Preprint, arXiv:1504.01608

NATURAL NUMBERS REPRESENTED BY $\lfloor x^2/a\rfloor+\lfloor y^2/b\rfloor+\lfloor z^2/c\rfloor$

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Abstract. Let a,b,c be positive integers. It is known that there are infinitely many positive integers not representated by $ax^2+by^2+cz^2$ with $x,y,z\in\mathbb{Z}$. In contrast, we conjecture that any natural number is represented by $\lfloor x^2/a\rfloor+\lfloor y^2/b\rfloor+\lfloor z^2/c\rfloor$ with $x,y,z\in\mathbb{Z}$ if $(a,b,c)\ne(1,1,1),(2,2,2)$, and that any natural number is represented by $\lfloor T_x/a\rfloor+\lfloor T_y/b\rfloor+\lfloor T_z/c\rfloor$ with $x,y,z\in\mathbb{Z}$, where T_x denotes the triangular number $x(x+1)/2$. We confirm this general conjecture in some special cases; in particular, we prove that

$$\left\{x^2+y^2+\left\lfloor\frac{z^2}{5}\right\rfloor:\ x,y,z\in\mathbb{Z}\text{ and }2\nmid y\right\}=\{1,2,3,\ldots\}$$

and

$$\left\{\left\lfloor\frac{x^2}{m}\right\rfloor+\left\lfloor\frac{y^2}{m}\right\rfloor+\left\lfloor\frac{z^2}{m}\right\rfloor:\ x,y,z\in\mathbb{Z}\right\}=\{0,1,2,\ldots\}\quad\text{for } m=5,6,15.$$

We also pose several conjectures for further research; for example, we conjecture that any integer can be written as $x^4-y^3+z^2$, where x, y and z are positive integers.

1. Introduction

Let $\mathbb{N}=\{0,1,2,\ldots\}$ be the set of all natural numbers (nonnegative integers). A well-known theorem of Lagrange asserts that each $n\in\mathbb{N}$ can be written as the sum of four squares. It is known that for any $a,b,c\in\mathbb{Z}^+=\{1,2,3,\ldots\}$ there are infinitely many positive integers not represented by $ax^2+by^2+cz^2$ with $x,y,z\in\mathbb{Z}$.

A classical theorem of Gauss and Legendre states that $n\in\mathbb{N}$ can be written as the sum of three squares if and only if it is not of the form $4^k(8l+7)$ with

2010 *Mathematics Subject Classification*. Primary 11E25; Secondary 11B75, 11D85, 11E20, 11P32.

Keywords. Representations of integers, the floor function, squares, polygonal numbers.

Supported by the National Natural Science Foundation (Grant No. 11171140) of China.

$k, l \in \mathbb{N}$. Consequently, for each $n \in \mathbb{N}$ there are $x, y, z \in \mathbb{Z}$ such that

$$8n+3 = (2x+1)^2+(2y+1)^2+(2z+1)^2, \text{ i.e., } n = \frac{x(x+1)}{2}+\frac{y(y+1)}{2}+\frac{z(z+1)}{2}.$$

Those $T_x = x(x+1)/2$ with $x \in \mathbb{Z}$ are called *triangular numbers*. For $m = 3, 4, \ldots$, those *m-gonal numbers* (or *polygonal numbers of order m*) are given by

$$p_m(n) := (m-2)\binom{n}{2}+n = \frac{(m-2)n^2-(m-4)n}{2} \ (n = 0, 1, 2, \ldots),$$

and those $p_m(x)$ with $x \in \mathbb{Z}$ are called *generalized m-gonal numbers*. Cauchy's polygonal number theorem states that for each $m = 5, 6, \ldots$ any $n \in \mathbb{N}$ can be written as the sum of m polygonals of order m (see, e.g., [N96, pp. 3-35] and [MW, pp. 54-57].)

For any $k \in \mathbb{Z}$, we clearly have

$$T_k = \frac{(2k+1)^2-1}{8} = \left\lfloor \frac{(2k+1)^2}{8} \right\rfloor.$$

As any natural number can be expressed as the sum of three triangular numbers, each $n \in \mathbb{N}$ can be written as $\lfloor x^2/8 \rfloor + \lfloor y^2/8 \rfloor + \lfloor z^2/8 \rfloor$ with $x, y, z \in \mathbb{Z}$. B. Farhi [F13] conjectured that any $n \in \mathbb{N}$ can be expressed the sum of three elements of the set $\{\lfloor x^2/3 \rfloor : x \in \mathbb{Z}\}$ and showed this for $n \not\equiv 2 \pmod{24}$. The conjecture was later proved by S. Mezroui, A. Azizi and M. Ziane [MAZ] in 2014 via the known formula for the number of ways to write n as the sum of three squares. In [F] Farhi provided an elementary proof of the conjecture and made a further conjecture that for each $a = 3, 4, 5, \ldots$ any $n \in \mathbb{N}$ can be written as the sum of three elements of the set $\{\lfloor x^2/a \rfloor : x \in \mathbb{Z}\}$. This general conjecture of Farhi has been solved for $a = 3, 4, 7, 8, 9$ (cf. [HKR]).

Motivated by the above work, we pose the following general conjecture based on our computation.

Conjecture 1.1. *Let $a, b, c \in \mathbb{Z}^+$ with $a \leqslant b \leqslant c$.*

(i) *If the triple (a, b, c) is neither $(1, 1, 1)$ nor $(2, 2, 2)$, then for any $n \in \mathbb{N}$ there are $x, y, z \in \mathbb{Z}$ such that*

$$n = \left\lfloor \frac{x^2}{a} \right\rfloor + \left\lfloor \frac{y^2}{b} \right\rfloor + \left\lfloor \frac{z^2}{c} \right\rfloor = \left\lfloor \frac{x^2}{a} + \frac{y^2}{b} \right\rfloor + \left\lfloor \frac{z^2}{c} \right\rfloor = \left\lfloor \frac{x^2}{a} \right\rfloor + \left\lfloor \frac{y^2}{b} + \frac{z^2}{c} \right\rfloor. \quad (1.1)$$

(ii) *For any $n \in \mathbb{N}$, there are $x, y, z \in \mathbb{Z}$ such that*

$$n = \left\lfloor \frac{T_x}{a} \right\rfloor + \left\lfloor \frac{T_y}{b} \right\rfloor + \left\lfloor \frac{T_z}{c} \right\rfloor = \left\lfloor \frac{T_x}{a} + \frac{T_y}{b} \right\rfloor + \left\lfloor \frac{T_z}{c} \right\rfloor = \left\lfloor \frac{T_x}{a} \right\rfloor + \left\lfloor \frac{T_y}{b} + \frac{T_z}{c} \right\rfloor. \quad (1.2)$$

Moreover, if the triple (a,b,c) *is not among*

$$(1,1,1),\ (1,1,3),\ (1,1,7),\ (1,3,3),\ (1,7,7),\ (3,3,3),$$

then for any $n\in\mathbb{N}$ *there are* $x,y,z\in\mathbb{Z}$ *such that*

$$\begin{aligned}n&=\left\lfloor\frac{x(x+1)}{a}\right\rfloor+\left\lfloor\frac{y(y+1)}{b}\right\rfloor+\left\lfloor\frac{z(z+1)}{c}\right\rfloor\\&=\left\lfloor\frac{x(x+1)}{a}+\frac{y(y+1)}{b}\right\rfloor+\left\lfloor\frac{z(z+1)}{c}\right\rfloor\\&=\left\lfloor\frac{x(x+1)}{a}\right\rfloor+\left\lfloor\frac{y(y+1)}{b}+\frac{z(z+1)}{c}\right\rfloor.\end{aligned}\tag{1.3}$$

In this paper we establish some results in the direction of Conjecture 1.1.

Theorem 1.1. (i) *For each* $m=4,6$, *any* $n\in\mathbb{N}$ *can be written as* $x^2+(2y)^2+\lfloor z^2/m\rfloor$ *with* $x,y,z\in\mathbb{Z}$. *Also, any* $n\in\mathbb{Z}^+$ *can be expressed as* $x^2+y^2+\lfloor z^2/5\rfloor$ *with* $x,y,z\in\mathbb{Z}$ *and* $2\nmid y$.

(ii) *For any* $\delta\in\{0,1\}$, *any* $n\in\mathbb{Z}^+$ *can be expressed as* $x^2+y^2+\lfloor z^2/8\rfloor$ *with* $x,y,z\in\mathbb{Z}$ *and* $y\equiv\delta\pmod 2$.

(iii) *For each* $m=2,3,9,21$, *any* $n\in\mathbb{N}$ *can be written as* $x^2+y^2+\lfloor z^2/m\rfloor$ *with* $x,y,z\in\mathbb{Z}$. *Also, for each* $m=3,4,6$ *we have*

$$\left\{x^2+y^2+\left\lfloor\frac{z(z+1)}{m}\right\rfloor:\ x,y,z\in\mathbb{Z}\right\}=\mathbb{N}.\tag{1.4}$$

(iv) *For each* $m=5,6,15$, *we have*

$$\left\{x^2+\left\lfloor\frac{y^2}{m}\right\rfloor+\left\lfloor\frac{z^2}{m}\right\rfloor:\ x,y,z\in\mathbb{Z}\right\}=\left\{\left\lfloor\frac{x^2}{m}\right\rfloor+\left\lfloor\frac{y^2}{m}\right\rfloor+\left\lfloor\frac{z^2}{m}\right\rfloor:\ x,y,z\in\mathbb{Z}\right\}=\mathbb{N}.\tag{1.5}$$

(v) *We have*

$$\left\{T_x+T_y+\left\lfloor\frac{T_z}{3}\right\rfloor:\ x,y,z\in\mathbb{Z}\right\}=\left\{\left\lfloor\frac{T_x}{3}\right\rfloor+\left\lfloor\frac{T_y}{3}\right\rfloor+\left\lfloor\frac{T_z}{3}\right\rfloor:\ x,y,z\in\mathbb{Z}\right\}=\mathbb{N}$$

and

$$\left\{x(x+1)+y(y+1)+\left\lfloor\frac{z(z+1)}{4}\right\rfloor:\ x,y,z\in\mathbb{Z}\right\}=\mathbb{N}.\tag{1.6}$$

Remark 1.1. As $x^2=(3x)^2/9$, Theorem 1.1(iii) with $m=9$ implies that any $n\in\mathbb{N}$ can be written as $\lfloor x^2/9\rfloor+\lfloor y^2/9\rfloor+\lfloor z^2/9\rfloor$ with $x,y,z\in\mathbb{Z}$. Theorem 1.1(iv) confirms Farhi's conjecture for $a=5,6,15$. The author [S15a, Remark 1.8] conjectured that for any $n\in\mathbb{N}$ we can write $20n+9$ as $5x^2+5y^2+(2z+1)^2$ with $x,y,z\in\mathbb{Z}$; it is easy to see that (1.4) for $m=5$ follows from this conjecture. As $\{2T_x+2T_y+T_z:\ x,y,z\in\mathbb{Z}\}=\mathbb{N}$ by Liouville's result, any $n\in\mathbb{N}$ can be written as $T_x+T_y+T_z/2$ with $x,y,z\in\mathbb{Z}$.

As a supplement to parts (i)-(iii) of Theorem 1.1, we pose the following conjecture.

Conjecture 1.2. (i) *Let $n \in \mathbb{Z}^+$. Then, for any integer $m > 6$ and $\delta \in \{0,1\}$, we have $n = x^2 + y^2 + \lfloor z^2/m \rfloor$ for some $x, y, z \in \mathbb{Z}$ with $y \equiv \delta \pmod 2$.*

(ii) *For any integer $m > 2$, we have*

$$\left\{x^2 + (2y)^2 + \left\lfloor \frac{z(z+1)}{m} \right\rfloor :\ x, y, z \in \mathbb{Z}\right\} = \mathbb{N}.$$

For each $m = 4, 5, \ldots$, any positive integer n can be represented by $x^2 + y^2 + \lfloor z(z+1)/m \rfloor$ with $x, y, z \in \mathbb{Z}$ and $2 \nmid y$.

Remark 1.2. It is known that $\{x^2 + (2y)^2 + T_z :\ x, y, z \in \mathbb{Z}\} = \{x^2 + (2y)^2 + 2T_z :\ x, y, z \in \mathbb{Z}\} = \mathbb{N}$ (cf. [S07, Section 4]).

