

AN ELEMENTARY SOLUTION OF GESSEL'S WALKS IN THE QUADRANT

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ABSTRACT. Around 2000, Ira Gessel conjectured that the number of lattice walks in the quadrant \mathbb{N}^2 , starting and ending at the origin $(0,0)$ and taking their steps in $\{\rightarrow, \nearrow, \leftarrow, \swarrow\}$ had a simple hypergeometric form. In the following decade, this problem was recast in the systematic study of walks with small steps (that is, steps in $\{-1, 0, 1\}^2$) confined to the quadrant. The generating functions of such walks are archetypal solutions of *partial discrete differential equations*.

A complete classification of quadrant walks according to the nature of their generating function (algebraic, D-finite or not) is now available, but Gessel's walks remained mysterious because they were the only model among the 23 D-finite ones that had not been given an elementary solution. Instead, Gessel's conjecture was first proved using an inventive computer algebra approach in 2008. A year later, the associated three-variate generating function was proved to be algebraic by a computer algebra *tour de force*. This was re-proved recently using elaborate complex analysis machinery. We give here an elementary and constructive proof. Our approach also solves other quadrant models (with multiple steps) recently proved to be algebraic via computer algebra.

1. INTRODUCTION

The enumeration of planar lattice walks confined to the quadrant has received a lot of attention in the past decade. The basic question reads as follows: given a finite step set $\mathcal{S} \subset \mathbb{Z}^2$ and a starting point $P \in \mathbb{N}^2$, what is the number $q(n)$ of n -step walks, starting from P and taking their steps in \mathcal{S} , that remain in the non-negative quadrant \mathbb{N}^2 ? This is a generic and versatile question, since such walks encode in a natural fashion many discrete objects (systems of queues, Young tableaux and permutations among others). More generally, the study of these walks fits in the larger framework of walks confined to cones. They are also much studied in probability theory, both in a discrete [22, 23] and a continuous [21, 27] setting. From a technical point of view, counting walks in the quadrant is part of a general program aiming at solving *partial discrete differential equations*, see (5) and (7) below. Details are given in Section 2, together with a more algebraic viewpoint involving division of power series.

On the combinatorics side, much attention has focused on the *nature* of the associated generating function $Q(t) = \sum_n q(n)t^n$. Is it rational in t , as for unconstrained walks? Is it algebraic over $\mathbb{Q}(t)$, as for walks confined to a (rational) half-space? More generally, is it D-finite, that is, a solution of a linear differential equation with polynomial coefficients? The answer depends on the step set, and, to a lesser extent, on the starting point.

A systematic study was initiated in [16, 40] for walks starting at the origin $(0,0)$ and taking only *small* steps (that is, $\mathcal{S} \subset \{-1, 0, 1\}^2$). For these walks, a complete classification is now available. In particular, the generating function $Q(t)$ (or rather, its three-variate refinement $Q(x, y; t)$ that also records the coordinates of the endpoint of the walk) is D-finite if and only if a certain group of rational

transformations is finite. The proof involves an attractive variety of tools, ranging from basic power series algebra [12, 16, 40] to complex analysis [38, 45], computer algebra [7, 31] and number theory [10].

In the D-finite class lie, up to symmetries, exactly 23 step sets \mathcal{S} (often called *models*) with small steps. A uniform approach, the *algebraic kernel method*, establishes the D-finiteness of 19 of them [16]. These 19 models are transcendental. The remaining 4 models are structurally simpler, since they are algebraic, but they are also harder to solve. Three of them, including the so-called Kreweras' model $\mathcal{S} = \{\leftarrow, \downarrow, \nearrow\}$, were solved in a uniform manner in [16], following several *ad hoc* proofs [35, 28, 12, 11]. The final one, $\mathcal{S} = \{\rightarrow, \nearrow, \leftarrow, \swarrow\}$, has been recognized as extremely challenging, and has been promoted recently as a popular model in the French magazine *Pour la science* [9]. It is named after Ira Gessel, who conjectured, around 2000, that for this model the number of walks of length $2n$ ending at the origin is

$$16^n \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n}, \quad (1)$$

where $(a)_n = a(a+1)\cdots(a+n-1)$ is the ascending factorial. This is sequence A135404 in Sloane's *Encyclopedia of integer sequences* [30]. The associated generating function is algebraic, but no one noticed it at that time. This attractive conjecture remained open for several years. In particular, it resisted the otherwise complete solution of D-finite models presented in [16]. Finally, Kauers, Koutschan and Zeilberger [31] succeeded in proving (1) via a clever computer-aided approach. About a year later, this was generalized by Bostan and Kauers [7], who proved, by a *tour de force* in computer algebra, that the *complete generating function* $Q(x, y; t)$ is also algebraic. This series is defined as

$$Q(x, y; t) = \sum_{i, j, n \geq 0} q(i, j; n) x^i y^j t^n, \quad (2)$$

where $q(i, j; n)$ is the number of n -step walks in the quarter plane that start from $(0, 0)$, end at (i, j) and take their steps in \mathcal{S} . A complete description, and rational parametrization of this series is given in Theorem 1 below. Both the result of [7] and the immense computational effort it required were remarkable, but they left open the problem of finding a "human" proof of Gessel's conjecture, and of its trivariate generalization. More recently, Bostan, Kurkova and Raschel [8] proposed another proof, which involves the deep machinery of complex analysis first developed for stationary random walks [23], and subsequently for counting walks in the quadrant [38, 45, 24]. The solution involves Weierstrass' \wp and ζ functions, which are transcendental, and may appear a surprisingly complicated detour to establish algebraicity. Let us also cite a number of other attempts [1, 24, 33, 37, 42, 45, 47]. Some of them can be seen as preliminary steps to the proofs of [31] or [8].

