EFFECTIVE RESISTANCES, KIRCHHOFF INDEX AND ADMISSIBLE INVARIANTS OF LADDER GRAPHS

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ABSTRACT. We explicitly compute the effective resistances between any two vertices of a ladder graph by using circuit reductions. Using our findings, we obtain explicit formulas for Kirchhoff index and admissible invariants of a ladder graph considering it as a model of a metrized graph. Comparing our formula for Kirchhoff index and previous results in literature, we obtain an explicit sum formula involving trigonometric functions. We also expressed our formulas in terms of certain generalized Fibonacci numbers that are the values of the Chebyshev polynomials of the second kind at 2.

1. INTRODUCTION

A ladder graph L_n is a planar graph that looks like a ladder with n rungs as shown in Figure 1. It has $2n$ vertices and $3n-2$ edges. Each of its edges has length 1, so the total length of L_n is $\ell(L_n) := 3n-2$. We label the vertices on the right and left as $\{q_1, q_2, \cdots, q_n\}$ and $\{p_1, p_2, \cdots, p_n\}$, respectively.

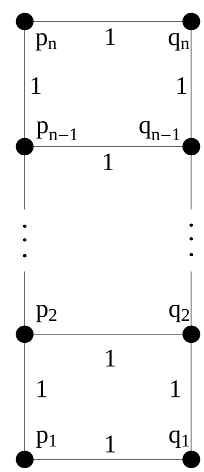

FIGURE 1. Ladder graph L_n with $2n$ vertices.

One can consider L_n as an electrical network in which the resistances along edges are given by the corresponding edge lengths. For the ladder graph L_n, Kirchhoff index and resistance values between vertices are studied in [3] by using the spectral properties of the discrete Laplacian of L_n, and closed form formulas are obtained in terms of Chebyshev polynomials.

In this paper, we obtained explicit formulas for Kirchhoff index and resistances between vertices of L_n with a rather elementary method. Namely, we used circuit reductions and solved a number of recurrence relations. Moreover, by considering L_n as a model of a metrized graph, we derived explicit formulas for its admissible invariants considered in [4], [5], [6], [15] and the references therein. At the end, we expressed these formulas in terms of a sequence of generalized Fibonacci numbers G_n defined by $G_{n+2} = 4G_{n+1} - G_n$ if $n \geq 2$, $G_1 = 1$ and $G_0 = 0$. The number G_n is known to be the number of spanning trees in L_n, and that $G_n = U_{n-1}(2)$, where $U_n(x)$ is the Chebyshev polynomial of the second kind.

Among other things, we showed that the Kirchhoff index of L_n satisfies the following equalities for each positive integer n (see Theorem 3.1 and Equation (23) below):

$$
\begin{aligned}
Kf(L_n) &= \frac{n^3}{3} + \frac{n^2 G_{2n}}{6G_n^2} \\
&= \frac{n^3}{3} - \frac{n^2}{\sqrt{3}}\Big[1 - \frac{2}{1-(2-\sqrt{3})^{2n}}\Big].
\end{aligned}
$$

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and we derived the following trigonometric sum formulas (see Equation (23) and Equation (25) below):

$$\sum_{k=0}^{n-1} \frac{1}{1+2\sin^2\left(\frac{k\pi}{2n}\right)} = \frac{1}{3} + \frac{nG_{2n}}{6G_n^2} \qquad \text{and} \qquad \sum_{k=1}^{n-1} \frac{1}{\sin^2\left(\frac{k\pi}{2n}\right)} = \frac{2(n^2-1)}{3}.$$

The resistance values on Wheel and Fan graphs are expressed in terms of generalized Fibonacci numbers in [1]. Our findings for resistance values on a Ladder graph are analogues of those results on Wheel and Fan graphs.

2. Resistances between any pairs of vertices in L_n

Let $r(p,q)$ be the effective resistance between the vertices p and q in L_n. We also use the notation $r_{L_n}(p,q)$ for this value to emphasize the graph the resistance being computed in. In this section, we find explicit formula of $r(p,q)$ for every pair of vertices p and q of L_n. Using the symmetry of the graph L_n, for all $i,\, j \in \{1,\, 2,\, \cdots,n\}$ we have

$$r(p_i,p_j) = r(q_i,q_j), \quad \text{and} \quad r(p_i,q_j) = r(q_i,p_j). \tag{1}$$

First, we compute effective resistances between the end vertices p_1, p_n, q_1 and q_n. Set $x_n := r_{L_n}(p_n,p_1)$, $y_n := r_{L_n}(p_n,q_1)$ and $z_n := r_{L_n}(p_n,q_n)$.

