TORUS FIXED POINTS IN SCHUBERT VARIETIES AND NORMALIZED MEDIAN GENOCCHI NUMBERS

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Abstract. We give a new proof for the fact that the number of torus fixed points for the degenerate flag variety is equal to the normalized median Genocchi number, using the identification with a certain Schubert variety. We further study the torus fixed points for the symplectic degenerate flag variety and develop a combinatorial model, symplectic Dellac configurations, so parametrize them. The number of these symplectic fixed points is conjectured to be the median Euler number.

Introduction

We consider the Schubert variety X_{τ_n} associated to the Weyl group element

$$\tau_n := (s_n s_{n+1} \cdots s_{2n-2}) \cdots (s_k s_{k+1} \cdots s_{2k-2}) \cdots (s_3 s_4) s_2 \in \mathfrak{S}_{2n}$$

in the partial flag variety SL_{2n}/P, where P is the standard parabolic subalgebra associated to the simple roots $\{\alpha_1, \alpha_3, \ldots, \alpha_{2n-1}\}$. Then there is a natural action of a $2n-1$-dimensional torus T_{2n-1} and we are mainly interested in the fixed points $X_{\tau_n}^{T_{2n-1}}$ of this torus action. It is well known that the fixed points are parametrized Weyl groups elements which are less or equal to τ_n in the Bruhat order (modulo the stabilizer of the parabolic, in this case, the subgroup generated by $s_1, s_3, \ldots, s_{2n-1}$). Our first result is

Theorem A. There is an explicit bijection $\mathbf{b}$ from Dellac configurations DC_n (Definition 1) of $2n$ columns and n rows to $X_{\tau_n}^{T_{2n-1}}$, hence the number of torus fixed points is equal to the normalized median Genocchi number (see Section 1 for definition).

Here is a an example of the Dellac configuration corresponding to a fixed point for $n = 3$:

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$\mapsto \quad \sigma = 124536$

We also consider Schubert varieties of the symplectic flag variety, e.g. the Schubert variety $X^{sp}_{\overline{\tau}_{2n}}$ corresponding to the element (of the symplectic Weyl group):

$$\overline{\tau}_{2n} := (r_{2n} \cdots r_{n+1}) \cdots (r_{2n} r_{2n-1} r_{2n-2})(r_{2n} r_{2n-1}) r_{2n} (r_n \cdots r_{2n-2}) \cdots (r_4 r_5 r_6)(r_3 r_4) r_2$$

in the symplectic partial flag variety. In this case, there is a natural action of T_{2n} on the Schubert variety and we are again interested in the fixed points of this torus action. To parametrize them similar to the non-symplectic case, we introduce symplectic Dellac configurations (Definition 2). These are Dellac configurations with $4n$ columns and $2n$ rows, which are invariant under the involution mapping the i-th row to the $2n-i+1$-st row. Our second result is

Theorem B. The torus fixed points in $X^{sp}_{\overline{\tau}_{2n}}$ are parametrized by the symplectic Dellac configurations SpDC_{2n}.

We conjecture that the number of symplectic Dellac configurations is equal to the normalized median Euler number ([K97]).

We should explain here why we are interested in these particular Schubert varieties. E. Feigin ([Fei11]) defined the degenerate flag variety

$$\mathcal{F}l_n^a := \{(U_1, \ldots, U_{n-1}) \in \prod_{i=1}^{n-1} \mathrm{Gr}_i(\mathbb{C}^n) \mid \mathrm{pr}_{i+1} U_i \subset U_{i+1}\}$$

where pr_i is the endomorphism of $\mathbb{C}^n$ setting the i-th coordinate to be zero. This is in fact a flat degeneration of the classical flag variety $\mathcal{F}l_n$, moreover it was shown in [CFR12, CLL15] that there is an action of T_{2n-1} on $\mathcal{F}l_n^a$. The symplectic degenerate flag variety $(\mathcal{F}l_{2n}^a)^{sp}$ has been defined in [FFiL14] in a similar way.

The degenerate flag variety is one of the main objects in the framework of PBW filtrations and degenerations on universal enveloping algebras of simple Lie algebras (see for various aspects [FFoL11a, FFoL11b, FFoL13, FFR15, Hag14, Fou14, Fou15, CFR12]). Here, one obtains *degenerate flag varieties* $\mathcal{F}l^a(\lambda)$ as highest weight orbits of PBW degenerate modules. In [Fei11, FFiL14] it has been shown that these highest weight orbits do have an interpretation as a variety of certain flags.

Recently, it was shown in [CL15] that these degenerate flag varieties are in fact our particular Schubert varieties:

Theorem. (Cerulli Irelli-Lanini)

1. In the $\mathfrak{sl}_n$-case, the degenerate flag variety $\mathcal{F}l_n^a$ is isomorphic to the Schubert variety X_{τ_n}, moreover the isomorphism $\zeta : \mathcal{F}l_n^a \xrightarrow{\sim} X_{\tau_n}$ is T_{2n-1}-equivariant.
2. In the $\mathfrak{sp}_{2n}$-case the degenerate symplectic flag variety is isomorphic to $X^{sp}_{\overline{\tau}_{2n}}$ and again the isomorphism $\zeta^{sp} : X^{sp}_{\overline{\tau}_{2n}} \xrightarrow{\sim} (\mathcal{F}l_{2n}^a)^{sp}$ is torus-equivariant.

