SOME ASYMPTOTIC RESULTS ON q-BINOMIAL COEFFICIENTS

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ABSTRACT. We look at the asymptotic behavior of the coefficients of the q-binomial coefficients (or Gaussian polynomials) $\binom{a+k}{k}_q$, when k is fixed. We give a number of results in this direction, some of which involve Eulerian polynomials and their generalizations.

1. INTRODUCTION

The purpose of this note is to investigate the asymptotic behavior of the coefficients of the q-binomial coefficient (or Gaussian polynomial) $\binom{a+k}{k}_q$. While much of the previous work in this area has focused on the case where both a and k get arbitrarily large (see e.g. [11]), in this paper we will be concerned with asymptotic estimates for the coefficients of $\binom{a+k}{k}_q$ when k is fixed.

Besides the intrinsic relevance of studying the combinatorial, analytic or algebraic properties of q-binomial coefficients, our work is also motivated by a series of recent papers that have revived the interest in analyzing the behavior of the coefficients of $\binom{a+k}{k}_q$, as well as their applications to other mathematical areas. See for instance [7], where I. Pak and G. Panova have first shown algebraically the strict unimodality of $\binom{a+k}{k}_q$, as well as the subsequent combinatorial proofs of the Pak-Panova result by the second author of this paper [13] and by V. Dhand [3]. See also another interesting recent work by Pak and Panova [8] (as well as their extensive bibliography), where the coefficients of $\binom{a+k}{k}_q$ have been investigated in relation to questions of representation theory concerning the growth of Kronecker coefficients. Further, one of the results of this note, Theorem 2.2, has also been motivated by, and finds a first useful application in the study of the unimodality of partitions with distinct parts that are contained inside certain Ferrers diagrams (see our own paper [10]).

For $m = \lfloor ak/2 \rfloor$ (the middle exponent of $\binom{a+k}{k}_q$ when k or a are even, and the smaller of the two middle exponents otherwise), define $g_{k,c}(a)$ to be the coefficient of degree $m-c$ of $\binom{a+k}{k}_q$, and let $f_{k,c}(a) = g_{k,c}(a) - g_{k,c+1}(a)$. Our first main result is a description of the generating functions (in two variables, referring to a and c) of $g_{k,c}(a)$ and $f_{k,c}(a)$. In

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particular, it follows from our result that both $g_{k,c}(a)$ and $f_{k,c}(a)$ are quasipolynomials in a, for any given k and c.

Our next result, Theorem 2.4, is an asymptotic estimate of the coefficient of degree $\lfloor \alpha a \rfloor - c$ of $\binom{a+k}{k}_q$, when $a \to \infty$, for any given integer c, positive integer k, and nonnegative real number α. Quite surprisingly, this result connects in a nice fashion to Eulerian numbers and, more generally, to Euler-Frobenius numbers, as we will discuss extensively after the proof of the theorem.

Finally, our last main result, Theorem 2.6, presents an asymptotic estimate of the difference between consecutive coefficients of $\binom{a+k}{k}_q$, again for k fixed.

We will wrap up this note with a brief remark, in order to highlight an interesting connection of our last result with Kostka numbers and to present some suggestions for further research.

2. Some asymptotic properties of the coefficients of $\binom{a+k}{k}_q$

In this section, we study the asymptotic behavior of the coefficients of $\binom{a+k}{k}_q$ for fixed k. Given $k \geq 1$, $c \geq 0$, and $a \geq 0$, set $m = \lfloor ak/2 \rfloor$. Define

$$\begin{aligned} g_{k,c}(a) &= [q^{m-c}]\binom{a+k}{k}_q, \\ f_{k,c}(a) &= g_{k,c}(a) - g_{k,c+1}(a), \end{aligned} \tag{1}$$

where $[q^n]F(q)$ denotes the coefficient of q^n in the polynomial (or power series) $F(q)$.

