DERIVATIVE POLYNOMIALS AND ENUMERATION OF PERMUTATIONS BY THEIR ALTERNATING DESCENTS

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Abstract. In this paper we present an explicit formula for the number of permutations with a given number of alternating descents. Moreover, we study the interlacing property of the real parts of the zeros of the generating polynomials of these numbers.

1. Introduction

Let $\mathfrak{S}_n$ denote the symmetric group of all permutations of $[n]$, where $[n] = \{1, 2, \ldots, n\}$. For a permutation $\pi \in \mathfrak{S}_n$, we define a *descent* to be a position i such that $\pi(i) > \pi(i+1)$. Denote by $\operatorname{des}(\pi)$ the number of descents of π. The classical Eulerian polynomials $A_n(x)$ are defined by

$$A_n(x) = \sum_{\pi\in\mathfrak{S}_n} x^{\operatorname{des}(\pi)} = \sum_{k=0}^{n-1} A(n,k)x^k.$$

As a variation of the descent statistic, the number of *alternating descents* of a permutation $\pi \in \mathfrak{S}_n$ is defined by

$$\operatorname{altdes}(\pi) = |\{2i : \pi(2i) < \pi(2i+1)\} \cup \{2i+1 : \pi(2i+1) > \pi(2i+2)\}|.$$

We say that π has a *3-descent* at index i if $\pi(i)\pi(i+1)\pi(i+2)$ has one of the patterns: 132, 213, or 321. Chebikin [2] showed that the alternating descent statistic of permutations in $\mathfrak{S}_n$ is equidistributed with the 3-descent statistic of permutations in $\{\pi \in \mathfrak{S}_{n+1} : \pi_1 = 1\}$. Then the equations

$$\widehat{A}_n(x) = \sum_{\pi\in\mathfrak{S}_n} x^{\operatorname{altdes}(\pi)} = \sum_{k=0}^{n-1} \widehat{A}(n,k)x^k$$

define the *alternating Eulerian polynomials* $\widehat{A}_n(x)$ and the *alternating Eulerian numbers* $\widehat{A}(n,k)$. The first few $\widehat{A}_n(x)$ are given as follows:

$$\widehat{A}_1(x) = 1,$$
$$\widehat{A}_2(x) = 1 + x,$$
$$\widehat{A}_3(x) = 2 + 2x + 2x^2,$$
$$\widehat{A}_4(x) = 5 + 7x + 7x^2 + 5x^3,$$
$$\widehat{A}_5(x) = 16 + 26x + 36x^2 + 26x^3 + 16x^4.$$

2010 *Mathematics Subject Classification.* Primary 05A15; Secondary 26C10.

Key words and phrases. Alternating Eulerian polynomials; Derivative polynomials; Zeros.

Chebikin [2] proved that

$$\sum_{n\geq 1}\widehat{A}_n(x)\frac{z^n}{n!}=\frac{\sec(1-x)z+\tan(1-x)z)-1}{1-x(\sec(1-x)z+\tan(1-x)z)}, \tag{1}$$

and the numbers $\widehat{A}(n,k)$ satisfy the recurrence relation

$$\sum_{i=0}^{n}\sum_{j=0}^{k}\binom{n}{i}\widehat{A}(i,j+1)\widehat{A}(n-i,k-j+1)=(n+1-k)\widehat{A}(n,k+1)+(k+1)\widehat{A}(n,k+2).$$