In this paper, we present the first elementary solution of Gessel's walks in the quadrant. It is elementary in the sense that it remains at the level of formal power series and polynomial equations. Also, it involves no guessing nor high-level computer algebra¹. Finally, it is constructive, in the sense that we construct, incrementally, the extensions of $\mathbb{Q}(x, y, t)$ that are needed to describe the series $Q(x, y; t)$. Even if this sounds a bit technical at this stage, we can inform the quadrant experts that the key ingredient is that the symmetric functions of the roots of the kernel are polynomials in $1/x$. This is also the case for Kreweras' walks, and our proof is in this sense close to one solution of Kreweras' model [11].

¹Although one feels better with a computer at hand when it comes to handling a system of three polynomial equations. An accompanying Maple session is available on the author's [webpage](#).

Theorem 1. *The generating function $Q(x, y; t)$ is algebraic over $\mathbb{Q}(x, y, t)$, of degree 72. The specialization $Q(0, 0; t)$ has degree 8, and can be written as*

$$Q(0, 0; t) = \frac{32 Z^3(3 + 3Z - 3Z^2 + Z^3)}{(1 + Z)(Z^2 + 3)^3},$$

where $Z = \sqrt{T}$ and T is the only power series in t with constant term 1 satisfying

$$T = 1 + 256 t^2 \frac{T^3}{(T + 3)^3}. \quad (3)$$

The series $Q(xt, 0; t)$ is an even series in t , with coefficients in $\mathbb{Q}[x]$, and is cubic over $\mathbb{Q}(Z, x)$. It can be written as

$$Q(xt, 0; t) = \frac{16 T(U + UT - 2T)M(U, Z)}{(1 - T)(T + 3)^3(U + Z)(U^2 - 9T + 8TU + T^2 - TU^2)}$$

where

$$\begin{aligned} M(U, Z) = & (T - 1)^2 U^3 + Z(T - 1)(T - 16Z - 1)U^2 \\ & - TU(T^2 + 16ZT - 82T - 16Z + 17) - ZT(T^2 - 18T + 128Z + 81), \end{aligned}$$

and U is the only power series in t with constant term 1 (and coefficients in $\mathbb{Q}[x]$) satisfying

$$16T^2(U^2 - T) = x(U + UT - 2T)(U^2 - 9T + 8TU + T^2 - TU^2).$$

Finally, the series $Q(0, y; t)$ is cubic over $\mathbb{Q}(Z, y)$. It can be written as

$$Q(0, y; t) = \frac{16 V Z^3(3 + V + T - VT)N(V, Z)}{(T - 1)(T + 3)^3(1 + V)^2(1 + Z + V - VZ)^2}$$

where

$$\begin{aligned} N(V, Z) = & (Z - 1)^2(T + 3)V^3 + (T - 1)(T + 2Z - 7)V^2 \\ & + (T - 1)(T - 2Z - 7)V + (Z + 1)^2(T + 3), \end{aligned}$$

and V is the only series in t , with constant term 0, satisfying

$$1 - T + 3V + VT = yV^2(3 + V + T - VT).$$

The algebraicity was established by Bostan and Kauers, and the parametrization was given by van Hoeij in the appendix of their paper [7] (with his notation, $T = v$, $U = u$ and $V = -w$). The parametrization that we construct through our proof is actually a bit different (and of course equivalent), but we prefer to give van Hoeij's to avoid confusion. Just for the record, ours reads as follows:

$$\begin{aligned} \tilde{T} &= t^2(1 - \tilde{T})(1 + 3\tilde{T})^3, & \tilde{Z} &= \tilde{T}(1 - \tilde{Z} + \tilde{Z}^2), & (4) \\ \tilde{U} &= 1 + x\tilde{Z} \frac{(\tilde{U} + \tilde{Z} - \tilde{U}\tilde{Z} + \tilde{U}\tilde{Z}^2)(\tilde{U} - \tilde{U}\tilde{Z} - \tilde{Z}^3 + \tilde{U}\tilde{Z}^2)}{\tilde{U}(\tilde{Z} - 1)(\tilde{Z} + 1)^3(\tilde{Z}^2 + \tilde{U})}, \\ \tilde{V} &= y \frac{(\tilde{V} - 1)^2(1 - \tilde{Z} - \tilde{V} + \tilde{Z}^2 - \tilde{Z}^3)}{(1 - \tilde{Z} + \tilde{V} - \tilde{Z}^2)^2}, \end{aligned}$$

with constant terms 0 for \tilde{T} and \tilde{Z} , 1 for \tilde{U} and $y/(1 + y)$ for \tilde{V} .

We conclude this introduction with some notation. For a ring R , we denote by $R[x]$ the ring of polynomials in x with coefficients in R . If R is a field, then $R(x)$ stands for the field of rational functions in x . Finally, if $F(x; t)$ is a power series in t whose coefficients are Laurent polynomials in x , say

$$F(x; t) = \sum_{n \geq 0, i \in \mathbb{Z}} f(i; n) t^n x^i,$$

we denote by $[x^{\geq}]F(x; t)$ the *non-negative part of F in x* :

$$[x^{\geq}]F(x; t) := \sum_{n \geq 0, i \in \mathbb{N}} f(i; n) t^n x^i.$$

This generalizes the standard notation $[x^i]F(x; t)$ for the coefficient of x^i in F . We similarly define the non-positive part of $F(x; t)$, denoted by $[x^{\leq}]F(x; t)$.