Lemma 2.1. *Let $F(q) \in \mathbb{C}[[q]]$, and $c, j, i \in \mathbb{Z}$ with $j > i \geq 0$. We have:*

(a)

$$\sum_{a \geq 0} [q^{aj-c}] q^{ai} F(q) x^a = \frac{1}{j-i} \sum_{\zeta^{j-i}=1} (\zeta x)^c F(\zeta x) \Bigg|_{x \to x^{1/(j-i)}}$$

(b)

$$\sum_{a \geq 0} \sum_{c \geq 0} [q^{aj-c}] q^{ai} F(q) x^a t^c = \frac{1}{j-i} \sum_{\zeta^{j-i}=1} \frac{F(\zeta x)}{1 - \zeta x t} \Bigg|_{x \to x^{1/(j-i)}}.$$

Proof. For any $G(q) = \sum a_i q^i \in \mathbb{C}[[q]]$ and $h \geq 1$, write

$$D^h G(q) = \sum a_{hi} x^{hi},$$

the hth *dissection* of $G(q)$. It is an elementary and standard result (see e.g. [9, Exercise 1.60]) that

$$D^h G(q) = \frac{1}{h} \sum_{\zeta^h = 1} G(\zeta x).$$

(The sum is over all h complex numbers ζ satisfying $\zeta^h = 1$.) Hence (a) follows.

Part (b) is the generating function (in t) with respect to c of the formula of part (a). We have:

$$\begin{aligned}\sum_{a\geq 0}\sum_{c\geq 0}[q^{aj-c}]q^{ai}F(q)x^at^c &= \sum_{a\geq 0}\sum_{c\geq 0}[q^{a(j-i)}]q^{c}F(q)x^at^c \\ &= \sum_{a\geq 0}[q^{a(j-i)}]\frac{F(q)}{1-qt}x^a \\ &= \frac{1}{j-i}\sum_{\zeta^{j-i}=1}\frac{F(\zeta x)}{1-\zeta xt}\Bigg|_{x\to x^{1/(j-i)}},\end{aligned}$$

and the proof follows. □

From Lemma 2.1, it is easy to describe the form of the generating functions for $g_{k,c}(a)$ and $f_{k,c}(a)$, when k and c are fixed. For this purpose, define a *quasipolynomial* to be a function $h\colon \mathbb{N}\to\mathbb{C}$ (where $\mathbb{N}=\{0,1,2,\dots\}$) of the form

$$h(n) = c_d(n)n^d + c_{d-1}(n)n^{d-1} + \cdots + c_0(n),$$

where each $c_i(n)$ is a periodic function of n. If $c_d(n)\neq 0$ then we call d the *degree* of h. For more information on quasipolynomials, see for instance [9, §4.4].

Write

$$\begin{aligned}F_k(x,t) &= \sum_{a\geq 0}\sum_{c\geq 0} f_{k,c}(a)x^at^c \\ G_k(x,t) &= \sum_{a\geq 0}\sum_{c\geq 0} g_{k,c}(a)x^at^c.\end{aligned}$$

Theorem 2.2. *Fix $k\geq 1$ and set $j=\lfloor k/2\rfloor$. If we denote both F_k and G_k by H_k, then*

$$H_k(x,t) = \begin{cases} \dfrac{N_k(x,t)}{D_k(x)(1-tx)(1-t^2x)(1-t^3x)\cdots(1-t^jx)}, & k \text{ even} \\ \dfrac{N_k(x,t)}{D_k(x)(1-tx^2)(1-t^3x^2)(1-t^5x^2)\cdots(1-t^kx^2)}, & k \text{ odd.}\end{cases}$$

where $N_k(x,t)\in\mathbb{Z}[x,t]$ and $D_k(x)$ is a product of cyclotomic polynomials. In particular, for fixed k and c we have that $g_{k,c}(a)$ and $f_{k,c}(a)$ are quasipolynomials.

Proof. Case 1: $k=2j$. We have $m=\lfloor ak/2\rfloor = aj$. Write

$$(1-q^{a+1})(1-q^{a+2})\cdots(1-q^{a+k}) = \sum_{i=0}^{k}(-1)^iP_i(q)q^{ai}, \tag{2}$$

where $P_i(q)$ is a polynomial in q independent of a. Specifically, we have

$$P_i(q) = \sum_{\substack{S\subseteq[k]\\ \#S=i}} q^{\sum_{s\in S}s}. \tag{3}$$

Writing $[k]! = (1-q)(1-q^2)\cdots(1-q^k)$, we get

$$\begin{aligned} G_k(x,t) &= \sum_{a\geq 0}\sum_{c\geq 0}[q^{m-c}]\binom{a+k}{k}_q x^a t^c \\ &= \sum_{a\geq 0}\sum_{c\geq 0}[q^{aj-c}]\frac{1}{[k]!}\sum_{i=0}^{k}(-1)^i P_i(q) q^{ai} x^a t^c \\ &= \sum_{i=0}^{k}\sum_{a\geq 0}\sum_{c\geq 0}[q^{aj-c}]\frac{1}{[k]!}(-1)^i P_i(q) q^{ai} x^a t^c. \end{aligned}$$

The proof now follows from Lemma 2.1(a). Note in particular that the expression $F(\zeta x)$ in Lemma 2.1 will produce cyclotomic polynomials in the denominator of $G_k(x,t)$, while the denominator $1-\zeta xt$ in part (b) will lead to the factor $1-t^{j-i}x$ in the denominator of $G_k(x,t)$. The proof for $F_k(x,t)$ is completely analogous.

