theta_x) + \theta_x \in (\psi_x + \theta_x,\ \psi_x + \pi/p_0 + \theta_x] \subset (\psi_x,\ \psi_x + \pi],$$

which implies that the angle $(n_0 + 1)\theta_x$ lies to the left of the line ψ_x, or on the line ψ_x. The latter possibility never occurs since the lines ψ_x and $n\theta_x$ do not coincide for any n. Now, by the periodicity obtained in Lemma 3.23, we see that there is an arbitrarily large integer n such that $\mathrm{pv}(n\theta_x) = \mathrm{pv}(n_0\theta_x)$, and thus, both the angles $n\theta_x$ and $(n + 1)\theta_x$ lie to the left of the line ψ_x.

Otherwise $\mathrm{pv}(n_0\theta_x) - \psi_x < 0$. Then Eq. (3.63) gives that

$$(3.64) \qquad\qquad \mathrm{pv}(n_0\theta_x) \in [\psi_x - \pi/p_0,\ \psi_x).$$

Since $\mathrm{pv}(\psi) \in [0,\ 2\pi)$ for all $\psi \in \mathbb{R}$, the above relation implies that $\mathrm{pv}(n_0\theta_x) \in [0,\ \psi_x)$. By Definition 3.19, we infer that the angle $n_0\theta_x$ lies to the right of the line ψ_x. On the other hand, by Relation (3.64) and Ineq. (3.62), we infer that

$$\mathrm{pv}(n_0\theta_x) - \theta_x \in [\psi_x - \pi/p_0 - \theta_x,\ \psi_x - \theta_x) \subset [\psi_x - \pi,\ \psi_x),$$

which implies that the angle $(n_0 - 1)\theta_x$ lies to the right of the line ψ_x, or on the line ψ_x. For the same reason, the second circumstance does not happen. Hence, by the periodicity, there is an arbitrarily large integer n such that the angles $n\theta_x$ and $(n + 1)\theta_x$ lie to the right of the line ψ_x. This completes the proof. $\qquad\square$

Now we are in a position to justify that the number x_Δ^- is a limit point.

Theorem 3.26. *Let $\{W_n(x)\}_{n\geq 0}$ be a $(1, 0)$-sequence, with ordered zero-set*

$$\{\xi_{n,1},\ \xi_{n,2},\ \ldots,\ \xi_{n,n}\}.$$

Then we have $\xi_{n,i} \searrow x_\Delta^-$ for all $i \geq 2$ if $f(x_\Delta^-) > 0$, and for all $i \geq 1$ otherwise.

Proof. Let $i \geq 1$. From the property $R_{n+1} \bowtie R_n$ obtained in Theorem 3.12, we see that the sequence $\xi_{n,i}$ $(n \geq i)$ decreases as $n \to \infty$. Since it is bounded by the interval J_0, the sequence $\xi_{n,i}$ converges. Suppose that $\lim_{n\to\infty} \xi_{n,i} = \ell_i$.

Suppose that $f(x_\Delta^-) > 0$ and $i \geq 2$. From Theorem 3.14 (i), we see that $\xi_{n,i} > x_\Delta^-$ for $n \geq i$, which implies that $\ell_i \geq x_\Delta^-$. Suppose, to the contrary, that $\ell_i > x_\Delta^-$. When n is large, the polynomial $W_n(x)$ has exactly $i - 2$ roots in the interval $(x_\Delta^-,\ \ell_i)$, that is, the roots $\xi_{n,2}, \xi_{n,3}, \ldots, \xi_{n,i-1}$. Thus, by the intermediate value theorem, we infer that

$$W_n(x_\Delta^-)W_n(\ell_i)(-1)^{i-2} > 0 \qquad \text{for large } n.$$

On the other hand, we have $A(x_\Delta^-) < 0$ by Ineq. (3.23). Since $f(x_\Delta^-) > 0$, by Eq. (3.22), we have $W_n(x_\Delta^-)(-1)^{n+1} > 0$ for large n. Multiplying it by the above

inequality results in that

$$(3.65) \qquad W_n(\ell_i)(-1)^{n+i} \; < \; 0 \qquad \text{for large } n.$$

To wit, the sign of the value $W_n(\ell_i)$ alternates as $n \to \infty$. By Theorem 3.21, the angle $n\theta_{\ell_i}$ moves between the two sides of the line ψ_{ℓ_i} alternatively for large n. Since $\ell_i \in J_\Delta$, we infer that $\Delta(\ell_i) < 0$, which contradicts Lemma 3.25. This proves that $\ell_i = x_\Delta^-$.

Now, suppose that $f(x_\Delta^-) \le 0$, and that $i \ge 1$. By Theorem 3.12, we have $\ell_i \ge x_\Delta^-$. Suppose, to the contrary, that $\ell_i > x_\Delta^-$. Along the same lines, we can show that

$$W_n(x_\Delta^-)W_n(\ell_i)(-1)^{i-1} \; > \; 0 \qquad \text{for large } n.$$

On the other hand, since $A(x_\Delta^-) < 0$ and $f(x_\Delta^-) \le 0$, we have $W_n(x_\Delta^-)(-1)^n > 0$ for large n. Multiplying it by the above inequality gives Ineq. (3.65), which causes the same contradiction. This completes the proof. $\qquad\square$

