Asymptotic expansions of zeros of a partial theta function

Vladimir Petrov Kostov

Université de Nice, Laboratoire de Mathématiques, Parc Valrose,
06108 Nice Cedex 2, France, e-mail: vladimir.kostov@unice.fr

Abstract

The bivariate series $\theta(q,x) := \sum_{j=0}^{\infty} q^{j(j+1)/2} x^j$ defines a *partial theta function*. For fixed q ($|q| < 1$), $\theta(q,.)$ is an entire function. We prove a property of stabilization of the coefficients of the Laurent series in q of the zeros of θ. The coefficients r_k of the stabilized series are positive integers. They are the elements of a known increasing sequence satisfying the recurrence relation $r_k = \sum_{\nu=1}^{\infty}(-1)^{\nu-1}(2\nu+1)r_{k-\nu(\nu+1)/2}$.

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1 Introduction

The bivariate series $\theta(q,x) := \sum_{j=0}^{\infty} q^{j(j+1)/2} x^j$ (where $(q,x) \in \mathbb{C}^2$, $|q| < 1$) defines a *partial theta function*. For fixed q, $\theta(q,.)$ is an entire function.

Different domains in which the partial theta function finds applications are asymptotic analysis (see [2]), the Ramanujan type q-series (see [16]), the theory of (mock) modular forms (see [3]), statistical physics and combinatorics (see [15]) and also some questions concerning hyperbolic polynomials (i.e. real polynomials with all roots real, see [8], [11] and [9]). The latter are connected to a problem considered by Hardy, Petrovitch and Hutchinson, see [5], [7], [12] and [13]. Other properties of θ are considered in [1].

In the article [10] the zeros of θ are presented in the form $-\xi_j = -1/q^j \Delta_j$. It is shown in [10] that Δ_j are *formal power series (FPS)* of the form $1 + O(q)$. For $|q| \leq 0.108$ all zeros $-\xi_j$ are distinct (see [10]), so they depend analytically on q and the series Δ_j converge. In the present paper we present the zeros of θ also as Laurent series of the form $-q^{-j} + a_j q^{\kappa_j} + o(q^{\kappa_j})$, where $\mathbb{Z} \ni \kappa_j > -j$.

Theorem 1. *(1) The series Δ_j is of the form $1 + (-1)^j q^{j(j+1)/2} \Phi_j(q)$, where $\Phi_j := 1 + O(q)$ is an FPS in q. Hence $a_j = (-1)^j$ and $\kappa_j = j(j-1)/2$.*

(2) Represent the zeros $-\xi_j$ in the form $-\xi_j = -q^{-j} + (-1)^j q^{j(j-1)/2}(1 + \sum_{k=1}^{\infty} g_{j,k} q^k)$. There exists an FPS of the form (H) : $1 + \sum_{k=1}^{\infty} r_k q^k$, $r_k \in \mathbb{Z}$, such that $g_{j,k} = r_k$ for $k = 1, \ldots, j$ and $j \geq 2$. The coefficients r_k satisfy the following recurrency relation (we set $r_k := 0$ for $k < 0$ and $r_0 := 1$):

$$r_k = \sum_{\nu=1}^{\infty}(-1)^{\nu-1}(2\nu+1)r_{k-\nu(\nu+1)/2} \ . \tag{1}$$

Remarks 2. (1) The relation (1) implies that all numbers r_k are integer. The first 20 of them are equal to

$$3 \ , \ 9 \ , \ 22 \ , \ 51 \ , \ 108 \ , \ 221 \ , \ 429 \ , \ 810 \ , \ 1479 \ , \ 2640 \ , \ 4599 \ , \ 7868 \ ,$$

$$13209 \ , \ 21843 \ , \ 35581 \ , \ 57222 \ , \ 90882 \ , \ 142769 \ , \ 221910 \ , \ 341649 \ .$$