Case 2: $k = 2j+1$. The proof is analogous to Case 1. Now we have to look at $a = 2b$ and $a = 2b+1$ separately. When $a = 2b$ we get that the part of $G_k(x,t)$ with even exponent of x is $G_k(x,t) = \sum_{a\geq 0}\sum_{c\geq 0}[q^{bk-c}]\binom{2b+k}{k}x^{2b}t^c$. When we apply Lemma 2.1, the denominator term becomes $1-\zeta x^2 t$, where $\zeta^{j-i} = 1$ and $j-i$ is odd. This produces a factor $1-t^{j-i}x^2$ (where $j-i$ is odd) in the denominator of $G_k(x,t)$. Exactly the same reasoning applies to $a = 2b+1$, so the proof follows. □

Example 2.3. Write $\Phi_m(x)$ for the mth cyclotomic polynomial normalized to have constant term 1. Hence $\Phi_1(x) = 1-x$, $\Phi_2(x) = 1+x$, $\Phi_3(x) = 1+x+x^2$, etc.. One can compute the following:

$$\begin{aligned} F_3(x,t) &= \frac{1+tx+tx^3+t^3x^4}{(1-x)(1+x)(1+x^2)(1-tx^2)(1-t^3x^2)} \\ G_3(x,t) &= \frac{N_3(x,t)}{(1-x)^2(1-x^4)(1-tx^2)(1-t^3x^2)} \\ F_4(x,t) &= \frac{1-tx+t^2x^2}{(1-x^2)(1-x^3)(1-tx)(1-t^2x)} \\ G_4(x,t) &= \frac{1-x+(1+t)x^2-(t+t^2)x^3}{(1-x)^2(1-x^2)(1-x^3)(1-tx)(1-t^2x)} \\ F_5(x,0) &= \frac{1-x^5-x^6+x^7+x^{12}}{\Phi_1^3\Phi_2^3\Phi_3\Phi_4^2\Phi_6\Phi_8} \\ G_5(x,0) &= \frac{B_5(x)}{(1-x)^2(1-x^4)(1-x^6)(1-x^8)} \\ F_6(x,t) &= \frac{M_6(x,t)}{\Phi_1^4\,\Phi_2^2\,\Phi_3\,\Phi_4\,\Phi_5\,(1-tx)(1-t^2x)(1-t^3x)} \\ G_6(x,t) &= \frac{N_6(x,t)}{\Phi_1^6\,\Phi_2^3\,\Phi_3\,\Phi_4\,\Phi_5\,(1-tx)(1-t^2x)(1-t^3x)}, \end{aligned} \tag{4}$$

where

$$N_3(x,t) = 1-(1-t)x+(1-t+t^2)x^2+(t-t^2)x^3-(t-t^2)x^4-(t^2+t^3)x^5$$

$$B_5(x) = 1-x+2x^2+x^3+2x^4+3x^5+x^6+5x^7+x^8+3x^9+2x^{10}+x^{11}+2x^{12}$$
$$-x^{13}+x^{14}+3x^{10}+x^{12}-x^{13}+2x^{14}-x^{15}+x^{17}-2x^{18}+x^{19}$$

$$M_6(x,t) = 1+(1-t-t^2)x-(t-t^3-t^4)x^2-(1-t-t^2-t^3-t^4+t^5)x^3$$
$$-(1-2t-t^2+t^5)x^4-(1-2t-t^2+t^3+t^4)x^5+(t+t^2-t^3-2t^4+t^5)x^6$$
$$+(1-t^3-2t^4+t^5)x^7+(1-t-t^2-t^3-t^4+t^5)x^8-(t+t^2-t^4)x^9$$
$$+(t^3+t^4-t^5)x^{10}-t^5x^{11}$$

$$N_6(x,t) = 1+(1+2t+t^2)x^2+(3+2t-t^2-2t^3-t^4)x^3+(4-2t^2-3t^3-t^4+t^5)x^4$$
$$+(4-3t^2-4t^3-t^4+2t^5)x^5+(4-t-4t^2-4t^3-t^4+3t^5)x^6$$
$$+(3-t-5t^2-4t^3+3t^5)x^7+(1-t-4t^2-3t^3+t^4+4t^5)x^8$$
$$-(2t^2+t^3-t^4-3t^5)x^9+(1-t^2-t^3+3t^5)x^{10}$$
$$-(t+t^2+t^3-t^4-2t^5)x^{11}+(t^3+t^4+t^5)x^{12}.$$