The torus fixed points of the degenerate flag variety in type A_n have been studied in [Fei11]. In that paper, an explicit bijection $\mathbf{f}$ to the set of Dellac configurations has been provided. Hence it was shown that the number of torus fixed points is equal to the normalized median Genocchi number.

Combining the theorem by Cerulli Irelli and Lanini with Theorem A, we obtain another proof of this fact, using the classical set up of Schubert varieties only. Moreover, we can show that the following diagram commutes (here α denotes the natural identification of $W^J_{\leq \tau_n}$ with $X^{T_{2n-1}}_{\tau_n}$)

$$\begin{tikzcd}
(\mathcal{F}l_n^a)^{T_n} \arrow[r, "\mathbf{f}"] \arrow[d, "\zeta"] & \mathrm{DC}_n \arrow[d, "\mathbf{b}"] \\
X^{T_{2n-1}}_{\tau_n} & W^J_{\leq \tau_n} \arrow[l, "\alpha"']
\end{tikzcd} .$$

In the symplectic case, the map $\mathbf{f}$ is not present, mainly because the construction of symplectic Dellac configurations has not been seen in the literature before. Nevertheless we obtain a similar picture, namely the number of torus fixed points in the symplectic degenerate flag variety are parametrized by SpDC_{2n}. We should mention here that E. Feigin (via the symplectic degenerate flag variety [FFiL14]) as well as G. Cerulli Irelli (via quiver Grassmannian [CFR12]) also conjectured the number of torus fixed points to be the normalized median Euler number.

This paper is organized as follow, in Section 1 we prove our first theorem for the $\mathfrak{sl}_n$, in Section 2 we consider the symplectic case. In Section 3 we relate our results to the framework of degenerate flag varieties.

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1. Symmetric groups and Median Genocchi numbers

1.1. Let $W = \mathfrak{S}_{2n}$ be the symmetric group generated by $S = \{s_1, s_2, \cdots, s_{2n-1}\}$ where $s_i = (i, i+1)$. Let $J = \{s_1, s_3, \cdots, s_{2n-1}\} \subset S$ and W_J be the subgroup generated by J, W^J be the set of minimal representatives of right cosets of W_J in W. We define

$$\tau_n = (s_n s_{n+1} \cdots s_{2n-2}) \cdots (s_k s_{k+1} \cdots s_{2k-2}) \cdots (s_3 s_4) s_2 \in W,$$

then for $t = 1, 2, \cdots, 2n$:

$$\tau_n(t) = \begin{cases} k, & t = 2k-1; \\ n+k, & t = 2k. \end{cases} \tag{1.1}$$

By construction, τ_n is a representative of minimal length in W/W_J, so $\tau_n \in W^J$. We define

$$W_{\leq \tau_n} = \{w \in W \mid w \leq \tau_n\}, \quad W^J_{\leq \tau_n} = \{w \in W^J \mid w \leq \tau_n\},$$

where $\leq$ is the Bruhat order.

Definition 1. A Dellac configuration C is a board of $2n$ columns and n rows with $2n$ marked cells such that

1. each column contains exactly one marked cell;
2. each row contains exactly two marked cells;
3. if the (i, j)-cell is marked, then $i \leq j \leq n + i$.

Let DC_n denote the set of such configurations.

It is worthy of pointing out that the definition of a Dellac configuration given above differs from that in [Fei11] by rotating the board by 90°.

The cardinality h_n of the set DC_n is called a normalized median Genocchi number (see [Fei11, Fei12] and the references therein). Consider the following polynomial defined by recursion: $H_0(x) = 1$,

$$H_n(x) = \frac{1}{2}(x+1)((x+1)H_{n-1}(x+1) - xH_{n-1}(x)).$$

Then it is proved in [DR94] that $h_n = H_n(1)$.

The following theorem is originally proved by Cerulli Irelli and Lanini in [CL15] as a corollary of their main result and a result of Feigin [Fei11] (see Remark 4 for details).

Theorem 1. For any integer $n \geq 1$, $h_n = \# W^J_{\leq \tau_n}$.

We provide in this section a purely combinatorial bijective proof of the theorem.

1.2. **Rook arrangements.** Consider a board of n rows and columns. A rook arrangement R is a filling of the cells by n marks such that each row and each column have exactly one mark. Let $\mathcal{R}_n$ denote the set of all rook arrangements. There is a bijection

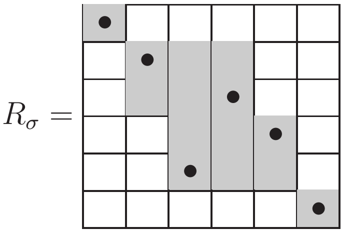
$$\varphi : \mathcal{R}_n \xrightarrow{\sim} \mathfrak{S}_n \tag{1.2}$$

sending a rook arrangement R to the permutation σ_R satisfying: for $i = 1, \cdots, n$, $\sigma_R(i) = j$ if and only if the cell (i, j) is marked in R. For $\sigma \in \mathfrak{S}_n$, we denote $R_\sigma := \varphi^{-1}(\sigma)$.