2. CONTEXT: PARTIAL DISCRETE DIFFERENTIAL EQUATIONS

The starting point of the systematic approach to the enumeration of quadrant walks is a functional equation that characterizes their complete generating function $Q(x, y; t) \equiv Q(x, y)$, defined as in (2). For instance, for square lattice walks (with steps $\rightarrow, \uparrow, \leftarrow, \downarrow$) starting at $(0, 0)$, this equation reads:

$$Q(x, y) = 1 + t(x + y)Q(x, y) + t \frac{Q(x, y) - Q(0, y)}{x} + t \frac{Q(x, y) - Q(x, 0)}{y}. \quad (5)$$

It translates the fact that, to construct a walk of length $n \geq 1$, we simply add a step to a shorter walk of length $n - 1$. The divided differences (or *discrete derivatives*)

$$\frac{Q(x, y) - Q(0, y)}{x} \quad \text{and} \quad \frac{Q(x, y) - Q(x, 0)}{y} \quad (6)$$

arise from the fact that one cannot add a West step if the shorter walk ends on the y -axis, nor a South step if it ends on the x -axis. For Gessel's walks, with steps $\rightarrow, \nearrow, \leftarrow, \swarrow$, the corresponding equation reads

$$Q(x, y) = 1 + t(x + xy)Q(x, y) + t \frac{Q(x, y) - Q(0, y)}{x} + t \frac{Q(x, y) - Q(x, 0) - Q(0, y) + Q(0, 0)}{xy}, \quad (7)$$

because the \swarrow step cannot be appended to a walk ending on the x - or y -axis. Referring to terms like (6) as *discrete derivatives*, it makes sense to call such equations *partial discrete differential equations*.

The more algebraically inclined reader will write (5) as

$$xy = (xy - t(x^2y + xy^2 + x + y)) Q(x, y) + txQ(x, 0) + tyQ(0, y),$$

and observe that we are just performing the formal division of xy by the polynomial $(xy - t(x^2y + xy^2 + x + y))$ with initial monomial xy (this is an instance of Grauert-Hironaka-Galligo division, see e.g. [46, Thm. 10.1]).

Discrete differential equations are ubiquitous in combinatorial enumeration. For instance, for the much simpler problem of counting walks with steps ± 1 starting at 0 and remaining at a non-negative level, the corresponding functional equation reads

$$F(x) \equiv F(x; t) = 1 + txF(x) + t \frac{F(x) - F(0)}{x},$$

an *ordinary* discrete differential equation (since the derivatives are only taken with respect to x). More generally, the enumeration of lattice walks confined to a rational half-space systematically yields *linear ordinary* discrete differential equations (DDEs) [3, 17]. Note that the order of the equation increases with the size of the down steps: if, in the above problem, we replace the step -1 by a step -2 , the equation reads

$$F(x) = 1 + txF(x) + t \frac{F(x) - F(0) - xF'(0)}{x^2},$$

and involves a discrete derivative of order 2. Moving beyond linear equations, the enumeration of *maps* (connected graphs properly embedded in a surface of prescribed genus) has produced since the sixties a rich collection of *non-linear* ordinary discrete differential equations (see e.g. [4, 20, 48]). For instance, the enumeration of planar maps is governed by the following equation:

$$F(x;t) = F(x) = 1 + tx^2F(x)^2 + tx \frac{x F(x) - F(1)}{x - 1}.$$

(Note that the discrete derivative is now taken at $x = 1$.)

An approach for solving linear ordinary DDEs was initiated by Knuth in the seventies [34, Section 2.2.1, Ex. 4] and is now known as the *kernel method* [2, 17]. Another approach, developed by Brown in the sixties and known as the *quadratic method*, solves quadratic ordinary DDEs of first order [19, 29]. More recently, this approach was generalized by the author and A. Jehanne [15] to ordinary DDEs of any degree and order, with the striking result that *any series solution of a (well-founded) equation of this type is algebraic*². This result is of particular importance to this paper, since the core of our proof is to derive from the *partial* DDE (7) satisfied by $Q(x, y)$ an *ordinary* DDE satisfied by $Q(x, 0)$. This is achieved in Section 3.3. This ordinary DDE (of order 3 and degree 3) is then solved using the method of [15], and found, of course, to have an algebraic solution.

If ordinary DDEs are now well-understood, much less is known about partial DDEs, even linear ones. The enumeration of quadrant walks with small steps appears as an essential step in their study, because the resulting equations are as simple as possible: linear, first order, and involving derivatives with respect to two variables only. This study started around 2000, and it was soon understood that the solutions would not always be algebraic, nor even D-finite [18, 41]. The classification of these problems is now complete. That is, one knows for every step set $\mathcal{S} \subset \{-1, 0, 1\}^2$ if the series $Q(x, y; t)$ is rational, algebraic, D-finite or not [7, 10, 16, 38, 41, 39]. A crucial role is played by a certain group of rational transformations associated with \mathcal{S} : the series $Q(x, y; t)$ turns out to be D-finite if and only if this group is finite. In this case, a generic method solves the partial DDE associated with the problem [16], except for four sets \mathcal{S} for which the series $Q(x, y; t)$ is in fact algebraic. Three of these four problems were solved in a uniform way in [16]. The fourth one is Gessel's model, the story of which we have told in the introduction. This paper thus adds a building block to the classification of quadrant walks, by deriving from the partial DDE (7), in a constructive manner, an algebraic equation for $Q(x, y)$.

To finish, let us mention a few investigations on more general partial DDEs: quadrant walks with larger backwards steps yield linear equations of larger order [25]; coloured planar maps yield non-linear equations [5, 14]; walks confined to the non-negative octant \mathbb{N}^3 yield linear equations with three differentiation variables [6]; the famous enumeration of permutations with bounded ascending subsequences (and in fact, many problems related to Young tableaux, plane partitions or alternating sign matrices [49]) yields linear equations with an arbitrary number of differentiation variables [13].

