

On Some Identities of Barred Preferential Arrangements

S.Nkonkobe, V.Murali,

Department of Mathematics (Pure & Applied)

Rhodes University

Grahamstown 6140 South Africa

snkonkobe@yahoo.com, v.murali@ru.ac.za

1. ABSTRACT

A preferential arrangement of a finite set is an ordered partition. Associated with each such ordered partition is a chain of subsets or blocks endowed with a linear order. The chain may be split into sections by the introduction of a vertical bar, leading to the notion of a barred preferential arrangements. In this paper we derive some combinatorial identities satisfied by the number of possible barred preferential arrangements of an n -element set. We illustrate with some suitable examples highlighting some important consequences of the identities.

Mathematics Subject Classifications:05A18,05A19,05A16, 2013

2. INTRODUCTION

The study of preferential arrangements in Combinatorics goes back to Cayley in [1]. Further studies of these objects were made by Gross[3], Mendelson[4] and others. More recently Pippenger studied special kind of preferential arrangements and called them barred preferential arrangements in [7]. The same further generalised in [8]. In this paper we derive several identities using combinatorial arguments, thus answering a comments raised by Pippenger in [7].In preliminaries we gather known results on preferential and barred preferential arrangements. In section 4 we state and prove several identities involving the number of barred preferential arrangements of an n -element set. In proving the identities we use combinatorial arguments. In section 5 we discuss some identities involving restricted barred preferential arrangements. Throughout the paper we illustrate the concept with some simple examples.

3. PRELIMINARIES

In this section we collect all the preliminaries and known results setting up convenient notations for the discussion of barred preferential arrangements. We refer to the papers [8] and [7] for discussions and results on barred preferential arrangements.

1° Preferential arrangement.

A partition of a set $X_n = \{1, 2, \dots, n\}$ with n elements is a collection of disjoint non-empty subsets whose union is X_n . The subsets forming the partition are called blocks. If the blocks are arranged in a linear order, the ordered partition is referred as a preferential arrangement of X_n . see [3], [4]. For example, $\{\{1\}, \{2, 3\}\}$ is a preferential arrangement of X_3 with two blocks with the natural ordering

$\{1\} \leq \{2, 3\}$. For convenience we write this preferential arrangement as 1 23, reading the block containing 1 comes first and the block containing 23 comes second.

From now onwards we will use the natural ordering of blocks as illustrated here.

A warning: The natural ordering of the numbers $1, 2, \dots, n$ is not relevant here but the ordering of the blocks are. Let Q_n and J_n denote the set of preferential arrangements of X_n and the number of preferential arrangements of X_n , so that $|Q_n| = J_n$. The numbers J_n for $n \in \mathbb{N}_0$ are also known as Fubini numbers and a list of values of J_n for $n = 0, 1, 2, \dots$ can be found in OEIS sequence number A000670, [6]. The number J_n is interpreted as the number of outcomes in a race with n participants assuming all of them finish the race and ties are allowed [4] or the number of restricted preferential fuzzy subsets of X_n , [5]. The numbers J_n satisfy a recurrence relation with the initial condition $J_0 = 1$, or identity as a finite sum, see [4] or an infinite sum, see [3] or a closed form involving Stirling numbers of the second kind [7], respectively

$$(1) \quad J_{n+1} = \sum_{s=0}^n \binom{n+1}{s} J_s, \quad \text{or} \quad J_n = \frac{1}{2} \sum_{s=0}^{\infty} s^n 2^{-s}, \quad \text{or} \quad J_n = \sum_{s=0}^n \left\{ \begin{matrix} n \\ s \end{matrix} \right\} s!.$$

2° Barred Preferential arrangements.

The idea of introducing bars in between the blocks of a preferential arrangement seems first to appear in [7] even though the numbers associated with counting barred preferential arrangements appear much earlier in the literature [2]. For example,

for the preferential arrangement $\mathfrak{F} = 1\ 23$ considered in $\mathbf{1}^\circ$, we can insert one bar $|$ in three different places in three different ways (what a coincidence!) as follows: $1\ | \ 23$ or $| \ 1\ 23$ or $1\ 23\ |$, whereas a two-bar insertion into the same \mathfrak{F} would give rise to 6 barred preferential arrangements. viz., $1\ || \ 23$, $|| \ 1\ 23$, $1\ 23\ ||$, $1\ | \ 23\ |$, $| \ 1\ 23\ |$ or $| \ 1\ | \ 23$. We now illustrate what we mean by sections associated with bars. Consider two 3-bar preferential arrangements of X_7 with 4 blocks.

$$\begin{array}{cccc} \underbrace{7\ 43\ 1}_{1^{st}\ section} & | & \underbrace{652}_{2^{nd}\ section} & | & \underbrace{\quad}_{3^{rd}\ section} & | & \underbrace{\quad}_{4^{th}\ section} \\ \underbrace{\quad}_{1^{st}\ section} & | & \underbrace{652\ 1}_{2^{nd}\ section} & | & \underbrace{7\ 43}_{3^{rd}\ section} & | & \underbrace{\quad}_{4^{th}\ section} \end{array}$$

We notice that in general m bars induce $m + 1$ sections with some of the sections possibly empty.