Suppose we make circuit reduction of L_{n-1} with respect to the vertices p_{n-1} and q_{n-1}. Since we obtain L_n by adding the vertices p_n and q_n, and the three edges with end points $\{p_{n-1},p_n\}$, $\{p_n,q_n\}$ and $\{q_n,q_{n-1}\}$, we have the circuit reduction of L_n as shown in Figure 2. Now, using the parallel circuit reduction in this graph, we can express z_n in terms of z_{n-1}. This gives us the following recurrence relation:

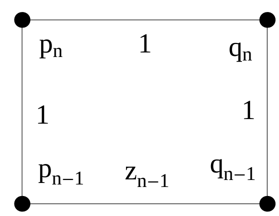

Figure 2. Ladder graph L_n with circuit reduction of L_{n-1} with respect to p_{n-1} and q_{n-1}, where $n \geq 2$.

$$\begin{aligned} z_n &= \frac{z_{n-1}+2}{z_{n-1}+3}, \quad \text{for all } n \geq 2. \\ z_1 &= 1. \end{aligned} \tag{2}$$

Now, we use Mathematica [14] to solve this recurrence relation. This gives

$$z_n = -1-\sqrt{3}+\frac{2\sqrt{3}}{1-(2-\sqrt{3})^{2n}}, \quad \text{for all } n \geq 1, \tag{3}$$

which indeed the solution of Equation (2). In particular, we have $z_1 = 1$, $z_2 = \frac{3}{4}$, $z_3 = \frac{11}{15}$, $z_4 = \frac{41}{56}$, $z_5 = \frac{153}{209}$, $z_6 = \frac{571}{780}$.

Other equivalent forms of z_n can be given as follows:

$$z_n = -1-\sqrt{3}+\frac{2\sqrt{3}(2+\sqrt{3})^n}{(2+\sqrt{3})^n-(2-\sqrt{3})^n}, \quad \text{or} \quad z_n = -1-\sqrt{3}\coth\big(n\ln(2-\sqrt{3})\big), \tag{4}$$

where coth is the hyperbolic cotangent function. Note that $(2-\sqrt{3})(2+\sqrt{3}) = 1$.

We can rewrite Equation (2) in the following form:

$$z_n = \frac{1}{1+\frac{1}{2+z_{n-1}}},$$

and if we use this equality to express z_{n-1} in terms of z_{n-2} and substitute it in this equality, we obtain

$$z_n = \frac{1}{1+\frac{1}{2+\frac{1}{1+\frac{1}{2+z_{n-2}}}}}.$$

We can repeat this process to express z_n in terms of z_k for any positive integer $k < n$. Since $0 < z_n < 1$ for each integer $n \geq 2$ and z_n is decreasing by Equation (2), we notice that z_n's must be part of the convergents of the number with continued fraction expansion $[0, 1, 2, 1, 2, 1, 2, \cdots]$. On the other hand, this is nothing but the every other terms in the continued fraction expansion of $\sqrt{3}-1$. Probabilistic explanation of these facts via spanning trees can be found in [10, page 11].

This kind of circuit reduction technique that we used to find z_n was used in the case of infinite ladder in [8, Chapter 22-Section 6].

Our next aim is to find explicit formulas for x_n and y_n as we did for z_n.

Now, suppose $n \geq 1$ and we make circuit reduction of the subgraph L_n of L_{n+1} with respect to the vertices p_n, q_n and p_1. That is, the part L_n in L_{n+1} is reduced to a Y-shaped graph with the outer vertices p_n, q_n and p_1, and having the effective resistances A, B and C between the end points of its edges. This is illustrated in Figure 3. Then we have $B + C = y_n$, $A + C = x_n$ and $A + B = z_n$. Solving these gives $A = \frac{x_n - y_n + z_n}{2}$, $B = \frac{-x_n + y_n + z_n}{2}$ and $C = \frac{x_n + y_n - z_n}{2}$. On the other hand, using parallel and series circuit reductions in Figure 3 we obtain $x_{n+1} = \frac{(A+1)(B+2)}{z_n+3} + C$ and $y_{n+1} = \frac{(B+1)(A+2)}{z_n+3} + C$. Therefore,

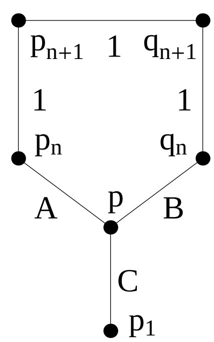

FIGURE 3. Ladder graph L_{n+1} with circuit reduction of L_n with respect to p_n, q_n and p_1, where $n \geq 1$.

$$\begin{aligned} x_{n+1} &= \frac{(x_n - y_n + z_n + 2)(-x_n + y_n + z_n + 4)}{4(z_n+3)} + \frac{x_n + y_n - z_n}{2}, \quad \text{if } n \geq 1. \\ y_{n+1} &= \frac{(-x_n + y_n + z_n + 2)(x_n - y_n + z_n + 4)}{4(z_n+3)} + \frac{x_n + y_n - z_n}{2}, \quad \text{if } n \geq 1. \\ x_1 &= 0 \quad \text{and} \quad y_1 = 1. \end{aligned} \tag{5}$$

If we subtract the second equation from the first one, we obtain $x_{n+1} - y_{n+1} = \frac{x_n - y_n}{z_n+3}$. Now, we set $t_n := x_n - y_n$ to obtain

$$t_{n+1} = \frac{t_n}{z_n+3}, \qquad \text{if } n \geq 1 \text{ and } t_1 = -1. \tag{6}$$