The sequence $\{r_k\}$ is known (see A000716 in Sloane's database of integer sequences [14]). Set $(q)_\infty := \prod_{k=1}^{\infty}(1 - q^k)$. It is shown in [4] that $1 + \sum_{k=1}^{\infty} r_k q^k = 1/(q)_\infty^3$ and that, using Jacobi's triple product (see [6], p. 377), one obtains the equality

$$(q)_\infty^3 = \sum_{j=0}^{\infty} (-1)^j (2j + 1) q^{j(j+1)/2} \ . \tag{2}$$

With the help of this equality the authors of [4] obtain the recurrence formula (1).

(2) The functions $M(q) := (q)_\infty^3 = 1 - 3q + 5q^3 - \cdots$ and $(q)_\infty = 1 - q - q^2 + \cdots$ are positive valued for $q \in (-1, 1)$ and flat at ± 1 (all this follows from $(q)_\infty := \prod_{k=1}^{\infty}(1 - q^k)$). They are decreasing on $[0, 1)$ because every factor $1 - q^k$ is positive valued and decreasing. Their derivatives at 0 are negative, therefore they attain their maximal values (which are > 1) at some point in $(-1, 0)$ (the same for both of them). Their second derivatives at negative points close to 0 are negative, so they have inflection points in $(-1, 0)$. At 0 the function M has an inflection point. The function $(q)_\infty$ has an inflection point in $(0, 1)$ (because it is flat at 1 and $((q)_\infty)''|_{q=0} < 0$).

The above remarks imply the following proposition:

Proposition 3. *(1) The integer sequences $\{r_k\}$ ($k \geq 0$), $\{r_{k+1} - r_k\}$ ($k \geq 0$) and $\{r_{k+2} - 2r_{k+1} + r_k\}$ ($k \geq 1$) are increasing.*
(2) The radius of convergence of the Taylor series (H) of Theorem 1 equals 1.

Proof of Proposition 3. Set $R(q) := \sum_{j=0}^{\infty} q^j$, $1/(q)_\infty^3 := R^3 W$, i. e. $W = (\prod_{k=2}^{\infty} R(q^k))^3$. The numbers r_k are the coefficients of the series $1/(q)_\infty^3$. Denote by s_k, t_k and u_k the coefficients of the series $R^2 W$, RW and W. The equality $R = 1 + qR$ implies

$$r_{k+1} = r_k + s_{k+1} \quad , \quad s_{k+1} = s_k + t_{k+1} \quad \text{and} \quad t_{k+1} = t_k + u_{k+1} \ .$$

As $r_k > 0$, $s_k > 0$, $t_k > 0$ and $u_k > 0$ for $k \geq 2$, this proves statement (1). The radius of convergence of the right-hand side of (2) is 1 (and $(q)_\infty^3 > 0$ for $q \in [0, 1)$), therefore this is the case of the series (H) as well. This is statement (2). $\quad\square$

Proposition 4. *The function M is convex on $(0, 1)$.*

The proposition is proved in Section 3.

Remark 5. The first 10 coefficients of the FPS Δ_1, Δ_2, Δ_3 are listed below:

1	-1	-1	-1	-2	-4	-10	-25	-66	-178
1	0	0	1	3	9	24	66	180	498
1	0	0	0	0	0	-1	-3	-9	-22

The reason why part (2) of the theorem does not hold true for $j = 1$ is explained in Remark 7. It would be interesting to (dis)prove that for $k \geq 1$ the sequence $\{r_{k+1}/r_k\}$ is decreasing.

2 Proof of Theorem 1

Prove part (1) of the theorem. Consider the condition $\theta(q, x) = 0$. If instead of $-\xi_j$ one substitutes just $-q^{-j}$ for x in θ, then the negative powers of q cancel in $\theta(q, -q^{-j})$. Indeed, denote by λ_s the degree of the sth monomial of the Laurent series $\theta(q, -q^{-j})$ (i.e. the degree of the monomial $q^{s(s+1)/2}x^s|_{x=-q^{-j}}$). Hence $\lambda_s = (s^2 + s)/2 - js$.