In general we denote the number of barred preferential arrangements of X_n with m bars ($m \geq 0$) by J_n^m . In addition to the numbers J_n^m , we use Q_n^m for the set of all barred preferential arrangements of X_n with m bars ($m \geq 0$). Hence $|Q_n^m| = J_n^m$. When $m = 0$, J_n^0 ($|Q_n^0|$) identified as simply the number (the set) of preferential arrangements without any bars and therefore is equal to J_n (Q_n) according to the notation in subsection $\mathbf{1}^\circ$. For instance $J_2^2 = 15$, $J_4^5 = 5340$ can be verified by direct manual counting of such barred preferential arrangements. The following closed form for J_n^m and the recurrence relation with the initial condition $J_0^m = 1$, $J_1^m = m + 1$ satisfied by it are obtained by Ahlback et al in [8]

$$(2) \quad J_n^m = \sum_{s=0}^n \left\{ \begin{matrix} n \\ s \end{matrix} \right\} s! \binom{m+s}{m}, \quad J_n^m = \sum_{s=0}^n \binom{n}{s} J_s^0 J_{n-s}^{m-1} \quad (m \geq 1)$$

The infinite sum representation for J_n^m corresponding to similar expression for preferential arrangement is bit more involved.

$$(3) \quad J_m^n = \frac{1}{2^{m+1} m!} \sum_{s=0}^{\infty} \frac{(s+1)^{\bar{m}} s^n}{2^s} \text{ where } (s+1)^{\bar{m}} = \sum_{t=1}^m s(m, t) \times (s+1)^m$$

again where $s(m, t)$ is the Stirling numbers of the first kind.

$\mathbf{3}^\circ$ Restricted Barred Preferential arrangements.

We also discuss some identities based on barred preferential arrangements with

some restrictions. Consider the following restriction on a preferential arrangement \mathfrak{k} of X_n with m bars. There are $m + 1$ sections. We fix a specific section, say the $(m + 1)^{th}$ section, namely the last section. After fixing the section, we impose the restriction that every other section to have at most one block. We then preferentially arrange the elements of the blocks of the fixed section. we then call such a barred preferential arrangement a restricted barred preferential arrangement. We denote the set of all restricted barred preferential arrangements of X_n having m bars by H_n^m . The order of the set H_n^m is denoted by the number I_n^m . In the proofs involving the set H_n^m , it is occasionally useful to consider one extra bar $|^*$ introduced on the elements of H_n^m at the far right hand end of the last section. It will be clear in the proofs that such an introduction of an extra bar does not affect counting. Therefore we use set $*\mathfrak{P}_n^m$ for elements of H_n^m with $|^*$. Both H_n^m and $*\mathfrak{P}_n^m$ have the same cardinality I_n^m .

4. THE IDENTITIES

In this section we develop a number of interesting identities satisfied by J_n^m and for various non-negative integral values of n and m . The proofs are combinatorial arguments based on preferential arrangements. The first identity was found in [7] without proof or reference. Further the author of that paper states that such an identity can be proved combinatorially or otherwise. Here below we state and prove that result as a lemma.

Lemma 1. *Let n be an non-negative integer. Then $J_{n+1}^0 = \sum_{s=0}^n \binom{n}{s} J_s^1$*

proof: What we are required to prove is that the number of preferential arrangements with no bars can be written as a sum of barred preferential arrangements with one bar.

The left hand side of the identity can be interpreted combinatorially as outcomes in a race of $n + 1$ people (assuming that all people finish the race)[4]. The terms on the right hand side would correspond to races of n people, of which the people are preferentially arranged with respect to a single bar (hence the superscript of J is 1). In proving the lemma, use a similar method to the one used in [4] in establishing equation (7)). We need to associate each term on the left with one term on the right and visa versa for the identity to hold. Therefore we first mark

a person out of $n + 1$ people who are to run a race. We interpret the block having the marked person as a bar. Clearly all outcomes of the race would not be affected whether a specific person is marked or not. In our notation the number of outcomes is captured by J_{n+1}^0 (by definition of J_{n+1}^0). We ask the following question in order to identify each outcome that is counted by J_{n+1}^0 is interpreted as an outcome in the sum on the right hand side and visa versa: How many people do not finish with the marked person? The block having the marked person would serve as the bar in a barred preferential arrangement counted on the right hand side of the identity. There are $\binom{n}{s}$ ways of choosing s people who do not finish with the marked person for each s such that $0 \leq s \leq n$. For each such s , there are J_s^1 ways of preferentially arranging the s people with the block having the marked person serving as the bar. Therefore there are $\binom{n}{s} \times J_s^1$ many number of barred preferential arrangement of the s people each of which is a preferential arrangement of $n + 1$ persons finishing a race. Clearly the preceding argument uniquely identifies each outcome of the LHS as an outcome of the RHS.