The denominator of $F_8(x,t)$ is given by

$$\Phi_1^6\,\Phi_2^3\,\Phi_3^2\,\Phi_4\,\Phi_5\,\Phi_7\,(1-tx)(1-t^2x)(1-t^3x)(1-t^4x),$$

and that of $G_8(x,t)$ by

$$\Phi_1^8\,\Phi_2^3\,\Phi_3^2\,\Phi_4\,\Phi_5\,\Phi_7\,(1-tx)(1-t^2x)(1-t^3x)(1-t^4x).$$

Let us also note that

$$\begin{aligned} F_8(x,0) &= \sum_{a\geq 0}([q^{4a}]-[q^{4a-1}])\binom{a+8}{8}_q x^a \\ &= \frac{1+x-x^3-x^4+x^6+x^7+x^8+x^9+x^{10}-x^{12}-x^{13}+x^{15}+x^{16}}{(1+x)(1-x^2)(1-x^3)^2(1-x^4)(1-x^5)(1-x^7)} \\ &= 1+x^2+x^3+2x^4+2x^5+4x^6+4x^7+7x^8+8x^9+12x^{10}+\cdots. \end{aligned}$$

This generating function appears in a paper [5, p. 847] of Igusa, stated in terms of the representation theory of $\mathrm{SL}(n,\mathbb{C})$. Igusa also computes $F_2(x,0)$, $F_4(x,0)$, and $F_6(x,0)$.

From the techniques for computing $F_k(x,t)$ and $G_k(x,t)$, we can determine asymptotic properties of some of the coefficients of $\binom{a+k}{k}_q$, for k fixed. The coefficients of $\binom{a+k}{k}_q$ have been considered for $a,k\to\infty$ by Takács [11] and others, but the computation for k fixed seems to be new. As usual, we define $f(x)=O(g(x))$ for $x\to x_0\le\infty$, if $|f(x)|\le C\cdot|g(x)|$ for some constant $C>0$, when x approaches x_0.

Theorem 2.4. *Fix $\alpha\ge 0$ ($\alpha\in\mathbb{R}$), $c\in\mathbb{Z}$, and k a positive integer. Then*

$$[q^{\lfloor\alpha a\rfloor-c}]\binom{a+k}{k}_q=\frac{1}{(k-1)!\,k!}C(\alpha,k)a^{k-1}+O(a^{k-2})$$

for $a\to\infty$, where

$$C(\alpha,k)=\sum_{i=0}^{\lfloor\alpha\rfloor}(-1)^i\binom{k}{i}(\alpha-i)^{k-1}.$$

Proof. First assume that α is rational, say $\alpha=u/v$. Fix $0\le r<v$ and consider only those a of the form $a=vb+r$. Set $d=\lfloor ur/v\rfloor$. Thus

$$[q^{\lfloor ua/v\rfloor-c}]\binom{a+k}{k}_q=[q^{ub+d-c}]\frac{(1-q^{vb+r+k})(1-q^{vb+r+k-1})\cdots(1-q^{vb+r+1})}{(1-q^k)(1-q^{k-1})\cdots(1-q)}.\tag{5}$$

Write

$$\begin{aligned}G_{\alpha,k,r}(x)&=\sum_{\substack{a\ge 0\\ a\equiv r\ (\mathrm{mod}\ v)}}[q^{\lfloor ua/v\rfloor-c}]\binom{a+k}{k}_q x^a\\ &=\sum_{b\ge 0}[q^{ub+d-c}]\binom{vb+r+k}{k}_q x^{vb+r}.\end{aligned}$$

