The Pascal Rhombus and the Stealth Configuration

Paul K. Stockmeyer
Department of Computer Science
The College of William and Mary
Williamsburg, Virginia, USA
Email: stockmeyer@cs.wm.edu

Abstract

The Pascal rhombus is a variant of Pascal's triangle in which each term is a sum of four earlier terms. Klostermeyer et al. made four conjectures about the Pascal rhombus modulo 2. In this paper we show how exploration of the stealth shape leads to unified proofs of all of these conjectures.

1 Introduction

The Pascal rhombus was introduced in 1997 by Klostermeyer *et al.* [4] as a variant of Pascal's triangle. The term rhombus does not refer to this figure's shape—as with Pascal's triangle, the Pascal rhombus forms an infinite triangular wedge. Rather, the term refers to the rule for recursively constructing the figure. Informally, each element in the Pascal rhombus is the sum of three adjacent elements in the row immediately above the element, plus one element from two rows above. The new element, together with the four elements that contribute to it, lie in the shape of a rhombus. The figure begins with a single 1 in the top row. Figure 1 shows the first six rows of the Pascal rhombus.

1
1 1 1
1 2 4 2 1
1 3 8 9 8 3 1
1 4 13 22 29 22 13 4 1
1 5 19 42 72 82 72 42 19 5 1

Figure 1: The Pascal rhombus.

More formally, we number the rows and columns of the Pascal rhombus, with $[n,\ k]$ denoting the cell in row n, column k, and $R[n,\ k]$ denoting its contents. The non-zero elements are those for which $-n < k < n$. The initial conditions are that $R[0,\ j] = 0$ for all j, $R[1,\ j] = 0$ for $j \neq 0$, and $R[1,\ 0] = 1$. Construction continues with

$$R[n,\ k] = R[n-1,\ k-1] + R[n-1,\ k] + R[n-1,\ k+1] + R[n-2,\ k]$$

for $n \geq 2$. (Note that this indexing is different from that of [4].)

Near the end of [4] the authors consider the Pascal rhombus with entries reduced modulo 2. Figure 2 shows the first 32 rows of the Pascal rhombus (mod 2), with odd entries in black and even entries in white. The construction rule now becomes: A cell at the bottom of a 5-cell rhombus is colored black if and only if an odd number of the other four cells are colored black. Thus every 5-cell rhombus contains an even number of black cells. The only exception to this rule in the entire infinite grid is the rhombus centered at cell [0, 0] which contains the single black cell at position [1, 0] that initiates the construction.

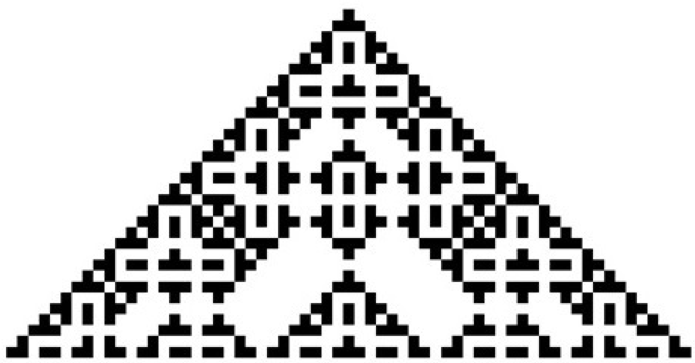

Figure 2: The Pascal rhombus (mod 2)

This is of course quite reminiscent of the well-known Pascal's triangle (mod 2), where the first 2^n rows form a structure consisting of three copies of the 2^{n-1} row structure. Properly scaled, these structures approach the fractal known as the Sierpiński gasket. See, for example, [1, Section 1.2]. No doubt motivated by the properties of this familiar triangle, Klostermeyer *et al.* [4] made four conjectures about the structure of the Pascal rhombus (mod 2).