This can be rewritten as follows

$$t_{n+1} = -\prod_{k=1}^{n} \frac{1}{z_k+3}. \tag{7}$$

Since $\frac{1}{z_k+3}=\frac{\left(2+\sqrt{3}\right)^k-\left(2-\sqrt{3}\right)^k}{\left(2+\sqrt{3}\right)^{k+1}-\left(2-\sqrt{3}\right)^{k+1}}$ by using the first equality in (4) and doing some algebra, we see that the product in Equation (7) can be simplified. This gives

$$t_n=\frac{-2\sqrt{3}}{(2+\sqrt{3})^n-(2-\sqrt{3})^n}, \qquad \text{for every } n\geq 1, \tag{8}$$

which can also be written as $t_n=-\frac{2\sqrt{3}\left(2-\sqrt{3}\right)^n}{1-\left(2-\sqrt{3}\right)^{2n}}$ for all $n\geq 1$. Now, we turn our attention back to the solutions of x_n and y_n. Using $x_n=t_n+y_n$, Equation (3), Equation (8) and doing some algebra, the second equality in (5) becomes

$$y_{n+1}=y_n+\frac{\sqrt{3}}{1-\left(2-\sqrt{3}\right)^{n+1}}-\frac{\sqrt{3}}{1-\left(2-\sqrt{3}\right)^{n}}+\frac{1}{2}, \qquad \text{for all } n\geq 1 \text{ and } y_1=1. \tag{9}$$

This can be solved as follows:

$$y_n=\frac{n-2-\sqrt{3}}{2}+\frac{\sqrt{3}}{1-\left(2-\sqrt{3}\right)^{n}}, \qquad \text{for all } n\geq 1. \tag{10}$$

Using Equation (10), Equation (8) and the fact that $x_n=t_n+y_n$, we obtain

$$x_n=\frac{n-2-\sqrt{3}}{2}+\frac{\sqrt{3}}{1+\left(2-\sqrt{3}\right)^{n}}, \qquad \text{for all } n\geq 1. \tag{11}$$

Note that for all $n\geq 1$ we have

$$\begin{aligned} x_n+y_n-z_n&=n-1,\\ x_n-y_n+z_n&=-1-\sqrt{3}+\frac{2\sqrt{3}}{1+(2-\sqrt{3})^n},\\ -x_n+y_n+z_n&=-1-\sqrt{3}+\frac{2\sqrt{3}}{1-(2-\sqrt{3})^n}. \end{aligned} \tag{12}$$

Next, we obtain formulas for $r_{L_n}(p_n,p_i)$, $r_{L_n}(p_n,q_i)$ and $r_{L_n}(p_i,q_i)$, where $n>i>1$. We can consider L_n as the union of three graphs; the upper part of p_{i+1} and q_{i+1}, the lower part of p_i and q_i, and the middle part consisting of p_{i+1}, q_{i+1}, p_i and q_i. These graphs are illustrated in Figure 4. Note that the graphs in the upper and the lower parts are nothing but the graphs L_{n-i} and L_i, respectively. We make the circuit reduction of the upper part with respect to p_n, p_{i+1} and q_{i+1} to obtain a Y-shaped graph having the resistances M, N and K along its edges. We make the circuit reduction of the lower part with respect to p_i and q_i. The resistance between p_i and q_i in the lower part, $r_{L_i}(p_i,q_i)$, is z_i by definition. Now, we have

$$M+N=x_{n-i}, \quad M+K=y_{n-i}, \quad N+K=z_{n-i}. \tag{13}$$

Solving these for M, N and K, and using Equations (12) give

$$\begin{aligned} M&=\frac{x_{n-i}+y_{n-i}-z_{n-i}}{2}=\frac{n-i-1}{2},\\ N&=\frac{x_{n-i}-y_{n-i}+z_{n-i}}{2}=\frac{-1-\sqrt{3}}{2}+\frac{\sqrt{3}}{1+(2-\sqrt{3})^{n-i}},\\ K&=\frac{-x_{n-i}+y_{n-i}+z_{n-i}}{2}=\frac{-1-\sqrt{3}}{2}+\frac{\sqrt{3}}{1-(2-\sqrt{3})^{n-i}}. \end{aligned} \tag{14}$$

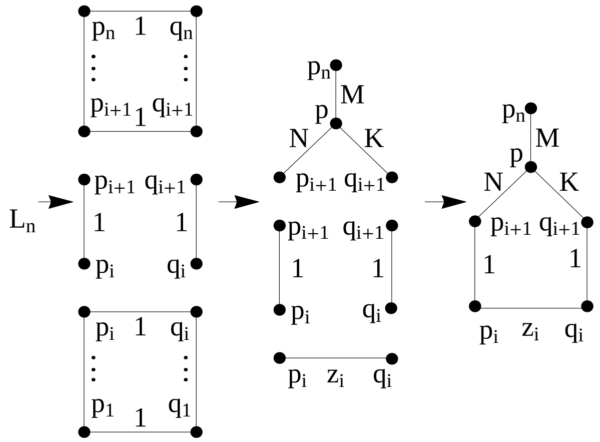

FIGURE 4. L_n and circuit reductions to find $r_{L_n}(p_n, p_i)$, $r_{L_n}(q_n, p_i)$ and $r_{L_n}(p_i, q_i)$.