Remark 6. The first j degrees λ_s decrease from $\lambda_0 = 0$ to $\lambda_{j-1} = -j(j-1)/2$. Starting from λ_j, the degrees increase. One has

$$(A) \; : \quad \lambda_\nu = \lambda_{2j-1-\nu} \;\; \text{for} \;\; \nu = 0, \ldots, j-1 \; .$$

$$(B) \; : \quad \lambda_{s+1} - \lambda_s = s + 1 - j \;\; (\text{for} \;\; s \geq j-1 \;\; \text{this gives} \;\; 0, 1, 3, 6, 10, \ldots) \; .$$

The expansion of $\theta(q, -q^{-j})$ contains the monomials $(-1)^\nu q^{\lambda_\nu}$ and $(-1)^{2j-1-\nu} q^{\lambda_\nu}$ which cancel. When one substitutes $-q^{-j} + a_j q^{\kappa_j} + o(q^{\kappa_j})$ for x in $\theta(q, x)$, one gets

$$\Psi_s := q^{s(s+1)/2} x^s|_{x=-q^{-j}+a_j q^{\kappa_j}+o(q^{\kappa_j})} = (-1)^s q^{\lambda_s} + (-1)^{s-1} s a_j q^{\mu_s} + o(q^{\mu_s}) \; ,$$

where $\mu_s = s(s+1)/2 + (s-1)(-j) + \kappa_j$. The lowest value of μ_s is attained for and only for $s = j-1$ and $s = j$. (One has $\mu_{j-1} = \mu_j = -j(j-3)/2 + \kappa_j$.) In the Taylor series expansion of Ψ_{j-1} and Ψ_j the sum of the coefficients of the two monomials with this power of q equals

$$S := (-1)^j (j-1) a_j + (-1)^{j-1} j a_j = (-1)^{j-1} a_j \; .$$

The quantity $S_1 := S q^{\mu_j}$ must cancel with other monomials or with just another monomial from the expansion of $\theta(q, -\xi_j)$. This must be just one monomial, and it has to be $q^{\lambda_{2j}}$. Indeed, any monomial T in the expansion of Ψ_s for $s \neq j-1$ and $s \neq j$ which is not q^{λ_s}, has a degree higher than μ_j. The same holds true for the monomials of Ψ_{j-1} and Ψ_j which are different from $\pm q^{\lambda_j}$ and $(-1)^{s-1} s a_j q^{\mu_j}$, $s = j-1$ or j. On the other hand, the monomials q^{λ_s} cancel for $s = 0, \ldots, 2j-1$. Hence S_1 can cancel with (a) monomial(s) of degree at least λ_{2j}. The only monomial of degree λ_{2j} is $q^{\lambda_{2j}}$ (where $\lambda_{2j} = (4j^2 + 2j)/2 - 2j^2 = j$), and S_1 must cancel with it. Indeed, otherwise no monomial T cancels with it either, i.e. the quantity $\theta(q, \xi_j)$ is not identically equal to 0 which is a contradiction. Hence $(-1)^{j-1} a_j + 1 = 0$, i.e. $a_j = (-1)^j$, and

$$-j(j-3)/2 + \kappa_j = j \;\; \text{hence} \;\; \kappa_j = j(j-1)/2 \; .$$

Thus the zero $\xi_j = -q^{-j}/\Delta_j$ is of the form

$$-q^{-j} + (-1)^j q^{j(j-1)/2} + o(q^{j(j-1)/2}) = (-q^{-j})(1 - (-1)^j q^{j(j+1)/2} + o(q^{j(j+1)/2}))$$

which means that $\Delta_j = 1 + (-1)^j q^{j(j+1)/2} + o(q^{j(j+1)/2})$. This proves part (1).