We generalize Lemma 1 to the case of multiple bars in the following way,

Theorem 1. For two integers $m, n \geq 0$, $J_{n+1}^m = (m + 1) \sum_{s=0}^n \binom{n}{s} J_s^{m+1}$

proof: In forming a barred preferential arrangement of an $n + 1$ -persons as in Lemma one above, with m bars, we mark one of the n persons. The position of the marked person would be determined by two quantities, one: the section in which the marked person finishes in and two: the block in which he finishes within the section. The block in which the marked person is in, is interpreted as the $m + 1^{th}$ bar on the right hand side. Let us assume that the marked person is in section i . The question is: how many people do not finish with the marked person? Let us say there are s of them, those s people can be selected in $\binom{n}{s}$ ways. There are J_s^{m+1} ways of preferentially arranging the s persons, where the $m + 1^{th}$ bar is the block having the marked person. Summing over s we have $\sum_{s=0}^n \binom{n}{s} J_s^{m+1}$ number of preferential arrangements. If the marked person finishes in section k instead of section i above, we would still have $\sum_{s=0}^n \binom{n}{s} J_s^{m+1}$ many preferential arrangements.

Since there are $m + 1$ sections hence the total number of preferential arrangements is $(m + 1) \sum_{s=0}^n \binom{n}{s} J_s^{m+1}$.

After putting the sums in theorem 1 in a Pascal like triangle for fixed values of m and n , we obtain the following corollaries.

Corollary 1. $J_2^1 = 4 \times J_1^1$

We prove the corollary combinatorially by establishing a bijection between a set Q_2^1 whose cardinality is J_2^1 and a set $\{0, 1\}_2 \times \{0, 1\}_1 \times Q_1^1$ whose cardinality is $4 \times J_1^1$. We denote by Q_n^m the set of barred preferential arrangements of a n -element set having m bar. We have, $Q_2^1 = \{|ab, ab|, a b|, b a|, |a b, |b a, a|b, b|a, \}$, $|Q_2^1| = J_2^1$ (by definition of Q_n^1). We say $Q_1^1 = \{|a, a|$. In proving the corollary we generalise the method used in [8] in proving theorem 2.1 from a single label of a bar to a double label of a bar, in the following way. We consider a map $f : \{0, 1\}_2 \times \{0, 1\}_1 \times Q_1^1 \rightarrow Q_2^1$. We construct the set $\{0, 1\}_2 \times \{0, 1\}_1 \times Q_1^1$ in the following way,

1. $\{0, 1\}_1$ labels one of the elements of Q_1^1 with a binary label 0 or 1 at the bottom. Hence from the two element of Q_1^1 there will result four labeled barred preferential arrangements. The set $\{0, 1\}_1 \times Q_1^1 = \{|a, |a, a|, a|$.

2. Then $\{0, 1\}_2$ labels at the top, the four elements of $\{0, 1\}_1 \times Q_1^1$ to form eight elements of $\{0, 1\}_2 \times \{0, 1\}_1 \times Q_1^1$ in the following way,

$\{0, 1\}_2 \times \{0, 1\}_1 \times Q_1^1 = \left\{ \begin{matrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ |a, a|, & |a, a|, & |a, a|, & |a, a| \end{matrix} \right\}$ We now define a mapping from $\{0, 1\}_2 \times \{0, 1\}_1 \times Q_1^1$ to Q_2^1 in the following way,

I. If the indexes on the bar are both 0, then for a barred preferential arrangement, by introducing a second element b , place a and b as a single block on the side of the bar where a was (in all cases remove the indices on the bars). An example of such an arrangement is $|ab$.

II. If the top index on the bar is 0 and the bottom index on the bar is 1, then remove the indexing on the bar and place b on opposite side of the bar from a in obtaining an element of Q_2^1 . An example of such an arrangement is $a|b$.

III. If the top index on the bar is 1 and the bottom index is 0, then place the two elements on the same side of the bar but placing the element b on its on block immediately before the bar then following by the block of the element a . An example of such an arrangement is, $a b|$.

IV. If both index are equal to 1, then place both elements on separate block on the same side of the bar, where now as opposed to 3 above the immediate block before the bar will be that of a then followed by the block of b . an example of such an arrangement is $|a \ b$.