Let R be a rook arrangement. The convex hull of the marked cells in R is the smallest right-aligned skew-Ferrers board containing all marks in R.

From now on we consider $\mathfrak{S}_{2n}$: R_{τ_n} is a board of $2n$ columns and rows. A restricted rook arrangement with respect to τ_n is a rook arrangement such that all marked cells in the board are contained in the convex hull (it is called the right hull in [Sjo07]) of the marked cells in R_{τ_n}. Let $R_{\leq \tau_n}$ denote the set of all restricted rook arrangements with respect to τ_n.

Example 1. We consider an example where $n = 3$, then $\tau_3 = 142536$ and the shadowed area is the called the convex hull of the marked cells in R_{τ_3}. We fix $\sigma = 124536$, then the rook arrangement of σ is (given by the dots):



R_σ is the restricted rook arrangement with respect to τ_3.

It is clear that τ_n avoids the patterns 4231, 35142, 42513, and 351624. The following result is a special case of Theorem 4 in [Sjo07].

Theorem 2 ([Sjo07])**.** The restriction of φ on $R_{\leq \tau_n}$ gives a bijection $R_{\leq \tau_n} \xrightarrow{\sim} W_{\leq \tau_n}$.

1.3. **From rook arrangements to Dellac configurations.** We define two maps $\mathbf{m} : R_{\leq \tau_n} \to \mathrm{DC}_n$ called the melt map and $\mathbf{b} : \mathrm{DC}_n \to R_{\leq \tau_n}$ called the blow map.

Let $R \in R_{\leq \tau_n}$ be a restricted rook arrangement. Consider a board C_R of $2n$ columns and n rows defined by: the cell (k, l) of C_R is marked if and only if either the cell $(2k-1, l)$ or the cell $(2k, l)$ is marked in R. Intuitively, the k-th row of C_R is obtained by merging the $(2k-1)$-th and the $2k$-th rows in R.

Lemma 1. The board C_R is a Dellac configuration.

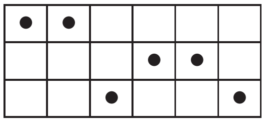
Proof. By the definition of a rook arrangement, each row of C_R has exactly two marked cells; each column of C_R has exactly one marked cell. When moreover R is restricted with respect to τ_n, by (1.1), C_R has the following property: if the cell (r, s) in C_R is marked, then $r \leq s \leq n + r$. $\square$

By using the lemma we obtain a well-defined melt map

$$\mathbf{m}(R) := C_R.$$

Let $C \in \mathrm{DC}_n$ be a Dellac configuration. A board R_C of $2n$ rows and columns is associated to C in the following way: the cells (i, j) and (i, k) with $j < k$ are marked in C if and only if the cells $(2i - 1, j)$ and $(2i, k)$ are marked in R_C. Intuitively, the i-th row in C is splitted into two rows where the first row bears the first marked point and the second row admits the second one.

Example 2. Let $\sigma = 124536$ be the permutation in Example 1. The corresponding Dellac configuration via the melt procedure is given by:



Lemma 2. The board R_C is a restricted rook arrangement with respect to τ_n.

Proof. Conditions (1) and (2) in the definition of the Dellac configuration guarantees that R_C is a rook arrangement. The condition (3) means that R_C is restricted with respect to τ_n. $\square$

By defining $\mathbf{b}(C) = R_C$, the blow map is well-defined by Lemma 2.

Lemma 3. The following statements hold:

(1) the map $\mathbf{b}$ is injective with $\mathrm{im}(\mathbf{b}) = \varphi^{-1}(W^J_{\leq \tau_n})$;
(2) we have $\mathbf{m} \circ \mathbf{b} = \mathrm{id}$.

Proof. By construction, the only thing to be prove is $\mathrm{im}(\mathbf{b}) = \varphi^{-1}(W^J_{\leq \tau_n})$. It holds by the following description of W^J:

$$W^J = \{\sigma \in W \mid \sigma(2k - 1) < \sigma(2k) \text{ for any } 1 \leq k \leq n\}.$$

$\square$

As an application of these maps, we give a bijective proof of Theorem 1:

Proof of Theorem 1. By Lemma 3, the blow map $\mathbf{b}$ induces a bijection $\mathrm{DC}_n \xrightarrow{\sim} W^J_{\leq \tau_n}$. By counting numbers we proved $h_n = \# W^J_{\leq \tau_n}$. $\square$

Remark 1. The normalized median Genocchi numbers h_n count a combinatorial structure in $\mathfrak{S}_{2n+2}$ called normalized Dumont permutation. Although *a posteriori* there exists a bijection between the normalized Dumont permutation and $W^J_{\leq \tau_n}$, our approach is different from the one in [K97], see also [Fei11].