3. THE PROOF

We begin, in a standard way, by writing a functional equation satisfied by $Q(x, y; t)$. It is based on a step by step construction of walks [16]:

$$Q(x, y) = 1 + t(\bar{x} + \bar{x}\bar{y} + x + xy)Q(x, y) - t\bar{x}(1 + \bar{y})Q(0, y) - t\bar{x}\bar{y}(Q(x, 0) - Q(0, 0)),$$

²As learnt recently by the author, this algebraicity also follows from a difficult theorem on Artin's approximation with *nested conditions*, see [36] or [43, Thm. 1.4].

where $Q(x, y) \equiv Q(x, y; t)$, $\bar{x} = 1/x$ and $\bar{y} = 1/y$. Equivalently,

$$xyK(x, y)Q(x, y) = xy - t(Q(x, 0) - Q(0, 0)) - t(1 + y)Q(0, y), \quad (8)$$

where

$$K(x, y) = 1 - t(\bar{x} + \bar{x}\bar{y} + x + xy)$$

is the *kernel* of the equation. This Laurent polynomial is left invariant by the following two rational transformations:

$$\Phi : (x, y) \mapsto (\bar{x}\bar{y}, y) \quad \text{and} \quad \Psi := (x, y) \mapsto (x, \bar{x}^2\bar{y}).$$

Both are involutions, and they generate a group G of order 8:

$$(x, y) \xleftarrow{\Phi} (\bar{x}\bar{y}, y) \xleftarrow{\Psi} (\bar{x}\bar{y}, x^2y) \xleftarrow{\Phi} (\bar{x}, x^2y) \xleftarrow{\Psi} (\bar{x}, \bar{y}) \xleftarrow{\Phi} (xy, \bar{y}) \xleftarrow{\Psi} (xy, \bar{x}^2\bar{y}) \xleftarrow{\Phi} (x, \bar{x}^2\bar{y}) \xleftarrow{\Psi} (x, y).$$

The construction of this group is also standard (see for instance [16]).

3.1. CANCELING THE KERNEL

As a polynomial in y , the kernel $K(x, y)$ has two roots, which are series in t with coefficients in $\mathbb{Q}[x, \bar{x}]$:

$$Y_0(x) = \frac{1 - t(x + \bar{x}) - \sqrt{(1 - t(x + \bar{x}))^2 - 4t^2}}{2tx} = \bar{x}t + O(t^2),$$

$$Y_1(x) = \frac{1 - t(x + \bar{x}) + \sqrt{(1 - t(x + \bar{x}))^2 - 4t^2}}{2tx} = \frac{\bar{x}}{t} - (1 + \bar{x}^2) - \bar{x}t + O(t^2).$$

Observe that the series $xY_i(x)$ are symmetric in x and \bar{x} :

$$\bar{x}Y_i(\bar{x}) = xY_i(x).$$

Moreover, the elementary symmetric functions of the Y_i , namely

$$Y_0 + Y_1 = -1 + \frac{\bar{x}}{t} - \bar{x}^2 \quad \text{and} \quad Y_0Y_1 = \bar{x}^2, \quad (9)$$

are polynomials in $\bar{x} = 1/x$. This property also holds for Kreweras walks [11, 12], and plays a crucial role in our proof. The following lemma tells us how to extract the constant term of a symmetric polynomial in Y_0 and Y_1 .

Lemma 2. *Let $P(u, v)$ be a symmetric polynomial in u and v with coefficients in \mathbb{Q} . Then $P(Y_0, Y_1)$ is a polynomial in \bar{x} with coefficients in $\mathbb{Q}[1/t]$. Its constant term in x (equivalently, its non-negative part in x) is $P(0, -1)$.*

Proof. The first statement follows from (9) and the fact that every symmetric polynomial in u and v is a polynomial in $u+v$ and uv . By linearity, it suffices to check the second statement when $P(u, v) = u^m v^n + u^n v^m$. If $\min(m, n) > 0$, then $P(Y_0, Y_1)$ has a factor $Y_0Y_1 = \bar{x}^2$ and its constant term is 0. If $P(u, v) = u^n + v^n$, then one proves by induction on n that the constant term is 2 if $n = 0$, and $(-1)^n$ otherwise. \square

Application. We will typically apply this lemma to prove that if $F(u, v; t)$ is a series in t whose coefficients are symmetric polynomials in u and v , with rational coefficients, and if $F(Y_0, Y_1; t)$ is well-defined as a series in t (with coefficients in $\mathbb{Q}[x, \bar{x}]$), then its coefficients actually lie in $\mathbb{Q}[\bar{x}]$.

Consider now the orbit of (x, Y_0) under the action of the group G :

$$(x, Y_0) \xleftarrow{\Phi} (xY_1, Y_0) \xleftarrow{\Psi} (xY_1, x^2Y_0) \xleftarrow{\Phi} (\bar{x}, x^2Y_0) \xleftarrow{\Psi} (\bar{x}, x^2Y_1) \xleftarrow{\Phi} \\ (xY_0, x^2Y_1) \xleftarrow{\Psi} (xY_0, Y_1) \xleftarrow{\Phi} (x, Y_1).$$

By construction, each pair (x', y') in this orbit cancels the kernel $K(x, y)$. If moreover the series $Q(x', y')$ is well-defined, then we derive from (8) that

$$R(x') + S(y') = x'y',$$

where we use the notation

$$R(x) = t(Q(x, 0) - Q(0, 0)), \quad S(y) = t(1 + y)Q(0, y). \quad (10)$$

Since Y_0 is a power series in t , the pairs (x, Y_0) and (\bar{x}, x^2Y_0) can obviously be substituted for (x, y) in $Q(x, y)$. Given that xY_0 is symmetric in x and \bar{x} , these pairs are derived from one another by replacing x by \bar{x} . One has to be more careful with pairs involving Y_1 , since this series contains a term \bar{x}/t . With the step set that we consider, and the non-negativity conditions imposed by the quadrant, one easily checks that each monomial $x^i y^j t^n$ occurring in the series $Q(x, y)$ satisfies $n + i - j \geq n/2$. Given that $Y_0 = \Theta(t)$ and $Y_1 = \Theta(1/t)$, this implies that the pair (xY_0, Y_1) and its companion (xY_0, x^2Y_1) (obtained by replacing x by \bar{x}) can be substituted for (x, y) in $Q(x, y)$, so that $Q(xY_0, Y_1)$ is a series in t with coefficients in $\mathbb{Q}[x, \bar{x}]$. We will not use the other pairs of the orbit, and the reader can check that they do not give well-defined series $Q(x', y')$.