From the way we have constructed the function f ,

two elements of $\{0, 1\}_2 \times \{0, 1\}_1 \times Q_1^1$ satisfy each of the four conditions above and each element of $\{0, 1\}_2 \times \{0, 1\}_1 \times Q_1^1$ is mapped to one element of Q_2^1 . Hence the mapping is 1-1.

Clear by marking the position of the elements a and b in an element $\mathfrak{k} \in Q_2^1$, we can identify an element of $\{0, 1\}_2 \times \{0, 1\}_1 \times Q_1^1$ which is mapped to \mathfrak{k} . Hence the map is onto. So $|\{0, 1\}_2 \times \{0, 1\}_1 \times Q_1^1| = |Q_2^1| \Rightarrow 2 \times 2 \times J_1^1 = J_2^1$ as required.

Corollary 2. *for $m = 1$ and $n = 1$ the identity holds,*

$$J_1^1 = J_0^1 + J_1^0$$

We prove the corollary by considering a mapping $f : Q_1^1 \rightarrow Q_0^1 \cup Q_1^0$, where Q_n^m is the set of all barred preferential arrangements of an $n \geq 1$ element set having $m \geq 0$ bars. The set $Q_1^1 = \{|a, a|\}$, also $Q_0^1 \cup Q_1^0 = \{|\, a\}$. We define a mapping from Q_1^1 in the following way,

I. If a is before the bar on $\mathfrak{k} \in Q_1^1$ then map \mathfrak{k} to a in $Q_0^1 \cup Q_1^0$ (this operation behaves as if you remove the bar from \mathfrak{k} , when mapping \mathfrak{k} to an element of $Q_0^1 \cup Q_1^0$).

II. If a is after the bar on $\mathfrak{k} \in Q_1^1$ then map \mathfrak{k} to $|\, a$ in $Q_0^1 \cup Q_1^0$ (this operation behaves as if a is removed from \mathfrak{k} when mapping \mathfrak{k} to an element of $Q_0^1 \cup Q_1^0$). Clearly this mapping is a one to one and onto mapping between the two sets of two elements each. So $|Q_1^1| = |Q_0^1 \cup Q_1^0|$. Hence $J_1^1 = J_0^1 + J_1^0$ ($|Q_0^1 \cup Q_1^0| = J_0^1 + J_1^0$ this is so, since the two sets are disjoint). Thus the corollary.

Corollary 3. *for $m = 1$ and $n = 2$ the identity holds,*

$$J_2^1 = 3 \times J_1^1 + 2 \times J_0^1$$

We prove the corollary by establishing a bijection between a set Q_2^1 whose cardinality is J_2^1 and a set $\{0, 1, 2\} \times Q_1^1 \cup \{0, 1\} \times Q_0^1$, whose cardinality is $3 \times J_1^1 + 2 \times J_0^1$. Where Q_n^m denotes the set of all barred preferential arrangements of an $n \geq 1$ element set having $m \geq 0$ bars. We say, $Q_1^1 = \{|a, a|\}$. We then construct the

set $\{0, 1, 2\} \times Q_1^1$ in the following way, $\{0, 1, 2\} \times Q_1^1 = \{a|, |a, a|, |a, a|, |a\}$, So the bar of each element of Q_1^1 is labeled by either 0, 1 or 2, in forming the set $\{0, 1, 2\} \times Q_1^1$. We construct the set $\{0, 1\} \times Q_0^1$ in a similar way to obtain $\{0, 1\} \times Q_0^1 = \{|\, |, |\}$. So we have, $\{0, 1, 2\} \times Q_1^1 \cup \{0, 1\} \times Q_0^1 = \{a|, |a, a|, |a, a|, |a, |\, |\}$. Also we have, $Q_2^1 = \{a|b, b|a, |a b, b a|, a b|, |b a, |ab, ab|\}$. We define a mapping $f : \{0, 1, 2\} \times Q_1^1 \cup \{0, 1\} \times Q_0^1 \rightarrow Q_2^1$ in the following way,

I. If $\mathfrak{X} \in \{0, 1, 2\} \times Q_1^1$ and the indexing on the bar being 0. Then place an element b on opposite side of a . An example of such an arrangement is $a|b$.

II. If $\mathfrak{X} \in \{0, 1, 2\} \times Q_1^1$ and the indexing on the bar being 1, then place an element b on it's own block on the same side as a but the block of a should be the one which is immediately before the bar. An example of such an arrangement is $|a b$.

III. If $\mathfrak{X} \in \{0, 1, 2\} \times Q_1^1$ and the indexing on the bar being 2. Then place an element b on its own block immediately before the bar, then followed by the block of a . An example of such an arrangement is $|b a$.

IV. If $\mathfrak{X} \in \{0, 1\} \times Q_0^1$ and the index on the bar is 0 then introduce write two elements a and b as a single block to the right of the bar. Hence that element will be mapped to the element $|ab$ in Q_2^1 .