2. Symplectic case

2.1. **Notations.** Let $\widetilde{W} = \mathfrak{S}_{4n}$ be the symmetric group, $\widetilde{J} = \{s_1, s_3, \cdots, s_{4n-1}\}$. Let ι be the involution of $\widetilde{W}$ defined by:

$$\iota(\sigma)(k) = 4n + 1 - \sigma(4n + 1 - k) \text{ for } \sigma \in \widetilde{W} \text{ and } 1 \leq k \leq 4n.$$

The Weyl group W of the symplectic group Sp_{4n} with generators $\{r_1, r_2, \cdots, r_{2n}\}$ can be embedded into $\widetilde{W}$ via the map $\kappa : W \to \widetilde{W}$, $r_i \mapsto s_i s_{4n-i}$ for $1 \leq i \leq 2n-1$ and $r_{2n} \mapsto s_{2n}$. The image of κ are the ι-fixed elements $\widetilde{W}^{\iota}$ in W. Let $J = \{r_1, r_3, \cdots, r_{2n-1}\}$. We denote

$$\overline{\tau}_{2n} = (r_{2n} \cdots r_{n+1}) \cdots (r_{2n} r_{2n-1} r_{2n-2})(r_{2n} r_{2n-1}) r_{2n} (r_n \cdots r_{2n-2}) \cdots (r_4 r_5 r_6)(r_3 r_4) r_2 \in W.$$

It is observed in [CLL15] that $\kappa(\overline{\tau}_{2n}) = \tau_{2n}$.

By Corollary 8.1.9 in [GTM05] (notice the differences between the indices here and those in the reference), the restriction of κ to $W_{\leq \overline{\tau}_{2n}}$ gives a bijection

$$\alpha : W_{\leq \overline{\tau}_{2n}} \xrightarrow{\sim} (\widetilde{W}_{\leq \tau_{2n}})^{\iota}.$$

By passing to the right cosets, α induces a bijection $\alpha' : W^{J}_{\leq \overline{\tau}_{2n}} \xrightarrow{\sim} (\widetilde{W}^{\widetilde{J}}_{\leq \tau_{2n}})^{\iota}$.

2.2. Symplectic Dellac configurations.

Definition 2. A symplectic Dellac configuration C is a board of $4n$ columns and $2n$ rows with $4n$ marked cells such that

(1) each column contains exactly one marked cell;
(2) each row contains exactly two marked cells;
(3) if the (i, j)-cell is marked, then $i \leq j \leq 2n + i$;
(4) for $1 \leq i, j \leq 2n$, the (i, j)-cell is marked if and only if the $(2n-i+1, 4n-j+1)$-cell is marked.

Let SpDC_{2n} denote the set of such configurations and e_n its cardinality.

We have $e_1 = 1, e_2 = 2, e_3 = 10, e_4 = 98, e_5 = 1594$. Consider the sequence of polynomials defined by recursion: $E_0(x) = 1$,

$$E_n(x) = \frac{1}{2}(x + 1)((x + 2)E_{n-1}(x + 2) - xE_{n-1}(x)).$$

Conjecture 1. For any $n \geq 0$, $e_{n+1} = E_n(1)$.

Remark 2. Giovanni Cerulli Irelli and Evgeny Feigin kindly informed us that they have also a similar conjecture.

If this conjecture were true, these numbers e_n coincide with the numbers r_n in [RZ96] (see A098279 in OEIS), where their continued fraction developments are studied (Théorème 29 in *loc. cit.*).

2.3. **Main result.** The main result of this section is the following

Theorem 3. For any integer $n \geq 1$, $e_n = \# W^J_{\leq \overline{\tau}_{2n}}$.

Proof. We prove the theorem by establishing a bijection between $W^J_{\leq \overline{\tau}_{2n}}$ and SpDC_{2n}, following the strategy in the proof of Theorem 1.

A symplectic rook arrangement C is a board of $4n$ columns and rows with $4n$ marked points satisfying:

(1) C is a rook arrangement;
(2) for any $1 \leq i \leq 4n$ and $1 \leq j \leq 2n$, the cell (i, j) is marked if and only if the cell $(4n+1-i, 4n+1-j)$ is marked.

The set of symplectic rook arrangements is denoted by $\mathcal{SR}_{4n}$. Similarly to Section 1.2 we can define the restricted symplectic rook arrangements with respect to τ_{2n}: $\mathcal{SR}_{\leq \tau_{2n}} := \mathcal{SR}_{4n} \cap \mathcal{R}_{\leq \tau_{2n}}$.

Consider the bijection $\varphi : \mathcal{R}_{4n} \xrightarrow{\sim} \mathfrak{S}_{4n}$ from (1.2).

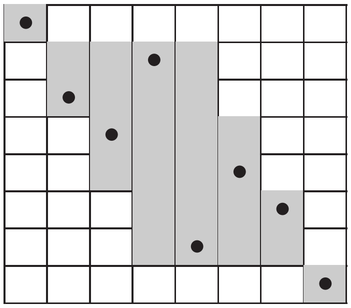
Lemma 4. (1) The restriction of the map φ induces a bijection $\varphi' : \mathcal{SR}_{4n} \xrightarrow{\sim} \widetilde{W}^\iota = W$.
(2) The restriction of the map φ' induces a bijection $\psi : \mathcal{SR}_{\leq \tau_{2n}} \xrightarrow{\sim} (\widetilde{W}_{\leq \tau_{2n}})^\iota$.