We prove part (2) for $j \geq 4$. For $j = 2$ and $j = 3$ its proof is contained in Remark 7 below. Recall that $-\xi_j$ is represented in the form $-q^{-j} + (-1)^j(q^{\kappa_j} + g_{j,1} q^{\kappa_j+1} + g_{j,2} q^{\kappa_j+2} + o(q^{\kappa_j+2}))$. When one computes the expansion of Ψ_s in powers of q, one applies the formula of the Newton binomial to the Laurent series of $-\xi_j$:

(i) The term containing the first power of q obtained in the expansion of Ψ_s is $(-1)^s q^{\lambda_s}$.

(ii) The next ones are of the form $(-1)^{s-1} s q^{\lambda_s+j+\kappa_j}$ and $(-1)^{s-1} s g_{j,\nu} q^{\lambda_s+j+\kappa_j+\nu}$, $\nu = 1, \ldots, j + \kappa_j - 1$.

(iii) To compute the higher powers of q one has to take into account the monomial $(-1)^{s-2}(s(s-1)/2)q^{\lambda_s+2j+2\kappa_j}$ and then other monomials in which participate two or more of the factors q^{κ_j}, $g_{j,1}q^{\kappa_j+1}$, $g_{j,2}q^{\kappa_j+2}$ etc.

(iv) Consider the sum $\Psi_{j-1-l}+\Psi_{j+l}$, $l = 0, 1, \ldots, j-1$. Its terms mentioned in (i) cancel. Its terms mentioned in (ii) equal $(-1)^{j+l-1}(2l+1)q^{\lambda_{j+l}+j+\kappa_j}$ and $(-1)^{j+l-1}(2l+1)g_{j,\nu}q^{\lambda_{j+l}+j+\kappa_j+\nu}$, $\nu = 1, \ldots, j+\kappa_j-1$.

(v) The lowest power of q in the expansion of Ψ_{2j+r}, $r = 0, 1, \ldots$, is $q^{\lambda_{2j+r}}$, where $\lambda_{2j+r} = jr + j + r(r+1)/2 \geq j$.

The following matrix illustrates the case $j = 4$. We present the quantity $-\xi_4$ in the form

$$-\xi_4 = -q^{-4} + aq^6 + bq^7 + cq^8 + dq^9 + hq^{10} + uq^{11} + \cdots$$

with $b = g_{j,1}$, $c = g_{j,2}$, ... (In fact, we know that $a = 1$, but for the moment we prefer to keep a as unknown quantity.) The first column indicates the power of q. The next columns show the coefficients of the corresponding powers of q in the expansions of Ψ_s for $s = 0, \ldots, 9$. This means, in particular, that

$$\Psi_3 = -q^{-6} + 3aq^4 + 3bq^5 + 3cq^6 + 3dq^7 + 3hq^8 + 3uq^9 + \cdots .$$

The entries ± 1 of the matrix are the coefficients of $(-1)^s q^{\lambda_s}$.

	Ψ_0	Ψ_1	Ψ_2	Ψ_3	Ψ_4	Ψ_5	Ψ_6	Ψ_7	Ψ_8	Ψ_9
9		c	$-2h$	$3u$	$-4u$	$5h$	$-6c$			-1
8		b	$-2d$	$3h$	$-4h$	$5d$	$-6b$			
7		a	$-2c$	$3d$	$-4d$	$5c$	$-6a$			
6			$-2b$	$3c$	$-4c$	$5b$				
5			$-2a$	$3b$	$-4b$	$5a$				
4				$3a$	$-4a$				1	
1 to 3										
0	1						-1			
-1										
-2										
-3		-1					1			
-4										
-5			1			-1				
-6				-1	1					