For any $a \in \mathbb{Z}^+$, clearly

$$\left\{\left\lfloor \frac{x^2}{a} \right\rfloor :\ x \in \mathbb{Z}\right\} \supseteq \left\{\left\lfloor \frac{(ax)^2}{a} \right\rfloor = ax^2 :\ x \in \mathbb{Z}\right\}.$$

Theorem 1.2. (i) *For each $m = 2, 3, 4, 5$ we have*

$$\left\{x^2 + 2y^2 + \left\lfloor \frac{z^2}{m} \right\rfloor :\ x, y, z \in \mathbb{Z}\right\} = \mathbb{N}.$$

(ii) *For each $m = 3, 4, 6, 8$, we have*

$$\left\{x^2 + 3y^2 + \left\lfloor \frac{z^2}{m} \right\rfloor :\ x, y, z \in \mathbb{Z}\right\} = \mathbb{N}.$$

(iii) *We have*

$$\left\{x^2 + 5y^2 + \left\lfloor \frac{z^2}{8} \right\rfloor :\ x, y, z \in \mathbb{Z}\right\} = \left\{x^2 + 6y^2 + \left\lfloor \frac{z^2}{4} \right\rfloor :\ x, y, z \in \mathbb{Z}\right\} = \mathbb{N}.$$

(iv) *We have*

$$\left\{2x^2 + 2y^2 + \left\lfloor \frac{z^2}{8} \right\rfloor :\ x, y, z \in \mathbb{Z}\right\} = \left\{2x^2 + 3y^2 + \left\lfloor \frac{z^2}{3} \right\rfloor :\ x, y, z \in \mathbb{Z}\right\} = \mathbb{N}$$

and

$$\left\{2x^2 + \left\lfloor \frac{y^2}{2} \right\rfloor + \left\lfloor \frac{z^2}{3} \right\rfloor :\ x, y, z \in \mathbb{Z}\right\} = \mathbb{N}.$$

Our following conjecture involving the ceiling function is quite similar to Conjecture 1.1.

Conjecture 1.3. *Let $a,b,c\in\mathbb{Z}^+$ with $a\leqslant b\leqslant c$.*

(i) *If the triple (a,b,c) is not among $(1,1,1),(1,1,2),(1,1,5)$, then for any $n\in\mathbb{N}$ there are $x,y,z\in\mathbb{Z}$ such that*

$$n=\left\lceil\frac{x^2}{a}\right\rceil+\left\lceil\frac{y^2}{b}\right\rceil+\left\lceil\frac{z^2}{c}\right\rceil.$$

(ii) *We have*

$$\left\{\left\lceil\frac{T_x}{a}\right\rceil+\left\lceil\frac{T_y}{b}\right\rceil+\left\lceil\frac{T_z}{c}\right\rceil:\ x,y,z\in\mathbb{Z}\right\}=\mathbb{N}.$$

Moreover, if the triple (a,b,c) is neither $(1,1,1)$ nor $(1,1,3)$, then for any $n\in\mathbb{N}$ there are $x,y,z\in\mathbb{Z}$ such that

$$n=\left\lceil\frac{x(x+1)}{a}\right\rceil+\left\lceil\frac{y(y+1)}{b}\right\rceil+\left\lceil\frac{z(z+1)}{c}\right\rceil.$$

We are also able to deduce some results similar to Theorems 1.1-1.2 in the direction of Conjecture 1.3. Here we just collect few results of this type.

Theorem 1.3. (i) *For each $m=2,3,4,5,6,15$, we have*

$$\left\{\left\lceil\frac{x^2}{m}\right\rceil+\left\lceil\frac{y^2}{m}\right\rceil+\left\lceil\frac{z^2}{m}\right\rceil:\ x,y,z\in\mathbb{Z}\right\}=\mathbb{N}. \tag{1.7}$$

(ii) *We have*

$$\left\{x^2+3y^2+\left\lceil\frac{z^2}{2}\right\rceil:\ x,y,z\in\mathbb{Z}\right\}=\left\{x^2+3y^2+\left\lceil\frac{z^2}{10}\right\rceil:\ x,y,z\in\mathbb{Z}\right\}=\mathbb{N}. \tag{1.8}$$

(iii) *For any $n\in\mathbb{N}$, there are $x,y,z\in\mathbb{Z}$ such that*

$$n=x(x+1)+\frac{y(y+1)}{3}+\left\lceil\frac{z(z+1)}{3}\right\rceil. \tag{1.9}$$

Also, any $n\in\mathbb{N}$ can be written as $x(3x+1)+y(3y+1)+\lceil z(z+1)/3\rceil$ with $x,y,z\in\mathbb{Z}$, and hence

$$\left\{\left\lceil\frac{x(x+1)}{3}\right\rceil+\left\lceil\frac{y(y+1)}{3}\right\rceil+\left\lceil\frac{z(z+1)}{3}\right\rceil:\ x,y,z\in\mathbb{Z}\right\}=\mathbb{N}. \tag{1.10}$$

Remark 1.3. In contrast with (1.8), we note that 20142 is the first natural number not represented by $x^2+3y^2+\lfloor z^2/10\rfloor$ with $x,y,z\in\mathbb{Z}$.

Now we state another theorem.

Theorem 1.4. (i) *For any integer $a>1$, we have*

$$\left\{\left\lfloor\frac{x^2+y^2+z^2}{a}\right\rfloor:\ x,y,z\in\mathbb{Z}\right\}=\mathbb{N}$$

and

$$\left\{\left\lfloor\frac{x(x+1)+y(y+1)+z(z+1)}{a}\right\rfloor:\ x,y,z\in\mathbb{Z}\right\}=\mathbb{N}.$$

(ii) *Let $a\in\mathbb{Z}^+$. If a is odd, then any $n\in\mathbb{N}$ can be written as $x^2+y^2+z^2+\lfloor\frac{a}{2}(x+y+z)\rfloor$ with $x,y,z\in\mathbb{Z}$. If $3\nmid a$, then any $n\in\mathbb{N}$ can be written as $x^2+y^2+z^2+\lfloor\frac{a}{3}(x+y+z)\rfloor$ with $x,y,z\in\mathbb{Z}$.*

(iii) *For any $n\in\mathbb{N}$, there are $x,y,z\in\mathbb{Z}$ such that*

$$n=\frac{p_8(x)}{2}+\left\lceil\frac{p_8(y)}{2}\right\rceil+\left\lceil\frac{p_8(z)}{2}\right\rceil.$$

Hence

$$\{s(x)+s(y)+s(z):\ x,y,z\in\mathbb{Z}\}=\mathbb{N},\tag{1.11}$$

where

$$s(x):=\left\lceil\frac{p_8(-x)}{2}\right\rceil=x+\left\lceil 1.5x^2\right\rceil.$$

Remark 1.4. For $m=19,20$, we have $111\neq x^2+y^2+z^2+\lfloor(x+y+z)/m\rfloor$ for any $x,y,z\in\mathbb{Z}$.

The generalized octagonal numbers $p_8(x)=x(3x-2)$ $(x\in\mathbb{Z})$ have some properties similar to certain properties of squares. For example, recently the author [S16] showed that any $n\in\mathbb{N}$ can be written as the sum of four generalized octagonal numbers; this result is quite similar to Lagrange's theorem on sums of four squares. Note that

$$\left\lfloor\frac{p_8(x)}{2m}\right\rfloor=\left\lfloor\frac{4p_8(x)+1}{4m}\right\rfloor=\left\lfloor\frac{p_8(1-2x)}{4m}\right\rfloor\ \text{and}\ \left\lfloor\frac{p_8(x)}{m}\right\rfloor=\left\lfloor\frac{(3x-1)^2}{3m}\right\rfloor\tag{1.12}$$

for any $m\in\mathbb{Z}^+$ and $x\in\mathbb{Z}$.

Theorem 1.5. (i) *$n\in\mathbb{N}$ can be written as $p_8(x)+p_8(y)+2p_8(z)$ with $x,y,z\in\mathbb{Z}$ if and only if n does not belong to the set*

$$\left\{4^{k+2}q-\frac{2}{3}(4^k+2):\ k\in\mathbb{N}\ \text{and}\ q\in\mathbb{Z}^+\right\}.$$

Also, each nonnegative even number can be represented by $p_8(x)+2p_8(y)+4p_8(z)$ with $x,y,z\in\mathbb{Z}$. Consequently,

$$\left\{p_8(x)+\left\lfloor\frac{p_8(y)}{2}\right\rfloor+\left\lfloor\frac{p_8(z)}{2}\right\rfloor:\ x,y,z\in\mathbb{Z}\right\}=\mathbb{N}\tag{1.13}$$

and

$$\left\{\left\lfloor\frac{x^2}{3}\right\rfloor+\left\lfloor\frac{y^2}{6}\right\rfloor+\left\lfloor\frac{z^2}{6}\right\rfloor:\ x,y,z\in\mathbb{Z}\right\}=\mathbb{N}. \tag{1.14}$$

(ii) *We have*

$$\left\{p_8(x)+p_8(y)+\left\lfloor\frac{p_8(z)}{2}\right\rfloor:\ x,y,z\in\mathbb{Z}\right\}=\mathbb{N}, \tag{1.15}$$

hence

$$\left\{p_8(x)+p_8(y)+\left\lfloor\frac{p_8(z)}{8}\right\rfloor:\ x,y,z\in\mathbb{Z}\right\}=\mathbb{N} \tag{1.16}$$

and

$$\left\{\left\lfloor\frac{x^2}{3}\right\rfloor+\left\lfloor\frac{y^2}{3}\right\rfloor+\left\lfloor\frac{z^2}{6}\right\rfloor:\ x,y,z\in\mathbb{Z}\right\}=\mathbb{N}. \tag{1.17}$$

(iii) *For $n\in\mathbb{N}$ there are $x,y,z\in\mathbb{Z}$ such that*

$$n=p_8(x)+p_8(y)+\frac{p_8(z)}{4}. \tag{1.18}$$

(iv) *We have*

$$\left\{p_8(x)+p_8(y)+\left\lfloor\frac{p_8(z)}{5}\right\rfloor:\ x,y,z\in\mathbb{Z}\right\}=\mathbb{N} \tag{1.19}$$

and hence

$$\left\{\left\lfloor\frac{x^2}{3}\right\rfloor+\left\lfloor\frac{y^2}{3}\right\rfloor+\left\lfloor\frac{z^2}{15}\right\rfloor:\ x,y,z\in\mathbb{Z}\right\}=\mathbb{N}. \tag{1.20}$$

We are going to prove Theorems 1.1-1.2 in the next section, and show Theorems 1.3-1.4 in Section 3. Section 4 is devoted to our proof of Theorem 1.5. We pose some further conjectures in Section 5.

2. Proofs of Theorems 1.1-1.2

Lemma 2.1. *Suppose that $n\in\mathbb{Z}^+$ is not a power of two. Then there are $x,y,z\in\mathbb{Z}$ with $|x|<n$, $|y|<n$ and $|z|<n$ such that $x^2+y^2+z^2=n^2$.*

Proof. In 1907 Hurwitz (cf. [D99, p. 271]) showed that

$$\begin{aligned}&|\{(x,y,z)\in\mathbb{Z}^3:\ x^2+y^2+z^2=n^2\}|\\ =&6\prod_{p>2}\left(\frac{p^{\mathrm{ord}_p(n)+1}-1}{p-1}+(-1)^{(p+1)/2}\frac{p^{\mathrm{ord}_p(m)}-1}{p-1}\right),\end{aligned} \tag{2.1}$$

where $\mathrm{ord}_p(n)$ is the order of n at the prime p. Note that

$$(\pm n)^2+0^2+0^2=0^2+(\pm n)^2+0^2=0^2+0^2+(\pm n)^2.$$

As n has an odd prime p, by (2.1) we have

$$|\{(x,y,z)\in\mathbb{Z}^3:\ x^2+y^2+z^2=n^2\}|\geqslant 6\frac{p^{\mathrm{ord}_p(n)+1}-p^{\mathrm{ord}_p(n)}}{p-1}\geqslant 6p>8$$

and hence there are $x,y,z\in\mathbb{Z}$ with $x^2,y^2,z^2\neq n^2$ such that $x^2+y^2+z^2=n^2$. This concludes the proof. □

Lemma 2.2. (i) *Let u and v be integers with u^2+v^2 a positive multiple of* 5. *Then $u^2+v^2=x^2+y^2$ for some $x,y\in\mathbb{Z}$ with $5\nmid xy$.*

(ii) *For any $n\in\mathbb{N}$ with $n\equiv\pm6\pmod{20}$, we can write n as $5x^2+5y^2+z^2$ with $x,y,z\in\mathbb{Z}$ and $2\nmid z$.*

Remark 2.1. Parts (i) and (ii) of Lemma 2.2 are Lemmas 2.1 and 2.2 of [S15b].