Proof. (1) Take a board R in $\mathcal{SR}_{4n}$, the condition (2) in its definition implies that $\varphi(R)$ is invariant under the involution ι. It suffices to show that φ' is surjective: let $\sigma \in \widetilde{W}$, by definition of ι, σ is fixed by the involution ι if and only if $\sigma(4n+1-k) = 4n+1-\sigma(k)$ for any $1 \leq k \leq 4n$, i.e., for any $1 \leq i \leq 4n$ and $1 \leq j \leq 2n$, $\sigma(i) = j$ if and only if $\sigma(4n+1-i) = 4n+1-j$. It implies that $\varphi^{-1}(\sigma)$ is in $\mathcal{SR}_{4n}$.
(2) Since $\mathcal{SR}_{\leq \tau_{2n}} = \mathcal{SR}_{4n} \cap \mathcal{R}_{\leq \tau_{2n}}$ and $(\widetilde{W}_{\leq \tau_{2n}})^\iota = \widetilde{W}^\iota \cap \widetilde{W}_{\leq \tau_{2n}}$, the bijectivity of ψ follows from (1) and Theorem 2.

□

Moreover, consider the restriction of the melt map $\mathbf{m} : \mathcal{R}_{\leq \tau_{2n}} \to \mathrm{DC}_{2n}$ on $\mathcal{SR}_{\leq \tau_{2n}}$. Since the condition (2) in the definition of the symplectic rook arrangement translates to the condition (4) in the definition of the symplectic Dellac configuration under the melt map, $\mathbf{m}$ induces a map $\mathbf{m}' : \mathcal{SR}_{\leq \tau_{2n}} \to \mathrm{SpDC}_{2n}$.

Example 3. Let us consider an example where $n = 2$ and the permutation is giving by the following rook arrangement:



where the shadowed area is the convex hull of the marked cells in $R_{\overline{\tau}_4}$. It is straightforward to see that the rook arrangement is fixed by ι and hence symplectic. The corresponding symplectic Dellac configuration via the melt map $\mathbf{m}$ is given by:

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continue the proof of Theorem 3:
The restriction of the blow map $\mathbf{b} : \mathrm{DC}_{2n} \to \mathcal{R}_{\le\tau_{2n}}$ to SpDC_{2n} gives a map $\mathbf{b}' : \mathrm{SpDC}_{2n} \to \mathcal{SR}_{\le\tau_{2n}}$. By Lemma 3, $\mathbf{b}$ is injective with $\mathrm{im}(\mathbf{b}) = \varphi^{-1}(\widetilde{W}^{\widetilde{J}}_{\le\tau_{2n}})$. It implies that $\mathbf{b}'$ is injective with

$$\mathrm{im}(\mathbf{b}') = \varphi^{-1}(\widetilde{W}^{\widetilde{J}}_{\le\tau_{2n}} \cap \widetilde{W}^{\iota}) = \psi^{-1}((\widetilde{W}^{\widetilde{J}}_{\le\tau_{2n}})^{\iota})$$

and $\mathbf{m}' \circ \mathbf{b}' = \mathrm{id}$.

By the above argument, the blow map $\mathbf{b}'$ gives a bijection $\mathrm{SpDC}_{2n} \xrightarrow{\sim} (\widetilde{W}^{\widetilde{J}}_{\le\tau_{2n}})^{\iota}$, composing with $(\varphi')^{-1}$ we get a bijection $\mathrm{SpDC}_{2n} \xrightarrow{\sim} W^J_{\le\overline{\tau}_{2n}}$. □

3. Application to torus fixed points

We show how the construction in Section 1 is related to the study of the torus fixed points in the degenerate flag variety.

3.1. **Schubert varieties.** Let $\sigma_n \in \mathfrak{S}_{2n}$ be the permutation defined as follows:

$$\sigma_n(r) = \begin{cases} k, & r = 2k; \\ n+1+r, & r = 2k+1. \end{cases} \tag{3.1}$$

We see that σ_n can be obtained by restricting $\tau_{n+1} \in S_{2n+2}$ to the set $\{2, \ldots, 2n+1\}$. We denote X_{σ_n} the Schubert variety corresponding to σ_n in the projective variety SL_n/P where P is the standard parabolic subalgebra defined as the stabilizer of the highest weight line of weight $\varpi_1 + \varpi_3 + \cdots + \varpi_{2n-1}$. The maximal torus T_{2n-1} of SL_{2n} acts naturally on X_{σ_n}: let $X_{\sigma_n}^{T_{2n-1}}$ be the set of torus fixed points.

It is a standard result that the torus fixed points $X_{\sigma_n}^{T_{2n-1}}$ can be identified with the quotient $W^J_{\le\sigma_n}$ where $W = \mathfrak{S}_{2n}$ and $J = \{2, 4, \cdots, 2n-2\}$: for $\tau \in W^J_{\le\sigma_n}$, the corresponding torus fixed point in $X_{\sigma_n}^{T_{2n-1}}$ is:

$$\langle e_{\tau(1)}\rangle_{\mathbb{C}} \subset \langle e_{\tau(1)}, e_{\tau(2)}, e_{\tau(3)}\rangle_{\mathbb{C}} \subset \cdots \subset \langle e_{\tau(1)}, e_{\tau(2)}, \cdots, e_{\tau(2n-1)}\rangle_{\mathbb{C}} \in X_{\sigma_n}$$

where $e_1, e_2, \cdots, e_{2n}$ is a fixed basis of $\mathbb{C}^{2n}$.