As $\Psi_8 = q^4 + o(q^9)$ and $\Psi_9 = -q^9 + o(q^9)$, one obtains the following system of equations from which one finds the quantities $a, \ldots, h$ (one writes the conditions that the sums of the coefficients of q^s are 0, $s = 9, 8, \ldots, 4$; the coefficient of q^5, for instance, equals $-2a+3b-4b+5a = -b+3a$):

$$
\begin{aligned}
-u \quad +3h \qquad -5c \qquad\qquad -1 &= 0 \qquad\qquad -c \quad +3b \qquad\qquad\qquad = 0 \\
-h \quad +3d \qquad -5b \qquad\qquad\quad\ &= 0 \qquad\qquad -b \quad +3a \qquad\qquad = 0 \qquad (3) \\
-d \quad +3c \qquad -5a \qquad\quad &= 0 \qquad\qquad\qquad -a \quad +1 = 0
\end{aligned}
$$

The system is triangular and one readily finds that

$$(a , b , c , d , h , u) = (1 , 3 , 9 , 22 , 51 , 107) .$$

4

Now we describe what the analogs of the above matrix and the above system of equations are when j is arbitrary. The analogs of the lines of the powers 4 and 9 of the matrix are the lines of the powers j and $2j+1$. Their rightmost indicated entries equal $(-1)^{2j} = 1$ and $(-1)^{2j+1} = -1$. The columns of Ψ_{j-3}, ..., Ψ_{j+2} are given in the next matrix (j is presumed greater than 4):

	Ψ_0	Ψ_{j-3}	Ψ_{j-2}	Ψ_{j-1}	Ψ_j	Ψ_{j+1}	Ψ_{j+2}	Ψ_{2j-1}	Ψ_{2j}
$j+5$		$(j-3)c$	$-(j-2)h$	$(j-1)u$	$-ju$	$(j+1)h$	$-(j+2)c$		
$j+4$		$(j-3)b$	$-(j-2)d$	$(j-1)h$	$-jh$	$(j+1)d$	$-(j+2)b$		
$j+3$		$(j-3)a$	$-(j-2)c$	$(j-1)d$	$-jd$	$(j+1)c$	$-(j+2)a$		
$j+2$			$-(j-2)b$	$(j-1)c$	$-jc$	$(j+1)b$			
$j+1$			$-(j-2)a$	$(j-1)b$	$-jb$	$(j+1)a$			
j				$(j-1)a$	$-ja$				1
1 to $j-1$									
0	1							-1	
λ_{j+2}		$(-1)^{j-3}$					$(-1)^{j+2}$		
$\lambda_{j+2} - 1$									
λ_{j+1}			$(-1)^{j-2}$			$(-1)^{j+1}$			
λ_j				$(-1)^{j-1}$	$(-1)^j$				

If one writes then the analog (E) of system (3), only its first equation will be slightly different: $-u + 3h - 5c = 0$. Hence

$$(\, a \, , \, b \, , \, c \, , \, d \, , \, h \, , \, u \,) \; = \; (\, 1 \, , \, 3 \, , \, 9 \, , \, 22 \, , \, 51 \, , \, 108 \,) \, .$$

Denote by (E_k) the equation of system (E) expressing the fact that the sum of the coefficients of q^k must be 0. (In system (3) we have written explicitly equations $(E_9) - (E_4)$.) Fix $j_0 \in \mathbb{N}$. It is easy to see that for $j \geq j_0$ the equations $(E_j) - (E_{j+j_0})$ do not depend on j. Their form is (1) (this follows from statement (B) of Remark 6). Therefore the values of the first $j_0 + 1$ quantities a, b, ..., do not depend on j for $j \geq j_0$. Part (2) is proved.