Now we sum up the results obtained in this section to complete the proof of Theorem 2.6. By Theorem 3.12, every polynomial $W_n(x)$ is real-rooted and $R_{n+1} \bowtie R_n$. By Theorem 3.22 and Theorem 3.26, we have $\xi_{n,j} \searrow x_\Delta^-$ and $\xi_{n,n+1-j} \nearrow x_\Delta^+$ for all $j \ge 2$. As will be seen in the following rearrangement, the limit of the smallest roots $\xi_{n,1}$ depends on the sign of the number $f(x_\Delta^-)$, while the limit of the largest roots $\xi_{n,n}$ depends on the sign of the number $f(x_\Delta^+)$. Recall from Definition (2.6) that $b_0 = |a-2|\sqrt{-d}$, and from Relations (3.24) and (3.25) that

$$f(x_\Delta^-) > 0 \quad \Longleftrightarrow \quad b < (a-2)\sqrt{-d}, \qquad \text{and}$$

$$f(x_\Delta^+) < 0 \quad \Longleftrightarrow \quad b > (2-a)\sqrt{-d}.$$

We are ready to state the remaining limits according to the ranges of the numbers a and b.

(i) When $a \le 2$ and $b \le b_0$, we have $f(x_\Delta^-) \le 0$ and $f(x_\Delta^+) < 0$. Therefore, Theorem 3.26 gives that $\xi_{n,1} \searrow x_\Delta^-$, while Theorem 3.22 gives that $\xi_{n,n} \nearrow x_\Delta^+$.

(ii) When $a > 2$ and $b < b_0$, we have $f(x_\Delta^-) > 0$ and $f(x_\Delta^+) < 0$. Therefore, Theorem 3.16 gives that $\xi_{n,1} \searrow x_g^-$, while Theorem 3.18 gives that $\xi_{n,n} \nearrow x_g^+$.

(iii) Otherwise, we have $b > b_0$ or "$b = b_0$ and $a > 2$". In either case, we have $f(x_\Delta^-) \le 0$ and $f(x_\Delta^+) < 0$. Therefore, Theorem 3.26 gives that $\xi_{n,1} \searrow x_\Delta^-$, and Theorem 3.18 gives that $\xi_{n,n} \nearrow x_g^+$.

This completes the proof of Theorem 2.6.

4. PROOF OF THEOREM 2.4

In this section we derive Theorem 2.4 by using Theorem 2.6. The proof is divided into two steps. First, we generalize Theorem 2.6 by dropping the restriction $b \geq 0$; see Proposition 4.1. Second, we extend Proposition 4.1 by allowing the polynomial $W_1(x) = t(x - r)$, by translation and magnification.

Proposition 4.1 (allowing $b < 0$). *Let $W_n(x)$ be polynomials defined by Recursion (2.1), where $a > 0$, $d < 0$, and $b \in \mathbb{R}$, with $W_0(x) = 1$ and $W_1(x) = x$. Then every polynomial $W_n(x)$ is real-rooted. Denote the zero-set of the polynomial $W_n(x)$ by $R_n = \{\xi_{n,1}, \xi_{n,2}, \ldots, \xi_{n,n}\}$ in increasing order. Then we have $R_{n+1} \bowtie R_n$ and Result (2.5). Moreover, we have the following.*

(i) *If $a \leq 2$ and $|b| \leq b_0$, then $\xi_{n,1} \searrow x_\Delta^-$ and $\xi_{n,n} \nearrow x_\Delta^+$.*

(ii) *If $a > 2$ and $|b| < b_0$, then $\xi_{n,1} \searrow x_g^-$ and $\xi_{n,n} \nearrow x_g^+$.*

(iii) *Otherwise, we have $b \neq 0$, and the following.*

    (iii)-1. *If $b < 0$, then $\xi_{n,1} \searrow x_g^-$ and $\xi_{n,n} \nearrow x_\Delta^+$.*

    (iii)-2. *If $b > 0$, then $\xi_{n,1} \nearrow x_\Delta^-$ and $\xi_{n,n} \nearrow x_g^+$.*

Proof. See Appendix F. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Now we are in a position to show Theorem 2.4. Suppose that all the hypotheses in Theorem 2.4 hold true. Consider the sequence $\tilde{W}_n(x)$ defined by

$$(4.1) \qquad\qquad\qquad \tilde{W}_n(x) = W_n(x/t + r).$$

Replacing x by $x/t + r$ in Recursion (2.1), we obtain that

$$\tilde{W}_n(x) = (\tilde{a}x + \tilde{b})\tilde{W}_{n-1}(x) + d \cdot \tilde{W}_{n-2}(x),$$

where $\tilde{a} = a/t$ and $\tilde{b} = A(r)$, with $\tilde{W}_0(x) = 1$, $\tilde{W}_1(x) = x$. It follows that $\tilde{a} > 0$. By Proposition 4.1, every polynomial $\tilde{W}_n(x)$ is distinct-real-rooted. From Definition (4.1), we see that every polynomial $W_n(x)$ is real-rooted. Let

$$\tilde{R}_n = \{\tilde{\xi}_{n,1}, \tilde{\xi}_{n,2}, \ldots, \tilde{\xi}_{n,n}\}$$

be the ordered zero-set of the polynomial $\tilde{W}_n(x)$. Then we have

$$(4.2) \qquad\qquad\qquad\qquad \xi_{n,i} = \tilde{\xi}_{n,i}/t + r.$$

It is clear that magnification and translation preserve the interlacing property. Thus $\tilde{R}_{n+1} \bowtie \tilde{R}_n$ implies that $R_{n+1} \bowtie R_n$.