Lemma 2.3. *Let $n>1$ be an integer with $n\equiv1,9\pmod{20}$ or $n\equiv11,19\pmod{40}$. Then we can write n as $5x^2+5y^2+z^2$ with $x,y,z\in\mathbb{Z}$ such that $x\not\equiv y\pmod 2$ if $n\equiv1,9\pmod{20}$, and $2\nmid y$ if $n\equiv11,19\pmod{40}$.*

Proof. As $n\equiv1\pmod4$ or $n\equiv3\pmod8$, by the Gauss-Legendre theorem n is the sum of three squares. As n is not a power of two, in view of Lemma 2.1 we can always write n as $w^2+u^2+v^2$ with $u,v,w\in\mathbb{Z}$ and $w^2,u^2,v^2\neq n$. Without loss of generality, we assume that $2\nmid w$ and $u\equiv v\pmod2$. Clearly, $u\equiv v\equiv0\pmod2$ if $n\equiv1\pmod4$. If $w^2\equiv-n\pmod5$, then $u^2+v^2\equiv2n\pmod5$ and hence $u^2\equiv v^2\equiv n\pmod5$. If $w^2\equiv n\pmod5$, then u^2+v^2 is a positive multiple of 5 and hence by Lemma 2.2 we can write it as s^2+t^2, where s and t are integers with $s^2\equiv-n\pmod5$ and $t^2\equiv n\pmod5$. When $n\equiv1\pmod4$, we have $s^2+t^2=u^2+v^2\equiv0\pmod4$, we have $s\equiv t\equiv0\pmod2$. If $5\mid w$, then one of u^2 and v^2 is divisible by 5 and the other is congruent to n modulo 5.

By the above, there always exist $x,y,z\in\mathbb{Z}$ with $z^2\equiv n\pmod5$ such that $n=x^2+y^2+z^2$ and that $2\mid z$ if $n\equiv1\pmod4$. Note that $x^2\equiv-y^2\equiv(\pm2y)^2\pmod5$. Without loss of generality, we assume that $x\equiv2y\pmod5$ and hence $2x\equiv-y\pmod5$. Set $\bar{x}=(x-2y)/5$ and $\bar{y}=(2x+y)/5$. Then

$$n=x^2+y^2+z^2=5\bar{x}^2+5\bar{y}^2+z^2.$$

If $n\equiv1\pmod4$, then $2\mid z$ and hence $\bar{x}\not\equiv\bar{y}\pmod2$. If $n\equiv3\pmod8$, then $z^2\not\equiv n\pmod4$ and hence $\bar{x}$ or $\bar{y}$ is odd. This concludes the proof. $\square$

Remark 2.1. Without using Lemma 2.1 and Lemma 2.2(i), the author [S15a, Theorem 1.7(iv)] showed by a different method that for any integer $n>1$ with $n\equiv1,9\pmod{20}$ we can write $n=5x^2+5y^2+(2z)^2$ with $x,y,z\in\mathbb{Z}$ if n is not a square.

For convenience, we define

$$E(f(x,y,z)):=\{n\in\mathbb{N}:\ n\neq f(x,y,z)\ \text{ for any } x,y,z\in\mathbb{Z}\}$$

for any function $f:\mathbb{Z}^3\to\mathbb{N}$.

Proof of Theorem 1.1. Let n be a fixed nonnegative integer.

(i) By Dickson [D39, pp. 112-113],

$$E(4x^2+16y^2+z^2)=\bigcup_{k\in\mathbb{N}}\{4k+2,4k+3,16k+12\}\cup\{4^k(8l+7):\ k,l\in\mathbb{N}\}.$$

So, there are $x,y,z\in\mathbb{Z}$ such that $4n+1=4x^2+16y^2+z^2$ and hence $n=x^2+(2y)^2+\lfloor z^2/4\rfloor$.

For $r\in\{1,4\}$, if $6n+r=6x^2+24y^2+z^2$ with $x,y,z\in\mathbb{Z}$, then $z^2\equiv r\pmod 6$ and $n=x^2+(2y)^2+\lfloor z^2/6\rfloor$. By Dickson [D39, pp. 112-113],

$$E(6x^2+24y^2+z^2)=\bigcup_{k\in\mathbb{N}}\{8k+3,8k+5,32k+12\}\cup\{9^k(3l+2):\ k,l\in\mathbb{N}\}.$$

If both $6n+1$ and $6n+4$ belong to this set, then one of them has the form $32k+12$ and hence we get a contradiction since $32k+12\pm 3\not\equiv 3,5\pmod 8$.

If $n\equiv 0,1\pmod 4$, then $5n+1\equiv 1,6\pmod{20}$ and hence by Lemmas 2.2 and 2.3 we have $5n+1=5x^2+5y^2+z^2$ with $x,y,z\in\mathbb{Z}$ and $x\not\equiv y\pmod 2$, thus x or y is odd and $n=x^2+y^2+\lfloor z^2/5\rfloor$. By Dickson [D39, pp. 112-113],

$$E(x^2+y^2+5z^2)=\{4^k(8l+3):\ k,l\in\mathbb{N}\}.$$

If $n\equiv 2\pmod 4$ or $n\equiv 7\pmod 8$, then there are $x,y,z\in\mathbb{Z}$ such that $n=x^2+y^2+5z^2=x^2+y^2+\lfloor(5z)^2/5\rfloor$ and one of x and y is odd since $5z^2\equiv z^2\not\equiv n\pmod 4$. If $n\equiv 3\pmod 8$, then $5n+4\equiv 19\pmod{40}$ and hence by Lemma 2.3 there are $x,y,z\in\mathbb{Z}$ with $2\nmid y$ such that $5n+4=5(x^2+y^2)+z^2$ and hence $n=x^2+y^2+\lfloor z^2/5\rfloor$ with y odd.

(ii) By [D39, pp. 112-113], there are $x,y,z\in\mathbb{Z}$ such that $8n+1=8x^2+32y^2+z^2$ and hence $n=x^2+(2y)^2+\lfloor z^2/8\rfloor$.

Suppose that $n\in\mathbb{Z}^+$. As conjectured by Sun [S07] and proved by Oh and Sun [OS], there are $x,y,z\in\mathbb{Z}$ with y odd such that $n=x^2+y^2+T_z$ and hence $n=x^2+y^2+\lfloor(2z+1)^2/8\rfloor$.

(iii) If $2n\equiv 6\pmod 8$, then $2n\notin\{4^k(8l+7):\ k,l\in\mathbb{N}\}$. If $2n\not\equiv 6\pmod 8$, then $2n+1\notin\{4^k(8l+7):\ k,l\in\mathbb{N}\}$. So, for some $\delta\in\{0,1\}$, we have $2n+\delta\notin\{4^k(8l+7):\ k,l\in\mathbb{N}\}$ and hence (by the Gauss-Legendre theorem) $2n+\delta=x^2+y^2+z^2$ for some $x,y,z\in\mathbb{Z}$ with $z\equiv\delta\pmod 2$. Note that $x\equiv y\pmod 2$ and

$$2n+\delta=2\left(\frac{x+y}{2}\right)^2+2\left(\frac{x-y}{2}\right)^2+z^2.$$

Therefore,

$$n=\left(\frac{x+y}{2}\right)^2+\left(\frac{x-y}{2}\right)^2+\frac{z^2-\delta}{2}=\left(\frac{x+y}{2}\right)^2+\left(\frac{x-y}{2}\right)^2+\left\lfloor\frac{z^2}{2}\right\rfloor.$$

By Dickson [D39, pp. 112-113],

$$E(3x^2+3y^2+z^2)=\{9^k(3l+2):\ k,l\in\mathbb{N}\}.$$

So, there are $x,y,z\in\mathbb{Z}$ such that $3n+1=3(x^2+y^2)+z^2$ and hence $n=x^2+y^2+\lfloor z^2/3\rfloor$.

Clearly $9n+1\equiv 9n+7\pmod 2$ but $9n+1\not\equiv 9n+7\pmod 4$. So, for some $r\in\{1,7\}$, we have $9n+r\notin\{4^k(8l+7):\ k,l\in\mathbb{N}\}$ and hence (by the Gauss-Legendre theorem) there are $x,y,z\in\mathbb{Z}$ such that $9n+r=(3x)^2+(3y)^2+z^2$ and therefore $n=x^2+y^2+\lfloor z^2/9\rfloor$.

By Dickson [D39, pp. 112-113],

$$E(21x^2+21y^2+z^2)=\bigcup_{k,l\in\mathbb{N}}\{4^k(8l+7),9^k(3l+2),49^k(7l+3),49^k(7l+5),49^k(7l+6)\}.$$

For each $r=1,4,16$, if $21n+r$ belongs to the above set then it has the form $4^k(8l+7)$ with $k,l\in\mathbb{N}$. If

$$\{21n+1,21n+4,21n+16\}\subseteq\{4^k(8l+7):\ k,l\in\mathbb{N}\},$$

then $21n+4$ and $21n+16$ are even since $21n+4\not\equiv 21n+16\pmod 8$, hence $21n+1\equiv 7\pmod 8$ and $21n+4\equiv 2\pmod 8$ which leads a contradiction. So, for some $r\in\{1,4,16\}$ and $x,y,z\in\mathbb{Z}$ we have $21n+r=21(x^2+y^2)+z^2$ and hence $n=x^2+y^2+\lfloor z^2/21\rfloor$.

By Dickson [D39, pp. 112-113],

$$E(12x^2+12y^2+z^2)=\bigcup_{k\in\mathbb{N}}\{(4k+2,\ 4k+3\}\cup\{9^k(3l+2):\ k,l\in\mathbb{N}\}.$$

So, for some $x,y,z\in\mathbb{Z}$ we have $12n+1=12(x^2+y^2)+(2z+1)^2$ and hence

$$n=x^2+y^2+\frac{z(z+1)}{3}=x^2+y^2+\left\lfloor\frac{z(z+1)}{3}\right\rfloor.$$

This proves (1.4) for $m=3$.

By Jones and Pall [JP], there are $x,y,z\in\mathbb{Z}$ such that $16n+1=16x^2+16y^2+(2z+1)^2$ and hence

$$n=x^2+y^2+\frac{(2z+1)^2-1}{16}=x^2+y^2+\left\lfloor\frac{z(z+1)}{4}\right\rfloor.$$

This proves (1.4) for $m=4$.

By [S15a, Theorem 1.7(ii)], n can be written as $x^2+y^2+p_5(z)$ with $x,y,z\in\mathbb{Z}$. Note that

$$p_5(z)=\frac{z(3z-1)}{2}=\frac{3z(3z-1)}{6}.$$

So (1.4) holds for $m=6$.

(iv) Now we prove (1.5) for $m=5$. By Dickson [D39, pp. 112-113],

$$E(5x^2+y^2+z^2)=\{4^k(8l+3):\ k,l\in\mathbb{N}\}.$$

As $5n+2\not\equiv 5n+4 \pmod 4$, for a suitable choice of $r\in\{2,4\}$ we can write $5n+r$ as $5x^2+y^2+z^2$ with $x,y,z\in\mathbb{Z}$. If $r=2$, then $y^2\equiv z^2\equiv 1 \pmod 5$ and hence

$$n=x^2+\frac{y^2-1}{5}+\frac{z^2-1}{5}=x^2+\left\lfloor\frac{y^2}{5}\right\rfloor+\left\lfloor\frac{z^2}{5}\right\rfloor.$$

If $r=4$, then we may assume that $y^2\equiv 0 \pmod 5$ and $z^2\equiv 4 \pmod 5$, hence

$$n=x^2+\frac{y^2}{5}+\frac{z^2-4}{5}=x^2+\left\lfloor\frac{y^2}{5}\right\rfloor+\left\lfloor\frac{z^2}{5}\right\rfloor.$$

If $\{5n+5,5n+6,5n+9\}\subseteq E:=\{4^k(8l+7):\ k,l\in\mathbb{N}\}$, then we must have $5n+6\equiv 7 \pmod 8$ and hence $5n+9\equiv 2 \pmod 8$ which leads a contradiction. Thus, by the Gauss-Legendre theorem, for some $r\in\{0,1,4\}$ the number $5n+5+r$ is the sum of three squares. If $5(n+1)+r=m^2$ for some $m\in\mathbb{Z}^+$ which is not a power of two, then by Lemma 2.1 we have $5(n+1)+r=x^2+y^2+z^2$ for some $x,y,z\in\mathbb{Z}$ with $x^2,y^2,z^2\neq 5(n+1)+r$. If $5(n+1)+r=(2^k)^2$ for some $k\in\mathbb{Z}^+$, then $r\in\{1,4\}$, $5(n+1)+(5-r)=4^k+5-2r\equiv 5-2r\equiv\pm3 \pmod 8$ and hence $5(n+1)+(5-r)\notin E$. So, for a suitable choice of $r\in\{0,1,4\}$, we can write $5(n+1)+r=x^2+y^2+z^2$ with $x,y,z\in\mathbb{Z}$ and $x^2,y^2,z^2\neq 5(n+1)+r$. Clearly, one of x^2,y^2,z^2, say z^2, is congruent to r modulo 5. Then x^2+y^2 is a positive multiple of 5. By Lemma 2.2, $x^2+y^2=\bar{x}^2+\bar{y}^2$ for some $\bar{x},\bar{y}\in\mathbb{Z}$ with $5\nmid\bar{x}\bar{y}$. Without loss of generality we may assume that $\bar{x}^2\equiv 1 \pmod 5$ and $\bar{y}^2\equiv 4 \pmod 5$. Therefore,

$$n=\frac{\bar{x}^2-1}{5}+\frac{\bar{y}^2-4}{5}+\frac{z^2-r}{5}=\left\lfloor\frac{\bar{x}^2}{5}\right\rfloor+\left\lfloor\frac{\bar{y}^2}{5}\right\rfloor+\left\lfloor\frac{z^2}{5}\right\rfloor.$$