In recent years, several authors pay attention to the alternating descent statistic and its associated permutation statistics. The reader is referred to [5, 9] for recent progress on this subject. For example, Gessel and Zhuang [5] defined an alternating run to be a maximal consecutive subsequence with no alternating descents. The purpose of this paper is to present an explicit formula for the numbers $\widehat{A}(n,k)$. In Section 2, we express the polynomials $\widehat{A}_n(x)$ in terms of the *derivative polynomials* $P_n(x)$ defined by Hoffman [4]:

$$P_n(\tan\theta)=\frac{d^n}{d\theta^n}\tan\theta.$$

2. An explicit formula

Let D denote the differential operator $d/d\theta$. Set $x=\tan\theta$. Then $D(x^n)=nx^{n-1}(1+x^2)$ for $n\geq 1$. Thus $D^n(x)$ is a polynomial in x. Let $P_n(x)=D^n(x)$. Then $P_0(x)=x$ and

$$P_{n+1}(x)=(1+x^2)P_n'(x). \tag{2}$$

Clearly, $\deg P_n(x)=n+1$. By definition, we have

$$\tan(\theta+z)=\sum_{n\geq 0}P_n(x)\frac{z^n}{n!}=\frac{x+\tan z}{1-x\tan z}, \tag{3}$$

Let $P_n(x)=\sum_{k=0}^{n+1}p(n,k)x^k$. It is easy to verify that

$$p(n,k)=(k+1)p(n-1,k+1)+(k-1)p(n-1,k-1).$$

The first few terms can be computed directly as follows:

$$\begin{aligned}
P_1(x)&=1+x^2,\\
P_2(x)&=2x+2x^3,\\
P_3(x)&=2+8x^2+6x^4,\\
P_4(x)&=16x+40x^3+24x^5.
\end{aligned}$$

Note that $P_n(-x)=(-1)^{n+1}P_n(x)$ and $x\|P_{2n}(x)$. Thus we have the following expression:

$$P_n(x)=\sum_{k=0}^{\lfloor (n+1)/2\rfloor}p(n,n-2k+1)x^{n-2k+1}.$$

There is an explicit formula for the numbers $p(n,n-2k+1)$.

Proposition 1 ([7, Proposition 1])**.** *For $n \geq 1$ and $0 \leq k \leq \lfloor (n+1)/2 \rfloor$, we have*

$$p(n, n-2k+1) = (-1)^k \sum_{i \geq 1} i! \left\{ {n \atop i} \right\} (-2)^{n-i} \left[\binom{i}{n-2k} - \binom{i}{n-2k+1} \right].$$

Now we present the first main result of this paper.

Theorem 2. *For $n \geq 1$, we have*

$$2^n (1+x^2) \widehat{A}_n(x) = (1-x)^{n+1} P_n \left(\frac{1+x}{1-x} \right). \tag{4}$$

Proof. It follows from (3) that

$$\begin{aligned}\sum_{n \geq 1} (1-x)^{n+1} P_n \left(\frac{1+x}{1-x} \right) \frac{z^n}{n!} &= (1-x) \sum_{n \geq 1} P_n \left(\frac{1+x}{1-x} \right) \frac{(z-xz)^n}{n!} \\ &= (1+x^2) \frac{2 \tan(z-xz)}{1-x-(1+x)\tan(z-xz)}.\end{aligned}$$

Comparing with (1), it suffices to show the following

$$\frac{\sec(2z-2xz)+\tan(2z-2xz)-1}{1-x(\sec(2z-2xz)+\tan(2z-2xz))} = \frac{2\tan(z-xz)}{1-x-(1+x)\tan(z-xz)}. \tag{5}$$

Set $t = \tan(z-xz)$. Using the tangent half-angle substitution, we have

$$\sec(2z-2xz) = \frac{1+t^2}{1-t^2},\ \tan(2z-2xz) = \frac{2t}{1-t^2}.$$

Then the left hand side of (5) equals

$$\frac{2t(1+t)}{1-t^2-x(1+t)^2} = \frac{2t}{1-x-(1+x)t}.$$

This completes the proof. $\square$

It follows from (2) that

$$2^n \widehat{A}_{n+1}(x) = (1-x)^n P_n' \left(\frac{1+x}{1-x} \right) = \sum_{k=0}^{\lfloor (n+1)/2 \rfloor} (n-2k+1) p(n, n-2k+1) (1-x)^{2k} (1+x)^{n-2k}. \tag{6}$$

Denote by $E(n,k,s)$ the coefficients x^s of $(1-x)^{2k}(1+x)^{n-2k}$. Clearly,

$$E(n,k,s) = \sum_{j=0}^{\min(\lfloor \frac{k}{2} \rfloor, s)} (-1)^j \binom{2k}{j} \binom{n-2k}{s-j}.$$

Then we get the following result.