By making parallel and series circuit reductions in the graph at the last column of Figure 4, for each i with $n > i > 1$, we obtain

$$\begin{aligned} r_{L_n}(p_n, p_i) &= \frac{(N+1)(K+z_i+1)}{z_{n-i}+z_i+2} + M, \\ r_{L_n}(p_n, q_i) &= \frac{(K+1)(N+z_i+1)}{z_{n-i}+z_i+2} + M, \qquad (15) \\ r_{L_n}(p_i, q_i) &= \frac{z_i(z_{n-i}+2)}{z_{n-i}+z_i+2}. \end{aligned}$$

We set

$$\alpha = 2 - \sqrt{3}.$$

Using Equation (3) and Equations (14), we can rewrite Equations in (15) as follows:

$$\begin{aligned} r_{L_n}(p_n, p_i) &= \frac{n-i}{2} + \frac{(1-\alpha^{n-i})}{4\sqrt{3}(1-\alpha^{2n})}\big(2 - 2\alpha^{n+i} - \alpha^{n+i-1} - \alpha^{n-i+1} + \alpha^{2i-1} + \alpha\big), \\ r_{L_n}(p_n, q_i) &= \frac{n-i}{2} + \frac{(1+\alpha^{n-i})}{4\sqrt{3}(1-\alpha^{2n})}\big(2 + 2\alpha^{n+i} + \alpha^{n+i-1} + \alpha^{n-i+1} + \alpha^{2i-1} + \alpha\big), \qquad (16) \\ r_{L_n}(p_i, q_i) &= \frac{(1+\alpha^{2n-2i+1})(1+\alpha^{2i-1})}{\sqrt{3}(1-\alpha^{2n})}. \end{aligned}$$

Although we obtained formulas in (16) under the condition $n > i > 1$, whenever $n = i$ or $i = 1$ these formulas are consistent with the ones given in Equations (3), (11) and (10). Therefore, formulas in (16) are valid for each integer n and i satisfying $n \geq i \geq 1$.

In the remaining part of this section, we obtain formulas for

$$r_{L_n}(p_i, q_j) \quad \text{and} \quad r_{L_n}(p_i, p_j), \quad \text{where } n > i \geq j \geq 1.$$

This time, we consider L_n as the union of two graphs; upper and lower parts of p_i and q_i as illustrated in the second stage in Figure 5. Note that the graph L_{n-i} appear in the upper part, and the lower part is nothing but L_i . Next, we can apply circuit reduction to reduce L_{n-i} into a line with the end points p_{i+1} and q_{i+1}, and this line has the resistance $r_{L_{n-i}}(p_{i+1}, q_{i+1}) = z_{n-i}$ between its end points. For the lower part, we apply circuit reduction to L_i fixing its points p_i, q_i and p_j so that we obtain a Y-shaped graph having the resistances D, E and F along its edges. These reductions are illustrated in the third

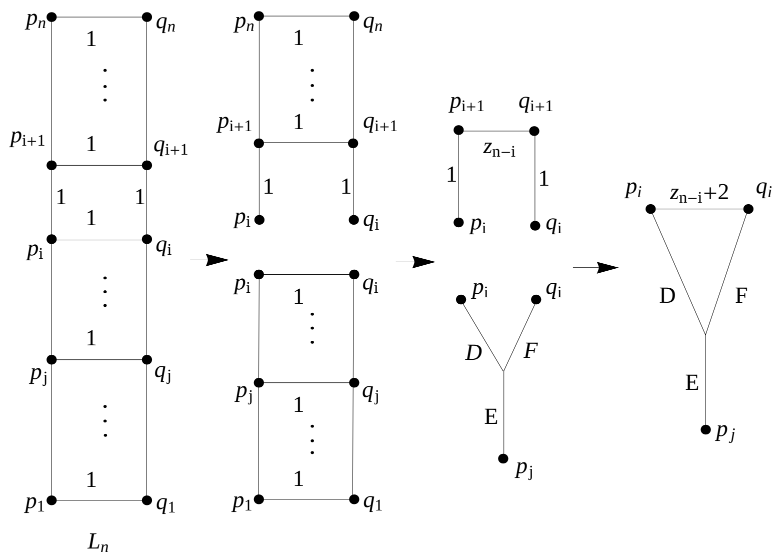

FIGURE 5. Circuit reductions applied to L_n to find $r_{L_n}(p_i,p_j)$ and $r_{L_n}(p_i,q_j)$.

stage in Figure 5, and the relations between D, E and F are given in Equations (17). Finally, we obtain the reduced graph as in the last stage in Figure 5.

$$D+E=r_{L_i}(p_i,p_j), \quad D+F=r_{L_i}(p_i,q_i)=z_i, \quad E+F=r_{L_i}(q_i,p_j). \tag{17}$$