Now we show (1.5) for $m=6$. By Dickson [D39, pp. 112-113],

$$E(6x^2+y^2+z^2)=\{9^k(9l+3):\ k,l\in\mathbb{N}\}.$$

So, there are $x,y,z\in\mathbb{Z}$ such that $6n+4=6x^2+y^2+z^2$. Clearly, exactly one of y and z, say y, is divisible by 3. Note that y and z have the same parity. If $y\equiv z\equiv 0 \pmod 2$, then $y^2\equiv 0 \pmod 6$ and $z^2\equiv 4 \pmod 6$. If $y\equiv z\equiv 1 \pmod 2$, then $y^2\equiv 3 \pmod 6$ and $z^2\equiv 1 \pmod 6$. Anyway, we have

$$n=x^2+\frac{y^2+z^2-4}{6}=x^2+\left\lfloor\frac{y^2}{6}\right\rfloor+\left\lfloor\frac{z^2}{6}\right\rfloor.$$

Assume that n is even. Then $6n+9\equiv 1 \pmod 4$ and hence by the Gauss-Legendre theorem and [S16, Lemma 2.2] we can write $6n+9=x^2+y^2+z^2$ with $x,y,z\in\mathbb{Z}$ and $3\nmid xyz$. Clearly, exactly one of x,y,z, say x, is odd. Thus $x^2\equiv 1 \pmod 6$ and $y^2\equiv z^2\equiv 4 \pmod 6$. Therefore

$$n=\frac{x^2-1}{6}+\frac{y^2-4}{6}+\frac{z^2-4}{6}=\left\lfloor\frac{x^2}{6}\right\rfloor+\left\lfloor\frac{y^2}{6}\right\rfloor+\left\lfloor\frac{z^2}{6}\right\rfloor.$$

Now suppose that n is odd. Then $3n+4\equiv 1 \pmod 6$, and hence by [S16, Lemma 4.3(ii)] we can write $3n+4=x^2+y^2+2z^2$ with $x,y,z\in\mathbb{Z}$ and $3\nmid xyz$. Without loss of generality, we may assume that $x\equiv y \pmod 3$ (otherwise we may use $-y$ to replace y). Clearly, $x\not\equiv y \pmod 2$. Thus $6n+8=(x+y)^2+(x-y)^2+(2z)^2$ with $(x+y)^2\equiv 1 \pmod 6$, $(x-y)^2\equiv 3 \pmod 6$ and $(2z)^2\equiv 4 \pmod 6$. Therefore

$$n=\frac{(x+y)^2-1}{6}+\frac{(x-y)^2-3}{6}+\frac{(2z)^2-4}{6}=\left\lfloor\frac{(x+y)^2}{6}\right\rfloor+\left\lfloor\frac{(x-y)^2}{6}\right\rfloor+\left\lfloor\frac{(2z)^2}{6}\right\rfloor.$$

Now we prove (1.5) for $m=15$. By Dickson [D39, pp. 112-113],

$$E(3x^2+y^2+z^2)=\{9^k(9l+6):\ k,l\in\mathbb{N}\}.$$

So, there are $x,y,z\in\mathbb{Z}$ such that $3n+1=3x^2+y^2+z^2$ and hence

$$15n+5=15x^2+(2^2+1^2)(y^2+z^2)=15x^2+(2y-z)^2+(y+2z)^2.$$

As $(2y-z)^2+(y+2z)^2=5(y^2+z^2)$ is a positive multiple of 5, by Lemma 2.2 there are $u,v\in\mathbb{Z}$ with $5\nmid uv$ such that $(2y-z)^2+(y+2z)^2=u^2+v^2$. Without loss of generality, we assume that $u^2\equiv 1 \pmod 5$ and $v^2\equiv 4 \pmod 5$. Then $15n+5=15x^2+u^2+v^2$ with $u^2\equiv 1 \pmod{15}$ and $v^2\equiv 4 \pmod{15}$. Therefore

$$n=x^2+\frac{u^2-1}{15}+\frac{v^2-1}{15}=x^2+\left\lfloor\frac{u^2}{15}\right\rfloor+\left\lfloor\frac{v^2}{15}\right\rfloor.$$

If $\{15n+6,15n+9,15n+15\}\subseteq E:=\{4^k(8l+7):\ k,l\in\mathbb{N}\}$, then we must have $15n+6\equiv 7 \pmod 8$ and hence $15n+9\equiv 2 \pmod 8$ which leads a contradiction. Thus, by the Gauss-Legendre theorem, for some $r\in\{1,4,10\}$ the number $15n+5+r$ is the sum of three squares. In view of [S16, Lemma 2.2], we can write $15n+5+r=x^2+y^2+z^2$ with $x,y,z\in\mathbb{Z}$ and $3\nmid xyz$. It is easy to see that one of x^2,y^2,z^2, say z^2, is congruent to r modulo 5. Then x^2+y^2 is a positive multiple of 5, and hence by Lemma 2.2 we can write $x^2+y^2=\bar{x}^2+\bar{y}^2$ with $\bar{x},\bar{y}\in\mathbb{Z}$ and $5\nmid\bar{x}\bar{y}$. Without loss of generality, we may assume that $\bar{x}^2\equiv 1 \pmod 5$ and $\bar{y}^2\equiv 4 \pmod 5$. Then $\bar{x}^2\equiv 1 \pmod{15}$, $\bar{y}^2\equiv 4 \pmod{15}$ and $z^2\equiv r \pmod{15}$. Therefore

$$n=\frac{\bar{x}^2-1}{15}+\frac{\bar{y}^2-4}{15}+\frac{z^2-r}{15}=\left\lfloor\frac{\bar{x}^2}{15}\right\rfloor+\left\lfloor\frac{\bar{y}^2}{15}\right\rfloor+\left\lfloor\frac{z^2}{15}\right\rfloor.$$

(v) Clearly,

$$\left\{\left\lfloor\frac{T_x}{3}\right\rfloor:\ x\in\mathbb{Z}\right\}\supseteq\left\{p_5(x)=\frac{T_{3x-1}}{3}:\ x\in\mathbb{Z}\right\}.$$

By [S15a, Theorem 1.14], $\{T_x+T_y+p_5(z):\ x,y,z\in\mathbb{Z}\}=\mathbb{N}$. It is also known that $\{p_5(x)+p_5(y)+p_5(z):\ x,y,z\in\mathbb{Z}\}=\mathbb{N}$ (cf. Guy [Gu] and [S15a]).

Now it remains to prove (1.6). Clearly, for some $r\in\{1,2\}$, $2n+r$ is not a triangular number. Hence, by [S07, Theorem 1(iii)] there are $x,y,z\in\mathbb{Z}$ with $x\not\equiv y \pmod 2$ such that $2n+r=x^2+y^2+T_z$. Thus $4n+2r=(x+y)^2+(x-y)^2+z(z+1)$ with $x\pm y$ odd and $z(z+1)\equiv 2(r-1) \pmod 4$. Write $x+y=2u+1$ and $x-y=2v+1$ with $u,v\in\mathbb{Z}$. Then

$$\begin{aligned} n=&\frac{(2u+1)^2-1}{4}+\frac{(2v+1)^2-1}{4}+\frac{z(z+1)-2(r-1)}{4}\\ =&u(u+1)+v(v+1)+\left\lfloor\frac{z(z+1)}{4}\right\rfloor. \end{aligned}$$

In view the above, we have completed the proof of Theorem 1.1. □

Proof of Theorem 1.2. Let n be a fixed natural number.

(i) By a known result first observed by Euler (cf. [D99, p. 260] and also [P]), there are $x,y,z\in\mathbb{Z}$ such that $2n+1=2x^2+4y^2+z^2$ and hence $n=x^2+2y^2+\lfloor z^2/2\rfloor$.

Suppose that $n\neq x^2+2y^2+\lfloor (3z)^2/3\rfloor=x^2+2y^2+3z^2$ for all $x,y,z\in\mathbb{Z}$. Then n is even by a known result (cf. [D39, p. 112-113] or [P]). By [D39, p. 112-113],

$$E(3x^2+6y^2+z^2)=\{3k+2:\ k\in\mathbb{N}\}\cup\{4^k(16l+14):\ k,l\in\mathbb{N}\}.$$

Since $3n+1$ is odd, for some $x,y,z\in\mathbb{Z}$ we have $3n+1=3x^2+6y^2+z^2$ and hence $n=x^2+2y^2+\lfloor z^2/3\rfloor$.

By [D39, p. 112-113],

$$E(4x^2+8y^2+z^2)=\bigcup_{k\in\mathbb{N}}\{4k+2,4k+3\}\cup\{4^k(16l+14):\ k,l\in\mathbb{N}\}.$$

So there are $x,y,z\in\mathbb{Z}$ such that $4n+1=4x^2+8y^2+z^2$ and hence $n=x^2+2y^2+\lfloor z^2/4\rfloor$.

By [D39, p. 112-113],

$$E(5x^2+10y^2+z^2)=\bigcup_{k,l\in\mathbb{N}}\{25^k(5l+2),25^k(5l+3)\}.$$

Thus, for some $x,y,z\in\mathbb{Z}$ we have $5n+1=5x^2+10y^2+z^2$ and hence $n=x^2+2y^2+\lfloor z^2/5\rfloor$.

(ii) By [D39, p. 112-113],

$$E(3x^2+y^2+z^2)=\{9^k(9l+6):\ k,l\in\mathbb{N}\}.$$

So, there are $x,y,z\in\mathbb{Z}$ such that $3n+1=3x^2+(3y)^2+z^2$ and hence $n=x^2+3y^2+\lfloor z^2/3\rfloor$.

By [D39, p. 112-113],

$$E(4x^2+12y^2+z^2)=\bigcup_{k\in\mathbb{N}}\{4k+2,4k+3\}\cup\{9^k(9l+6):\ k,l\in\mathbb{N}\}.$$

Choose $\delta\in\{0,1\}$ such that $4n+\delta\not\equiv 0\pmod 3$. Then, for some $x,y,z\in\mathbb{Z}$ we have $4n+\delta=4x^2+12y^2+z^2$ and hence $n=x^2+3y^2+\lfloor z^2/4\rfloor$.

If $6n+r=6x^2+18y^2+z^2$ for some $r\in\{0,1,3,4\}$ and $x,y,z\in\mathbb{Z}$, then $n=x^2+3y^2+\lfloor z^2/6\rfloor$. Now suppose that $6n+r\neq 6x^2+18y^2+z^2$ for any $r\in\{0,1,3,4\}$ and $x,y,z\in\mathbb{Z}$. By [D39, p. 112-113],

$$S:=E(6x^2+18y^2+z^2)=\bigcup_{k\in\mathbb{N}}\{3k+2,9k+3\}\cup\{4^k(8l+5):\ k,l\in\mathbb{N}\}.$$

So $6n+1$ or $6n+4$ is congruent to 5 modulo 8. If $6n+4\equiv 5\pmod 8$, then $6n+1\equiv 2\pmod 8$ which contradicts that $6n+1\in S$. So, $6n+1\equiv 5\pmod 8$ and hence $6n+3\equiv 7\pmod 8$. By $6n+3\in S$, we must have $3\mid n$. As $6n\equiv 0\pmod 9$ and $6n\equiv 4\pmod 8$, by $6n\in S$ we have $6n=4(8q+5)$ for some $q\in\mathbb{Z}$. As $6n+4=4(8q+6)\notin S$, we get a contradiction.

As conjectured by Sun [S07] and confirmed in [GPS], there are $x,y,z\in\mathbb{Z}$ such that $n=x^2+3y^2+T_z$ and hence $n=x^2+3y^2+\lfloor(2z+1)^2/8\rfloor$.