V. If $\mathfrak{X} \in \{0, 1\} \times Q_0^1$ and the index on the bar is 1 then introduce write two elements a and b as a single block to the left of the bar. Hence that element will be mapped to the element $ab|$ in Q_2^1 .

From the way we have defined f above, the 1st and the 2nd elements of Q_2^1 , satisfy property I. The 3rd and the 4th element of Q_2^1 satisfy property II. The 5th and the 6th elements of Q_2^1 satisfy property III. The 7th element of Q_2^1 satisfy property IV above. The 8th element of Q_2^1 satisfy property V above. From the way we have defined f above, each element of $\{0, 1, 2\} \times Q_1^1 \cup \{0, 1\} \times Q_0^1$ is mapped to a unique element of Q_2^1 . Also using f above, we can move from elements of Q_2^1 to elements of $\{0, 1, 2\} \times Q_1^1 \cup \{0, 1\} \times Q_0^1$ with out ambiguity. Hence the mapping is bijective. Thus $|\{0, 1, 2\} \times Q_1^1 \cup \{0, 1\} \times Q_0^1| = |Q_2^1|$ i.e $3 \times J_1^1 + 2 \times J_0^1 = J_2^1$ as required (note the sets $\{0, 1, 2\} \times Q_1^1$ and the set $\{0, 1\} \times Q_0^1$ are disjoint hence order of their union is the sum of the orders).

Theorem 2. for $m, n \geq 1$,

$$J_n^k = J_n^{k-1} + \sum_{s=0}^{n-1} \binom{n}{s} J_s^{k-1} J_{n-s}^0$$

We recall J_n^k is the cardinality of a set Q_n^k which is the collection of all possible barred preferential arrangements of an n -element having k bars. Each barred preferential arrangement in Q_n^k has $k + 1$ section since there are k bars. We prove the theorem by partitioning Q_n^k into disjoint subsets μ_1 and μ_2 . Where μ_1 is the collection of all those elements from Q_n^k whose first section is empty and μ_2 is the collection of all those elements from Q_n^k whose first section is non-empty. In finding the number of element in μ_1 we argue as follows: the first section of $\Phi \in Q_n^k$ being empty means the n elements are distributed among the k other sections. A distribution of n elements among k sections is a distribution of the elements among $k - 1$ bars. As a result $|\mu_1| = J_n^{k-1}$.

For the case the first section of each element of Q_n^k being required to be non-empty we argue as follows in finding the number of elements of μ_2 : There can be a minimum of 0 elements not in the first section and there can be a maximum of $n - 1$ elements which are not in the first section on all barred preferential arrangements which are to be in μ_2 . Lets assume there are s elements which are not in the first section of each barred preferential arrangement. There are $\binom{n}{s}$ ways of selecting the s elements. There are J_s^{k-1} ways of preferentially arranging the s elements among the other k sections. There $n - s$ remaining elements can be preferentially arranged on the first section in J_{n-s}^0 ways. Taking the product and summing over s we obtain $\mu_2 = \sum_{s=0}^{n-1} \binom{n}{s} J_s^{k-1} J_{n-s}^0$. When we combine μ_1 and μ_2 we obtain the result of the theorem.

5. IDENTITIES OF RESTRICTED BARRED PREFERENTIAL ARRANGEMENTS

We denote by H_n^m the set of all barred preferential arrangements of an n element set having m bars, in which m fixed sections (out of $m + 1$ sections) are allowed to have a maximum of only one block and the other section can have one or more blocks (we refer to this section as the free section). The elements of the free section can have any possible number of blocks of a preferential arrangement of given $r \geq 0$ elements. So H_n^m represents the set of all barred preferential arrangements of an n element set having m bars in which the m fixed (first) sections have a maximum of one block but one fixed section can have more than one block. The free section can be any of the section as long its fixed. We denote by $|H_n^m| = I_n^m$.

We prove the identity in the following way,

Lemma 2. *for $m = 0$ and $n \geq 0$ we have,*

$$2I_n^0 = I_n^1$$

We consider the set H_n^0 which is the set of all barred preferential arrangements of an n element set with no bars. Hence all elements of H_n^0 have only one section since there are no bars. We also consider the set H_n^1 , which is the set of all barred preferential arrangement of an n element set having one bar, in which the first section has a at most one block and the second section is just a preferential arrangement of given elements not necessarily into one block.