Conjecture 1 (Klostermeyer *et al.* [4])**.** *For any* $n \geq 1$ *the triangle in the Pascal rhombus* (mod 2) *with corners at* $[1,\ 0]$, $[2^{n-1},\ -(2^{n-1}-1)]$, *and* $[2^{n-1},\ 2^{n-1}-1]$ *is identical to the triangle with corners at* $[2^n+1,\ -2^n]$, $[2^n+2^{n-1},\ -(2^n+2^{n-1}-1)]$, *and* $[2^n+2^{n-1},\ -(2^n-2^{n-1}+1)]$, *and also identical to the triangle with corners at* $[2^n+1,\ 2^n]$, $[2^n+2^{n-1},\ 2^n-2^{n-1}+1]$, *and* $[2^n+2^{n-1},\ 2^n+2^{n-1}-1]$.

For example, with $n = 4$ in Figure 2, the triangle with corners at $[1,\ 0]$, $[8,\ -7]$ and $[8,\ 7]$ (the top quarter of the figure) is identical to the triangle with corners at $[17,\ -16]$, $[24,\ -23]$ and $[24,\ -9]$ (the third quarter down, on the left), and to the triangle with corners at $[17,\ 16]$, $[24,\ 9]$, and $[24,\ 23]$ (the third quarter down, on the right). The conjecture makes no mention of a fourth such triangle, with corners at $[25,\ 0]$, $[32,\ -7]$ and $[32,\ 7]$ (the middle of the bottom quarter of the figure).

Conjecture 2 (Klostermeyer *et al.* [4]). *Let I_n be the number of ones in row $m = 2^n$ of the Pascal rhombus* (mod 2). *Then*

$$I_n = \frac{1}{3}\left(2^{n+2} - (-1)^n\right).$$

Conjecture 3 (Klostermeyer *et al.* [4]). *For each $k \geq 1$, diagonal D_k of the Pascal rhombus* (mod 2) *consists of cells $[n,\ -n+k+1]$ for $n \geq (k+1)/2$. Each such diagonal is periodic with period length 2^p, where $p = \lceil \log_2(k) \rceil + 1$. The period of the $[n,\ -n+1]$ diagonal D_0 is 1.*

Note that the periods claimed here are not necessarily minimal. The authors illustrate the conjecture by observing that diagonal D_6 begins 11011011, and presumably repeats these eight values. Conjecture 3, however, only asserts a period of 16.

Conjecture 4 (Klostermeyer *et al.* [4]). *Let G_n and H_n be the number of odd and even entries, respectively, in the first n rows of the Pascal rhombus. Then* $\lim_{n\to\infty} G_n/H_n = 0$.

In a subsequent paper by many of the same authors, Goldwasser *et al.* [3] proved the correctness of Conjectures 2 and 4, using an elaborate decomposition of the Pascal rhombus based on odd and even values of the indices n and k. In the next two sections we show how one key observation leads to unified proofs of all four conjectures.

2 The stealth shape

A standard way to analyze structures such as the Pascal triangle (mod 2) and their fractal limits is to exploit the fact that such structures can be decomposed into a finite union of lower order structures. Unfortunately, there is no way to decompose the first 2^n rows of the Pascal rhombus (mod 2) into a finite number of smaller, similar copies of these triangles. Conjecture 1 suggests how far we are from this desirable situation. However, truncating the Pascal rhombus (mod 2) along a zig-zag line produces a structure that *can* be decomposed into five lower order copies of itself. We call this resulting structure a stealth configuration because of its vague resemblance to the B2 stealth bomber. See Figure 3.

Definition 1. The *order n stealth configuration* S_n, for $n \geq 2$, is that part of the Pascal rhombus (mod 2) strictly within the octagon with edges linking cells $[0,\ 0]$, $[2^n,\ -2^n]$, $[2^n + 2^{n-1},\ -2^{n-1}]$, $[2^n + 2^{n-2},\ -2^{n-2}]$ $[2^n + 2^{n-1},\ 0]$, $[2^n + 2^{n-2},\ 2^{n-2}]$, $[2^n + 2^{n-1},\ 2^{n-1}]$, $[2^n, 2^n]$, and back to $[0, 0]$. The order 1 stealth configuration S_1 consists of cells $[1, 0]$, $[2, -1]$, $[2,\ 0]$ and $[2,\ 1]$ in the Pascal rhombus (mod 2), and the order 0 stealth configuration S_0 consist of just cell $[1,\ 0]$.