Solving these for D, E and F gives

$$\begin{aligned} D &= \frac{r_{L_i}(p_i,p_j)+z_i-r_{L_i}(q_i,p_j)}{2}, \\ E &= \frac{r_{L_i}(p_i,p_j)-z_i+r_{L_i}(q_i,p_j)}{2}, \\ F &= \frac{-r_{L_i}(p_i,p_j)+z_i+r_{L_i}(q_i,p_j)}{2}. \end{aligned} \tag{18}$$

By making parallel and series circuit reductions in the graph at the last column of Figure 5, for each i with $n > i \geq j \geq 1$, we obtain

$$\begin{aligned} r_{L_n}(p_i,p_j) &= \frac{D(z_{n-i}+F+2)}{z_{n-i}+z_i+2}+E, \\ r_{L_n}(q_i,p_j) &= \frac{F(z_{n-i}+D+2)}{z_{n-i}+z_i+2}+E, \end{aligned} \tag{19}$$

Now, we use Equation (3), Equations (16), (18) and (19) and do some algebra using Mathematica [14] to derive the following resistance values:

(20)

$$r_{L_n}(p_i,p_j)=\frac{i-j}{2}+\frac{(1-\alpha^{i-j})}{4\sqrt{3}(1-\alpha^{2n})}\big(2-\alpha^{i+j-1}+\alpha^{2j-1}+\alpha^{2n-2i+1}(1-\alpha^{i-j}-2\alpha^{i+j-1})\big),$$

$$r_{L_n}(q_i,p_j)=\frac{i-j}{2}+\frac{(1+\alpha^{i-j})}{4\sqrt{3}(1-\alpha^{2n})}\big(2+\alpha^{i+j-1}+\alpha^{2j-1}+\alpha^{2n-2i+1}(1+\alpha^{i-j}+2\alpha^{i+j-1})\big).$$

In spite of the fact that we obtained formulas in (20) under the condition $n > i \geq j \geq 1$, when $n = i$ these formulas are consistent with the ones given in Equations (16). Therefore, formulas in (20) are valid for each integers i, j and n satisfying $n \geq i \geq j \geq 1$. That is, we can use the explicit formulas in (20) to find the resistances between any pair of vertices in L_n.

3. Kirchhoff Index of L_n

In this section, we obtain an explicit formula for Kirchhoff index of L_n by using our explicit formulas derived in §2 for the resistances between any pairs of vertices of L_n. Moreover, we obtain an interesting summation formula by combining our findings and what is known in the literature about Kirchhoff index of L_n.

Recall that Kirchhoff index of a graph Γ, $Kf(\Gamma)$, is defined [11] as follows:

$$Kf(\Gamma) = \frac{1}{2} \sum_{p,\, q \in V(\Gamma)} r(p, q).$$

Theorem 3.1. *For any positive integer n, we have*

$$Kf(L_n) = \frac{n^3}{3} - \frac{n^2}{\sqrt{3}} \Big[1 - \frac{2}{1 - (2 - \sqrt{3})^{2n}} \Big].$$

Proof. With the notation of vertices as in Figure 1, using Equation (1) gives

$$Kf(L_n) = \frac{1}{2} \sum_{p,\, q \in V(\Gamma)} r(p, q) = 2 \sum_{1 \leq j < i \leq n} r(p_i, p_j) + 2 \sum_{1 \leq j < i \leq n} r(p_i, q_j) + \sum_{i=1}^{n} r(p_i, q_i).$$

Then the result follows if we use Equations (20) and doing some algebra [14]. □

Note that the Kirchhoff index formula in Theorem 3.1 can also be expressed as follows:

$$Kf(L_n) = \frac{n^2}{3} \big[n - \sqrt{3} \coth \big(n \ln(2 - \sqrt{3}) \big) \big].$$

The values of $Kf(L_n)$ are rational numbers. For example, its values for $1 \leq n \leq 8$ are as follows: 1, 5, $\frac{71}{5}$, $\frac{214}{7}$, $\frac{11725}{209}$, $\frac{6031}{65}$, $\frac{415177}{2911}$, $\frac{140972}{679}$.

Theorem 3.2. *For any positive integer n, we have*

$$\sum_{k=0}^{n-1} \frac{1}{1 + 2\sin^2(\frac{\pi k}{2n})} = \frac{n}{\sqrt{3}} \Big[\frac{2}{1 - (2 - \sqrt{3})^{2n}} - 1 \Big] + \frac{1}{3}.$$

Proof. We recall the following result [13, Theorem 4.1] obtained by using the relation between the Kirchhoff index and the eigenvalues of the discrete Laplacian matrix of L_n.

$$Kf(L_n) = \frac{n(n^2 - 1)}{3} + n \sum_{k=0}^{n-1} \frac{1}{1 + 2\sin^2(\frac{\pi k}{2n})}. \tag{21}$$

Note that Equation (21) is also a particular case of [3, Corollary 12] (namely, when $c = 1$). Then the proof is completed by combining Equation (21) and the result in Theorem 3.1. □

Since $(2 - \sqrt{3})^2 \approx 0.071796$, for large values of n we have $Kf(L_n) \approx \frac{n^2(n+\sqrt{3})}{3}$ by Theorem 3.1.