Adding an extra bar $|$ to the far right of each element of the set H_n^0 does not affect counting. We add an extra bar $|$ to the far right of each element of the set H_n^0 to form the set $*\mathfrak{A}_n^0$, with the extra bar on the far right of each element. In proving the lemma, use a similar method to the one used in [8] in proving theorem 2.1 . We have $|*\mathfrak{A}_n^0| = I_n^0 = |H_n^0|$. We define the set $R_n^*(0) = \{0, 1\} \times *\mathfrak{A}_n^0$ as containing the same elements as $*\mathfrak{A}_n^0$ but with an indexing on the bar $|$ which is either 0 or 1, hence $R_n^*(0)$ has twice the number of elements as $*\mathfrak{A}_n^0$ (half of them having the index 0 and the other half the index 1). We now use the elements of $R_n^*(0)$ to reconstruct the set H_n^1 . We do it as follows:

I. If the indexing on the bar $|$ on an element $\mathfrak{k} \in R_n^*(0)$ is 0 then such an element will be interpreted in H_n^1 , as an element of H_n^1 whose 2^{nd} section is empty. We collect all such elements to form the set W . The set W has I_n^0 elements (half of the elements in $R_n^*(0)$ have the index 0) .

II If the indexing on the bar $|$ on an element $\mathfrak{k} \in R_n^*(0)$ is 1 then we shift the last block of section 1 to be the only block to the right of the bar $|$ to form the set K . The set K has I_n^0 elements (half of the elements in $R_n^*(0)$ have the indexing 1)

Clearly the sets K and W are disjoint. So we have $|K \cup W| = |K| + |W| \Rightarrow |K \cup W| = I_n^0 + I_n^0$. Now what we are having is two sets $K \cup W$ and H_n^1 have the same definition (one fixed section can have a maximum of one block and the other section is a free section) and the same number of elements in the underlying sets, So they must be of the same size. Hence $|K \cup W| = I_n^1 + I_n^1 = I_n^1 = |H_n^1|$. Thus the

lemma.

For the value $m = 1$, we have the following lemma,

Lemma 3. *for $m = 1$ and $n \geq 0$ we have,*

$$2I_n^1 - 1 = I_n^2$$

We consider a set H_n^1 , which is the set of all barred preferential arrangements of an n element set having 1 bar, in which the first section has a maximum of one block each and the 2^{nd} section to be just a preferential arrangement of given elements not necessarily into one block (this is the only section with this property). So the 2^{nd} section of each element of H_n^1 is a free section. We also consider the set H_n^2 , which is the set of all barred preferential arrangements of an n -element set, having 2 bars in which 2 fixed sections have a maximum of one block each and one section can have more than one block (the free section).

We now want to reconstruct H_n^2 using elements of H_n^1 . Adding an extra bar $\bar{}$ to the far right of each element of H_n^1 does not affect counting. We do that to form the set $*\bar{\mathfrak{A}}_n^1$. Now we have $|*\bar{\mathfrak{A}}_n^1| = I_n^1 = |H_n^1|$. The bar $\bar{}$ is to the right of the free sections of elements of H_n^1 . On each element $\mathfrak{A} \in *\bar{\mathfrak{A}}_n^1$ to the left of the bar $\bar{}$ is the free section of \mathfrak{A} and the section to the right of $\bar{}$ is empty.

We define the set $R_n^*(1) = \{0, 1\} \times *\bar{\mathfrak{A}}_n^1$ as containing the same elements as $*\bar{\mathfrak{A}}_n^1$ but with an indexing on the bar $\bar{}$ which is either 0 or 1, hence $R_n^*(1)$ has twice the number of elements as $*\bar{\mathfrak{A}}_n^1$ (of which half of them having the indexing 0 and the other half having the index 1). We now use elements of $R_n^*(1)$ to reconstruct the set H_n^2 . We construct as follows,

I. If the indexing on the bar $\bar{}$ on an element $\mathfrak{A} \in R_n^*(1)$ is 0 then such an element will be interpreted in H_n^2 , as an element of H_n^2 whose 3^{rd} section is empty. We collect all such elements to form the set W . The set W has I_n^1 elements (half of the elements in $R_n^*(1)$ have the indexing 0)

II If the indexing on the bar $\bar{}$ on an element $\mathfrak{A} \in R_n^*(1)$ is 1 then we shift the last block of the 3^{rd} section of \mathfrak{A} to be the only block to the right of $\bar{}$ (i.e the block closest to the bar $\bar{}$ on \mathfrak{A}) to form the set K . There are I_n^1 elements having index 1 in $R_n^*(1)$ (half of the elements in $R_n^*(1)$ have the indexing 1). On the construct of the set K some elements of K also appear as elements of the set W . Those

redundant elements (elements having identical interpretation in H_n^2) occur when there free section on elements of $R_n^*(1)$ is empty. In that case there is no block to put to the right of the bar $|^*$ when constructing the set K . In that case we get common elements between K and W . Common elements between K and W occur when the n elements are distributed on the first m sections of H_n^1 of which there is only one way of doing that. So there is one element in the intersection of K and W (when elements of K and W are interpreted as elements of).