Corollary 3. *For $n \geq 2$ and $1 \leq s \leq n$, we have*

$$\widehat{A}(n+1, s) = \frac{1}{2^n} \sum_{k=0}^{\lfloor (n+1)/2 \rfloor} (n-2k+1) p(n, n-2k+1) E(n,k,s).$$

Let $\pi = \pi(1)\pi(2)\cdots\pi(n) \in \mathfrak{S}_n$. An *interior peak* in π is an index $i \in \{2, 3, \ldots, n-1\}$ such that $\pi(i-1) < \pi(i) > \pi(i+1)$. Let $\mathrm{pk}\,(\pi)$ denote the number of interior peaks of π. Let $W_n(x) = \sum_{\pi\in\mathfrak{S}_n} x^{\mathrm{pk}\,(\pi)}$. It is well known that the polynomials $W_n(x)$ satisfy the recurrence relation

$$W_{n+1}(x) = (nx - x + 2)W_n(x) + 2x(1-x)W_n'(x),$$

with initial conditions $W_1(x) = 1, W_2(x) = 2$ and $W_3(x) = 4 + 2x$. By the theory of *enriched P-partitions*, Stembridge [11, Remark 4.8] showed that

$$W_n\left(\frac{4x}{(1+x)^2}\right) = \frac{2^{n-1}}{(1+x)^{n-1}}A_n(x). \tag{7}$$

From [6, Theorem 2], we have

$$P_n(x) = x^{n-1}(1+x^2)W_n(1+x^{-2}). \tag{8}$$

Therefore, combining (4) and (8), we get the counterpart of (7):

$$W_n\left(\frac{2+2x^2}{(1+x)^2}\right) = \frac{2^{n-1}}{(1+x)^{n-1}}\widehat{A}_n(x).$$

3. Zeros of the alternating Eulerian polynomials

Combining (2) and (4), it is easy to derive that

$$2\widehat{A}_{n+1}(x) = (1 + n + 2x + nx^2 - x^2)\widehat{A}_n(x) + (1-x)(1+x^2)\widehat{A}_n'(x)$$

for $n \geq 1$. The bijection $\pi \mapsto \pi^c$ on $\mathfrak{S}_n$ defined by $\pi^c(i) = n + 1 - \pi(i)$ shows that $\widehat{A}_n(x)$ is symmetric. Hence $(x+1)\|\widehat{A}_{2n}(x)$ for $n \geq 1$. It is well known that the classical Eulerian polynomials $A_n(x)$ have only real zeros, and the zeros of $A_n(x)$ separates that of $A_{n+1}(x)$ (see Bóna [1, p. 24] for instance). Now we present the corresponding result for $\widehat{A}_n(x)$.

Theorem 4. *For $n \geq 1$, the zeros of $\widehat{A}_{2n+1}(x)$ and $\widehat{A}_{2n+2}(x)/(1+x)$ are imaginary with multiplicity 1, and the moduli of all zeros of $\widehat{A}_n(x)$ are equal to 1. Furthermore, the sequence of real parts of the zeros of $\widehat{A}_n(x)$ separates that of $\widehat{A}_{n+1}(x)$. More precisely, suppose that $\{r_j \pm \ell_j \mathrm{i}\}_{j=1}^{n-1}$ are all zeros of $\widehat{A}_{2n}(x)/(1+x)$, $\{s_j \pm t_j \mathrm{i}\}_{j=1}^{n}$ are all zeros of $\widehat{A}_{2n+1}(x)$ and $\{p_j \pm q_j \mathrm{i}\}_{j=1}^{n}$ are all zeros of $\widehat{A}_{2n+2}(x)/(1+x)$, where $-1 < r_1 < r_2 < \cdots < r_{n-1} < 0$, $-1 < s_1 < s_2 < \cdots < s_n < 0$ and $-1 < p_1 < p_2 < \cdots < p_n < 0$. Then we have*