We thus obtain a total of four equations:

$$R(x) + S(Y_0) = xY_0 \quad (11)$$

$$R(xY_0) + S(Y_1) = \bar{x}, \quad (12)$$

$$R(\bar{x}) + S(x^2Y_0) = xY_0, \quad (13)$$

$$R(xY_0) + S(x^2Y_1) = x. \quad (14)$$

3.2. AN EQUATION RELATING $R(x)$ AND $R(\bar{x})$

We will now construct from the above system two identities that are symmetric in Y_0 and Y_1 , and extract their non-negative parts in x .

We first sum the first two equations, and subtract the last two:

$$R(x) - S(x^2Y_0) - S(x^2Y_1) + x = R(\bar{x}) - S(Y_0) - S(Y_1) + \bar{x}.$$

For two indeterminates u and v , the coefficient of t^n in the series $S(u) + S(v)$ is a symmetric polynomial in u and v . Applying Lemma 2 to this coefficient (for any n) shows that the right-hand side of the above identity is a series in t with coefficients in $\mathbb{Q}[\bar{x}]$. But the left-hand side is obtained by replacing x by \bar{x} in the right-hand side. This implies that both sides are independent of x and equal to their constant term, that is, to $-S(0) - S(-1)$ (by (10) and Lemma 2 again). Finally, since $S(y)$ is a multiple of $(1 + y)$ (see (10)), this constant term is simply $-S(0)$.

We have thus obtained a new equation,

$$S(Y_0) + S(Y_1) = R(\bar{x}) + \bar{x} + S(0).$$

Combined with (11) and (9), it gives

$$S(Y_1) - xY_1 = R(x) + R(\bar{x}) + 2\bar{x} - 1/t + x + S(0). \quad (15)$$

For our second symmetric function of Y_0 and Y_1 , we take a product derived from (11) and (15):

$$(S(Y_0) - xY_0)(S(Y_1) - xY_1) = -R(x)(R(x) + R(\bar{x}) + 2\bar{x} - 1/t + x + S(0)). \quad (16)$$

We want to extract the non-negative part in x . Let us focus first on the left-hand side. By Lemma 2, the term $S(Y_0)S(Y_1)$ contributes $S(0)S(-1)$, which is zero since $S(y)$ is a multiple of $(1+y)$. Then $x^2Y_0Y_1$ equals 1 by (9). We are left with the term $-x(Y_0S(Y_1) + Y_1S(Y_0))$. By Lemma 2, the factor between parentheses is a series in t with polynomial coefficients in \bar{x} , and its constant term in x is $-S(0)$. But we also need to determine the coefficient of \bar{x} in this series. Expanding $S(y)$ in powers of y shows that

$$Y_0S(Y_1) + Y_1S(Y_0) = (Y_0 + Y_1)S(0) + Y_0Y_1F(Y_0, Y_1),$$

for some series $F(u, v)$ in t with symmetric coefficients in u and v . Since $Y_0Y_1 = \bar{x}^2$, the coefficient of \bar{x} in $Y_0S(Y_1) + Y_1S(Y_0)$ is the same as in $(Y_0 + Y_1)S(0)$, namely $S(0)/t$. We can now extract the non-negative part from (16), and this allows us to express the non-negative part of $R(x)R(\bar{x})$:

$$1 + (x - 1/t)S(0) = -R(x)^2 - [x^{\geq}]R(x)R(\bar{x}) - (2\bar{x} - 1/t + x + S(0))R(x).$$

(Recall that $R(x)$ is a multiple of x .) Extracting the constant term in x gives

$$1 - S(0)/t = -[x^0]R(x)R(\bar{x}) - 2R'(0).$$

Since $R(x)R(\bar{x})$ is symmetric in x and \bar{x} , we can now reconstruct it:

$$\begin{aligned} R(x)R(\bar{x}) &= [x^{\geq}]R(x)R(\bar{x}) + [x^{\leq}]R(x)R(\bar{x}) - [x^0]R(x)R(\bar{x}) \\ &= -R(x)^2 - (2\bar{x} - 1/t + x + S(0))R(x) - R(\bar{x})^2 \\ &\quad - (2x - 1/t + \bar{x} + S(0))R(\bar{x}) - 1 - (\bar{x} + x - 1/t)S(0) + 2R'(0). \end{aligned}$$

That is,

$$\begin{aligned} R(x)^2 + R(x)R(\bar{x}) + R(\bar{x})^2 + (2\bar{x} - 1/t + x + S(0))R(x) \\ + (2x - 1/t + \bar{x} + S(0))R(\bar{x}) = 2R'(0) - (\bar{x} + x - 1/t)S(0) - 1. \end{aligned} \quad (17)$$

3.3. AN EQUATION FOR $R(x)$ ONLY

We cannot extract the positive part of the above equation explicitly because of the ‘‘hybrid’’ term $R(x)R(\bar{x})$. However, the form of the first three terms suggests a multiplication by $R(x) - R(\bar{x})$ to decouple the series in x from the series in \bar{x} . More precisely, if we multiply (17) by $R(x) - R(\bar{x}) + \bar{x} - x$, we find a decoupled equation

$$P(x) = P(\bar{x}), \quad (18)$$

with

$$\begin{aligned} P(x) &= R(x)^3 + (S(0) + 3\bar{x} - 1/t)R(x)^2 \\ &\quad + (2\bar{x}^2 - \bar{x}/t + x/t - x^2 - 2R'(0) + (2\bar{x} - 1/t)S(0))R(x) \\ &\quad - x^2S(0) + x(2R'(0) + S(0)/t - 1). \end{aligned} \quad (19)$$

Given that $R(x)$ is a multiple of x , all terms in the expansion of $P(x)$ have an exponent (of x) at least -1 . But then (18) implies that $P(x)$ is a symmetric Laurent polynomial in x , of degree 1 and valuation -1 . More precisely,

$$P(x) = [x^0]P(x) + (x + \bar{x})[x]P(x) = [x^0]P(x) + (x + \bar{x})[\bar{x}]P(x).$$