3.2. **Degenerate flag varieties.** We fix a basis $\{f_1, f_2, \cdots, f_{n+1}\}$ of $\mathbb{C}^{n+1}$. Let $\mathcal{F}l^a_{n+1}$ be the degenerate flag variety of SL_{n+1} (see [Fei11] for details):

$$\mathcal{F}l^a_{n+1} = \{(V_1, V_2, \cdots, V_n) \in \prod_{i=1}^{n} \mathrm{Gr}_i(\mathbb{C}^{n+1}) |\ \mathrm{pr}_{i+1}(V_i) \subset V_{i+1} \text{ for any } i = 1, \cdots, n\},$$

where $\mathrm{pr}_i : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ is the linear projection along the line generated by f_i. By [CFR12], the torus T_{2n-1} acts on $\mathcal{F}l^a_{n+1}$: let $(\mathcal{F}l^a_{n+1})^{T_{2n-1}}$ be the corresponding set of torus fixed points.

In [CL15], it is shown that there exists a T_{2n-1}-equivariant isomorphism of projective varieties $\zeta : \mathcal{F}l^a_{n+1} \xrightarrow{\sim} X_{\sigma_n} \subset \mathrm{SL}_{2n}/P$. We are especially interested in the image of torus fixed points under ζ:
Fix a basis $\{e_1, e_2, \cdots, e_{2n}\}$ of $\mathbb{C}^{2n}$. For any $i = 1, 2, \cdots, n$, we denote the coordinate subspace $U_{n+i} = \langle e_1, e_2, \cdots, e_{n+i}\rangle \subset W$. The surjection $\pi_i : U_{n+i} \to \mathbb{C}^{n+1}$ is defined by:

$$\pi_i(e_k) = \begin{cases} 0 & \text{if } 1 \le k \le i-1; \\ f_k & \text{if } i \le k \le n+1; \\ f_{k-n-1} & \text{if } n+2 \le k \le n+i. \end{cases} \tag{3.2}$$

Define $\zeta_i : \mathrm{Gr}_i(\mathbb{C}^{n+1}) \to \mathrm{Gr}_{2i-1}(\mathbb{C}^{2n})$ to be the concatenation of the following maps:

$$\mathrm{Gr}_i(\mathbb{C}^{n+1}) \to \mathrm{Gr}_{2i-1}(U_{n+i}) \to \mathrm{Gr}_{2i-1}(\mathbb{C}^{2n}), \quad U \mapsto \pi_i^{-1}(U) \mapsto \pi_i^{-1}(U).$$

Then $\zeta : \mathcal{F}l^a_{n+1} \to X_{\sigma_n}$ is given by $\prod_{i=1}^n \zeta_i$ (see Section 2 of [CL15] for details).

It is clear that the torus T_n of SL_{n+1} acts naturally on $\mathcal{F}l^a_{n+1}$. By results in Section 7.2 of [CFR12], any T_{2n-1} fixed point in $\mathcal{F}l^a_{n+1}$ is in fact a T_n-fixed point. In [Fei11], an explicit bijection $\mathbf{f}$ between the T_{2n-1}-fixed points and Dellac configuration is provided.

3.3. **A commutative diagram.** As a summary, starting with a T_n-fixed point in $\mathcal{F}l^a_{n+1}$, there are two ways to obtain a Dellac configuration:

(1) via the bijection $\mathbf{f}$ given by [Fei11];
(2) consider this fixed point as a fixed point in the Schubert variety X_{σ_n}, hence identify it with an element in $W^J_{\le\sigma_n}$, then melt the corresponding rook arrangement to get a Dellac configuration.

It is natural to ask whether the following diagram commutes:

$$\begin{tikzcd}
(\mathcal{F}l^a_{n+1})^{T_{2n-1}} = (\mathcal{F}l^a_{n+1})^{T_n} \arrow[r, "\mathbf{f}"] \arrow[d, "\beta"] & \mathrm{DC}_{n+1} \arrow[d, "\mathbf{b}"] \\
X^{T_{2n+1}}_{\tau_{n+1}} = X^{T_{2n-1}}_{\sigma_n} & W^J_{\le\tau_{n+1}} \arrow[l, "\alpha"']
\end{tikzcd}.$$

where the map α is given as follows:
for $\sigma \in W^J_{\le\tau_{n+1}}$ where $W = \mathfrak{S}_{2n+2}$, we define the map α as follows: $\alpha(\sigma)$ is the sequence of subspaces $W_1 \subset W_2 \subset \cdots \subset W_n$ such that W_i is the subspace of $\mathbb{C}^{2n}$ generated by $e_{\overline{\sigma}(1)}, e_{\overline{\sigma}(2)}, \cdots, e_{\overline{\sigma}(2i-1)}$, where $\overline{\sigma}$ is the (well-defined) restriction of σ to $\mathfrak{S}_{2n}$. We can identify this element in $X^{T_{2n-1}}_{\sigma_n}$ with n subsets $J_1, \cdots, J_n$ of $\{1, 2, \cdots, 2n\}$ such that $J_i = \{\overline{\sigma}(1), \overline{\sigma}(2), \cdots, \overline{\sigma}(2i-1)\}$.