The elements of both sets K and W have 3 sections, of which for those with a fixed $\mathfrak{k} \in K \cup W$, 2 sections of \mathfrak{k} have a maximum of one block and one section is a free section. We have, $|K \cup W| = |K| + |W| - |K \cap W| \Rightarrow |K \cup W| = I_n^1 + I_n^1 - 1$. Now what we are having is two sets $K \cup W$ and H_n^2 have the same definition and the same number of elements in the underlying sets, hence must be of the same size. That is, $|K \cup W| = I_n^1 + I_n^1 - 1 = I_n^2 = |H_n^2|$. Thus the lemma.

The case $m = 2$ differs from the above two cases in the sense that a non-trivial constant term arises in the form of 2^n . Therefore we state and prove for the value $m = 2$, the following lemma,

Lemma 4. *for $m = 2$ and $n \geq 0$ we have,*

$$2I_n^2 - 2^n = I_n^3$$

We consider a set H_n^2 , which is the set of all barred preferential arrangements of an n element set having 2 bars, in which the first 2 sections have a maximum of one block each and the 3^{rd} section to be just a preferential arrangement of given elements not necessarily into one block (this is the only section with this property). So the 3^{rd} section of each element of H_n^2 is the free section. We also consider the set H_n^3 , which is the set of all barred preferential arrangements of an n element set, having 3 bars in which 3 fixed sections have a maximum of one block each and one section can have more than one block (the free section).

We now want to reconstruct H_n^3 using elements of H_n^2 . Adding an extra bar $|^*$ to the far right of each element of H_n^2 does not affect counting. We do that to form the set $*\mathfrak{H}_n^2$. Now we have $|*\mathfrak{H}_n^2| = I_n^2 = |H_n^2|$. Where the bar $|^*$ is to the right of the free sections of elements of H_n^2 . On each element $\mathfrak{k} \in *\mathfrak{H}_n^2$ to the left of the bar $|^*$ is the free section of \mathfrak{k} and the section to the right of $|^*$ is empty.

We define the set $R_n^*(2) = \{0, 1\} \times * \mathfrak{A}_n^2$ as containing the same elements as $* \mathfrak{A}_n^2$ but with an indexing on the bar $|$ which is either 0 or 1, hence $R_n^*(2)$ has twice the number of elements as $* \mathfrak{A}_n^2$ (of which half of them having the indexing 0 and the other half having the index 1). We now use elements of $R_n^*(2)$ to reconstruct the set H_n^3 . We construct as follows,

I. If the indexing on the bar $|$ on an element $\mathfrak{k} \in R_n^*(2)$ is 0 then such an element will be interpreted in H_n^3 , as an element of H_n^3 whose 4th section is empty. We collect all such elements to form the set W . The set W has I_n^2 elements (half of the elements in $R_n^*(2)$ have the indexing 0)

II If the indexing on the bar $|$ on an element $\mathfrak{k} \in R_n^*(2)$ is 1 then we shift the last block of the 3rd section of \mathfrak{k} to be the only block to the right of $|$ (i.e the block closest to the bar $|$ on \mathfrak{k}) to form the set K . There are I_n^2 elements having index 1 in $R_n^*(2)$ (half of the elements in $R_n^*(2)$ have the indexing 1). On the construct of the set K some elements of K also appear as elements of the set W . Those redundant elements occur when there free section on elements of $R_n^*(2)$ is empty. In that case there is no block to put to the right of the bar $|$ when constructing the set K . In that case we get common elements between K and W . Common elements between K and W occur when the n elements are distributed on the first m sections of H_n^2 of which there are 2^n ways of doing that. So the number of elements in the intersection of K and W is 2^n .

The elements both sets K and W have 4 sections. Of which for a fixed $\mathfrak{k} \in K \cup W$, 3 sections of \mathfrak{k} have a maximum of one block and one section is a free section. We have, $|K \cup W| = |K| + |W| - |K \cap W| \Rightarrow |K \cup W| = I_n^2 + I_n^2 - 2^n$. Now what we are having is two sets $K \cup W$ and H_n^3 have the same definition and the same number of elements in the underlying sets, So they must be of the same size. Hence $|K \cup W| = I_n^2 + I_n^2 - 2^n = I_n^3 = |H_n^3|$. Thus the lemma.

We generalise lemmas 2,3 and 4 into the following theorem,

Theorem 3. *for $m, n \geq 0$, we have*

$$2I_n^m - m^n = I_n^{m+1}$$

We consider a set H_n^m , which is the set of all barred preferential arrangements of an n element set having m bars, in which the first m sections have a maximum of one block each and the $(m + 1)^{th}$ section to be just a preferential arrangement

of given elements not necessarily into one block(this is the only section with this property). So the $(m + 1)^{th}$ section of each element of H_n^m is the free section. We also consider the set H_n^{m+1} , which is the set of all barred preferential arrangements of an n element set, having $m+1$ bars in which $m+1$ fixed sections have a maximum of one block each and one section can have more than one block(the free section).