We will refer to translations and rotations of stealth configurations as stealth configurations as well, and will sometimes refer to those growing down from cell $[1,\ 0]$ as being in standard position.

Figure 3 shows the stealth configuration S_5, strictly inside the octagon passing through $[0,\ 0]$, $[32,\ -32]$, $[48,\ -16]$, $[40,\ -8]$, $[48,\ 0]$, $[40,\ 8]$, $[48,\ 16]$, $[32,\ 32]$, and back to $[0,\ 0]$. It consists of an order 4 stealth configuration growing down from $[1,\ 0]$ (the nose), an order 4 stealth configuration growing to the right from $[32,\ -31]$ (the left wing), an order 4 stealth

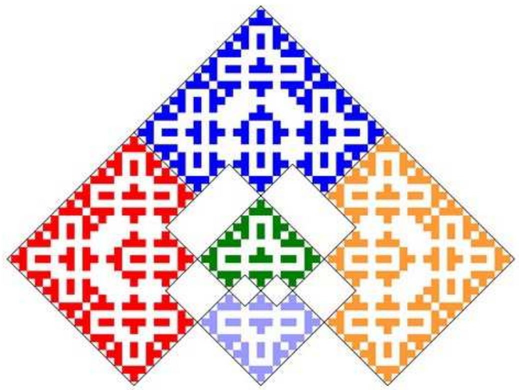

Figure 3: The Stealth Configuration and Stealth Decomposition.

configuration growing to the left from [32, 31] (the right wing), an order 3 stealth configuration growing down from [25, 0] (the main body), and an order 3 stealth configuration growing up from [47, 0] (the tail).

Theorem 1. *For all $n \geq 2$, the order n stealth configuration S_n in standard position can be decomposed into the disjoint union of 5 smaller stealth configurations: an order $n-1$ stealth configuration growing down from $[1, 0]$ (the nose), an order $n-1$ stealth configuration growing to the right from $[2^n, -(2^n-1)]$ (the left wing), an order $n-1$ stealth configuration growing to the left from $[2^n, 2^n-1]$ (the right wing), an order $n-2$ stealth configuration growing down from $[2^n-2^{n-2}+1, 0]$ (the main body), and an order $n-2$ stealth configuration growing up from $[2^n+2^{n-1}-1, 0]$ (the tail).*

We need two temporary definitions before we prove Theorem 1. First, the *order n pseudo-stealth configuration S'_n* is defined as follows: For $n = 0$ and $n = 1$, the order n pseudo-stealth configuration S'_n is the same as the order n stealth configuration S_n from Definition 1. For $n \geq 2$, S'_n is the configuration formed from copies of lower order pseudo-stealth configurations S'_{n-1} and S'_{n-2} in accordance with Theorem 1. Our task is to prove that the cut-and-paste pseudo-stealth configuration S'_n is identical to the rhombus rule stealth configuration S_n for all $n \geq 0$.

Second, we call a grid cell *exceptional* with respect to a stealth or pseudo-stealth configuration if it is the center cell of a 5-cell rhombus containing an odd number of ones. We will show, for example, that that the exceptional cells with respect to the order 5 configuration in Figure 3 are [0, 0]. [32, −32], [48, −16], [48, 0], [48, 16], and [32, 32] (the acute corners of the bounding octagon).

Lemma 1. *The exceptional cells with respect to the order n pseudo-stealth configuration S'_n in standard position, for $n \geq 1$, are at $[0, 0]$, $[2^n, -2^n]$, $[2^n+2^{n-1}, -2^{n-1}]$, $[2^n+2^{n-1}, 0]$, $[2^n+2^{n-1}, 2^{n-1}]$, and $[2^n, 2^n]$.*

Proof of Lemma 1. The proof is by induction. The cases $n = 1$ and $n = 2$ can be confirmed by hand. We assume that the statement is true for all pseudo-stealth configurations of order less than n, for some arbitrary $n \geq 3$, and consider an order n pseudo-stealth configuration