It remains to consider restriction of the map ζ to fixed points. Here we have to include an extra twist, since the definition of the degenerate flag variety is slightly different in [Fei11] and [CL15]: let $(V_1, V_2, \cdots, V_n) \in (\mathcal{F}l^a_{n+1})^{T_n}$, it can be identified ([Fei11], Corollary 2.11) with n subsets $I_1, I_2, \cdots, I_n$ of $\{1, 2, \cdots, n+1\}$ such that $\#I_k = k$ and for any $k = 1, 2, \cdots, n$, $I_k \backslash \{k+1\} \subset I_{k+1}$.

We denote $\kappa = (12 \cdots n+1)^{-1}$ be the inverse of the longest cycle in $\mathfrak{S}_{n+1}$. Suppose that $I_l = \{i_{l,1}, i_{l,2}, \cdots, i_{l,l}\}$, we denote $I^\kappa_l = \{\kappa(i_{l,1}), \kappa(i_{l,2}), \cdots, \kappa(i_{l,l})\}$. We define a map $p_l : \{1, 2, \cdots, n+l\} \to \{1, 2, \cdots, n+1\}$ by

$$p_l(s) = \begin{cases} 0 & \text{if } 1 \leq s \leq l-1; \\ s & \text{if } l \leq k \leq n+1; \\ s-n-1 & \text{if } n+2 \leq k \leq n+l. \end{cases} \tag{3.3}$$

Then $\beta((I_1, I_2, \cdots, I_n)) = (T_1, T_2, \cdots, T_n)$ where $T_l = p_l^{-1}(I_l^\kappa)$.

Theorem 4. The diagram above commutes, i.e., $\zeta = \alpha \circ \mathbf{b} \circ \mathbf{f}$.

The proof is given by a case-by-case examination, we will only give a sketch.

Proof. We pick $\mathbf{I} = (I_1, I_2, \cdots, I_n) \in (\mathcal{F}l^a_{n+1})^{T_{n+1}}$. Recall that the map $\mathbf{f}$ is given in [Fei11, Proposition 3.1].

(1) Suppose that $l \notin I_{l-1}$, then $I_l \backslash I_{l-1} = \{j\}$. We consider the case $j > l$: in the Dellac configuration $f(\mathbf{I})$, the cells (l, l) and (l, j) are marked. Then by definition, $\sigma = \mathbf{b}(f(\mathbf{I}))$ satisfies $\sigma(2l-1) = l$ and $\sigma(2l) = j$. Hence in $\alpha(\sigma)$, $J_l \backslash J_{l-1} = \{l-1, j-1\}$.

   We compute $\beta(\mathbf{I})$: it is clear that $I_l^\kappa \backslash I_{l-1}^\kappa = \{j-1\}$, then $p_l^{-1}(I_l^\kappa) \backslash p_{l-1}^{-1}(I_{l-1}^\kappa) = p_l^{-1}(\{l-1, j-1\}) = \{l-1, j-1\}$. Therefore $T_l \backslash T_{l-1} = \{l-1, j-1\}$, i.e., $J_l = T_l$.

   It is similar to deal with the case $j < l$.

(2) Suppose that $l \in I_{l-1}$ and $l \in I_l$, then $I_l \backslash I_{l-1} = \{j\}$. We study the case $j < l$: in the corresponding Dellac configuration, the cells $(l, l+n+1)$ and $(l, j+n+1)$ are marked. The associated permutation $\sigma = \mathbf{b}(f(\mathbf{I}))$ satisfies $\sigma(2l-1) = j+n+1$ and $\sigma(2l) = l+n+1$. Hence in $\alpha(\sigma)$, $J_l \backslash J_{l-1} = \{j+n, l+n\}$.

   For $\beta(\mathbf{I})$: $l \in I_{l-1} \cap I_l$ and $I_l \backslash I_{l-1} = \{j\}$ imply that $l-1 \in I_{l-1}^\kappa \cap I_l^\kappa$ and $I_l^\kappa \backslash I_{l-1}^\kappa = \{\kappa(j)\}$. Notice that no matter $j = 1$ or $j > 1$, $p_l^{-1}(\kappa(j)) = j+n$. By the assumption $j < l$,

$$p_l^{-1}(I_l^\kappa) \backslash p_{l-1}^{-1}(I_{l-1}^\kappa) = p_l^{-1}(\{l-1, \kappa(j)\}) = \{j+n, l+n\},$$

   which proved $J_l = T_l$.

   The case where $j > l$ can be similarly proved.