We now want to reconstruct H_n^{m+1} using elements of H_n^m . Adding an extra bar $\bar{\quad}^*$ to the far right of each element of H_n^m does not affect counting. We do that to form the set $^*\mathfrak{P}_n^m$. Now we have $|^*\mathfrak{P}_n^m| = I_n^m = |H_n^m|$. Where the bar $\bar{\quad}^*$ is to the right of the free sections of elements of H_n^m . On each element $\mathfrak{F} \in ^*\mathfrak{P}_n^m$ to the left of the bar $\bar{\quad}^*$ is the free section of \mathfrak{F} and the section to the right of $\bar{\quad}^*$ is empty.

We define the set $R_n^*(m) = \{0, 1\} \times ^*\mathfrak{P}_n^m$ as containing the same elements as $^*\mathfrak{P}_n^m$ but with an indexing on the bar $\bar{\quad}^*$ which is either 0 or 1, hence $R_n^*(m)$ has twice the number of elements as $^*\mathfrak{P}_n^m$ (of which half of them having the indexing 0 and the other half having the index 1). We now use elements of $R_n^*(m)$ to reconstruct the set H_n^{m+1} . We construct as follows,

I.If the indexing on the bar $\bar{\quad}^*$ on an element $\mathfrak{F} \in R_n^*(m)$ is 0 then such an element will be interpreted in H_n^{m+1} , as an element of H_n^{m+1} whose $(m + 2)^{th}$ section is empty. We collect all such elements to form the set W . The set W has I_n^m elements(half of the elements in $R_n^*(m)$ have the indexing 0)

II If the indexing on the bar $\bar{\quad}^*$ on an element $\mathfrak{F} \in R_n^*(m)$ is 1 then we shift the last block of the $(m + 1)^{th}$ section of \mathfrak{F} to be the only block to the right of $\bar{\quad}^*$ (i.e the block closest to the bar $\bar{\quad}^*$ on \mathfrak{F}) to form the set K . There are I_n^m elements having index 1 in $R_n^*(m)$ (half of the elements in $R_n^*(m)$ have the indexing 1). On the construct of the set K some elements of K also appear as elements of the set W . Those redundant elements occur when there free section on elements of $R_n^*(m)$ is empty. In that case there is no block to put to the right of the bar $\bar{\quad}^*$ when constructing the set K . In that case we get common elements between K and W . Common elements between K and W occur when the n elements are distributed on the first m sections of H_n^m of which there are m^n ways of doing that. So the number of elements in the intersection of K and W is m^n .

The elements both sets K and W have $m + 2$ sections. Of which for a fixed $\mathfrak{F} \in K \cup W$, $m + 1$ sections of \mathfrak{F} have a maximum of one block and one section is a

free section. We have, $|K \cup W| = |K| + |W| - |K \cap W| \Rightarrow |K \cup W| = I_n^m + I_n^m - m^n$. Now what we are having is two sets $K \cup W$ and H_n^{m+1} have the same definition and the same number of elements in the underlying sets, So they must be of the same size. Hence $|K \cup W| = I_n^m + I_n^m - m^n = I_n^{m+1} = |H_n^{m+1}|$. Thus the theorem.

6. ACKNOWLEDGMENTS

Both authors acknowledge the support from Rhodes University. The first author would like also to acknowledge financial support from the DAAD-NRF scholarship of South Africa, the Levenstein Bursary of Rhodes University and the NRF-Innovation doctoral scholarship of South Africa.

REFERENCES

- [1] A. Cayley, *LVIII. On the analytical forms called trees.Part II*, The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science 18,121 (1859): 374-378.
- [2] L. Comtet, *Advanced Combinatorics*, Reidel, 1974, p. 294.
- [3] O. A. Gross, *Preferential arrangements*, Amer. Math. Monthly, 69 (1962) 4-8.
- [4] Mendelson Elliott, *Races with Ties*, Mathematics Magazine, Vol.55, No.3(May,1982), 170-175.
- [5] V. Murali, *Ordered Partitions and Finite Fuzzy Sets*, Far East J. Math.Sci.(FJMS), 21(2) (2006), 121-132.
- [6] <https://oeis.org/>, The OEIS Foundation, 1964.
- [7] Pippenger Nicholas, *The Hypercube of resistors, Asymptotic Expansions and Preferential Arrangements*, The American mathematical monthly,83(2010) pp.331-346.
- [8] C. Ahlbach, J. Usatine, and N. Pippenger. *Barred Preferential Arrangement*, Electronic Journal of Combinatorics, 20:2 (2013), 1-18