(iii) By [D39, p. 112-113], $E(8x^2+40y^2+z^2)$ coincides with

$$\bigcup_{k\in\mathbb{N}}\{4k+2,4k+3,8k+5,32k+28\}\cup\bigcup_{k,l\in\mathbb{N}}\{25^k(25l+5),25^k(25l+20)\}.$$

Choose $\delta\in\{0,1\}$ such that $8n+\delta\not\equiv 0\pmod 5$. Then $8n+\delta\notin E(8x^2+40y^2+z^2)$. So, for some $x,y,z\in\mathbb{Z}$ we have $8n+\delta=8x^2+40y^2+z^2$ and hence $n=x^2+5y^2+\lfloor z^2/8\rfloor$.

By [D39, p. 112-113],

$$E(4x^2+24y^2+z^2)=\bigcup_{k\in\mathbb{N}}\{4k+2,4k+3\}\cup\{9^k(9l+3):\ k,l\in\mathbb{N}\}.$$

Choose $\delta\in\{0,1\}$ such that $4n+\delta\not\equiv 0\pmod 3$. Then $4n+\delta\notin E(4x^2+24y^2+z^2)$. Hence there are $x,y,z\in\mathbb{Z}$ such that $4n+\delta=4x^2+24y^2+z^2$ and thus $n=x^2+6y^2+\lfloor z^2/4\rfloor$.

(iv) By [JP] or [D39, p. 112-113], for some $x,y,z\in\mathbb{Z}$ we have $8n+1=16x^2+16y^2+z^2$ and hence $n=2x^2+2y^2+\lfloor z^2/8\rfloor$.

In view of [D39, p. 112-113],

$$E(6x^2+9y^2+z^2)=\{3k+2:\ k\in\mathbb{N}\}\cup\{9^k(9l+3):\ k,l\in\mathbb{N}\}.$$

So, there are $x,y,z\in\mathbb{Z}$ such that $3n+1=6x^2+9y^2+z^2$ and hence $n=2x^2+3y^2+\lfloor z^2/3\rfloor$.

By [D39, p. 112-113],

$$E(3x^2+3y^2+2z^2)=\{9^k(3l+1):\ k,l\in\mathbb{N}\}.$$

So there are $x,y,z\in\mathbb{Z}$ such that $6n+5=3x^2+3y^2+2z^2$. Since $x\not\equiv y\pmod 2$, without loss of generality we may assume that $2\mid x$ and $2\nmid y$. Thus

$$n=2\left(\frac{x}{2}\right)^2+\frac{y^2-1}{2}+\frac{z^2-1}{3}=2\left(\frac{x}{2}\right)^2+\left\lfloor\frac{y^2}{2}\right\rfloor+\left\lfloor\frac{z^2}{3}\right\rfloor.$$

So far we have completed the proof of Theorem 1.2. □

3. Proofs of Theorems 1.3-1.4

Proof of Theorem 1.3. (i) Clearly, $0=\lceil 0^2/m\rceil+\lceil 0^2/m\rceil+\lceil 0^2/m\rceil$, $1=\lceil 1^2/3\rceil+\lceil 0^2/3\rceil+\lceil 0^2/3\rceil$ and $2=\lceil 1^2/3\rceil+\lceil 1^2/3\rceil+\lceil 0^2/3\rceil$. for any $m\in\{2,3,4,5\}$. So we just consider required representations for $n\in\{3,4,5,\ldots\}$.

If n is even, then $2n-2\equiv 2\pmod 4$, hence by the Gauss-Legendre theorem there are integers x,y,z with $2\nmid yz$ such that $2n-2=(2x)^2+y^2+z^2$ and thus

$$n=2x^2+\frac{y^2+1}{2}+\frac{z^2+1}{2}=\frac{(2x)^2}{2}+\left\lceil\frac{y^2}{2}\right\rceil+\left\lceil\frac{z^2}{2}\right\rceil.$$

When $n\equiv 1\pmod 4$, we have $2n-1\equiv 1\pmod 8$ and hence by the Gauss-Legendre theorem there are $x,y,z\in\mathbb{Z}$ with $2\nmid z$ such that $2n-1=(2x)^2+(2y)^2+z^2$ and thus

$$n=2x^2+2y^2+\frac{z^2+1}{2}=\frac{(2x)^2}{2}+\frac{(2y)^2}{2}+\left\lceil\frac{z^2}{2}\right\rceil.$$

If $n\equiv 3\pmod 4$, then $2n-3\equiv 3\pmod 8$, hence there are odd integers x,y,z such that $2n-3=x^2+y^2+z^2$ and thus

$$n=\frac{x^2+1}{2}+\frac{y^2+1}{2}+\frac{z^2+1}{2}=\left\lceil\frac{x^2}{2}\right\rceil+\left\lceil\frac{y^2}{2}\right\rceil+\left\lceil\frac{z^2}{2}\right\rceil.$$

This proves (1.7) for $m=2$.

Now we show (1.7) for $m=3$. Clearly we cannot have $\{3n-4,3n-6\}\subseteq\{4^k(8l+7):\ k,l\in\mathbb{N}\}$ and hence either $3n-4$ or $3n-6$ can be written as the sum of three squares. If $3n-4=x^2+y^2+z^2$ for some $x,y,z\in\mathbb{Z}$, then exactly one of x,y,z (say, x) is divisible by 3, hence

$$n=3\left(\frac{x}{3}\right)^2+\frac{y^2+2}{3}+\frac{z^2+2}{3}=\left\lceil\frac{x^2}{3}\right\rceil+\left\lceil\frac{y^2}{3}\right\rceil+\left\lceil\frac{z^2}{3}\right\rceil.$$

When $3n-6=x^2+y^2+z^2$ with $x,y,z\in\mathbb{Z}$ not all zero, by [S16, Lemma 2.2] there are $u,v,w\in\mathbb{Z}$ with $3\nmid uvw$ such that $3n-6=u^2+v^2+w^2$ and hence

$$n=\frac{u^2+2}{3}+\frac{v^2+2}{3}+\frac{w^2+2}{3}=\left\lceil\frac{u^2}{3}\right\rceil+\left\lceil\frac{v^2}{3}\right\rceil+\left\lceil\frac{w^2}{3}\right\rceil.$$

As $4n-3\not\in\{4^k(8l+7):\ k,l\in\mathbb{N}\}$, by the Gauss-Legendre theorem there are $x,y,z\in\mathbb{Z}$ such that $4n-3=(2x)^2+(2y)^2+(2z+1)^2$ and hence $n=x^2+y^2+\lceil(2z+1)^2/4\rceil$. This proves (1.7) for $m=4$.

Now we prove (1.7) for $m=5$ by modifying our proof of the last equality in (1.5). If $\{5n-5,5n-6,5n-9\}\subseteq E:=\{4^k(8l+7):\ k,l\in\mathbb{N}\}$, then $5n-6\equiv 7\pmod 8$ and hence $5n-5\equiv 6\pmod 8$ which leads a contradiction. So, for some $r\in\{0,1,4\}$ we can write $5n-5-r>5$ as the sum of three squares. If $5n-5-r=m^2$ for some integer $m>2$ which is not a power of two, then by Lemma 2.1 we have $5(n-1)-r=x^2+y^2+z^2$ for some $x,y,z\in\mathbb{Z}$ with $x^2,y^2,z^2\neq 5n-5-r$. If $5(n-1)-r=(2^k)^2$ for some $k\in\mathbb{Z}^+$, then $r\in\{1,4\}$, and $5(n-1)-(5-r)=4^k+2r-5\equiv 2r-5\equiv\pm3\pmod 8$ and hence $5(n-1)-(5-r)\not\in E$. So, for a suitable choice of $r\in\{0,1,4\}$, we can write $5(n-1)-r=x^2+y^2+z^2$ with $x,y,z\in\mathbb{Z}$ and $x^2,y^2,z^2\neq 5(n-1)-r$. Clearly, one of x^2,y^2,z^2, say z^2, is congruent to $-r$ modulo 5. Then x^2+y^2 is a positive multiple of 5. By Lemma 2.2, $x^2+y^2=\bar{x}^2+\bar{y}^2$ for some $\bar{x},\bar{y}\in\mathbb{Z}$ with $5\nmid\bar{x}\bar{y}$. Without loss of generality, we may assume that $\bar{x}^2\equiv 1\pmod 5$ and $\bar{y}^2\equiv 4\pmod 5$. Therefore,

$$n=\frac{\bar{x}^2+4}{5}+\frac{\bar{y}^2+1}{5}+\frac{z^2+r}{5}=\left\lceil\frac{\bar{x}^2}{5}\right\rceil+\left\lceil\frac{\bar{y}^2}{5}\right\rceil+\left\lceil\frac{z^2}{5}\right\rceil.$$

Now we show (1.7) for $m=6$. If n is odd, then $6n-9\equiv 1\pmod 4$, hence by the Gauss-Legendre theorem and [S16, Lemma 2.2] we can write $6n-9=x^2+y^2+z^2$ with $x,y,z\in\mathbb{Z}$, $2\nmid x$, $2\mid y$, $2\mid z$ and $3\nmid xyz$, therefore

$$n=\frac{x^2+5}{6}+\frac{y^2+2}{6}+\frac{z^2+2}{6}=\left\lceil\frac{x^2}{6}\right\rceil+\left\lceil\frac{y^2}{6}\right\rceil+\left\lceil\frac{z^2}{6}\right\rceil.$$

Now assume that n is even. Then $6n-10\equiv 2\pmod{12}$. By the Gauss-Legendre theorem we can write $6n-10=x^2+y^2+z^2$ with $x,y,z\in\mathbb{Z}$, $2\nmid xy$ and $2\mid z$. Note that exactly one of x,y,z is divisible by 3. If $3\nmid xy$ and $3\mid z$, then $x^2\equiv y^2\equiv 1\pmod 6$ and $z^2\equiv 0\pmod 6$, hence

$$n=\frac{x^2+5}{6}+\frac{y^2+5}{6}+\frac{z^2}{6}=\left\lceil\frac{x^2}{6}\right\rceil+\left\lceil\frac{y^2}{6}\right\rceil+\left\lceil\frac{z^2}{6}\right\rceil.$$

If $3\nmid z$, then exactly one of x and y, say x, is divisible by 3, hence $x^2\equiv 3\pmod 6$, $y^2\equiv 1\pmod 6$ and $z^2\equiv 4\pmod 6$, and thus

$$n=\frac{x^2+3}{6}+\frac{y^2+5}{6}+\frac{z^2+2}{6}=\left\lceil\frac{x^2}{6}\right\rceil+\left\lceil\frac{y^2}{6}\right\rceil+\left\lceil\frac{z^2}{6}\right\rceil.$$

Now we prove (1.7) for $m=15$. By the proof of the last equality in (1.5) for $m=15$, for a suitable choice of $r\in\{1,4,10\}$ we have $15(n-3)+5+r=x^2+y^2+z^2$ for some $x,y,z\in\mathbb{Z}$ with $x^2\equiv 1\pmod{15}$, $y^2\equiv 4\pmod{15}$ and $z^2\equiv r\pmod{15}$. It follows that

$$n=\frac{x^2+14}{15}+\frac{y^2+11}{15}+\frac{z^2+15-r}{15}=\left\lceil\frac{x^2}{15}\right\rceil+\left\lceil\frac{y^2}{15}\right\rceil+\left\lceil\frac{z^2}{15}\right\rceil.$$

(ii) Now we turn to prove (1.8). Apparently, $0=0^2+3\times 0^2+\lceil 0^2/2\rceil$. Let $n\in\mathbb{Z}^+$. If $2n-1\equiv 5\pmod 8$ then $4\nmid 2n$. So, we may choose $\delta\in\{0,1\}$ such that $2n-\delta\not\in\{4^k(8l+5):\ k,l\in\mathbb{N}\}$. By [D39, p. 112-113],

$$E(2x^2+6y^2+z^2)=\{4^k(8l+5):\ k,l\in\mathbb{N}\}.$$

So there are $x,y,z\in\mathbb{Z}$ such that $2n-\delta=2x^2+6y^2+z^2$ and hence $n=x^2+3y^2+\lceil z^2/2\rceil$.

Obviously, $0=0^2+3\times 0^2+\lceil 0^2/10\rceil$. Let $n\in\mathbb{Z}^+$. By [D39, p. 112-113],

$$T:=E(10x^2+30y^2+z^2)=\bigcup_{k,l\in\mathbb{N}}\{4^k(8l+5),9^k(9l+6),25^k(5l+2),25^k(5l+3)\}.$$

If $10n-r\not\in T$ for some $r\in\{0,1,4,5,6,9\}$, then there are $x,y,z\in\mathbb{Z}$ such that $10n-r=10x^2+30y^2+z^2$ and hence $n=x^2+3y^2+\lceil z^2/10\rceil$. Now we suppose that $10n-r\in T$ for all $r=0,1,4,5,6,9$ and want to deduce a contradiction. If $3\mid n(n+1)$, then by $10n-1\in T$ we have $10n-1\equiv 5\pmod 8$ and hence $10n-4\equiv 2\pmod 8$ which contradicts $10n-4\in T$. When $n\equiv 1\pmod 3$, by $10n-9\in T$ we must have $10n-9\equiv 5\pmod 8$ and thus $10n\equiv 6\pmod 8$, hence $10n\equiv 0\not\equiv 5\pmod{25}$ by $10n\in T$, and thus by $10n-5\in T$ we have $10n-5\equiv 5\pmod 8$ which contradicts $10n\equiv 6\pmod 8$.