We now apply equation (5), expand the numerator and apply Lemma 2.1(a). We obtain a linear combination of expressions like

$$\frac{1}{s}\sum_{\zeta^s=1}\left.\frac{(\zeta x)^e}{(1-\zeta x)(1-\zeta^2x^2)\cdots(1-\zeta^k x^k)}\right|_{x\to x^{1/s}}=G(x)|_{x\to x^{1/s}}\,,\tag{6}$$

say. Let $\zeta_s=e^{2\pi i/s}$, a primitive sth root of unity. The order to which 1 is a pole in equation (6) is thus at most the order to which ζ_s is a pole of $G(x)$. Now any term indexed by $\zeta\ne 1$ has ζ_s as a pole of $G(x)$ of order less than k, while the term indexed by $\zeta=1$ has a pole of order at most k at $x=1$. Hence if in the end we have a pole of order k, then it suffices to retain only the term in (6) indexed by $\zeta=1$. Therefore if, for any integer e,

$$\frac{1}{s}\sum_{\zeta^s=1}\left.\frac{(\zeta x)^e}{(1-\zeta x)(1-\zeta^2x^2)\cdots(1-\zeta^k x^k)}\right|_{x\to x^{1/s}}=\frac{c_0}{(1-x)^k}+O\left(\frac{1}{(1-x)^{k-1}}\right)$$

for $x\to 1$, then

$$c_0=\lim_{x\to 1}(1-x^s)^k\frac{x^e}{s(1-x)(1-x^2)\cdots(1-x^k)}=\frac{s^{k-1}}{k!}.$$

Write

$$(1-q^{vb+r+k})(1-q^{vb+r+k-1})\cdots(1-q^{vb+r+1}) = \frac{\sum_{i=0}^{k}(-1)^i Q_i(q) q^{bvi}}{[k]!},$$

where $Q_i(q)$ is a polynomial independent of b and v, so $Q_i(1)=\binom{k}{i}$. Note that $u-vi\geq 0$ if and only if $i\leq\lfloor\alpha\rfloor$. It follows that

$$\begin{aligned}
G_{\alpha,k,r}(x) &= \sum_{b\geq 0}\left[q^{ub+d-c}\right]\frac{\sum_{i=0}^{k}(-1)^i Q_i(q) q^{bvi}}{[k]!}x^{bv+r} \\
&= \sum_{b\geq 0}[q^{(u-vi)b}]\frac{\sum_{i=0}^{k}(-1)^i Q_i(q) q^{c-d}}{[k]!}x^{bv+r} \\
&= \left(\frac{1}{k!}\sum_{i=0}^{\lfloor\alpha\rfloor}(-1)^i\binom{k}{i}(u-vi)^{k-1}\right)\frac{x^r}{(1-x^v)^k}+O\left(\frac{1}{(1-x)^{k-1}}\right) \\
&= \left(\frac{1}{k!}\sum_{i=0}^{\lfloor\alpha\rfloor}(-1)^i\binom{k}{i}(u-vi)^{k-1}\right)\frac{1}{v^k(1-x)^k}+O\left(\frac{1}{(1-x)^{k-1}}\right).
\end{aligned}$$

Now sum over $0\leq r<v$. Since we have v terms in the sum, we pick up an extra factor of v on the right, giving

$$\begin{aligned}
\sum_{a\geq 0}[q^{\lfloor ua/v\rfloor}]\binom{a+k}{k}_q x^a &= \left(\frac{1}{k!}\sum_{i=0}^{\lfloor\alpha\rfloor}(-1)^i\binom{k}{i}(u-vi)^{k-1}\right)\frac{1}{v^{k-1}(1-x)^k}+O\left(\frac{1}{(1-x)^{k-1}}\right) \\
&= \left(\frac{1}{k!}\sum_{i=0}^{\lfloor\alpha\rfloor}(-1)^i\binom{k}{i}(\alpha-i)^{k-1}\right)\frac{1}{(1-x)^k}+O\left(\frac{1}{(1-x)^{k-1}}\right).
\end{aligned}$$

Now

$$[x^a]\frac{1}{(1-x)^k} = \binom{k+a-1}{k-1} = \frac{a^{k-1}}{(k-1)!}+O(a^{k-2}),$$

completing the proof for α rational.

The proof for general α now follows by a simple continuity argument, using the unimodality and symmetry of the coefficients of $\binom{a+k}{k}_q$. □

The numbers $C(\alpha,k)$ have appeared before and are known as *Euler-Frobenius numbers*, denoted $A_{k-1,\lfloor\alpha\rfloor,\alpha-\lfloor\alpha\rfloor}$. For a discussion of the history and properties of these numbers, see Janson [6]. Some special cases are of interest. Recall that the *Eulerian number* $A(d,i)$ can be defined as the number of permutations w of $1,2,\ldots,d$ with $i-1$ descents (e.g. [9, §1.4]). Similarly the *MacMahon number* $B(d,i)$ can be defined as the number of elements in the hyperoctahedral group B_n according to the number of type B descents. For further information, see [1]. Standard results about these numbers imply that for integers $1 \leq j < k$,

$$\begin{aligned} C(j,k) &= A(k-1,j), \\ 2^{k-1}C((2j-1)/2,k) &= B(k-1,j). \end{aligned}$$