Next, we give a geometric interpretation of the summation that appears in Equation (21). Let $P = \{0, \frac{\pi}{2n}, \frac{2\pi}{2n}, \frac{3\pi}{2n}, \cdots, \frac{(n-1)\pi}{2n}, \frac{n\pi}{2n} = \frac{\pi}{2}\}$ be a partition of the interval

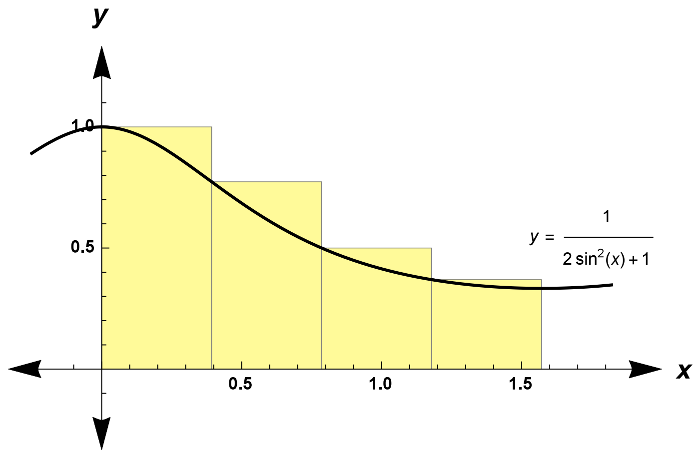

FIGURE 6. A Riemann sum approximation using left points.

$[0, \frac{\pi}{2}]$. Then the Riemann sum of $f(x) = \frac{1}{1+2\sin^2(x)}$ on $[0, \frac{\pi}{2}]$ that uses left points in each subinterval is nothing but

$$\frac{\pi}{2n}\sum_{k=0}^{n-1}\frac{1}{1+2\sin^2(\frac{\pi k}{2n})}.$$

Figure 6 illustrates the case with $n = 4$ subintervals. Note that

$$\lim_{n\to\infty}\frac{\pi}{2n}\sum_{k=0}^{n-1}\frac{1}{1+2\sin^2(\frac{\pi k}{2n})} = \int_0^{\frac{\pi}{2}}\frac{dx}{1+2\sin^2 x} = \frac{\pi}{2\sqrt{3}},$$

which is consistent with our findings in Theorem 3.2.

4. Admissible Invariants of L_n

In this section, we give explicit formulas for the following admissible invariants of L_n: $\tau(L_n)$, $\theta(L_n)$, $\lambda(L_n)$, $\varphi(L_n)$ and $\epsilon(L_n)$ when L_n is considered as a model of a metrized graph. These invariants were studied in [4], [5], [6], [15] and the references therein.

Theorem 4.1. *For any positive integer n, we have*

$$\theta(L_n) = \frac{2(n-2)}{3}\Big[n^2 - 4n + 10 - (n-6)\sqrt{3}(1 - \frac{2}{1-(2-\sqrt{3})^{2n}})\Big].$$

Proof. By definition [4, Section 4], $\theta(L_n) = \sum_{p,\,q\in V(L_n)}(\upsilon(p)-2)(\upsilon(q)-2)r(p,q)$, where $\upsilon(p)$ is the degree of the vertex p. Therefore,

$$\begin{aligned}2Kf(L_n) - \theta(L_n) &= 8\sum_{p\in V(L_n)} r(p,p_n) - 4\big[r(p_n,p_1) + r(p_n,q_1) + r(p_n,q_n)\big]\\ &= 8\sum_{i=2}^{n-1}\big(r(p_i,p_n) + r(q_i,p_n)\big) + 4\big[r(p_n,p_1) + r(p_n,q_1) + r(p_n,q_n)\big],\end{aligned}$$

where the vertices p_i and q_i with $i \in \{1, \ldots, n\}$ are as in Figure 1. Thus, the result follows by using this equality, Equations (20), Theorem 3.1 and doing some algebra [14]. ☐

Theorem 4.2. *For any positive integer n, we have*

$$\tau(L_n) = \frac{9n-20}{36} + \frac{n-6}{6\sqrt{3}}\Big[1 - \frac{2}{1-(2-\sqrt{3})^{2n}}\Big].$$

Proof. Let Γ be a graph with set of vertices $V(\Gamma)$ such that each edge length in Γ is 1. Suppose Γ is a model of a metrized graph. If we use [5, Proposition 2.6], [5, Equation (3)] and [5, Proof of Lemma 4.9], we obtain the following formula of the tau constant $\tau(\Gamma)$ of Γ for every $s \in V(\Gamma)$:

$$\tau(\Gamma) = \frac{1}{12} \sum_{\substack{p \sim q \\ p,\, q \in V(\Gamma)}} (1 - r(p,q))^2 + \frac{1}{4} \sum_{\substack{p \sim q \\ p,\, q \in V(\Gamma)}} \big(r(s,p) - r(s,q)\big)^2, \tag{22}$$

where $p \sim q$ means p and q are adjacent, i.e., connected by an edge in Γ.