Regarding the numbers $x_{\Delta}^{\pm} = x_{\Delta}^{\pm}(a, b)$ as functions of a and b, we can define that $x_{\tilde{\Delta}}^{\pm} = x_{\Delta}^{\pm}(\tilde{a}, \tilde{b})$. Similarly, we define $x_{\tilde{g}}^{\pm} = x_g^{\pm}(\tilde{a}, \tilde{b})$, and $\tilde{b}_0 = b_0(\tilde{a})$ Then we have

$$(4.3) \qquad x_{\tilde{\Delta}}^{\pm} = \frac{-\tilde{b} \pm 2\sqrt{-d}}{\tilde{a}} = \frac{-A(r) \pm 2\sqrt{-d}}{a/t}, \qquad \text{and}$$

$$x_{\tilde{g}}^{\pm} = \begin{cases} \dfrac{-\tilde{b} \pm \sqrt{\tilde{b}^2 - 4d(\tilde{a} - 1)}}{2(\tilde{a} - 1)}, & \text{if } \tilde{a} \neq 1 \\[3mm] -\dfrac{d}{\tilde{b}}, & \text{if } \tilde{a} = 1 \text{ and } \tilde{b} \neq 0 \end{cases}$$

$$(4.4) \qquad = \begin{cases} \dfrac{-A(r) \pm \sqrt{A(r)^2 - 4d(a/t - 1)}}{2(a/t - 1)}, & \text{if } a \neq t \\[3mm] -\dfrac{d}{A(r)}, & \text{if } a = t \text{ and } A(r) \neq 0, \end{cases}$$

$$(4.5) \qquad \tilde{b}_0 = |\tilde{a} - 2|\sqrt{-d} = \frac{|a - 2t|\sqrt{-d}}{t}.$$

By Proposition 4.1, we have

$$\tilde{\xi}_{n,j} \searrow x_{\tilde{\Delta}}^- \quad \text{and} \quad \tilde{\xi}_{n,n+1-j} \nearrow x_{\tilde{\Delta}}^+ \qquad \text{for all } j \geq 2.$$

By Eqs. (4.2) and (4.3), the above relations can be rewritten as

$$t(\xi_{n,j} - r) \searrow \frac{-A(r) - 2\sqrt{-d}}{a/t} \quad \text{and} \quad t(\xi_{n,n+1-j} - r) \nearrow \frac{-A(r) + 2\sqrt{-d}}{a/t},$$

namely, Result (2.5) got proved. The remaining cases are shown individually below.

(i) When $a \leq 2t$, we have $\tilde{a} \leq 2$. Suppose that $r \in [r^-, r^+]$. Then $(r - r^-)(r - r^+) \leq 0$. Substituting Definition (2.2) into it gives that

$$t^2 A(r)^2 + d(a - 2t)^2 \leq 0.$$

By Eq. (4.5), the above inequality is equivalent to that $|\tilde{b}| \leq \tilde{b}_0$. By Proposition 4.1, we have

$$\tilde{\xi}_{n,1} \searrow x_{\tilde{\Delta}}^- \quad \text{and} \quad \tilde{\xi}_{n,n} \nearrow x_{\tilde{\Delta}}^+.$$

Along the same lines for proving Result (2.5), we find $\xi_{n,1} \searrow x_{\Delta}^-$ and $\xi_{n,n} \nearrow x_{\Delta}^+$.

(ii) When $a > 2t$, we have $\tilde{a} > 2$. Suppose that $r \in (r^-, r^+)$. Same to the above proof for (i), we obtain that $|\tilde{b}| < \tilde{b}_0$. By Proposition 4.1, we have

$$\tilde{\xi}_{n,1} \searrow x_{\tilde{g}}^- \quad \text{and} \quad \tilde{\xi}_{n,n} \nearrow x_{\tilde{g}}^+.$$

By Eqs. (4.2) and (4.4), the above relations can be recast as

$$t(\xi_{n,1} - r) \searrow \frac{-A(r) - \sqrt{A(r)^2 - 4d(a/t - 1)}}{2(a/t - 1)} \quad \text{and}$$

$$t(\xi_{n,n} - r) \nearrow \frac{-A(r) + \sqrt{A(r)^2 - 4d(a/t - 1)}}{2(a/t - 1)}.$$

In view of Definition (2.4), the above relations reduce to $\xi_{n,1} \searrow \xi^-$ and $\xi_{n,n} \nearrow \xi^+$ respectively.

(iii) Suppose that "$r = r^-$ and $a > 2t$", or $r < r^-$. Then we have $r \leq r^-$. By using Definition (2.2), we infer that

$$\tilde{b} = A(r) \leq -|a - 2t|\sqrt{-d}/t \leq 0.$$

Assume that $\tilde{b} = 0$. Then we have $a = 2t$ and thus get into Case (i), a contradiction. Therefore, $\tilde{b} < 0$. By Proposition 4.1 (iii)-1, we have

$$\tilde{\xi}_{n,1} \searrow x_{\tilde{g}}^- \quad \text{and} \quad \tilde{\xi}_{n,n} \nearrow x_{\tilde{\Delta}}^+.$$

By Eqs. (4.2) — (4.4), the above relations can be rewritten as

$$t(\xi_{n,1} - r) \searrow \frac{-A(r) - \sqrt{A(r)^2 - 4d(a/t - 1)}}{2(a/t - 1)} \quad \text{and}$$

$$t(\xi_{n,n} - r) \nearrow \frac{-A(r) \pm 2\sqrt{-d}}{a/t}.$$

Same to the proofs of (i) and (ii), we deduce that $\xi_{n,1} \searrow \xi^-$ and $\xi_{n,n} \nearrow x_{\Delta}^+$.