(iii) Choose $\delta\in\{0,1\}$ with $n\equiv\delta\pmod 2$. Then $12n+5-4\delta\not\equiv 0\pmod 3$. By [D39, pp. 112-113],

$$E(3x^2+y^2+z^2)=\{9^k(9l+6):\ k,l\in\mathbb{N}\}.$$

So, there are $u,v,w\in\mathbb{Z}$ such that $12n+5-4\delta=3u^2+v^2+w^2$. If v and w are both even, then $5\equiv 3u^2\pmod 4$ which is impossible. Without loss of generality, we assume that $w=2z+1$ with $z\in\mathbb{Z}$. Then

$$3u^2+v^2\equiv 12n+5-4\delta-1\equiv 4\pmod 8.$$

Hence, by [S15a, Lemma 3.2] we can write $3u^2+v^2$ as $3(2x+1)^2+(2y+1)^2$ with $x,y\in\mathbb{Z}$. Therefore,

$$12n+5-4\delta=3(2x+1)^2+(2y+1)^2+(2z+1)^2=12x(x+1)+4y(y+1)+4z(z+1)+5$$

and hence

$$3n-\delta = 3x(x+1)+y(y+1)+z(z+1).$$

Note that $m(m+1)\not\equiv 1 \pmod 3$ for any $m\in\mathbb{Z}$. If $y(y+1),z(z+1)\not\equiv 0 \pmod 3$, then $-\delta\equiv 2+2 \pmod 3$ which is impossible. Without loss of generality we assume that $3\mid y(y+1)$. Then

$$n = x(x+1)+\frac{y(y+1)}{3}+\frac{z(z+1)+\delta}{3} = x(x+1)+\frac{y(y+1)}{3}+\left\lceil\frac{z(z+1)}{3}\right\rceil.$$

Let $\delta\in\{0,1\}$ with $n\equiv\delta \pmod 2$. Then $12n+3-4\delta$ is congruent to 0 or 2 modulo 3. As $12n+3-4\delta\equiv 3 \pmod 8$, there are odd integers u,v,w such that $12n+3-4\delta = u^2+v^2+w^2$. If $\delta=0$, then by [S16, Lemma 2.2] we can write $u^2+v^2+w^2$ as $r^2+s^2+t^2$ with $r,s,t\in\mathbb{Z}$ and $\gcd(rst,6)=1$. So, there are $x,y,z\in\mathbb{Z}$ such that

$$12n+3-4\delta = (2x+1)^2+(2y+1)^2+(2z+1)^2 = 4x(x+1)+4y(y+1)+4z(z+1)+3$$

and $2x+1,2y+1\not\equiv 0 \pmod 3$. As $x,y\not\equiv 1 \pmod 3$, both $x(x+1)$ and $y(y+1)$ are divisible by 3. Thus

$$n = \frac{x(x+1)}{3}+\frac{y(y+1)}{3}+\frac{z(z+1)+\delta}{3} = \left\lceil\frac{x(x+1)}{3}\right\rceil+\left\lceil\frac{y(y+1)}{3}\right\rceil+\left\lceil\frac{z(z+1)}{3}\right\rceil.$$

Note that $\{m(m+1)/3:\ m\in\mathbb{Z}\ \&\ 3\mid m(m+1)\} = \{q(3q+1):\ q\in\mathbb{Z}\}$.

The proof of Theorem 1.3 is now complete. □

Proof of Theorem 1.4. (i) Let $n\in\mathbb{N}$. If $2n+1\in\{4^k(8l+7):\ k,l\in\mathbb{N}\}$, then $2n\equiv 6 \pmod 8$ and hence $2n\not\in\{4^k(8l+7):\ k,l\in\mathbb{N}\}$. So, for some $\delta\in\{0,1\}$ we have $2n+\delta\not\in\{4^k(8l+7):\ k,l\in\mathbb{N}\}$, and hence by the Gauss-Legendre theorem there are $x,y,z\in\mathbb{Z}$ such that $2n+\delta = x^2+y^2+z^2$ and hence $n=\lfloor(x^2+y^2+z^2)/2\rfloor$. Note also that $n=T_x+T_y+T_z$ for some $x,y,z\in\mathbb{Z}$. This proves the desired result for $a=2$.

Now we handle the case $a>2$. Clearly, for some $r\in\{0,2\}$ we have $an+r\not\in\{4^k(8l+7):\ k,l\in\mathbb{N}\}$, hence for some $x,y,z\in\mathbb{Z}$ we have $an+r = x^2+y^2+z^2$ and thus $n=\lfloor(x^2+y^2+z^2)/a\rfloor$. Take $\delta\in\{0,1\}$ with $an\equiv\delta \pmod 2$. Then, there exist $x,y,z\in\mathbb{Z}$ such that $(an+da)/2 = T_x+T_y+T_z$ and hence $n=\lfloor(x(x+1)+y(y+1)+z(z+1))/a\rfloor$.

(ii) Suppose that a is odd. As $16n+3a^2\equiv 3 \pmod 8$, by the Gauss-Legendre symbol $16n+3a^2$ can be expressed as the sum of three odd squares. For any odd integer w, either w or $-w$ is congruent to a modulo 4. Thus, there are $x,y,z\in\mathbb{Z}$ such that

$$16n+3a^2 = (4x+a)^2+(4y+a)^2+(4z+a)^2,\ \text{i.e.,}\ 2n = 2(x^2+y^2+z^2)+a(x+y+z).$$

Hence $n = x^2+y^2+z^2+\lfloor\frac{a}{2}(x+y+z)\rfloor$ as desired.

Now assume that $\gcd(a,6)=1$. Choose $\delta\in\{0,1\}$ such that $n\equiv\delta\pmod 2$. As $12(3n+\delta)+3a^2\equiv 3\pmod 8$, there are odd integers u,v,w such that $12(3n+\delta)+3a^2=u^2+v^2+w^2$. Applying [S16, Lemma 2.2], we can write $u^2+v^2+w^2$ as $r^2+s^2+t^2$, where r,s,t are integers with

$$r\equiv u_0\equiv u\equiv 1\pmod 2,\ s\equiv v\equiv 1\pmod 2,\ t\equiv w\equiv 1\pmod 2,\ \text{and } 3\nmid rst.$$

Thus r or $-r$ has the form $6x+a$, s or $-s$ has the form $6y+a$, and t or $-t$ has the form $6z+a$, where $x,y,z\in\mathbb{Z}$. Therefore,

$$\begin{aligned}12(3n+\delta)+3a^2=&(6x+a)^2+(6y+a)^2+(6z+a)^2\\ =&12(3x^2+ax+3y^2+ay+3z^2+3z)+3a^2\end{aligned}$$

and hence

$$n=x^2+y^2+z^2+\frac{a(x+y+z)-\delta}{3}=x^2+y^2+z^2+\left\lfloor\frac{a}{3}(x+y+z)\right\rfloor.$$

Now we suppose that $2\mid a$ and $3\nmid a$. If $9n+3(a/2)^2+3r\in\{4^k(8l+7):\ k,l\in\mathbb{N}\}$ for all $r=1,2,3$, then $9n+3(a/2)^2+6\equiv 7\pmod 8$ and hence $9n+3(a/2)^2+9\equiv 2\pmod 8$ which leads a contradiction. So, by the Gauss-Legendre theorem, for some $r\in\{1,2,3\}$ and $u,v,w\in\mathbb{Z}$ we have $9n+3(a/2)^2+3r=u^2+v^2+w^2$. By [S16, Lemma 2.2] we can write $9n+3(a/2)^2+3r=\bar{u}^2+\bar{v}^2+\bar{w}^2$, where $\bar{u},\bar{v},\bar{w}\in\mathbb{Z}$ and $3\nmid\bar{u}\bar{v}\bar{w}$. So there are $x,y,z\in\mathbb{Z}$ such that

$$9n+3r+3\left(\frac{a}{2}\right)^2=\left(3x+\frac{a}{2}\right)^2+\left(3y+\frac{a}{2}\right)^2+\left(3z+\frac{a}{2}\right)^2,$$

i.e.,

$$3n+r-1=x(3x+a)+y(3y+a)+z(3z+a).$$

It follows that

$$n=x^2+y^2+z^2+\frac{a(x+y+z)-(r-1)}{3}=x^2+y^2+z^2+\left\lfloor\frac{a}{3}(x+y+z)\right\rfloor.$$

(iii) Obviously, $0=p_8(0)/2+\lceil p_8(0)/2\rceil+\lceil p_8(0)/2\rceil$. Now we let $n>0$ and choose $\delta\in\{0,1\}$ with $n\not\equiv\delta\pmod 2$. As $6n-3\delta$ is congruent to 1 or 2 modulo 4, by the Gauss-Legendre theorem we can write $6n-3\delta$ as the sum of three squares and hence by [S16, Lemma 2.2] there are $x,y,z\in\mathbb{Z}$ such that

$$6n-3\delta=(3x-1)^2+(3y-1)^2+(3z-1)=3p_8(x)+1+(3p_8(y)+1)+(3p_8(z)+1).$$

Clearly, $3x-1,3y-1,3z-1$ cannot be all odd or all even. Without loss of generality, we may assume that

$$3x-1\equiv 1\pmod 2,\ 3y-1\equiv 0\pmod 2\ \text{and}\ 3z-1\equiv 1-\delta\equiv n\pmod 2.$$

Then $p_8(x)=((3x-1)^2-1)/3$ is even, $p_8(y)$ is odd, and $p_8(z)\equiv-\delta\pmod 2$. Therefore

$$n=\frac{p_8(x)}{2}+\frac{p_8(y)+1}{2}+\frac{p_8(z)+\delta}{2}=\frac{p_8(x)}{2}+\left\lceil\frac{p_8(y)}{2}\right\rceil+\left\lceil\frac{p_8(z)}{2}\right\rceil.$$

This concludes our proof. $\square$

4. Proof of Theorem 1.5

For $a,b,c,n\in\mathbb{Z}^+$, define

$$r_{(a,b,c)}(n)=|\{(x,y,z)\in\mathbb{Z}^3:\ ax^2+by^2+cz^2=n\}| \tag{4.1}$$

and

$$H_{(a,b,c)}(n):=\prod_{p\nmid 2abc}\left(\frac{p^{\mathrm{ord}_p(n)+1}-1}{p-1}-\left(\frac{-abc}{p}\right)\frac{p^{\mathrm{ord}_p(n)}-1}{p-1}\right), \tag{4.2}$$

where $(\frac{\cdot}{p})$ is the Legendre symbol. Clearly,

$$H_{(a,b,c)}(n)\geqslant\prod_{p\nmid 2abc}\frac{p^{\mathrm{ord}_p(n)+1}-1-(p^{\mathrm{ord}_p(n)}-1)}{p-1}=\prod_{p\nmid 2abc}p^{\mathrm{ord}_p(n)}. \tag{4.3}$$

In 1907 Hurwitz (cf. [D99, p. 271]) showed that $r_{(1,1,1)}(n^2)=6H_{(1,1,1)}(n)$ which is just (2.1). In 2013 S. Cooper and H. Y. Lam [CL] deduced some similar formulas for

$$r_{(1,1,2)}(n^2),\ r_{(1,1,3)}(n^2),\ r_{(1,2,2)}(n^2),\ r_{(1,3,3)}(n^2).$$

Lemma 4.1. *For any integer $n>1$, there are $x,y,z\in\mathbb{Z}$ with $|x|<n$ and $|y|<n$ such that $x^2+y^2+2z^2=n^2$.*

Proof. By Cooper and Lam [CL, Theorem 1.2],

$$r_{(1,1,2)}(n^2)=\begin{cases}4H_{(1,1,2)}(n) & \text{if } 2\nmid n,\\ 12H_{(1,1,2)}(n) & \text{if } 2\mid n.\end{cases} \tag{4.4}$$

If n is odd, then there is an odd prime p dividing n, hence $r_{(1,1,2)}(n^2)=4H_{(1,1,2)}(n)>4$ with the help of (4.3). If n is even, then $r_{(1,1,2)}(n^2)=12H_{(1,1,2)}(n)\geqslant 12>4$. Clearly, $x^2+y^2+2z^2=n^2$ for $(x,y,z)=(\pm n,0,0),(0,\pm n,0)$. So, there are $x,y,z\in\mathbb{Z}$ with $x^2,y^2\neq n^2$ such that $x^2+y^2+2z^2=n^2$. This concludes the proof. □

Lemma 4.2. *Suppose that $n\in\mathbb{Z}^+$ is not a power of two. Then there are $x,y,z\in\mathbb{Z}$ with $|x|<n$ and $|y|<n$ such that $x^2+y^2+5z^2=n^2$.*

Proof. As conjectured by Cooper and Lam [CL, Conjecture 8.1] and proved by Guo et al. [GPQ],

$$r_{(1,1,5)}(n^2)=2(5^{\mathrm{ord}_5(n)+1}-3)H_{(1,1,5)}(n). \tag{4.5}$$

If $5\mid n$, then $2(5^{\mathrm{ord}_5(n)+1}-3)>4$. If n has a prime divisor $p\neq 2,5$, then $H_{(1,1,5)}(n)>1$ by (4.3). Since $n>1$ is not a power of two, we have $r_{(1,1,5)}(n^2)>4$. Clearly, $x^2+y^2+5z^2=n^2$ for $(x,y,z)=(\pm n,0,0),(0,\pm n,0)$. So, there are $x,y,z\in\mathbb{Z}$ with $x^2,y^2\neq n^2$ such that $x^2+y^2+5z^2=n^2$. This ends the proof. □

Remark 4.1. Note that Lemmas 4.1 and 4.2 are similar to Lemma 2.1.