There is an alternative way to show the above formula for $C(\alpha,k)$ (done with assistance from Fu Liu). Write $\beta = \lfloor \alpha \rfloor$. Since the coefficient of $q^{a\beta}$ in $\binom{a+k}{k}_q$ is the number of partitions of $a\beta$ into at most a parts of length at most k, equivalently, it is equal to the number of solutions $(m_1,\dots,m_k)$ in nonnegative integers to

$$\begin{aligned} m_1 + 2m_2 + \cdots + km_k &= a\beta, \\ m_1 + \cdots + m_k &\leq a. \end{aligned}$$

Set $x_i = m_i/a$ and let $a \to \infty$. Standard arguments (see e.g., [9, Proposition 4.6.13]) show that $C(\alpha,k)$ is the $(k-1)$-dimensional relative volume (as defined in [9, p. 497]) of the convex polytope:

$$\begin{aligned} x_i &\geq 0, \quad 1 \leq i \leq k, \\ x_1 + 2x_2 + \cdots + kx_k &= \beta, \\ x_1 + x_2 + \cdots + x_k &\leq 1. \end{aligned}$$

Set $y_i = x_i + x_{i+1} + \cdots + x_k$. The matrix of this linear transformation has determinant 1, so it preserves the relative volume. We get the new polytope $\mathcal{P}_k$ defined by

$$y_1 + y_2 + \cdots + y_k = \beta,$$
$$0 \leq y_1 \leq y_2 \leq \cdots \leq y_k \leq 1.$$

By symmetry, the relative volume of $\mathcal{P}_k$ is $1/k!$ times the relative volume of the polytope

$$y_1 + y_2 + \cdots + y_k = \beta,$$
$$0 \leq y_i \leq 1, \quad 1 \leq i \leq k.$$

This polytope is a cube cross-section, whose relative volume is computed e.g. in [6, Theorem 2.1], completing the proof.

When $\alpha \in \mathbb{Q}$, $C(\alpha,k)$ is related to the Eulerian polynomial $A_{k-1}(x)$ via the following result.

Proposition 2.5. *Let $v \in \mathbb{P}$. Then*

$$v^{k-1}\sum_{u\geq 0} C(u/v,k)x^u = (1+x+x^2+\cdots+x^{v-1})^k A_{k-1}(x). \tag{7}$$

Proof. We have

$$v^{k-1}\sum_{u\geq 0} C(u/v,k)x^u = \sum_{u\geq 0}\sum_{i=0}^{\lfloor u/v\rfloor}(-1)^i\binom{k}{u}(u-vi)^{k-1}x^i. \tag{8}$$

A fundamental property of Eulerian polynomials is the identity (see [9, Proposition 1.4.4])

$$\sum_{n\geq 0} n^{k-1}x^n = \frac{A_{k-1}(x)}{(1-x)^k}.$$

Hence,

$$A_{k-1}(x)(1+x+\cdots+x^{v-1})^k = (1-x^v)^k \sum_{n\geq 0} n^{k-1}x^n. \tag{9}$$

It is now routine to compute the coefficient of x^m on the right-hand sides of equations (8) and (9) and see that they agree term by term. □

Note that if $j\in\mathbb{P}$ and we take the coefficient of x^{jv} on both sides of equation (7), then we obtain the identity

$$v^{k-1}A(k-1,j) = [x^{vj}](1+x+x^2+\cdots+x^{v-1})^k A_{k-1}(x).$$

It is not difficult to give a direct proof of this identity.

Let us now turn to the *difference* between two consecutive coefficients of $\binom{a+k}{k}_q$, i.e., the function $f_{k,c}(a)$ of equation (1). We consider here only the coefficients near the middle (i.e., q^{aj}) when $k=2j$, though undoubtedly our results can be extended to other coefficients. Note that, by the previous theorem, we have

$$[q^{aj-c}]\binom{a+k}{k}_q \sim [q^{aj-c-1}]\binom{a+k}{k}_q \sim \frac{1}{(k-1)!\,k!}C(\alpha,k)a^{k-1}, \quad a\to\infty.$$

Thus we might expect that the difference $([q^{aj-c}]-[q^{aj-c-1}])\binom{a+k}{k}_q$ grows like a^{k-2}. However, the next result shows that the correct growth rate is a^{k-3}.