(iv) It is highly similar to Case (iii), and we omit it. This completes the proof of Theorem 2.4.

5. Concluding Remarks

This section explains why we set $a > 0$, $d < 0$, and $t > 0$ in Theorem 2.4.

First of all, the restriction $adt \neq 0$ is without loss of generality. In fact, the sign restriction $a \neq 0$ comes from the type $(1, 0)$ of recursive polynomials as the topic of this paper. When $d = 0$, it is clear that $W_n(x) = A(x)^{n-1}W_1(x)$ by Recursion (2.1), and the root geometry problem becomes trivial. When $t = 0$, the polynomial $W_1(x) = 0$, and one may consider the sequence $\{W_{n+2}(x)/d\}_{n \geq 0}$.

Second, with the assumption $adt \neq 0$ in hand, the real-rootedness for every polynomial $W_n(x)$ is still not true in general. In fact, consider the case $b = 0$, $t = 1$, $W_0(x) = 1$ and $W_1(x) = x$.

- When $ad > 0$, the polynomial $W_2(x) = ax^2 + d$ has no real roots since its sign is same to the sign of the number d.
- When $a < 0 < d$, the polynomial $W_3(x) = x(a^2x^2 + d(a + 1))$ has non-real roots as if $a + 1 > 0$.

The remaining case except that we handled in this paper is that all the parameters a, d, and t are negative. In this case, by considering the sequence $\{(-1)^n W_n(r - x/t)\}_{n \geq 0}$ with the aid of Theorem 2.4, one may derive the real-rootedness of every polynomial $W_n(x)$, as well as the interlacing property and the limit points of the union of the zero-sets. Yet in applications, it is not often to meet such situation that all a, d, and t are negative.

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APPENDIX A. PROOF OF LEMMA 3.9

Let $x \in \{x_g^{\pm}\}$. Then we have $g(x) = 0$. From Definition (3.18) of the function $g(x)$ as $g^-(x)g^+(x)$, we have either $g^-(x) = 0$ or $g^+(x) = 0$. To show that $W_n(x) = x^n$, we split into two cases.

Assume that $g^-(x) = 0$. From Definition (3.17) of the function $g^-(x)$ as $\big(f(x) - \sqrt{\Delta(x)}\big)/2$, and from Definition (3.16) of the function $f(x)$ as $2x - A(x)$, we have

$$(A.1) \qquad A(x) + \sqrt{\Delta(x)} = 2x.$$

It follows from Definition (3.17) of the function $g^+(x)$ that

$$(A.2) \qquad g^+(x) = \frac{2x - A(x) + \sqrt{\Delta(x)}}{2} = \sqrt{\Delta(x)}.$$

If $\Delta(x) \neq 0$, then Lemma 3.2 gives that

$$(A.3) \qquad W_n(x) = \frac{g^+(x)(A(x) + \sqrt{\Delta(x)})^n - g^-(x)(A(x) - \sqrt{\Delta(x)})^n}{2^n \sqrt{\Delta(x)}}.$$

Substituting the condition $g^-(x) = 0$, Eqs. (A.1) and (A.2) into the above equation, we obtain that

$$W_n(x) = \frac{\sqrt{\Delta(x)}\,(2x)^n - 0}{2^n \sqrt{\Delta(x)}} = x^n.$$

Otherwise $\Delta(x) = 0$, then Eq. (A.1) reduces to $A(x) = 2x$. Since $W_1(x) = x$, by Lemma 3.2, we deduce that

$$W_n(x) = \left(1 + \frac{n(2W_1(x) - A(x))}{A(x)}\right)\left(\frac{A(x)}{2}\right)^n = x^n.$$

When $g^+(x) = 0$, from Definition (3.17) of the function $g^+(x)$ as $\big(f(x) + \sqrt{\Delta(x)}\,\big)/2$, and from Definition (3.16), we have

$$(A.4) \qquad\qquad A(x) - \sqrt{\Delta(x)} \;=\; 2x.$$

It follows from Definition (3.17) that

$$(A.5) \qquad\qquad g^-(x) \;=\; \frac{2x - A(x) - \sqrt{\Delta(x)}}{2} \;=\; -\sqrt{\Delta(x)}.$$

If $\Delta(x) \neq 0$, substituting the condition $g^+(x) = 0$, Eqs. (A.4) and (A.5) into Eq. (A.3), we obtain that

$$W_n(x) \;=\; \frac{0 - (-\sqrt{\Delta(x)}\,)(2x)^n}{2^n\sqrt{\Delta(x)}} \;=\; x^n.$$

This completes the proof of the identity $W_n(x_g^{\pm}) = (x_g^{\pm})^n$.

Below we show the results (i) and (ii) in Lemma 3.9.