Thus, by expanding the expression (19) of $P(x)$ in x at order 0, we find:

$$P(x) = 2(x + \bar{x})R'(0) + R'(0)(2S(0) - 1/t) + R''(0).$$

Returning to (19), this gives

$$\begin{aligned} R(x)^3 + (S(0) + 3\bar{x} - 1/t)R(x)^2 \\ + (2\bar{x}^2 - \bar{x}/t + x/t - x^2 - 2R'(0) + (2\bar{x} - 1/t)S(0))R(x) \\ = R''(0) + R'(0)(2S(0) + 2\bar{x} - 1/t) + xS(0)(x - 1/t) + x. \end{aligned} \quad (20)$$

Thus $R(x)$ satisfies a cubic equation over $\mathbb{Q}(t, x, S(0), R'(0), R''(0))$,

$$\text{Pol}(R(x), S(0), R'(0), R''(0), t, x) = 0,$$

with

$$\begin{aligned} \text{Pol}(x_0, x_1, x_2, x_3, t, x) = x_0^3 + (x_1 + 3\bar{x} - 1/t)x_0^2 \\ + (2\bar{x}^2 - \bar{x}/t + x/t - x^2 - 2x_2 + (2\bar{x} - 1/t)x_1)x_0 \\ - x_3 - x_2(2x_1 + 2\bar{x} - 1/t) - xx_1(x - 1/t) - x. \end{aligned} \quad (21)$$

In particular, if we prove that the one-variable series $S(0)$, $R'(0)$ and $R''(0)$ are algebraic over $\mathbb{Q}(t)$, then (20) shows that $R(x)$ is algebraic over $\mathbb{Q}(t, x)$. Observe, as an encouraging sign, that $R(x) = t(Q(x, 0) - Q(0, 0))$ has degree (at most) 3 over $\mathbb{Q}(t, x, S(0), R'(0), R''(0))$, in accordance with Theorem 1. Equation (20) is, in disguise, an ordinary discrete differential equation of degree 3 and order 3, as discussed in Section 2.

3.4. THE GENERALIZED QUADRATIC METHOD

We have described in [15] how to study equations of the form

$$\text{Pol}(R(x), A_1, \dots, A_k, t, x) = 0, \quad (22)$$

where $\text{Pol}(x_0, x_1, \dots, x_k, t, x)$ is a polynomial with (say) rational coefficients, $R(x) \equiv R(x; t)$ is a formal power series in t with coefficients in $\mathbb{Q}[x]$, and A_1, \dots, A_k are k auxiliary series depending on t only (in the above example, Pol is a *Laurent* polynomial in t and x , but this makes no difference). The strategy of [15] instructs us to look for power series $X(t) \equiv X$ satisfying

$$\frac{\partial \text{Pol}}{\partial x_0}(R(X), A_1, \dots, A_k, t, X) = 0. \quad (23)$$

Indeed, by differentiating (22) with respect to x , we see that any such series also satisfies

$$\frac{\partial \text{Pol}}{\partial x}(R(X), A_1, \dots, A_k, t, X) = 0, \quad (24)$$

and we thus obtain three polynomial equations, namely Eq. (22) written for $x = X$, Eqs. (23) and (24), that relate the $(k+2)$ unknown series $R(X)$, A_1, \dots, A_k and X . If we can prove the existence of k distinct series X_1, \dots, X_k satisfying (23), we will have $3k$ equations between the $3k$ unknown series $R(X_1), \dots, R(X_k)$, A_1, \dots, A_k , X_1, \dots, X_k . If there is no redundancy in this system, we will have proved that each of the $3k$ unknown series is algebraic over $\mathbb{Q}(t)$.

We apply this strategy to (20), with $A_1 = S(0)$, $A_2 = R'(0)$ and $A_3 = R''(0)$. Equation (23) reads:

$$\begin{aligned} 3R(X)^2 + 2(S(0) + 3/X - 1/t)R(X) \\ + 2/X^2 - 1/(tX) + X/t - X^2 - 2R'(0) + (2/X - 1/t)S(0) = 0. \end{aligned} \quad (25)$$

Recall that $R(x) := t(Q(x, 0) - Q(0, 0))$ and $S(0) := tQ(0, 0)$ are multiples of t . Hence, once multiplied by tX^2 , this equation has the following form:

$$X(1 - X)(1 + X) = t \text{Pol}_1(Q(X, 0), R'(0), Q(0, 0), t, X).$$

This shows that there exists exactly three series in t , denoted X_0 , X_1 and X_2 , that cancel (25). Their constant terms are respectively 0, 1 and -1 . Due to the special form of our polynomial Pol (given by (21)), it is in fact simple to determine these series. Indeed, we observe that

$$tx^2 \left(\frac{\partial \text{Pol}}{\partial x_0} + x^2 \frac{\partial \text{Pol}}{\partial x} \right) = (1-x)(1+x)(2tx^2 + 2t - x)(x_0x + x_1x + 1), \quad (26)$$

and since the three series X_i cancel both partial derivatives of Pol , we conclude that $X_1 = 1$, $X_2 = -1$, while X_0 is the only formal power series in t that cancels $(2tX^2 + 2t - X)$ (a Catalan-ish series, the exact expression of which we will not use). The fourth factor in (26) cannot vanish for $x_0 = R(X)$ and $x_1 = S(0)$ because of the factor t occurring in these series.