Using the notations in Figure 1 and the symmetry in L_n, we can rewrite Equation (22) for L_n with $s = p_n$ as follows:

$$\tau(L_n) = \frac{1}{6}\sum_{i=1}^{n-1}(1 - r(p_i, p_{i+1}))^2 + \frac{1}{12}\sum_{i=1}^{n}(1 - r(p_i, q_i))^2 + \frac{1}{4}\sum_{i=1}^{n-1} \big(r(p_n, p_i) - r(p_n, p_{i+1})\big)^2$$
$$+ \frac{1}{4}\sum_{i=1}^{n} \big(r(p_n, p_i) - r(p_n, q_i)\big)^2 + \frac{1}{4}\sum_{i=1}^{n-1} \big(r(p_n, q_i) - r(p_n, q_{i+1})\big)^2.$$

Therefore, the proof follows if we use this equality, Equations (20) and doing some algebra [14]. ∎

Theorem 4.3. *For any positive integer n, we have*

$$\varphi(L_n) = \frac{3n^3 - 9n^2 - 5n + 1}{18(n-1)} + \frac{(n-6)(2n-1)}{6\sqrt{3}(n-1)}\Big[1 - \frac{2}{1 - (2-\sqrt{3})^{2n}}\Big],$$
$$\lambda(L_n) = \frac{n(n+4)(n-1)}{12(2n-1)},$$
$$\epsilon(L_n) = \frac{(3n^2 - 3n + 10)(n-2)}{9(n-1)} - \frac{(n-2)(n-6)}{3\sqrt{3}(n-1)}\Big[1 - \frac{2}{1 - (2-\sqrt{3})^{2n}}\Big],$$
$$Z(L_n) = \frac{(3n^2 - 13)n}{36(n-1)^2} + \frac{n(n-6)}{12\sqrt{3}(n-1)^2}\Big[1 - \frac{2}{1 - (2-\sqrt{3})^{2n}}\Big].$$

Proof. Since each of $\varphi(L_n)$, $\lambda(L_n)$, $\epsilon(L_n)$ and $Z(L_n)$ can be expressed in terms of $\tau(L_n)$, $\theta(L_n)$ and $\ell(L_n)$ [4, Propositions 4.6, 4.7, 4.8 and 4.9] with $g(L_n) = (3n-2) - (2n) + 1 = n - 1$, the results follow from Theorem 4.1 and Theorem 4.2. ∎

Note that our findings in this section are consistent with the numeric results given in [6, Table 5] for $n \in \{5,\ 10,\ 15,\ 20\}$.

Finally, we observe the following behavior of these invariants:

$$\lim_{n->\infty} \frac{\tau(L_n)}{\ell(L_n)} = \frac{1}{108}(9 - 2\sqrt{3}), \qquad \lim_{n->\infty} \frac{Z(L_n)}{\ell(L_n)} = \frac{1}{36},$$
$$\lim_{n->\infty} \frac{1}{g(L_n)}\frac{\varphi(L_n)}{\ell(L_n)} = \frac{1}{18}, \qquad \lim_{n->\infty} \frac{1}{g(L_n)}\frac{\epsilon(L_n)}{\ell(L_n)} = \frac{1}{9},$$
$$\lim_{n->\infty} \frac{1}{g(L_n)}\frac{\lambda(L_n)}{\ell(L_n)} = \frac{1}{72}, \qquad \lim_{n->\infty} \frac{1}{g^2(L_n)}\frac{\theta(L_n)}{\ell(L_n)} = \frac{2}{9}.$$

5. Connection to Generalized Fibonacci Numbers

We note that the powers of $2 - \sqrt{3}$ appear in the binet formula of certain generalized Fibonacci numbers [9]. Namely, for the sequence of integers G_n defined by the following recurrence relation

$$G_{n+2} = 4G_{n+1} - G_n, \quad \text{if } n \geq 2, \text{ and } G_0 = 0,\, G_1 = 1,$$

we have

$$G_n = \frac{(2-\sqrt{3})^{-n} - (2-\sqrt{3})^n}{2\sqrt{3}}, \quad \text{for each integer } n \geq 0.$$

The values of G_n with $0 \leq n \leq 10$ are as follows: 0, 1, 4, 15, 56, 209, 780, 2911, 10864, 40545, 151316.

Various properties of the sequence G_n are well-known in the literature [12]. For example, we recognize the number G_n as the number of spanning trees of L_n [2].