(i) Suppose that $a \leq 1$. If $a < 1$, from Definition (2.7) of the numbers $x_g^{\pm}$, the desired inequalities $0 < x_g^+ \leq x_g^-$ can be rewritten as

$$0 \;<\; \frac{-b + \sqrt{b^2 - 4d(a-1)}}{2(a-1)} \;\leq\; \frac{-b - \sqrt{b^2 - 4d(a-1)}}{2(a-1)}.$$

Since $a < 1$, the first inequality holds because $d < 0$, and the second inequality holds trivially. Otherwise $a = 1$ and $b \neq 0$. By Definition (2.7), the desired inequalities $0 < x_g^+ \leq x_g^-$ become

$$0 < -d/b \leq -d/b,$$

which is also trivial since $d < 0 < b$. The remaining inequalities $W_n(x_g^{\pm}) > 0$ follow immediately from the equations $W_n(x_g^{\pm}) = x_g^{\pm}$ and the inequalities $x_g^{\pm} > 0$.

(ii) Suppose that $a > 1$. From Definition (2.7) of the numbers $x_g^{\pm}$, the desired inequalities $x_g^- < 0 < x_g^+$ can be rewritten as

$$\frac{-b - \sqrt{b^2 - 4d(a-1)}}{2(a-1)} \;<\; 0 \;<\; \frac{-b + \sqrt{b^2 - 4d(a-1)}}{2(a-1)}.$$

Both of them hold trivially since $d < 0 \leq b$. Consequently, since $W_n(x_g^{\pm}) = (x_g^{\pm})^n$, we have

$$\begin{aligned} W_n(x_g^-)(-1)^n &= (x_g^-)^n(-1)^n > 0 \quad \text{and} \\ W_n(x_g^+) &= (x_g^+)^n > 0. \end{aligned}$$

This completes the proof of Lemma 3.9.

Appendix B. Proof of Lemma 3.10

Suppose that

$$(B.1) \qquad\qquad f(x_\Delta^+) < 0.$$

Substituting Eq. (3.21) into Ineq. (B.1), and canceling the positive factor $2/a$, we obtain that

$$(B.2) \qquad\qquad b > (2-a)\sqrt{-d}.$$

When $a = 1$, Ineq. (B.2) reduces to $b > \sqrt{-d}$, which implies that $b \neq 0$. It follows from Definition (2.7) that the numbers $x_g^\pm$ are well-defined. We shall show that

(i) the numbers $x_g^\pm$ are real;

(ii) we have $x_g^+ > x_\Delta^+$;

(iii) if additionally we have $f(x_\Delta^-) > 0$, then $x_g^- < x_\Delta^-$.

For (i) and (ii), we split into the cases $a = 1$ and $a \neq 1$. The desired result (iii) will be shown individually.

Assume that $a = 1$. From Definition (2.7) of the numbers $x_g^\pm$ as the real number $-d/b$ identically, and from Definition (2.3) of the number x_Δ^+ as $-b + 2\sqrt{-d}$, it is routine to compute that

$$x_g^+ - x_\Delta^+ \;=\; -\frac{d}{b} - (-b + 2\sqrt{-d}) \;=\; \frac{(b - \sqrt{-d}\,)^2}{b} \;>\; 0.$$

In other words, the desired results (i) and (ii) are true when $a = 1$.

Suppose that $a \neq 1$. From Definition (2.7), we have

$$x_g^\pm = \frac{-b \pm \sqrt{\Delta_g}}{2(a-1)}, \qquad \text{where } \Delta_g = b^2 - 4d(a-1).$$

To show the realness of the numbers $x_g^\pm$, it suffices to prove that $\Delta_g > 0$. In fact, when $a > 1$, one has $\Delta_g > 0$ since $d < 0$. When $0 < a < 1$, squaring both sides of Ineq. (B.2) gives that $b^2 > -d(2-a)^2$. Therefore, we infer that

$$\Delta_g \;=\; b^2 - 4d(a-1) \;>\; -d(2-a)^2 - 4d(a-1) \;=\; -da^2 \;>\; 0.$$

This confirms the desired result (i).

In order to show the desired result (ii) for $a \neq 1$, as well as the desired result (iii), we make some preparations uniformly.

Definitions (3.17) and (3.18) imply that

$$(B.3) \qquad \Delta(x_g^\pm) \;=\; f^2(x_g^\pm) - 4g(x_g^\pm) \;=\; f^2(x_g^\pm) \;\geq\; 0.$$

Since the function $\Delta(x)$ is a quadratic polynomial with positive leading coefficient, whose roots are $x_\Delta^\pm$ with $x_\Delta^- < x_\Delta^+$, we deduce that

$$(B.4) \qquad \text{either} \quad x_g^- \leq x_\Delta^- \quad \text{or} \quad x_g^- \geq x_\Delta^+, \qquad \text{and that}$$

$$(B.5) \qquad \text{either} \quad x_g^+ \leq x_\Delta^- \quad \text{or} \quad x_g^+ \geq x_\Delta^+.$$

On the other hand, let $X = (a-2)b$ and $Y = a\sqrt{\Delta_g}$. Since $a > 0$, we have

(B.6) $$Y > 0.$$

By Definition (2.7) of the numbers $x_g^{\pm}$, it is direct to calculate that

(B.7) $$-\frac{b}{a} - x_g^{\pm} = \frac{X \pm Y}{2a(1-a)}, \quad \text{and}$$

(B.8) $$X^2 - Y^2 = 4(1-a)(b^2 - a^2 d).$$

Since $d < 0 < a$, we have $b^2 - a^2 d > 0$. Since $a \neq 1$, Eq. (B.8) implies that