(3) Suppose that $l \in I_{l-1}$ and $l \notin I_l$, then there exists j_1 and j_2 such that $I_l \backslash I_{l-1} = \{j_1, j_2\}$. We assume that $j_1 < l$ and $j_2 > l$, in the corresponding Dellac configuration, the cells (l, j_1+n+1) and (l, j_2) are marked, hence in $\alpha(\mathbf{b}(f(\mathbf{I})))$, $J_l \backslash J_{l-1} = \{j_1+n, j_2-1\}$.

   For $\beta(\mathbf{I})$, we have

$$p_l^{-1}(I_l^\kappa) \backslash p_{l-1}^{-1}(I_{l-1}^\kappa) = p_l^{-1}(\{\kappa(j_1), j_2-1\}) = \{j_1+n, j_2-1\},$$

   therefore $J_l = T_l$.

   All other cases can be proved in the same way.

□

Remark 3. A similar diagram without the map $\mathbf{f}$ exists in the symplectic case by changing

(1) the degenerate flag variety to the symplectic degenerate flag variety (see [FFiL14]);
(2) the Schubert variety of SL_{2n} by the Schubert variety in the symplectic group (see [CL15]);

(3) the Dellac configuration by the symplectic Dellac configuration;
(4) the set $W^J_{\leq \tau_{n+1}}$ by $W^J_{\leq \overline{\tau}_{2n+2}}$.

Remark 4. The original proof of Theorem 1 is given by showing the composition $\alpha^{-1} \circ \beta \circ \mathbf{f}^{-1}$ is a bijection: $\mathbf{f}$ is a bijection is shown in [Fei11]; by the main theorem of [CL15], β is a bijection; α is a well-known bijection. Our proof of the theorem uses an intuitive map $\mathbf{b}$ to avoid the geometrical proof.

References

[GTM05] A. Björner and F. Brenti, *Combinatorics of Coxeter groups.* Graduate Texts in Mathematics, 231. Springer, New York, 2005. xiv+363 pp.

[CL15] G. Cerulli Irelli and M. Lanini, *Degenerate flag varieties of type A and C are Schubert varieties.* Int. Math. Res. Not. (2015), 6353–6374.

[CLL15] G. Cerulli Irelli, M. Lanini, and P. Littelmann, *Degenerate flag varieties and Schubert varieties: a characteristic free approach.* to appear in Pac. J. Math, 2016.

[CFR12] G. Cerulli Irelli, E. Feigin, and M. Reineke, *Quiver Grassmannians and degenerate flag varieties.* Algebra Number Theory 6 (2012), no. 1, 165-194.

[DR94] D. Dumont and A. Randrianarivony, *Dérangements et nombres de Genocchi.* Discrete Math. 132 (1994), no. 1-3, 37-49.

[FFR15] X. Fang, G. Fourier, and M. Reineke, *PBW-type filtration on quantum groups of type A_n.* Journal of Algebra 449 (2016) 321-345.

[Fei11] E. Feigin, *Degenerate flag varieties and the median Genocchi numbers.* Math. Res. Lett. 18 (2011), no. 6, 1163-1178.

[Fei12] E. Feigin, *The median Genocchi numbers, q-analogues and continued fractions.* European J. Combin. 33 (2012), no. 8, 1913-1918.

[FFiL14] E. Feigin, M. Finkelberg, and P. Littelmann, *Symplectic degenerate flag varieties.* Canad. J. Math. 66 (2014), no. 6, 1250-1286.

[FFoL11a] E. Feigin, G. Fourier, and P. Littelmann, *PBW filtration and bases for irreducible modules in type A_n*, Transformation Groups: Volume 16, Issue 1 (2011), 71-89.

[FFoL11b] E. Feigin, G. Fourier, and P. Littelmann, *PBW filtration and bases for symplectic Lie algebras*, International Mathematics Research Notices 2011 (24), pp. 5760–5784.

[FFoL13] E. Feigin, G. Fourier, and P. Littelmann, *Favourable modules: Filtrations, polytopes, Newton-Okounkov bodies and flat degenerations*, Transformation Groups, 2016, DOI: 10.1007/S00031-016-9389-2.

[Fou15] G. Fourier, *New homogeneous ideals for current algebras: filtrations, fusion products, and the Pieri rules*, Moscow Mathematical Journal, Volume 15 (1), 2015 pp. 49-72.

[Fou14] G. Fourier, *Marked poset polytopes: Minkowski sums, indecomposables, and unimodular equivalence*, Journal of Pure and Applied Algebra, 220, Issue 2, 2016, 606–620.

[Hag14] C. Hague, *Degenerate coordinate rings of flag varieties and Frobenius splitting*, Selecta Math. (N.S.) 20 (2014), no. 3, 823-838.

[K97] G. Kreweras, *Sur les permutations comptées par les nombres de Genocchi de 1-ière et 2-ième espèce.* European J. Combin. 18 (1997), no. 1, 49-58.

[RZ96] A. Randrianarivony and J. Zeng, *Une famille de polynômes qui interpole plusieurs suites classiques de nombres.* Adv. in Appl. Math. 17 (1996), no. 1, 1-26.

[Sjo07] J. Sjöstrand, *Bruhat intervals as rooks on skew Ferrers boards.* Journal of Combinatorial Theory, Series A 114 (2007), 1182-1198.

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