Since we have

$$(2-\sqrt{3})^n = \frac{1}{G_{n+1} - (2-\sqrt{3})G_n}, \quad \text{for each integer } n \geq 0$$

and

$$G_{2n} = G_n\big((2-\sqrt{3})^{-n} + (2-\sqrt{3})^n\big) = -2\sqrt{3}G_n^2 \coth\left(n \ln(2-\sqrt{3})\right),$$

we can rewrite our findings in the previous sections in terms of G_n. Namely, we obtained the following results in this paper:

For every integer $n \geq 1$,

$$t_n = -\frac{1}{G_n}, \qquad z_n = -1 + \frac{G_{2n}}{2G_n^2},$$

$$y_n = \frac{n-2}{2} + 3\frac{G_n^2}{G_{2n} - 2G_n}, \qquad x_n = \frac{n-2}{2} + \frac{G_{2n} - 2G_n}{4G_n^2}.$$

If we let $g_n := \frac{1}{G_{n+1} - (2-\sqrt{3})G_n}$, we can rewrite Equation (20) in the following form

$$\begin{aligned} r_{L_n}(p_i, p_j) = {} & \frac{i-j}{2} + \frac{1-g_{i-j}}{8\sqrt{3}}\Big(1 + \frac{G_{2n}}{2\sqrt{3}G_n^2}\Big)\Big[(1-g_{i+j-1})(1+g_{2n-2i+1}) \\ & + (1+g_{2j-1})(1-g_{2n-i-j+1})\Big] \\ r_{L_n}(q_i, p_j) = {} & \frac{i-j}{2} + \frac{1+g_{i-j}}{8\sqrt{3}}\Big(1 + \frac{G_{2n}}{2\sqrt{3}G_n^2}\Big)\Big[(1+g_{i+j-1})(1+g_{2n-2i+1}) \\ & + (1+g_{2j-1})(1+g_{2n-i-j+1})\Big], \end{aligned}$$

where $n \geq i \geq j \geq 1$.

Here is how we can express the results given in Theorem 3.1 and Theorem 3.2 in terms of G_n:

$$Kf(L_n) = \frac{n^3}{3} + \frac{n^2 G_{2n}}{6G_n^2}, \quad \text{and} \quad \sum_{k=0}^{n-1} \frac{1}{1 + 2\sin^2\left(\frac{k\pi}{2n}\right)} = \frac{1}{3} + \frac{nG_{2n}}{6G_n^2}. \tag{23}$$

If λ_1, λ_2, $\dots$, λ_m are nonzero eigenvalues of a connected graph Γ with m vertices, then $Kf(\Gamma) = m\sum_{i=1}^m \frac{1}{\lambda_i}$ ([7] and [16]). Since 2, $2 - 2\cos\left(\frac{k\pi}{n}\right) = 4\sin^2\left(\frac{k\pi}{2n}\right)$ and $4 - 2\cos\left(\frac{k\pi}{n}\right) = 2 + 4\sin^2\left(\frac{k\pi}{2n}\right)$ for $k = 1, 2, \dots, n-1$ are the nonzero eigenvalues of the discrete Laplacian matrix of L_n [2, Proof of Theorem 6], we have

$$Kf(L_n) = n + \frac{n}{2}\sum_{k=1}^{n-1} \frac{1}{\sin^2\left(\frac{k\pi}{2n}\right)} + n\sum_{k=1}^{n-1} \frac{1}{1 + 2\sin^2\left(\frac{k\pi}{2n}\right)}. \tag{24}$$

Then the following equality follows from Equations (23) and Equation (24),

$$\sum_{k=1}^{n-1} \frac{1}{\sin^2\left(\frac{k\pi}{2n}\right)} = \frac{2(n^2-1)}{3}. \tag{25}$$

Since the Chebyshev polynomial of the second kind $U_n(x)$ is given by the relation $U_{n+2}(x) = 2xU_{n-1}(x) - U_n(x)$ for $n \geq 0$ and the initial values $U_1(x) = 1$ and $U_0(x) = 0$, we have $G_n = U_{n-1}(2)$. That is, the formulas we found are nothing but expressions involving Chebyshev polynomials. Therefore, combining the formulas in Equation (23) with the ones given in [3, Corollary 12] (when $a = c = 1$), we obtain the following equality:

$$6U'_{n-1}(2) = n\frac{U_{2n-1}(2)}{U_{n-1}(2)} - 4U_{n-1}(2).$$

Next, we express the admissible invariants of L_n in terms of the numbers G_n:

$$\tau(L_n) = \frac{9n-20}{36} - \frac{(n-6)G_{2n}}{36G_n^2}, \quad \theta(L_n) = \frac{2(n-2)}{3}\Big[n^2 - 4n + 10 + \frac{(n-6)G_{2n}}{2G_n^2}\Big],$$

$$\lambda(L_n) = \frac{n(n+4)(n-1)}{12(2n-1)}, \qquad \varphi(L_n) = \frac{3n^3 - 9n^2 - 5n + 1}{18(n-1)} - \frac{(n-6)(2n-1)G_{2n}}{36(n-1)G_n^2},$$

and similarly we have

$$\epsilon(L_n) = \frac{(n-2)}{9(n-1)}\Big[3n^2 - 3n + 10 + \frac{(n-6)G_{2n}}{2G_n^2}\Big],$$

$$Z(L_n) = \frac{n}{36(n-1)^2}\Big[3n^2 - 13 - \frac{(n-6)G_{2n}}{2G_n^2}\Big].$$

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