(B.9) $$(X+Y)(X-Y)(1-a) > 0.$$

Now we are in a position to show the desired result (ii) for $a \neq 1$. Suppose, to the contrary, that $x_g^+ \leq x_\Delta^+$. Then, by Relation (B.5), we deduce that either $x_g^+ = x_\Delta^+$, or

$$x_g^+ \leq x_\Delta^- = \frac{-b - 2\sqrt{-d}}{a} < -\frac{b}{a}.$$

In the former case, we infer from Ineq. (B.3) that

$$f^2(x_\Delta^+) = f^2(x_g^+) = \Delta(x_g^+) = \Delta(x_\Delta^+) = 0.$$

It follows that $f(x_\Delta^+) = 0$, contradicting Ineq. (B.1). In the latter case, Eq. (B.7) implies that

(B.10) $$(X+Y)(1-a) > 0.$$

Multiplying Ineq. (B.10) by Ineq. (B.9), we find that $X - Y > 0$. Together with Ineq. (B.6), we conclude that

$$X > Y > 0.$$

On one hand, since $X = (a-2)b$ and $X > 0$, we obtain that $a > 2$. On the other hand, since $X > 0$ and $Y > 0$, from Ineq. (B.10), we deduce that $a < 1$, a contradiction. This proves (ii).

At last, let us show the desired result (iii). Suppose that

(B.11) $$f(x_\Delta^-) > 0.$$

By Eq. (3.20), Ineq. (B.11) can be rewritten as $b < (a-2)\sqrt{-d}$. Since $b \geq 0$, we have

$$a > 2.$$

Suppose, by way of contradiction, that $x_g^- \geq x_\Delta^-$. Then, by Relation (B.4), we deduce that either $x_g^- = x_\Delta^-$, or

$$x_g^- \geq x_\Delta^+ = \frac{-b + 2\sqrt{-d}}{a} > -\frac{b}{a}.$$

In the former case, we infer from Ineq. (B.3) that

$$f^2(x_\Delta^-) = f^2(x_g^-) = \Delta(x_g^-) = \Delta(x_\Delta^-) = 0.$$

It follows that $f(x_\Delta^-) = 0$, contradicting Ineq. (B.11). In the latter case, Eq. (B.7) implies that $(X - Y)(1 - a) < 0$. Since $a > 2$, we infer that $X > Y$. Together with Ineq. (B.6), we conclude that

$$X > Y > 0,$$

which contradicts Ineq. (B.9). This completes the proof of Lemma 3.10.

Appendix C. Proof of Lemma 3.17

Suppose that $f(x_\Delta^+) < 0$. Then we have Ineq. (B.2).

When $a \geq 2$, from Definition (3.16), the function $f(x) = (2 - a)x - b$ is decreasing. Together with Ineq. (B.1), we infer that $f(x) < 0$ for all $x > x_\Delta^+$.

Let $0 < a < 2$ below. By the monotonicity of the function $f(x)$, it suffices to show that $f(x_g^+) < 0$.

When $a = 1$, Ineq. (B.2) reduces to $b > \sqrt{-d}$, that is, $b^2 + d > 0$. On the other hand, from Definition (2.7) of the number x_g^+, we have $x_g^+ = -d/b$. It follows that

$$f(x_g^+) = x_g^+ - b = -\frac{d + b^2}{b} < 0.$$

When $a \neq 1$, recall that $\Delta_g = b^2 - 4d(a-1)$ is the discriminant of the quadratic function polynomial $g(x)$. By Definition (2.7) of the number x_g^+, it is direct to compute that

$$2(1 - a)f(x_g^+) = (a - 2)\sqrt{\Delta_g} + ab,$$

which implies the equivalence relation

$$(1 - a)f(x_g^+) > 0 \quad \Longleftrightarrow \quad ab > (2 - a)\sqrt{\Delta_g}.$$

Since $a < 2$, squaring both sides of the above rightmost inequality gives another equivalence relation:

$$ab > (2 - a)\sqrt{\Delta_g} \quad \Longleftrightarrow \quad (1 - a)(b^2 + d(a - 2)^2) < 0.$$

By Ineq. (B.2), we have $b^2 + d(a - 2)^2 > 0$. Therefore, transiting the above two equivalence relations gives the following equivalence relation:

$$(1 - a)f(x_g^+) > 0 \quad \Longleftrightarrow \quad 1 - a < 0.$$

Hence, we infer that $f(x_g^+) < 0$. This completes the proof.

Appendix D. Proof of Lemma 3.23

Let p_0 be the integer defined by Eq. (3.59). From Definition 3.19, we have $\mathrm{pv}(\theta + 2k\pi) = \mathrm{pv}(\theta)$ for any $\theta \in \mathbb{R}$ and $k \in \mathbb{Z}$. Thus, we have

$$\mathrm{pv}((p_0 + j)\theta) = \mathrm{pv}(p_0 q\pi/p + j\theta) = \mathrm{pv}(j\theta).$$

In other words, the sequence $\{\mathrm{pv}(n\theta)\}_{n \geq 1}$ is periodic with a period p_0.

To show the minimality of the period p_0, it suffices to show that the numbers $\mathrm{pv}(n\theta)$ for $n \in [p_0]$ are pairwise distinct. In fact, otherwise, there are three integers j_1, j_2, and k, such that $1 \le j_1 < j_2 \le p_0$, and that $(j_2 - j_1)\theta = 2k\pi$, that is, $(j_2 - j_1)q = 2kp_0$. Since $1 \le j_2 - j_1 \le p_0 - 1$ and $(p_0, q) = 1$, the left hand side $(j_2 - j_1)q$ is not divided by the factor p_0 of the right hand side, which is absurd.