Proof of Theorem 1.5. (i) Let $n \in \mathbb{N}$. Clearly, $n = p_8(x) + p_8(y) + 2p_8(z)$ if and only if $3n + 4 = (3x - 1)^2 + (3y - 1)^2 + 2(3z - 1)^2$. In view of [D39, pp. 112-113],

$$E(x^2 + y^2 + 2z^2) = \{4^k(16l + 14) : \ k, l \in \mathbb{N}\}.$$

If $3n+4 = 4^k(16l+14)$ for some $k, l \in \mathbb{N}$, then for some $q \in \mathbb{Z}^+$ we have $l = 3q-1$ and hence

$$n = \frac{4^k(16(3q - 1) + 14) - 4}{3} = 4^{k+2}q - \frac{2}{3}(4^k + 2).$$

If n has the form $4^{k+2}q - \frac{2}{3}(4^k + 2)$ with $k \in \mathbb{N}$ and $q \in \mathbb{Z}^+$, then $n \neq p_8(x) + p_8(y) + 2p_8(z)$ for all $x, y, z \in \mathbb{Z}$.

Now assume that n is not of the form $4^{k+2}q - \frac{2}{3}(4^k + 2)$ with $k \in \mathbb{N}$ and $q \in \mathbb{Z}^+$. Then there are $r, s, t \in \mathbb{Z}$ such that $3n + 4 = r^2 + s^2 + 2t^2$. In view of Lemma 4.1, we may assume that $r^2, s^2 \neq 3n + 4$. Clearly r and s cannot be both divisible by 3. Without loss of generality, we assume that $3 \nmid r$. As $s^2 + 2t^2 = 3n + 4 - r^2$ is a positive multiple of 3, by [S15a, Lemma 2.1] we can rewrite it as $u^2 + 2v^2$ with $u, v \in \mathbb{Z}$ and $3 \nmid uv$. Thus there are $x, y, z \in \mathbb{Z}$ such that

$$\begin{aligned} 3n + 4 =& r^2 + u^2 + 2v^2 = (3x - 1)^2 + (3y - 1)^2 + 2(3z - 1)^2 \\ =& 3p_8(x) + 1 + (3p_8(y) + 1) + 2(3p_8(z) + 1) \end{aligned}$$

and hence $n = p_8(x) + p_8(y) + 2p_8(z)$.

By the above, there are $x, y, z \in \mathbb{Z}$ with $2n+1 = p_8(x)+p_8(y)+2p_8(z)$. Without loss of generality, we may assume that $p_8(x)$ is even and $p_8(y) = y(3y-2)$ is odd. Clearly, $w = (1 - y)/2 \in \mathbb{Z}$ and $p_8(y) - 1 = 4p_8(w)$. So, $2n = p_8(x) + 2p_8(z) + 4p_8(w)$. Note also that

$$n = \frac{p_8(x)}{2} + \frac{p_8(y) - 1}{2} + p_8(z) = \left\lfloor \frac{p_8(x)}{2} \right\rfloor + \left\lfloor \frac{p_8(y)}{2} \right\rfloor + p_8(z).$$

Therefore (1.13) and (1.14) hold.

(ii) Fix a nonnegative integer n. If $6n + 5 \equiv 7 \pmod 8$, then $6n + 8 \equiv 2 \pmod 8$. So, for a suitable choice of $\delta \in \{0, 1\}$ we have $6n + 5 + 3\delta \notin E(x^2 + y^2 + z^2) = \{4^k(8l + 7) : \ k, l \in \mathbb{N}\}$ and hence $6n + 5 + 3\delta = u^2 + v^2 + w^2$ for some $u, v, w \in \mathbb{Z}$. Two of u, v, w have the same parity. Without loss of generality, we assume that $u + v = 2s$ and $u - v = 2t$ for some $s, t \in \mathbb{Z}$. Hence $6n + 5 + 3\delta = w^2 + 2s^2 + 2t^2$. If $(6n + 5 + 3\delta) = 2m^2$ for some $m \in \mathbb{Z}^+$, then by Lemma 4.1 there are $r, s_1, t_1 \in \mathbb{Z}$ with $s_1^2, t_1^2 \neq m^2$ such that $m^2 = s_1^2 + t_1^2 + 2r^2$ and hence $6n+5+3\delta = (2r)^2 + 2s_1^2 + 2t_1^2$ with $2s_1^2, 2t_1^2 \neq 6n+5+3\delta$. So, we may simply suppose that $6n + 5 + 3\delta = w^2 + 2s^2 + 2t^2$ with $2s^2, 2t^2 \neq 6n + 5 + 3\delta$. Clearly, one of s and t is not divisible by 3. Without loss of generality we assume that $t^2 = (3x - 1)^2$ with $x \in \mathbb{Z}$ As $w^2 + 2s^2 = 6n + 5 + 3\delta - 2t^2$ is a positive multiple of 3, by [S15a, Lemma 2.1] we can write w^2+2s^2 as $(3z-1)^2+2(3y-1)^2$ with $y,z\in\mathbb{Z}$. Thus

$$6n+5+3\delta=(3z-1)^2+2(3y-1)^2+2(3x-1)^2=3p_8(z)+1+2(3p_8(x)+3p_8(y)+2)$$

and hence

$$n=p_8(x)+p_8(y)+\frac{p_8(z)-\delta}{2}=p_8(x)+p_8(y)+\left\lfloor\frac{p_8(z)}{2}\right\rfloor.$$

This proves (1.15). In view of (1.12), both (1.16) and (1.17) follow from (1.15).

(iii) Let $n\in\mathbb{N}$. As $12n+9\equiv 1 \pmod 4$, by the Gauss-Legendre theorem we can write $12n+9$ as the sum of three squares. In view of [S16, Lemma 2.2], there are $u,v,w\in\mathbb{Z}$ with $3\nmid uvw$ such that $12n+9=u^2+v^2+w^2$. Clearly, exactly one of u,v,w is odd. Without loss of generality we may assume that $u=2(3x-1)$, $v=2(3y-1)$ and $w=3z-1$ with $x,y,z\in\mathbb{Z}$. Thus

$$12n+9=4(3x-1)^2+4(3y-1)^2+(3z-1)^2=12p_8(x)+12p_8(y)+3p_8(z)+9$$

and hence (1.18) follows.

(iv) By Dickson [D39, pp. 112-113],

$$E(5x^2+5y^2+z^2)=\bigcup_{k\in\mathbb{N}}\{5k+2,\ 5k+3\}\cup\{4^k(8l+7):\ k,l\in\mathbb{N}\}.$$

If $15n+11+3r$ belongs to this set for all $r=0,1,3$, then $15n+11$ is odd, hence $15n+11\equiv 7 \pmod 8$ and $15n+11+3\equiv 2 \pmod 8$ which leads a contradiction. So, there is a choice of $r\in\{0,1,3\}$ such that $15n+11+3r\notin E(5x^2+5y^2+z^2)$. Hence, for some $u,v,w\in\mathbb{Z}$ we have $15n+11+3r=5u^2+5v^2+w^2$. If $15n+11+3r=5m^2$ for some positive integer m which is not a power of two, then by Lemma 4.2 there are $u_1,v_1,w_1\in\mathbb{Z}$ with $u_1^2,v_1^2\neq m^2$ such that $m^2=u_1^2+v_1^2+5w_1^2$ and hence $15n+11+3r=5u_1^2+5v_1^2+(5w_1)^2$ with $5u_1^2,5v_1^2\neq 15n+11+3r$. If $15n+11+3r=5\times 2^a$ for some $a\in\mathbb{N}$, then $a\geqslant 2$, $r=3$, $15n+11+3\times 1=5\times 2^a-6\equiv 2 \pmod 4$ and hence $15n+11+3\notin E(5x^2+5y^2+z^2)$. So, we may simply assume that $15n+11+3r=5u^2+5v^2+w^2$ with $5u^2,5v^2<15n+11+3r$. Clearly, u or v is not divisible by 3. Without loss of generality we suppose that $u^2=(3x-1)^2$ for some $x\in\mathbb{Z}$. As $5v^2+w^2=15n+11+3r-5u^2>0$ is a positive multiple of 3, by [S15a, Lemma 2.1] we can write $5v^2+w^2$ as $5(3y-1)^2+(3z-1)^2$ with $y,z\in\mathbb{Z}$. Thus

$$\begin{aligned}15n+11+3r=&5(3x-1)^2+5(3y-1)^2+(3z-1)^2\\=&5(3p_8(x)+1)+5(3p_8(y)+1)+3p_8(z)+1\end{aligned}$$

and hence

$$n=p_8(x)+p_8(y)+\frac{p_8(z)-r}{5}=p_8(x)+p_8(y)+\left\lfloor\frac{p_8(z)}{5}\right\rfloor.$$

This proves (1.19). In view of (1.12), (1.20) follows from (1.19).

The proof of Theorem 1.5 is now complete. □

5. Some further conjectures

Conjecture 5.1. *For any $n \in \mathbb{N}$, there are $x, y, z \in \mathbb{N}$ such that $8n + 3 = x^2 + y^2 + z^2$ and $x \equiv 1, 3 \pmod 8$. Also, for any $n \in \mathbb{N}$ with $n \neq 20$, there are $x, y, z \in \mathbb{Z}$ with $x \equiv \pm 3 \pmod 8$ such that $x^2 + y^2 + z^2 = 8n + 3$.*

Remark 5.1. In [S15a] the author conjectured that any $n \in \mathbb{N}$ can be written as the sum of two triangular numbers and a hexagonal number, equivalently, $8n + 3 = (4x - 1)^2 + y^2 + z^2$ for some $x, y, z \in \mathbb{N}$.

Conjecture 5.2. *Let $a > 2$ be an integer with $a \neq 4, 6$. Then any positive integer can be written as the sum of three elements of the set $\{\lfloor x^2/a\rfloor : x \in \mathbb{Z}\}$ one of which is odd.*

Remark 5.2. This is a refinement of Farhi's conjecture for $a \neq 4, 6$.

Conjecture 5.3. *Let*

$$T := \left\{x^2 + \left\lfloor \frac{x}{2} \right\rfloor :\ x \in \mathbb{Z}\right\} = \left\{\left\lfloor \frac{k(k+1)}{4} \right\rfloor :\ k \in \mathbb{N}\right\}.$$

Then each $n = 2, 3, 4, \ldots$ can be expressed as $r + s + t$, where r, s, t are elements of T with $r \leqslant s \leqslant t$ and $2 \nmid s$. Also, for any ordered pair (b, c) among

$$(1, 2),\ (1, 3),\ (1, 4),\ (1, 5),\ (1, 6),\ (1, 8),\ (1, 9),\ (2, 2),\ (2, 3),$$

each $n \in \mathbb{N}$ can be written as $x + by + cz$ with $x, y, z \in T$.

Remark 5.3. It is easy to see that $\{T_x :\ x \in \mathbb{Z}\} = \{p_6(-x) = x(2x + 1) :\ x \in \mathbb{Z}\}$.

Conjecture 5.4. (i) *Let α be a positive real number with $\alpha \neq 1$ and $\alpha \leqslant 1.5$. Define*

$$S(\alpha) := \{x^2 + \lfloor \alpha x\rfloor :\ x \in \mathbb{Z}\}.$$

Then any positive integer can be written as the sum of three elements of $S(\alpha)$ one of which is odd.