Let $\text{Disc}(x)$ be the discriminant of $\text{Pol}(x_0, S(0), R'(0), R''(0), t, x)$ with respect to x_0 . It is obtained by eliminating x_0 between this polynomial and its derivative with respect to x_0 . Then $\text{Disc}(X_i) = 0$ for $i = 0, 1, 2$, and this gives three polynomial equations relating $S(0)$, $R'(0)$, $R''(0)$, t and X_0 . In fact, we observe that $\text{Disc}(x)$ is symmetric in x and \bar{x} (this comes from the construction of (20)) and is thus a polynomial in $s = x + \bar{x}$. So we can alternatively write that this polynomial vanishes at $s = \pm 2$ and $s = 1/(2t)$. Systematically eliminating two of the three series $S(0)$, $R'(0)$ and $R''(0)$ in the resulting system of three equations yields an algebraic equation for each of them: for $R'(0)$ this equation has degree 4, and degree 8 for the other two series. The equations for $R'(0)$ and $S(0)$ are shown below. From this, one can derive an algebraic equation for $R(x)$ (using (20)), that is, for $Q(x, 0)$, and then one for $S(y) = t(1 + y)Q(0, y)$, as explained below. Finally, the original functional equation (8) proves the algebraicity of $Q(x, y)$.

3.5. RATIONAL PARAMETRIZATION

To avoid handling big polynomials, it is convenient to use rational parametrizations of all these series. To begin with, the equation satisfied by $R'(0)$ is:

$$\begin{aligned} &729t^6 R'(0)^4 + 243t^4(4t^2 + 1)R'(0)^3 - 27t^2(14t^4 + 19t^2 - 1)R'(0)^2 \\ &\quad - (20t^2 - 1)(7t^2 - 6t + 1)(7t^2 + 6t + 1)R'(0) - t^2(343t^4 - 37t^2 + 1) = 0. \end{aligned}$$

It has genus 0 —so Maple tells us— and can be parametrized by introducing the unique formal power series $T \equiv T(t)$ in t with constant term 1 satisfying (3). Then

$$R'(0) = \frac{(T-1)(21-6T+T^2)}{(T+3)^3}.$$

Rational parametrizations can be computed via the Maple command `parametrization`. The one that we originally obtained has a slightly different form, see (4).

The equation satisfied by $S(0)$ is:

$$\begin{aligned} &27t^7 S(0)^8 + 108t^6 S(0)^7 + 189t^5 S(0)^6 + 189t^4 S(0)^5 - 9t^3(32t^4 + 28t^2 - 13)S(0)^4 \\ &\quad - 9t^2(64t^4 + 56t^2 - 5)S(0)^3 - 2t(256t^6 - 312t^4 + 156t^2 - 5)S(0)^2 \\ &\quad - (32t^2 - 1)(4t^2 - 6t + 1)(4t^2 + 6t + 1)S(0) - t(256t^6 + 576t^4 - 48t^2 + 1) = 0. \end{aligned}$$

Since $S(0) = tQ(0, 0)$, this gives an equation for $Q(0, 0)$, involving only even powers of t . If we replace t^2 by its rational expression in T derived from (3), this equation of degree 8 in $Q(0, 0)$ factors into a quadratic term and one of degree 6. Injecting the first few coefficients of $Q(0, 0)$ shows that the term that vanishes is the quadratic one. So $Q(0, 0)$ has degree 2 over $\mathbb{Q}(T)$, and can be written in terms of $Z = \sqrt{T}$ as stated in Theorem 1:

$$Q(0, 0) = \frac{32Z^3(3 + 3Z - 3Z^2 + Z^3)}{(1 + Z)(Z^2 + 3)^3}.$$

The series $R''(0)/t$ is also found to have a rational expression in Z :

$$R''(0) = 2t[x^2]Q(x, 0) = 1024t \frac{Z^3(Z-1)(1+2Z+7Z^2-Z^4-2Z^5+Z^6)}{(1+Z)(3+Z^2)^6}.$$

We now go back to the equation (20) satisfied by $R(x) = t(Q(x, 0) - Q(0, 0))$. Injecting the above expressions of $S(0) = tQ(0, 0)$, $R'(0)$ and $R''(0)$ in it gives a cubic equation for $Q(x, 0)$ over $\mathbb{Q}(t, Z, x)$. We now consider $Q(xt, 0)$ instead of $Q(x, 0)$, since it is an even series in t . Using (3) to express t^2 in terms of $T = Z^2$, we see that this series is cubic over $\mathbb{Q}(Z, x)$.

We finally construct an equation for $Q(0, y)$ thanks to the kernel equation (11). Written for xt instead of x , it reads

$$t(Q(xt, 0) - Q(0, 0)) + t(1+y)Q(0, y) = xty,$$

with $y = Y_0(xt)$. Thus the equation we have obtained for $Q(xt, 0)$ gives a cubic equation for $Q(0, y)$, and eliminating x between this equation and $K(xt, y) = 0$ gives a (still cubic) equation for $Q(0, y)$ over $\mathbb{Q}(Z, y)$, valid for a generic value of y .

The equations for $Q(xt, 0)$ and $Q(0, y)$ take a few Maple lines each (four or five), and we do not give them explicitly. They can be seen in the accompanying [Maple session](#). They factor if we express x and y in terms of van Hoeij's series U and V , as described in Theorem 1. Our alternative parametrization, given below the theorem, was obtained thanks to the `parametrization` command in Maple. One can use this command with Z as a parameter, but it is faster to compute a parametrization for a few values of Z (e.g. $Z = 10$, $Z = 100$) and then reconstruct a generic one. \square

4. MORE ALGEBRAIC MODELS

It is natural to ask when the symmetric functions of the roots of the kernel are polynomials in \bar{x} , since this property plays a crucial role in our proof. Let us consider each of the 23 models with small steps and a finite group [16], and denote by Y_0 and Y_1 the roots of the kernel

$$K(x, y) = 1 - t \sum_{(i,j) \in \mathcal{S}} x^i y^j.$$

It is not hard to see that $Y_0 + Y_1$ and $Y_0 Y_1$ are polynomials in \bar{x} if and only if the only step of the form $(i, 1)$ is $(1, 1)$. We find 4 models having this property.