Let L (resp., R) denote the set on the left (resp., right) hand side of Eq. (3.60). We have just proved that the set L has cardinality p_0, as same as the set R. Therefore, to justify $L = R$ as sets, it suffices to show that the set L is contained in the set R. In fact, we have

$$\mathrm{pv}(n\theta) = \mathrm{pv}(nq\pi/p) = \begin{cases} \mathrm{pv}(2(nq/2)\pi/p_0), & \text{if } q \text{ is even;} \\ \mathrm{pv}(2nq\pi/p_0), & \text{if } q \text{ is odd.} \end{cases}$$

Since $\mathrm{pv}(\psi) \in [0, 2\pi)$ for any angle ψ, the rightmost expression in the above formula implies that the number $\mathrm{pv}(n\theta)$ has the form $2j\pi/p_0$, where $j \in [0, p_0 - 1]$. Namely, we have $L \subseteq R$. This completes the proof.

APPENDIX E. PROOF OF LEMMA 3.24

We will need Dirichlet's approximation theorem.

Theorem E.1 (Dirichlet's approximation theorem). *For any real number μ and any positive integer N, there exist integers p and q such that $q \in [N]$ and $|q\mu - p| \le 1/(N + 1)$.*

For the sake of proving Lemma 3.24, we show the following stronger proposition.

Proposition E.2. *Let $\theta \in \mathbb{R}$ such that $\theta \notin \pi\mathbb{Q}$. Then for any positive integer M, and for any nonempty open interval $I \subset (0, 2\pi)$, there exist positive integers m and q such that*

$$\mathrm{pv}((m + j)q\theta) \in I \qquad \text{for all } j \in [M].$$

Proof. Let M be a positive integer. Let $I = (u, v)$ be an open interval such that $0 < u < v < 2\pi$. Let

$$\epsilon = \frac{v - u}{M + 1}.$$

Take an integer N such that $2\pi/(N + 1) < \epsilon$. By Dirichlet's approximation theorem, there exist integers p and q' such that $q' \in [N]$ and $|q'\theta/(2\pi) - p| < 1/(N + 1)$. Therefore, we have

$$|q'\theta - 2p\pi| < \frac{2\pi}{N + 1} < \epsilon.$$

In other words, we have either $\mathrm{pv}(q'\theta) \in [0, \epsilon)$ or $2\pi - \mathrm{pv}(q'\theta) \in [0, \epsilon)$. Since $\theta \notin \pi\mathbb{Q}$, the principal value $\mathrm{pv}(q'\theta)$ does not vanish, that is, either $\mathrm{pv}(q'\theta) \in (0, \epsilon)$ or $\mathrm{pv}(q'\theta) \in (2\pi - \epsilon, 2\pi)$. For the latter case, there is a positive integer k such that $\mathrm{pv}(kq'\theta) \in (0, \epsilon)$, and we define $q = kq'$. In the former case, we define $q = q'$. In summary, we obtain a positive integer q such that $\mathrm{pv}(q\theta) \in (0, \epsilon)$.

Now, let m be the minimum positive integer such that

(E.1)
$$(m+1)\mathrm{pv}(q\theta) \;>\; u.$$

Since $\mathrm{pv}(q\theta) < \epsilon = (v-u)/(M+1)$, we have $M\mathrm{pv}(q\theta) < v-u-\mathrm{pv}(q\theta)$. It follows that

(E.2)
$$(m+M)\mathrm{pv}(q\theta) \;<\; u+v-u-\mathrm{pv}(q\theta) \;<\; v.$$

Hence, by Ineqs. (E.1) and (E.2), we conclude that

$$u \;<\; (m+1)\mathrm{pv}(q\theta) \;<\; (m+2)\mathrm{pv}(q\theta) \;<\; \cdots \;<\; (m+M)\mathrm{pv}(q\theta) \;<\; v.$$

Let $j \in [M]$. Since $(m+j)\mathrm{pv}(q\theta) \in (u,v) \subset (0,2\pi)$, we infer that $\mathrm{pv}((m+j)q\theta) = (m+j)\mathrm{pv}(q\theta)$, which implies the desired relation. This completes the proof of Proposition E.2. $\qquad\square$

Appendix F. Proof of Proposition 4.1

Since Case (iii) is the exclusive case of (i) and (ii), we have either $|b| > b_0$, or "$|b| = b_0$ and $a > 2$". Assume that $b = 0$, then the former possibility is impossible, and the latter possibility $|b| = b_0$ implies that $b_0 = 0$ and thus $a = 2$, a contradiction. Thus we have $b \neq 0$ for Case (iii).