S'_n of order n, in standard position. Configuration S'_n is the union of the following: An order $n-1$ pseudo-stealth configuration (the nose) with exceptional cells at [0, 0], $[2^{n-1}, -2^{n-1}]$, $[2^n - 2^{n-2}, -2^{n-2}]$, $[2^n - 2^{n-2}, 0]$, $[2^n - 2^{n-2}, 2^{n-2}]$, and $[2^{n-1}, 2^{n-1}]$; an order $n-1$ pseudo-stealth configuration (the left wing) with exceptional cells at $[2^n, -2^n]$, $[2^n + 2^{n-1}, -2^{n-1}]$, $[2^n + 2^{n-2}, -2^{n-2}]$, $[2^n, -2^{n-2}]$, $[2^n - 2^{n-2}, -2^{n-2}]$, and $[2^{n-1}, -2^{n-1}]$; an order $n-1$ pseudo-stealth configuration (the right wing) with exceptional cells at $[2^n, 2^n]$, $[2^{n-1}, 2^{n-1}]$, $[2^n - 2^{n-2}, 2^{n-2}]$, $[2^n, 2^{n-2}]$, $[2^n + 2^{n-2}, 2^{n-2}]$, and $[2^n + 2^{n-1}, 2^{n-1}]$; an order $n-2$ pseudo-stealth configuration (the main body) with exceptional cells at $[2^n - 2^{n-2}, 0]$, $[2^n, -2^{n-2}]$, $[2^n + 2^{n-3}, -2^{n-3}]$, $[2^n + 2^{n-3}, 0]$, $[2^n + 2^{n-3}, 2^{n-3}]$, and $[2^n, 2^{n-2}]$; and on order $n-2$ pseudo-stealth configuration (the tail) with exceptional cells at $[2^n + 2^{n-1}, 0]$, $[2^n + 2^{n-2}, 2^{n-2}]$, $[2^n + 2^{n-3}, 2^{n-3}]$, $[2^n + 2^{n-3}, 0]$, $[2^n + 2^{n-3}, -2^{n-3}]$, and $[2^n + 2^{n-2}, -2^{n-2}]$.

It is easy to see that a cell in the union of several pseudo-stealth configurations is exceptional if and only if it is exceptional in an odd number of the component pseudo-stealth configurations. Of the 30 exceptional cells listed above, 24 pair off and cancel out, leaving configuration S'_n with just the six exceptional cells specified in the statement of the lemma. □

Proof of Theorem 1. We will show that for all $n \geq 0$, the order n stealth configuration S_n is identical to the order n pseudo-stealth configuration S'_n. For $n = 0$ and $n = 1$, this is true by definition. For arbitrary $n \geq 2$, let S'_{n+1} be the order $n+1$ pseudo-stealth configuration in standard position. Consider the nose section of S'_{n+1}. On the one hand, from Lemma 1 we know all the exceptional cells of S'_{n+1}, and all the cells in the nose of S'_{n+1} are closer to the exceptional cell [0, 0] than to any other. Thus the nose of S'_{n+1} is contained in the Pascal Rhombus (mod 2) that grows down from cell [1, 0] without interruption, using the rhombus rule. That is, it is the order n stealth configuration S_n. On the other hand, the nose of S'_{n+1} is, by construction, the order n pseudo-stealth configuration S'_n as well. □

3 Proving the conjectures

Armed with Theorem 1, we can readily prove the conjectures of Klostermeyer *et al.*[4] stated in Section 1.

Proof of Conjecture 1. Let S_{n+1} be the order $n+1$ stealth configuration in standard position for some $n \geq 1$. It contains an order $n-1$ stealth configuration growing down from [1, 0] (the nose of the nose of S_{n+1}), an order $n-1$ stealth configuration growing down from $[2^n + 1, -2^n]$ (the right wing component of the left wing of S_{n+1}), and an order $n-1$ stealth configuration growing down from $[2^n + 1, 2^n]$ (the left wing component of the right wing of S_{n+1}). The three triangles of Conjecture 1 consist of the first 2^{n-1} rows of these three order $n-1$ stealth configurations. □

The fourth identical triangle noted in Section 1 consists of the first 2^{n-1} rows of the order $n-2$ stealth configuration growing down from $[2^n + 2^{n-1} + 1, 0]$ (the body of configuration S_{n+1}).