Theorem 2.6. *Let $c\in\mathbb{N}$ and $k=2j$, where $j\in\mathbb{P}$. Then for $j\geq 3$ we have*

$$\left([q^{aj-c}]-[q^{aj-c-1}]\right)\binom{a+k}{k}_q = \frac{2c+1}{(k-3)!\,k!}D(k)a^{k-3}+O(a^{k-4}),$$

where

$$D(k) = \frac{1}{2}\sum_{i=0}^{j-1}(-1)^{i+1}\binom{k}{i}(j-i)^{k-3}.$$

Proof. Write

$$\begin{aligned} F_k(x,t) &= \sum_{a\geq 0}\sum_{c\geq 0}([q^{aj-c}]-[q^{aj-c-1}])\binom{a+k}{k}_q x^a t^c \\ &= \sum_{a\geq 0}[q^{aj}]\frac{(1-q^{a+k})\cdots(1-q^{a+1})}{(1-q^k)\cdots(1-q^2)(1-qt)}x^a. \end{aligned}$$

When $k\geq 6$, the order to which a primitive root of unity $x\neq 1$ is a pole of $F_k(x,t)$ is at most $k-3$. Thus we need to show that the pole at $x=1$ contributes the stated result.

Let

$$F_k(x,t) = \alpha_k(t)\frac{1}{(1-x)^{k-1}} + \beta_k(t)\frac{1}{(1-x)^{k-2}} + O\left(\frac{1}{(1-x)^{k-3}}\right).$$

First we show that $\alpha_k(t) = 0$. Reasoning as in the proof of Theorem 2.4 gives

$$\alpha_k(t) = \frac{1}{(k-2)!\,k!(1-t)}\sum_{i=0}^{j-1}(-1)^i\binom{k}{i}(j-i)^{k-2}.$$

Since k is even, the summand $(-1)^i\binom{k}{i}(j-i)^{k-2}$ remains the same when we substitute $k-i$ for i. Moreover, when $i=j$ the summand is 0. Hence

$$\alpha_k(t) = \frac{1}{2(k-2)!\,k!(1-t)}\sum_{i=0}^{k}(-1)^i\binom{k}{i}(j-i)^{k-2}.$$

This sum is the kth difference at 0 of a polynomial of degree $k-2$, and is therefore equal to 0 (see [9, Proposition 1.9.2]), as desired.

We now need to find the coefficient β of $(1-x)^{k-2}$ in the Laurent expansion at $x=1$ of linear combinations of rational functions of the type

$$H = \frac{P(x)}{(1-x^2)\cdots(1-x^k)(1-xt)} = \frac{\alpha}{(1-x)^{k-1}} + \frac{\beta}{(1-x)^{k-2}} + \cdots,$$

where $P(x)$ is a polynomial in x. Write $(i)_x = 1+x+x^2+\cdots+x^{i-1}$. It is easy to see that $\alpha = P(1)/k!(1-t)$. Thus

$$\begin{aligned}
\beta &= \lim_{x\to 1}(1-x)^{k-2}\left(\frac{P(x)}{(1-x^2)\cdots(1-x^k)(1-xt)} - \frac{P(1)}{k!(1-t)(1-x)^{k-1}}\right)\\
&= \lim_{x\to 1}\frac{1}{1-x}\cdot\frac{P(x)k!(1-t) - P(1)(2)_x\cdots(k)_x(1-xt)}{(2)_x\cdots(k)_x(1-xt)k!(1-t)}\\
&= -\frac{1}{k!^2(1-t)^2}\frac{d}{dx}\left(P(x)k!(1-t) - P(1)(2)_x\cdots(k)_x(1-xt)\right)\Big|_{x=1}\\
&= -\frac{1}{k!^2(1-t)^2}\left(P'(1)k!(1-t) - k!P(1)\left(\frac{1}{2}+\frac{3}{3}+\cdots+\frac{\binom{k}{2}}{k}\right)(1-t) + P(1)t\right)\\
&= -\frac{1}{k!(1-t)^2}\left(P'(1)(1-t) - \frac{1}{2}P(1)\binom{k}{2}(1-t) + P(1)t\right)\\
&= \frac{1}{k!(1-t)^2}\left(-P'(1)(1-t) + \frac{1}{4}P(1)(-k+kt+k^2-k^2t) - P(1)t\right).
\end{aligned}$$