The results for $b \geq 0$ are exactly those in Theorem 2.6. Let $b < 0$. Define

(F.1)
$$\tilde{W}_n(x) = (-1)^n W_n(-x).$$

The the polynomials $\tilde{W}_n(x)$ satisfy the recursion

$$\tilde{W}_n(x) = (ax-b)\tilde{W}_{n-1}(x) + d\tilde{W}_{n-2}(x).$$

with same initiations $\tilde{W}_0(x) = 1$ and $\tilde{W}_1(x) = x$. By Theorem 2.6, every polynomial $\tilde{W}_n(x)$ is distinct-real-rooted. Let

$$\tilde{R}_n = \{\tilde{\xi}_{n,1},\, \tilde{\xi}_{n,2},\, \ldots,\, \tilde{\xi}_{n,n}\}$$

be the ordered zero-set of the polynomial $\tilde{W}_n(x)$. From Definition (F.1), we see that the roots of the polynomial $W_n(x)$ are the opposites of the roots of the polynomial $\tilde{W}_n(x)$. It follows that every polynomial $W_n(x)$ is real-rooted, with the ordered zero-set

$$R_n = \{-\tilde{\xi}_{n,n},\, -\tilde{\xi}_{n,\,n-1},\, \ldots,\, -\tilde{\xi}_{n,1}\}.$$

In other words, we have

(F.2)
$$\xi_{n,i} = -\tilde{\xi}_{n,\,n+1-i}, \qquad \text{for all } i \in [n].$$

Since $\tilde{R}_{n+1} \bowtie \tilde{R}_n$, we infer that $R_{n+1} \bowtie R_n$.

Regarding the numbers $x_{\widetilde{\Delta}}^{\pm} = x_{\Delta}^{\pm}(b)$ as a functions in b, we can define the numbers $x_{\widetilde{\Delta}}^{\pm} = x_{\Delta}^{\pm}(-b)$. Similarly, we define $x_{\widetilde{g}}^{\pm} = x_g^{\pm}(-b)$. Then we have

$$(\text{F.3}) \qquad x_{\widetilde{\Delta}}^{\pm} = \frac{-(-b) \pm 2\sqrt{-d}}{a} = -\frac{-b \mp 2\sqrt{-d}}{a} = -x_{\Delta}^{\mp}, \qquad \text{and}$$

$$x_{\widetilde{g}}^{\pm} = \begin{cases} \dfrac{-(-b) \pm \sqrt{(-b)^2 - 4d(a-1)}}{2(a-1)}, & \text{if } a \neq 1, \\[3mm] -\dfrac{d}{-b}, & \text{if } a = 1 \text{ and } b \neq 0, \end{cases}$$

$$= \begin{cases} -\dfrac{-b \mp \sqrt{b^2 - 4d(a-1)}}{2(a-1)}, & \text{if } a \neq 1, \\[3mm] -\left(-\dfrac{d}{b}\right), & \text{if } a = 1 \text{ and } b \neq 0, \end{cases}$$

$$(\text{F.4}) \qquad = -x_g^{\mp}.$$

By Theorem 2.6, we have

$$\widetilde{\xi}_{n,j} \searrow x_{\widetilde{\Delta}}^{-} \quad \text{and} \quad \widetilde{\xi}_{n,n+1-j} \nearrow x_{\widetilde{\Delta}}^{+} \qquad \text{for all } j \geq 2.$$

By Eqs. (F.2) and (F.3), the above relations can be rewritten as

$$-\xi_{n,n+1-j} \searrow -x_{\Delta}^{+} \quad \text{and} \quad -\xi_{n,j} \nearrow -x_{\Delta}^{-},$$

namely, Relation (2.5) got proved. The remaining cases are shown individually below.

(i) When $a \leq 2$ and $|b| \leq b_0$, we have $-b \leq b_0$. By Theorem 2.6 (i), we have

$$\widetilde{\xi}_{n,1} \searrow x_{\widetilde{\Delta}}^{-} \quad \text{and} \quad \widetilde{\xi}_{n,n} \nearrow x_{\widetilde{\Delta}}^{+}.$$

By using Eqs. (F.2) and (F.3), the above limits can be rewritten as

$$-\xi_{n,n} \searrow -x_{\Delta}^{+} \quad \text{and} \quad -\xi_{n,1} \nearrow -x_{\Delta}^{-},$$

that is the desired limits $\xi_{n,1} \searrow x_{\Delta}^{-}$ and $\xi_{n,n} \nearrow x_{\Delta}^{+}$.

(ii) When $a > 2$ and $|b| < b_0$, we have $-b < b_0$. By Theorem 2.6 (ii), we have

$$\widetilde{\xi}_{n,1} \searrow x_{\widetilde{g}}^{-} \quad \text{and} \quad \widetilde{\xi}_{n,n} \nearrow x_{\widetilde{g}}^{+}.$$

By using Eqs. (F.2) and (F.4), the above limits can be rewritten as

$$-\xi_{n,n} \searrow -x_g^{+} \quad \text{and} \quad -\xi_{n,1} \nearrow -x_g^{-},$$

that is the desired limits $\xi_{n,1} \searrow x_g^{-}$ and $\xi_{n,n} \nearrow x_g^{+}$.

(iii) Otherwise, we have either $|b| > b_0$ or "$|b| = b_0$ and $a > 2$". Since $b < 0$, we only need to show (iii)-1. In this case, we have either $-b > b_0$ or "$-b = b_0$ and $a > 2$". By Theorem 2.6 (iii), we have

$$\tilde{\xi}_{n,1} \searrow x_{\tilde{\Delta}}^- \quad \text{and} \quad \tilde{\xi}_{n,n} \nearrow x_{\tilde{g}}^+.$$

By using Eqs. (F.2) — (F.4), the above limits can be rewritten as

$$-\xi_{n,n} \searrow -x_{\Delta}^+ \quad \text{and} \quad -\xi_{n,1} \nearrow -x_g^-,$$

that is the desired limits $\xi_{n,1} \searrow x_g^-$ and $\xi_{n,n} \nearrow x_{\Delta}^+$. This completes the proof of Proposition 4.1.

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