Proof of Conjecture 2. For $n \geq 1$ let A_n denote the number of ones (black cells) in the central column of the order n stealth configuration in standard position, from [1, 0] down to $[2^n + 2^{n-1},\ 0]$. By inspection we have $A_1 = 2$ and $A_2 = 4$. From Theorem 1 we have that $A_n = A_{n-1} + 2A_{n-2}$ for $n \geq 3$. It is easy to see that

$$A_n = 2^n \tag{1}$$

is the solution to this recurrence relation. For completeness we set $A_0 = 1$. This is sequence A000079 in the OEIS [7].

Now for $n \geq 1$ let B_n denote the number of ones (black cells) on the row from left wingtip to right wingtip of the order n stealth configuration in standard position, from $[2^n,\ -(2^n-1)]$ to $[2^n,\ 2^n-1]$. By inspection we have $B_1 = 3$ and $B_2 = 5$. From Theorem 1 we have that $B_n = 2A_{n-1} + B_{n-2}$ for $n \geq 3$, or $B_n = 2^n + B_{n-2}$. Standard methods for recurrence relations yield the solution

$$B_n = \frac{1}{3}\left(2^{n+2} - (-1)^n\right), \tag{2}$$

which can be confirmed by induction. For completeness we set $B_0 = 1$. This result was first derived, using different methods and different recurrences, by Goldwasser *et al.* [3] as a special case of their Theorem 1. This is sequence A001045 in the OEIS; its first few terms are 1, 3, 5, 11, 21, 43, 85, 171, ... $\square$

Let S_n be the stealth configuration of order $n \geq 1$. We call the set of cells strictly within the rhombus linking cells $[0,\ 0]$, $[2^{n-1},\ -2^{n-1}]$, $[2^n,\ 0]$, and $[2^{n-1},\ 2^{n-1}]$ the *upper rhombus* of S_n. Likewise, we call the set of cells strictly within the rhombus linking cells $[2^{n-1},\ -2^{n-1}]$, $[2^n,\ -2^n]$, $[2^n + 2^{n-1},\ -2^{n-1}]$, and $[2^n,\ 0]$ the *left rhombus* of S_n.

Lemma 2. *For all $n \geq 1$, the upper rhombus of stealth configuration S_n is the mirror image of the left rhombus of S_n, reflected along the line of cells $[2^{n-1}+k,\ -2^{n-1}+k]$ for $0 \leq k \leq 2^{n-1}$.*

Proof. The proof is by induction on n. The claim is easily confirmed for $n = 1$ and 2. We suppose the claim is true for all stealth configurations of order less than n, for some arbitrary $n \geq 2$, and examine the stealth configuration S_n of order n. Clearly the upper rhombus of S_n consists of the order $n-1$ stealth configuration forming the nose section of S_n, together with the upper rhombus of the order $n-2$ configuration forming the main body section of S_n. Likewise, the left rhombus of S_n consists of the order $n-1$ configuration forming the left wing section of S_n, together with the left rhombus of the order $n-2$ configuration forming the main body section of S_n. The basic bilateral symmetry of stealth configurations insures that the nose section of S_n reflects onto the left wing section of S_n, and the inductive hypothesis insures that the upper rhombus of the main body section of S_n reflects onto the left rhombus of the main body section of S_n. $\square$

Proof of Conjecture 3. We will prove the somewhat stronger claim that for all $k \geq 1$, diagonal D_k is periodic with period 2^{m+1}, where $m = \lfloor \log_2(k) \rfloor$. In particular, we will show that within every stealth configuration S_{m+n} with $n \geq 1$, diagonal D_k is both periodic with period 2^{m+1} and a palindrome. The proof is by induction on n.

Let D_k be a diagonal, with $m = \lfloor \log_2(k) \rfloor$. We assume k is even; the odd case is slightly different and left to the reader. D_k contains 2^m cells within the upper rhombus of configuration S_{m+1}. These are reflected onto the next 2^m cells of D_k, lying within the left rhombus of S_{m+1}. Together these 2^{m+1} cells, the intersection of D_k and S_{m+1}, form a palindrome and by default are periodic of period 2^{m+1}. This establishes the claim within S_{m+1}.