Let us apply this result to $P(x) = P_i(x)$, where P_i is defined by equation (3). Clearly $P_i(1) = \binom{k}{i}$, while

$$P_i'(1) = \sum_{\substack{S\subseteq[k]\\ \#S=i}}\sum_{s\in S} s.$$

The element $i \in [k]$ appears in $\binom{k-1}{i-1}$ i-element subsets of $[k]$. Hence

$$P_i'(1) = \sum_{i=1}^{k} i\binom{k-1}{i-1} = \binom{k+1}{2}\binom{k-1}{i-1},$$

where when $i=0$ we set $\binom{k-1}{-1}=0$. Arguing as in the proof of Theorem 2.4 now gives

$$\begin{aligned}\beta_k(t) &= \frac{1}{k!(1-t)^2}\sum_{i=0}^{j-1}(-1)^{i+1}(j-i)^{k-3} \\ &\times \left(\binom{k}{i}\left(\frac{1}{2}(1-t)(k-1)(j-i-1)+t-\frac{1}{4}(1-t)k(k-1)\right)\right. \\ &\left. + (1-t)\binom{k+1}{2}\binom{k-1}{i-1}\right). \qquad (10)\end{aligned}$$

If we set $t=-1$ on the right-hand-side of equation (10), then a straightforward computation shows that the sum is 0. If we set $t=1$, then another computation gives

$$\sum_{i=0}^{j-1}(j-i)^{k-3}(-1)^{i+1}\binom{k}{i}.$$

Since

$$\frac{1+t}{(1-t)^2} = \sum_{c\geq 0}(2c+1)t^c,$$

the proof now follows. □

Remark 2.7. (a) It follows from work of Verma [12] and of Hering and Howard [4] that $D(k)$ also satisfies

$$K_{a(k/2,k/2),a\cdot 1^k} = \frac{1}{(k-3)!}D(k)a^{k-3} + O(a^{k-4}), \qquad (11)$$

where $K_{\lambda\mu}$ is a Kostka number and $a \cdot 1^k$ denotes the partition of ak with k a's. Is the appearance of $D(k)$ in both Theorem 2.6 and equation (11) just a coincidence?

(b) Theorem 2.6 is false for $j=2$. Indeed, it follows from equation (4) that

$$F_4(x,t) = \frac{1-t+t^2}{6(1-t)(1-t^2)}\cdot\frac{1}{(1-x)^2} + O\left(\frac{1}{1-x}\right)$$

and

$$([q^{2a-c}]-[q^{2a-c-1}])\binom{a+4}{4}_q = \frac{1}{24}(2c+1+3\cdot(-1)^c)a + O(1), \quad a\to\infty.$$

(c) An obvious problem arising from our work is the extension of Theorem 2.4 to additional terms. Can such a computation be automated?

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References

[1] A060187, *On-Line Encyclopedia of Integer Sequences*. Available at http://oeis.org/A060187.

[2] G. E. Andrews, P. Paule and A. Riese: *MacMahon's partition analysis: the Omega Package*, European J. Combin. **22** (2001), 887–904.

[3] V. Dhand: *A combinatorial proof of strict unimodality for q-binomial coefficients*, Discrete Math. **335** (2014), 20–24.

[4] M. Hering and B. Howard: *The ring of evenly weighted points on the line*, Math. Z. **277** (2014), no. 3–4, 691–708.

[5] J.-I. Igusa: *Modular forms and projective invariants*, Amer. J. Math. **89** (1967), 817–855.

[6] S. Janson: *Euler-Frobenius numbers and rounding*, Online J. Analytic Comb. **8** (2013), 34 pp..

[7] I. Pak and G. Panova: *Strict unimodality of q-binomial coefficients*, C. R. Math. Acad. Sci. Paris **351** (2013), no. 11–12, 415–418.

[8] I. Pak and G. Panova: *Bounds on the Kronecker coefficients*, preprint. Available on the arXiv.

[9] R. Stanley: "Enumerative Combinatorics", Vol. I, Second Ed., Cambridge University Press, Cambridge, U.K. (2012).

[10] R. Stanley and F. Zanello: *Unimodality of partitions with distinct parts inside Ferrers shapes*, European J. Combin. **49** (2015), 194–202.

[11] L. Takács: *Some asymptotic formulas for lattice paths*, J. Stat. Planning and Inference **14** (1986), 123–142.

[12] D.-N. Verma: *Toward classifying finite point-set configurations*, preprint (1997).

[13] F. Zanello: *Zeilberger's KOH theorem and the strict unimodality of q-binomial coefficients*, Proc. Amer. Math. Soc. **143** (2015), no. 7, 2795–2799.

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