$$-1 < s_1 < r_1 < s_2 < r_2 < \cdots < r_{n-1} < s_n, \tag{9}$$

$$-1 < s_1 < p_1 < s_2 < p_2 < \cdots < s_n < p_n. \tag{10}$$

Proof. Define $\widetilde{P}_n(x) = \mathrm{i}^{n-1}P_n(\mathrm{i}x)$. Then

$$\widetilde{P}_{n+1}(x) = (1-x^2)\widetilde{P}_n'(x).$$

From [8, Theorem 2], we get that the polynomials $\widetilde{P}_n(x)$ have only real zeros, belong to $[-1,1]$ and the sequence of zeros of $\widetilde{P}_n(x)$ separates that of $\widetilde{P}_{n+1}(x)$. From [3, Corollary 8.7], we see that the zeros of the derivative polynomials $P_n(x)$ are pure imaginary with multiplicity 1, belong to the line segment $[-\mathrm{i}, \mathrm{i}]$. In particular, $(1+x^2)\|P_n(x)$. Therefore, the polynomials $P_{2n+1}(x)$ and $P_{2n+2}(x)$ have the following expressions:

$$P_{2n+1}(x) = (1+x^2)\prod_{i=1}^{n}(x^2+a_i), P_{2n+2}(x) = x(1+x^2)\prod_{i=1}^{n}(x^2+b_i),$$

where

$$0 < a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n < 1. \tag{11}$$

Using (4), we get

$$2^{2n}\widehat{A}_{2n+1}(x) = \prod_{i=1}^{n}((1+x)^2 + a_i(1-x)^2),$$

$$2^{2n+1}\widehat{A}_{2n+2}(x) = (1+x)\prod_{i=1}^{n}((1+x)^2 + b_i(1-x)^2).$$

Hence

$$2^{2n}\widehat{A}_{2n+1}(x) = \prod_{i=1}^{n}(1+a_i)\left(x + \frac{1-a_i}{1+a_i} + \frac{2\mathrm{i}\sqrt{a_i}}{1+a_i}\right)\left(x + \frac{1-a_i}{1+a_i} - \frac{2\mathrm{i}\sqrt{a_i}}{1+a_i}\right),$$

$$2^{2n+1}\widehat{A}_{2n+2}(x) = (1+x)\prod_{i=1}^{n}(1+b_i)\left(x + \frac{1-b_i}{1+b_i} + \frac{2\mathrm{i}\sqrt{b_i}}{1+b_i}\right)\left(x + \frac{1-b_i}{1+b_i} - \frac{2\mathrm{i}\sqrt{b_i}}{1+b_i}\right).$$

Since

$$\left(\frac{1-a_i}{1+a_i}\right)^2 + \left(\frac{2\sqrt{a_i}}{1+a_i}\right)^2 = \left(\frac{1-b_i}{1+b_i}\right)^2 + \left(\frac{2\sqrt{b_i}}{1+b_i}\right)^2 = 1,$$

the moduli of all zeros of $\widehat{A}_n(x)$ are equal to 1. Note that

$$s_j = -\frac{1-a_j}{1+a_j},\ p_j = -\frac{1-b_j}{1+b_j}.$$

So (10) is immediate. Along the same lines, one can get (9). □

From Theorem 4, we immediately get that the sequence of imaginary parts of the zeros of $\widehat{A}_n(x)$ also separates that of $\widehat{A}_{n+1}(x)$.

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