Now assume the claim is true within configuration S_{m+n} for some arbitrary $n \geq 1$. By hypothesis, we know that within configuration S_{m+n}, diagonal D_k is a palindrome and periodic of period 2^{m+1}. Now configuration S_{m+n} is contained within the upper rhombus of configuration S_{m+n+1} and thus reflects onto the left rhombus of S_{m+n+1} forming a palindrome twice as long. But the mirror image of a palindrome is itself, so the intersection of D_k and S_{m+n+1} is just the doubling of D_k within S_{m+n}. It is both periodic with period 2^{m+1} and a palindrome, confirming that the claim is true within S_{m+n+1}. □

Before proving Conjecture 4, we examine the density of ones in the order n stealth configuration.

Theorem 2. *The density of ones in the order n stealth configuration approaches a limit of 0 as n approaches infinity.*

Proof. Let C_n denote the total number of ones (black entries) in the order n stealth configuration. By inspection we have $C_0 = 1$ and $C_1 = 4$. From Theorem 1 we know that $C_n = 3C_{n-1} + 2C_{n-2}$ for all $n \geq 2$. The solution to this recurrence relation is

$$C_n = \frac{17 + 5\sqrt{17}}{34}\left(\frac{3+\sqrt{17}}{2}\right)^n + \frac{17 - 5\sqrt{17}}{34}\left(\frac{3-\sqrt{17}}{2}\right)^n \tag{3}$$

which can be confirmed by induction. This is sequence A055099 in the OEIS; its first few terms are 1, 4, 14, 50, 178, 634, 2258, 8042, ...

Now let D_n denote the total number of entries, both zeros and ones, in the order n stealth configuration S_n. Note that a triangle of k rows contains $1 + 3 + 5 + \cdots + (2k-1) = k^2$ cells. For $n \geq 2$, the top 2^n rows of S_n contain $(2^n)^2$ cells. The bottom part of S_n consists of a triangle of $2^n - 1$ rows, minus a triangle of 2^{n-1} rows, minus two triangles of 2^{n-2} rows. Adding and subtracting all the parts, we have

$$\begin{aligned} D_n &= (2^n)^2 + (2^n-1)^2 - (2^{n-1})^2 - 2(2^{n-2})^2 \\ &= \frac{13}{8}(4^n) - 2(2^n) + 1 \end{aligned} \tag{4}$$

for $n \geq 2$, with $D_0 = 1$ and $D_1 = 4$. This is sequence A256959 in the OEIS; its first few terms are 1, 4, 19, 89, 385, 1601, 6529, 26369, ...

The density of the order n stealth configuration is C_n/D_n, which clearly approaches a limit of 0. □

Proof of Conjecture 4. Let E_n denote the number of ones in the first 2^n rows of the order n stealth configuration. From Theorem 1 we know that this triangle consists of an order $n-1$ stealth configuration for its nose section, containing C_{n-1} ones; the first 2^{n-2} rows of an order $n-2$ stealth configuration for its middle section, containing E_{n-2} ones; approximately half of an order $n-1$ stealth configuration for its left part; and approximately half of an

Figure 4: Decomposing the first 2^n rows

order $n-1$ stealth configuration for its right part. We say "approximately" here because both the left part and the right part contain the spine of the order $n-1$ stealth configuration, so the left and right parts together contain $C_{n-1} + A_{n-1}$ ones. See Figure 4.

Summing, we have

$$E_n = 2C_{n-1} + E_{n-2} + A_{n-1}$$

for $n \geq 2$, with $E_0 = 1$ and $E_1 = 4$. The solution to this recurrence relation is

$$\begin{aligned} E_n = {} & \frac{17+7\sqrt{17}}{68}\left(\frac{3+\sqrt{17}}{2}\right)^n + \frac{17-7\sqrt{17}}{68}\left(\frac{3-\sqrt{17}}{2}\right)^n \\ & + \frac{2^{n+2}-(-1)^n}{6}. \end{aligned} \tag{5}$$

This result was first derived, using different methods and different recurrences, by Goldwasser *et al.* [3, Equation (21)]. This is sequence A256960 in the OEIS; its first few terms are 1, 4, 11, 36, 119, 408, 1419, 4988, 17631, ...