(ii) *Let $0 < \alpha \leqslant \beta \leqslant \gamma \leqslant 1.5$ such that two of α, β, γ are different from 1 or $\{\alpha, \beta, \gamma\} = \{1, 1/m\}$ for some $m = 2, 3, 4, \ldots$. Then any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + \lfloor \alpha x\rfloor + \lfloor \beta y\rfloor + \lfloor \gamma z\rfloor$ with $x, y, z \in \mathbb{Z}$. In particular, if $a, b, c \in \mathbb{Z}^+$ are not all equal to one, then*

$$\left\{x^2 + y^2 + z^2 + \left\lfloor \frac{x}{a} \right\rfloor + \left\lfloor \frac{y}{b} \right\rfloor + \left\lfloor \frac{z}{c} \right\rfloor :\ x, y, z \in \mathbb{Z}\right\} = \mathbb{N}.$$

Remark 5.4. Note that 2 cannot be written as the sum of three elements of $S(11/4)$, and 4 cannot be written as the sum of three elements of $S(8/5)$ one of which is odd.

Conjecture 5.5. *Any integer $n>1$ can be written as $p+\lfloor k(k+1)/4\rfloor$, where p is a prime and k is a positive integer.*

Remark 5.5. The author [S09] conjectured that 216 is the only natural number not representable by $p+T_x$, where p is prime or zero, and x is an integer.

Motivated by Theorem 1.5, we pose the following conjecture.

Conjecture 5.6. *Let $a,b,c\in\mathbb{Z}^+$. Then*

$$\left\{\left\lfloor\frac{p_5(x)}{a}\right\rfloor+\left\lfloor\frac{p_5(y)}{b}\right\rfloor+\left\lfloor\frac{p_5(z)}{c}\right\rfloor:\ x,y,z\in\mathbb{Z}\right\}=\mathbb{N}.$$

When $(a,b,c)\neq(1,1,1),(1,1,2),(2,2,2)$, we have

$$\left\{\left\lfloor\frac{p_7(x)}{a}\right\rfloor+\left\lfloor\frac{p_7(y)}{b}\right\rfloor+\left\lfloor\frac{p_7(z)}{c}\right\rfloor:\ x,y,z\in\mathbb{Z}\right\}=\mathbb{N}.$$

If $(a,b,c)\neq(1,1,1),(2,2,2)$, then

$$\left\{\left\lfloor\frac{p_8(x)}{a}\right\rfloor+\left\lfloor\frac{p_8(y)}{b}\right\rfloor+\left\lfloor\frac{p_8(z)}{c}\right\rfloor:\ x,y,z\in\mathbb{Z}\right\}=\mathbb{N}.$$

Now we present a general conjecture related to Theorems 1.1-1.3.

Conjecture 5.7. (i) *Let a and b be positive integers. If $c\in\mathbb{Z}^+$ is large enough, then*

$$\left\{ax^2+by^2+\left\lfloor\frac{z^2}{c}\right\rfloor:\ x,y,z\in\mathbb{Z}\right\}=\left\{ax^2+by^2+\left\lceil\frac{z^2}{c}\right\rceil:\ x,y,z\in\mathbb{Z}\right\}=\mathbb{N}.$$

Also, for any sufficiently large $c\in\mathbb{Z}^+$ we have

$$\left\{ax^2+by^2+\left\lfloor\frac{z(z+1)}{c}\right\rfloor:\ x,y,z\in\mathbb{Z}\right\}=\mathbb{N}$$

and

$$\left\{ax^2+by^2+\left\lceil\frac{z(z+1)}{c}\right\rceil:\ x,y,z\in\mathbb{Z}\right\}=\mathbb{N}.$$

(ii) *For $a,b,c\in\mathbb{Z}^+$ with $2a\leqslant b+c$, if $(a,b,c)\neq(1,1,1),(3,3,3),(4,2,6)$ then*

$$\left\{ax^2+\left\lfloor\frac{y^2}{b}\right\rfloor+\left\lfloor\frac{z^2}{c}\right\rfloor:\ x,y,z\in\mathbb{Z}\right\}=\mathbb{N}.$$

For $a,b\in\mathbb{Z}^+$, we define

$$S^*(a,b):=\left\{c\in\mathbb{Z}^+:\ \left\{ax^2+by^2+\left\lceil\frac{z^2}{c}\right\rceil:\ x,y,z\in\mathbb{Z}\right\}\neq\mathbb{N}\right\},$$
$$S_*(a,b):=\left\{c\in\mathbb{Z}^+:\ \left\{ax^2+by^2+\left\lfloor\frac{z^2}{c}\right\rfloor:\ x,y,z\in\mathbb{Z}\right\}\neq\mathbb{N}\right\},$$
$$T^*(a,b):=\left\{c\in\mathbb{Z}^+:\ \left\{ax^2+by^2+\left\lceil\frac{z(z+1)}{c}\right\rceil:\ x,y,z\in\mathbb{Z}\right\}\neq\mathbb{N}\right\},$$
$$T_*(a,b):=\left\{c\in\mathbb{Z}^+:\ \left\{ax^2+by^2+\left\lfloor\frac{z(z+1)}{c}\right\rfloor:\ x,y,z\in\mathbb{Z}\right\}\neq\mathbb{N}\right\}.$$

Based on our computation we conjecture that

$$S^*(1,1)=\{1,2,5\},\ S^*(1,2)=\{1,3\},\ S^*(1,3)=\{1,4\},\ S^*(1,4)=\{1,2,3,5\},$$
$$S^*(1,5)=\{1,2,3,5\},\ S^*(1,6)=\{1,2,3,4\},\ S^*(1,7)=\{1,2,4,8\},$$
$$S^*(1,8)=\{1,\ldots,6,9\},\ S^*(1,9)=\{1,\ldots,6\},\ S^*(1,10)=\{1,\ldots,6,8,12\},$$
$$S^*(2,2)=\{1,\ldots,5,9,10\},\ S^*(2,3)=\{1,2,8\};$$
$$S_*(1,2)=\{1\},\ S_*(1,3)=\{1,2,10\},\ S_*(1,4)=\{1,2,3,5\},\ S_*(1,5)=\{1,2,3,4,5\},$$
$$S_*(1,6)=\{1,3\},\ S_*(1,7)=\{1,2,3,4,5\},\ S_*(1,8)=\{1,2,3,5,9\},$$
$$S_*(1,9)=\{1,2,3,4,5,7\},\ S_*(1,10)=\{1,2,3,4,12\},$$
$$S_*(1,11)=\{1,2,3,4,5,6,9\},\ S_*(1,12)=\{1,2,3,4,5,6,10\}$$
$$S_*(2,2)=\{1,2,3,4,5,6,10\},\ S_*(2,3)=\{1,2,8\},$$
$$S_*(2,4)=\{1,2,5,6\},\ S_*(2,5)=\{1,2,3,5\};$$

$$T^*(1,1)=T^*(1,2)=\emptyset,\ T^*(1,3)=1,\ T^*(1,4)=\{3\},\ T^*(1,5)=T^*(1,6)=\{1,2\},$$
$$T^*(1,7)=\{1,2,4\},\ T^*(1,8)=\{1\},\ T^*(1,9)=T^*(1,10)=T^*(1,11)=\{1,2,3\},$$
$$T^*(2,2)=\{1,3\},\ T^*(2,3)=\{1,2\},\ T^*(2,4)=\{1,2,3\},\ T^*(3,4)=\{1,2,3\};$$
$$T_*(1,2)=\emptyset,\ T_*(1,3)=\{1\},\ T_*(1,5)=\{1,2,3\},\ T_*(1,6)=\{1,2\},$$
$$T_*(1,7)=\{1,2,4\},\ T_*(1,8)=\{1\},\ T_*(1,10)=T_*(2,3)=\{1,2,3\}.$$

Also, our computation suggests that

$$\left\{4x^2+4y^2+\left\lfloor\frac{z^2}{c}\right\rfloor:\ x,y,z\in\mathbb{Z}\right\}=\mathbb{N}$$

for any integer $c>42$, and that

$$\left\{4x^2+4y^2+\left\lfloor\frac{z(z+1)}{c}\right\rfloor:\ x,y,z\in\mathbb{Z}\right\}=\mathbb{N}$$

for any integer $c>27$. Note that $179\neq 4x^2+4y^2+\lfloor z^2/42\rfloor$ for any $x,y,z\in\mathbb{Z}$ and that $29\neq 4x^2+4y^2+\lfloor z(z+1)/27\rfloor$ for all $x,y,z\in\mathbb{Z}$.

Motivated by Theorem 1.4(i), we pose the following conjecture similar to Conjecture 1.1.

Conjecture 5.8. *Let a, b, c be positive integers with $a \leqslant b \leqslant c$. If $c > 1$, then*

$$\left\{\left\lfloor \frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} \right\rfloor :\ x, y, z \in \mathbb{Z}\right\} = \mathbb{N}$$

If $(a, b, c) \neq (1, 1, 1), (1, 1, 3), (1, 1, 7), (1, 3, 3)$, then

$$\left\{\left\lfloor \frac{x(x+1)}{a} + \frac{y(y+1)}{b} + \frac{z(z+1)}{c} \right\rfloor :\ x, y, z \in \mathbb{Z}\right\} = \mathbb{N}.$$

Conjecture 5.9. *We have*

$$\left\{w^3 + \left\lfloor \frac{x^3}{2} \right\rfloor + \left\lfloor \frac{y^3}{3} \right\rfloor + \left\lfloor \frac{z^3}{4} \right\rfloor :\ x, y, z \in \mathbb{N}\right\} = \mathbb{N}$$

and

$$\left\{w^3 + \left\lfloor \frac{x^3}{2} \right\rfloor + \left\lfloor \frac{y^3}{4} \right\rfloor + \left\lfloor \frac{z^3}{8} \right\rfloor :\ x, y, z \in \mathbb{N}\right\} = \mathbb{N}.$$

Our following conjecture is a natural extension of Goldbach's Conjecture.

Conjecture 5.10. *For any positive integers a and b with $a + b > 2$, any integer $n > 2$ can be written as $\lfloor p/a \rfloor + \lfloor q/b \rfloor$ with p and q both prime.*

Remark 5.6. In the case $\{a, b\} = \{1, 2\}$, Conjecture 5.10 reduces to Lemoine's Conjecture which states that any odd number greater than 5 can be written as $p + 2q$ with p and q both prime. In the case $a = b = 2$, Conjecture 5.10 reduces to the Goldbach Conjecture.

Let us conclude this paper with one more conjecture.

Conjecture 5.11. *Let*

$$\begin{aligned} S &= \left\{\left\lfloor \frac{x}{9} \right\rfloor :\ x - 1 \text{ and } x + 1 \text{ are twin prime}\right\} \\ &= \left\{\left\lfloor \frac{x}{3} \right\rfloor :\ 3x - 1 \text{ and } 3x + 1 \text{ are twin prime}\right\}. \end{aligned}$$

Then, any positive integer can be written as the sum of two distinct elements of S one of which is even. Also, any positive integer can be expressed as the sum of an element of S and a positive generalized pentagonal number.

Remark 5.7. Clearly either of the two assertions in Conjecture 5.11 implies the Twin Prime Conjecture.

Conjecture 5.12. *Any integer $n > 1$ can be written as $x^2 + y^2 + \varphi(z^2)$ with $x, y \in \mathbb{N}$, $z \in \mathbb{Z}^+$, and $\max\{x, y\}$ or z prime. Also, any $n \in \mathbb{Z}^+$ can be written as $x^3 + y^2 + T_z$ with $x, y \in \mathbb{N}$ and $z \in \mathbb{Z}^+$.*

Remark 5.8. We have verified this for all $n = 1, \ldots, 10^5$. See [S15c, A262311 and A262813] for related data.

Conjecture 5.13. *Any integer m can be written as $x^4-y^3+z^2$ with $x,y,z\in\mathbb{Z}^+$.*

Remark 5.9. We have verified this for all $m\in\mathbb{Z}$ with $|m|\leqslant 10^5$, see [S15c] for related data. For example,

$$0=4^4-8^3+16^2,\ \ 6=36^4-139^3+1003^2,\ \ \text{and } 11019=4325^4-71383^3+3719409^2.$$

Conjecture 5.14. *Any $n\in\mathbb{N}$ can be written as $w^2+x^3+y^4+2z^4$ with $w,x,y,z\in\mathbb{N}$. Also, any $n\in\mathbb{N}$ can be written as $w^2+2x^2+y^3+2z^3$ with $w,x,y,z\in\mathbb{N}$.*

Remark 5.10. We have verified this for all $n=1,\ldots,4\times10^6$, see [S15c, A262827 and A262857] for related data.

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