- The first one is Gessel's model $\mathcal{S} = \{\rightarrow, \nearrow, \leftarrow, \swarrow\}$, which we have just solved.
- Then comes Kreweras' model $\mathcal{S} = \{\leftarrow, \downarrow, \nearrow\}$, which is also algebraic, and was solved in [11] using the same principles as in this paper (see [12] for a variant). The solution is simpler than in Gessel's case in two aspects: first, the diagonal symmetry implies that $Q(x, 0) = Q(0, x)$, so that we have only one unknown series $R(x)$, not $R(x)$ and $S(y)$ as before; then, and more importantly, the equation obtained by forming a symmetric function of Y_0 and Y_1 only involves $R(x)$, not $R(\bar{x})$.
- Finally, we also have the models $\{\leftarrow, \searrow, \nearrow\}$ and $\{\leftarrow, \rightarrow, \searrow, \nearrow\}$, which are known to be D-finite [16] but transcendental (this can be derived from [10, Thm. 4]).

More recently, Kauers and Yatchak initiated a study of walks in the quadrant with multiple steps [32]. Such walks naturally arise, after projection, in the study of 3D walks confined to the first octant [6]. In particular, it was proved in [6], using computer algebra, that the model $\{\leftarrow, \swarrow, \searrow, \rightarrow, \nearrow\}$ is algebraic (note the double East step). This was generalized by Kauers and Yatchak [32], who proved

that algebraicity persists if one includes a South step, with arbitrary multiplicity λ . For this model, the symmetric functions of the roots of the kernel are polynomials in \bar{x} , and we can solve it using the tools of this paper. In fact, the proof is only marginally more difficult than in Kreweras' case: there are two unknown series $R(x)$ and $S(y)$, but the equations obtained by forming symmetric functions of Y_0 and Y_1 only involve $R(x)$, not $R(\bar{x})$.

Let us briefly sketch the main steps of the solution. The basic equation reads:

$$K(x, y)xyQ(x, y) = xy - R(x) - S(y), \quad (27)$$

where $K(x, y) = 1 - t(\bar{x} + \bar{x}\bar{y} + \lambda\bar{y} + x\bar{y} + 2x + xy)$ is the kernel,

$$R(x) = t(1 + \lambda x + x^2)Q(x, 0) - tQ(0, 0) \quad \text{and} \quad S(y) = t(1 + y)Q(0, y). \quad (28)$$

The group of this walk has order 6, and the orbit of (x, Y_0) contains exactly 4 pairs (x', y') for which $Q(x', y')$ is well defined:

$$(x, Y_0), \quad \left(\frac{t(1 + Y_1)}{1 + \lambda t}, Y_0 \right), \quad \left(\frac{t(1 + Y_0)}{1 + \lambda t}, Y_1 \right), \quad \left(\frac{t(1 + Y_0)}{1 + \lambda t}, x/t + \lambda x - 1 \right).$$

Each such pair gives an equation $x'y' = R(x') + S(y')$. The first equation expresses $S(Y_0)$ in terms of $R(x)$, and a combination of the last two expresses $S(Y_1)$ in terms of $S(x/t + \lambda x - 1)$. By extracting the positive part in the resulting expression of $S(Y_0) + S(Y_1)$, we conclude that

$$S(Y_0) + S(Y_1) = \bar{x} \quad \text{and} \quad x = S(x/t + \lambda x - 1) - R(x). \quad (29)$$

From this we form a second symmetric function of Y_0 and Y_1 :

$$(S(Y_0) - xY_0)(S(Y_1) - xY_1) = -R(x)(R(x) + 2x + 2\bar{x} - 1/t).$$

Extracting the positive part in x gives

$$\lambda x + x^2 = -R(x)(R(x) + 2x + 2\bar{x} - 1/t) + 2R'(0).$$

This discrete differential equation can be solved using the method of Section 3.4. One obtains a cubic equation for $R'(0)$, and one of degree 6 for $R(x)$. Finally, saying that $R(x)$ equals $-tQ(0, 0)$ modulo $(1 + \lambda x + x^2)$ (see (28)) gives, by taking the above equation modulo $(1 + \lambda x + x^2)$,

$$t^2Q(0, 0)^2 + (2\lambda t + 1)Q(0, 0) = 2R'(0) + 1,$$

and proves the algebraicity of $Q(0, 0)$. One can then compute an equation for $S(y)$ using the second part of (29). The algebraicity of $Q(x, y)$ follows using the equation (27) we started from. \square

To complete our guided tour of (conjecturally) algebraic models, let us mention three models that were discovered by Kauers and Yatchak [32]:

$$\begin{array}{ccc} \begin{array}{c} 2 \\ \swarrow \uparrow \\ 1 \leftarrow \oplus \rightarrow 2 \\ \searrow \downarrow \\ 1 \end{array} & \begin{array}{c} 1 \\ \swarrow \uparrow \\ 2 \leftarrow \oplus \rightarrow 1 \\ \searrow \downarrow \\ 2 \end{array} & \begin{array}{c} 2 \\ \swarrow \uparrow \\ 2 \leftarrow \oplus \rightarrow 1 \\ \searrow \downarrow \\ 1 \end{array} \end{array}$$

The weights indicate multiplicities. All three models have a group of order 10. We can solve the first two using half-orbit sums, as in [16, Sec. 6], and we find them to be algebraic indeed. Regarding the third model, its algebraicity should follow from a complex analysis approach [44], as for Gessel's model in [8].

Finally, let us recall that another natural way of weighting steps comes from the study of Markov chains in the quadrant [23]. In this setting each step is weighted by its probability. It may be possible to construct a Markov chain with Gessel's steps and stationary distribution given by an algebraic generating function, provable by the tools of this paper. This would parallel the case of Kreweras' walks: they first appeared as a counting problem in 1965 [35], and then 20 years later, independently,

as a Markov chain with algebraic stationary distribution [26]. Many years later, the algebraicity of both problems was proved by a common elementary approach [12], using ideas that we have extended in this paper.

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