From the discussion above we know that F_n, the total number of cells in the first 2^n rows of the Pascal rhombus (mod 2), is

$$F_n = 4^n, \tag{6}$$

which is sequence A000302. The density of ones in the first 2^n rows is then E_n/F_n, which approaches a limit of 0.

Now let m be any positive integer, and let $n = \lfloor \log_2(m) \rfloor$, so that $2^n \leq m < 2^{n+1}$. The first m rows of the Pascal rhombus (mod 2) contain fewer than E_{n+1} ones out of at least F_n total cells. The density of ones in the first m rows is thus less than E_{n+1}/F_n, which approaches 0 as a limit. This is equivalent to the statement of Conjecture 4. $\square$

4 Other work

Moshe [5] placed the Pascal rhombus (mod 2) in the general context of double linear recurrence sequences over finite fields. Using extensions of the methods of Goldwasser *et al.* [3], he proved that the number of ones, say, in the first q^n rows of such a structure, where q is the order of the field, can be determined from the n-th power of a certain square matrix. The 5×5 matrix corresponding to the Pascal rhombus (mod 2) is explicitly displayed in

Finch [2]. The expression in Equation (5) for the sequence E_n can be derived from these two works.

In a later paper, Moshe [6] proved that the number of ones in row m of the structure can be determined from a product of matrices, one for each digit in the base-q representation of m. Again, the relevant 5×5 matrices are displayed in Finch [2]. From these it is possible to compute the sequence 1, 3, 2, 5, 5, 6, 3, 11, 4, 15, 7, 10, … , (sequence A059319), but no nice closed form expression for these numbers is known.

More recently, stealth configurations have been independently discovered by Sloane [8] in his exploration of 2-dimensional cellular automata. In examining the evolution of the "odd-rule" cellular automaton using the centered von Neumann neighborhood and starting with a single ON cell at time 0, he found that the ON cells in generation $2^n - 1$ form a diamond shaped pattern H_n consisting of four order $n-1$ stealth configurations S_{n-1}, one in each corner facing out, together with a copy of H_{n-2} in the center. Alternatively, we can view Sloane's H_n pattern as the diamond formed from the first 2^n rows of the Pascal rhombus (mod 2) and its reflection in its bottom row. Sloane calls our stealth configurations *haystacks*, and indicates the decomposition of a haystack into five smaller haystack in [8, Line (35)]. The appearance of our stealth configurations and his haystacks in rather different settings, analyzing different problems, seems quite remarkable.

References

[1] Gerald A. Edgar *Measure, Topology, and Fractal Geometry* Springer-Verlag, New York, 1990.

[2] S. R. Finch, P. Sebah and Z.-Q. Bai, Odd Entries in Pascal's Trinomial Triangle; arXiv:0802.2654, 2008.

[3] John Goldwasser, William F. Klostermeyer, Michael E. Mays, and George Trapp, The density of ones in Pascal's rhombus, *Discrete Math.* **204** (1999), 231–236.

[4] William F. Klostermeyer, Michael E. Mays, Lubomir Soltes, and George Trapp, A Pascal Rhombus, *Fibonacci Quart.* **35** (1997), 318–328.

[5] Yossi Moshe, The density of 0's in recurrence double sequences, *J. Number Theory* **103** (2003) 109–121.

[6] Yossi Moshe, The distribution of elements in automatic double sequences, *Discrete Math.* **297** (2005) 91–103.

[7] The OEIS Foundation Inc., *The On-Line Encyclopedia of Integer Sequences,* 1996–present; https://oeis.org.

[8] N. J. A. Sloane, On the Number of ON Cells in Cellular Automata; arXiv:1503.01168, 2015.

2010 *Mathematics Subject Classification*: Primary 05A10, Secondary 11B50.
Keywords: Pascal, triangle, rhombus.