Tabulation of Noncrossing Acyclic Digraphs

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1 Introduction

A *noncrossing graph* is a graph with its vertices drawn on a circle and its edges drawn in the interior such that no two edges cross each other. This note is concerned with *noncrossing acyclic digraphs*. Examples of such structures are given in Figure 1. I present an algorithm that, given a number $n \geq 1$, computes a compact representation of the set of all noncrossing acyclic digraphs with n nodes. This compact representation can be used as the basis for a wide range of dynamic programming algorithms on these graphs. As an illustration, along with this note I am releasing the implementation of an algorithm for counting the number of noncrossing acyclic digraphs of a given size.[1] This number is given by the following integer sequence (starting with $n = 1$); this is A246756 in the OEIS [OEIS Foundation Inc., 2011]:

$$1, 3, 25, 335, 5521, 101551, 1998753, 41188543, 877423873, 19166868607, \ldots$$

Another application of the tabulation technique is in semantic dependency parsing [Oepen et al., 2014], where it can be used to compute the highest-scoring dependency graph for a given sentence under an edge-factored model.

[1] `https://github.com/khlmnn/ncdags`

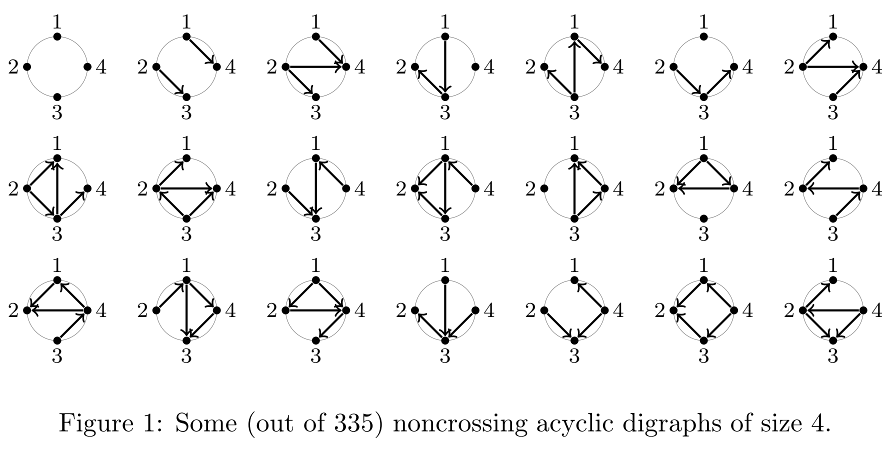


Figure 1: Some (out of 335) noncrossing acyclic digraphs of size 4.

2 Preliminaries

Before presenting the tabulation I introduce some terminology for special types of noncrossing acylic digraphs, as well as a set of operations on graphs that will be useful for the understanding of the technique.

2.1 Classification

The proposed tabulation is based on a classification of noncrossing acyclic digraphs into 7 different types. To simplify the presentation I only consider graphs with at least 2 nodes.

A graph is called *edge-covered* if there is an edge connecting its extremal vertices. Thus there are 2 types of edge-covered graphs:

- The covering edge goes from the minimal vertex to the maximal vertex. In this case, I say that the graph is *minmax-covered.*

- The covering edge goes from the minimal vertex to the maximal vertex. In this case, I say that the graph is *maxmin-covered.*

If a graph is not edge-covered, I distinguish two cases depending on whether or not the graph is weakly connected – that is, whether there exists a path (consisting of two or more edges) between the extremal vertices. In the following I use the term *connected* in the sense 'weakly connected but not edge-covered'. I distinguish 3 types of connected graphs:

- There is a directed path from the minimal vertex to the maximal vertex. In this case, I say that the graph is *minmax-connected.*

- There is a directed path from the maximal vertex to the minimal vertex. In this case, I say that the graph is *maxmin-connected.* Note that because of acyclicity, a graph cannot be both minmax-connected and maxmin-connected.

- There is no directed path between the two extremal vertices, implying that there is a path consisting of edges with mixed directions. In this case, I say that the graph is *mix-connected.*

The last two types are the graphs that are neither edge-covered nor connected. In the following I refer to these graphs as *unconnected.* I distinguish 2 types:

- The graph has 2 nodes. In this case, the graph is uniquely determined; it is the graph with 2 nodes and no edges. I refer to this graph as the *elementary graph.*

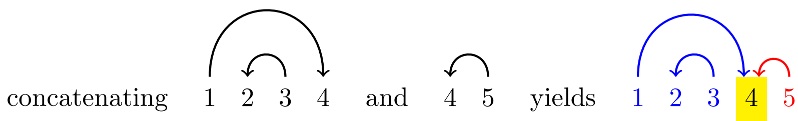
- The graph has more than 2 nodes. I say that the graph is *non-elementary.*

Note that this classification is exhaustive, meaning that every noncrossing acyclic digraph falls into (exactly) one of the 7 classes.

2.2 Operations

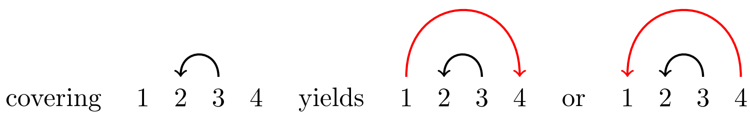
The proposed tabulation takes an algebraic view on noncrossing acyclic digraphs where every graph is composed from 'smaller' graphs by means of three operations:

- *Concatenate* two graphs, identifying the last vertex of the first graph with the first vertex of the second graph. This operation is perhaps easiest illustrated by drawing the graphs on a straight line rather than on a circle. (With this layout, the non-crossing condition means that the edges can be drawn in the half-plane above the line without crossings.)