Remark 7. Part (2) of the theorem does not hold true for $j = 1$ for the following reason. Consider the matrices of coefficients of the Laurent series Ψ_j. Recall that $\Psi_s = (-1)^s + (-1)^{s-1} s a_j q^{\mu_s} + \cdots$ with $\mu_s \geq j(j+1)/2$ for $s \geq 2j - 1$. For $j > 3$ the inequality $j(j+1)/2 > 2j$ holds true which means that the columns of Ψ_{2j-1} and Ψ_{2j} (considered only for degrees of q ranging from $-j(j-1)/2$ to $2j$) contain as only nonzero entries ± 1 in the rows corresponding respectively to degree 0 and j. (The columns of Ψ_ν for $\nu > 2j$ contain only zeros in these rows.) Thus the system to which $(a, b, \dots)$ is solution is completely defined (and according to one and the same rule) by the columns of the matrix containing the coefficients of Ψ_0, ..., Ψ_{2j}.

For $j = 2$ and $j = 3$ one has to check directly that the system is of the same form, i. e.

$$-a + 1 = 0 \quad -b + 3a = 0 \quad -c + 3b = 0 \quad \text{and}$$
$$-a + 1 = 0 \quad -b + 3a = 0 \quad -c + 3b = 0 \quad -d + 3c - 5a = 0 \, .$$

For $j = 1$ the system becomes $-a + 1 = 0$, $-b + 2a = 0$ and the value of b is not 3, but 2.

3 Proof of Proposition 4

Set $M = e^{3L}$, where $3L := \ln M = 3\sum_{k=1}^{\infty} \ln(1 - q^k)$. Hence $M'' = (9(L')^2 + 3L'')e^{3L}$ and it suffices to show that $3(L')^2 + L'' > 0$ on $(0, 1)$. Using Taylor series at 0 of the logarithm one

gets $L = -\sum_{j=1}^{\infty}(1/j)\sum_{k=1}^{\infty}q^{jk} = -\sum_{j=1}^{\infty}q^j/j(1-q^j) = -\sum_{j=1}^{\infty}(-1/j + 1/j(1-q^j))$, so

$$L' = -\sum_{j=1}^{\infty}q^{j-1}/(1-q^j)^2 \qquad \text{and} \qquad L'' = U + 2V \qquad , \text{ where}$$

$$U := -\sum_{j=2}^{\infty}(j-1)q^{j-2}/(1-q^j)^2 \quad , \quad V := -\sum_{j=1}^{\infty}jq^{2j-2}/(1-q^j)^3 \ .$$

In the expansion of $3(L')^2$ all terms are of the form $q^s/(1-q^k)^2(1-q^l)^2$. For s fixed denote by q^sS_s the sum of all these terms, i.e. $3(L')^2 = \sum_{s=0}^{\infty}q^sS_s$. One has

$$S_s = \begin{cases} 6\sum_{\nu=1}^{s/2}1/(1-q^\nu)^2(1-q^{s+2-\nu})^2 + 3/(1-q^{s/2+1})^4 & \text{for } s \text{ even} \\[2mm] 6\sum_{\nu=1}^{(s+1)/2}1/(1-q^\nu)^2(1-q^{s+2-\nu})^2 & \text{for } s \text{ odd.} \end{cases}$$

On the other hand, $L'' = -\sum_{s=0}^{\infty}q^sT_s$, where $T_s := (s+1)/(1-q^{s+2})^2 + 2(s+1)q^s/(1-q^{s+1})^3$. To prove that $3(L')^2 + L'' > 0$ it suffices to show that for $s = 0, 1, \ldots$ one has $S_s > T_s$ for $q \in (0,1)$.

Observe that $T_s < 3(s+1)/(1-q^{s+1})^3$. One has $1/(1-q^\nu)^2(1-q^{s+2-\nu})^2 > 1/(1-q^{s+1})^3$, $1/(1-q^{s/2+1})^4 > 1/(1-q^{s+1})^3$ and $1/(1-q^\nu)^2(1-q^{s+2-\nu})^2 > 1/(1-q^{s+1})^3$ for the values of the indices s and ν as indicated above. Hence $S_s > 3(s+1)/(1-q^{s+1})^3 > T_s$ from which the proposition follows.

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