  Here the vertices and edges contributed by the first graph are drawn in blue, those contributed by the second graph are drawn in red, and the joint vertex (simultaneously the last vertex of the first graph and the first vertex of the second graph) is highlighted in yellow.

- *Cover* a graph by adding a new edge (with two possible directions) between the first vertex and the last vertex. In the following illustration, the new edges are drawn in red:



Note that the set of noncrossing acyclic digraphs is not closed under these operations. In particular, the cover operations may introduce cycles and even multiple edges.

3 Tabulation

I present the proposed tabulation as a deduction system in the sense of Shieber et al. [1995]. Tabulation is viewed as a deductive process in which rules of inference are used to derive statements about sets of graphs from other such statements. Statements are represented by formulas called *items*.

Notation Recall that I assume that $n \geq 2$. In the following, $1 \leq i \leq j \leq k \leq n$.

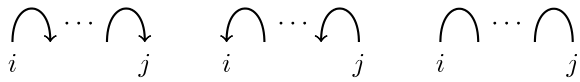
3.1 Items

Following the classification given in Section 2.1, the items of the deduction system take one of 7 possible forms. I represent these items using a graphical notation that is intended to be mnemonic.

- *Items for edge-covered graphs.* For $j - i \geq 1$:

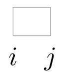
  The intended interpretation of these items is: 'It is possible to construct an edge-covered noncrossing acyclic digraph on the vertices $i, \ldots, j$.'

- *Items for connected graphs.* For $j - i \geq 2$:



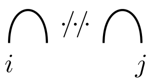
  The intended interpretation of these items is: 'It is possible to construct an minmax-connected, maxmin-connected, mix-connected noncrossing acyclic digraph on the vertices $i, \ldots, j$.'

- *Items for elementary graphs.* For $j - i = 1$:



  The intended interpretation of these items is: 'The elementary graph on the vertices i, j is a noncrossing acyclic digraph.'

- *Items for unconnected graphs.* For $j - i \geq 2$:



  The intended interpretation of these items is: 'It is possible to construct an unconnected noncrossing acyclic digraph on the vertices $i, \ldots, j$.'

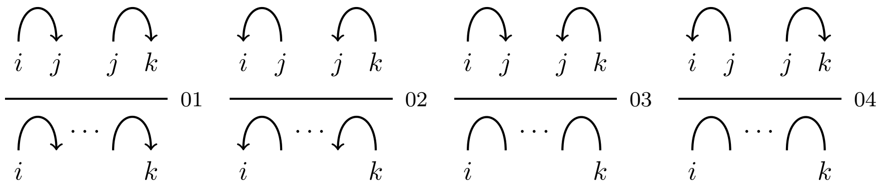
3.2 Axioms

The axioms of the deduction system are the items for the elementary graphs.

3.3 Rules

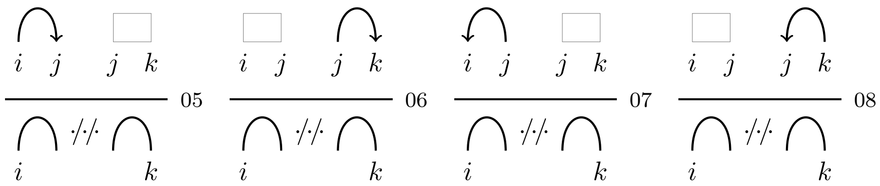
The deduction system has 26 rules. Each of these rules simulates a concatenation or cover operation on the 7 different types of noncrossing acyclic digraphs specified in Section 2.1.

Concatenate two edge-covered graphs The first four rules simulate the concatenation of two edge-covered graphs. The result of such a concatenation is a connected graph:



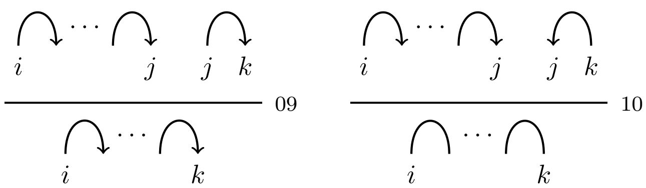
For instance, rule 03 states that the concatenation of a minmax-covered graph on the vertices $i, \ldots, j$ and an maxmin-covered graph on the vertices $j, \ldots, k$ yields a mix-connected graph on the vertices $i, \ldots, k$.

Concatenate an edge-covered graph and the elementary graph The next rules simulate the concatenation of an edge-covered graph and the elementary graph. The result of such a concatenation is an unconnected graph. There are 4 cases:

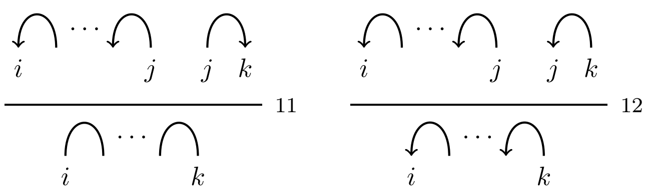


Concatenate a connected graph and an edge-covered graph The following rules simulate the concatenation of a connected graph and an edge-covered graph. The result of such a concatenation is a connected graph. There are 6 cases; I group them based on the type of the first argument of the concatenation operation.

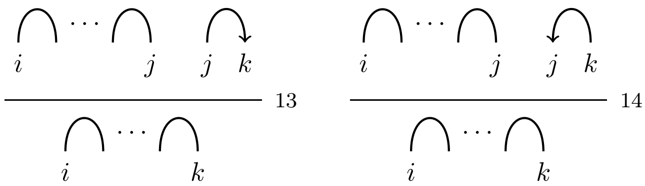
Group 1: The first argument is minmax-connected



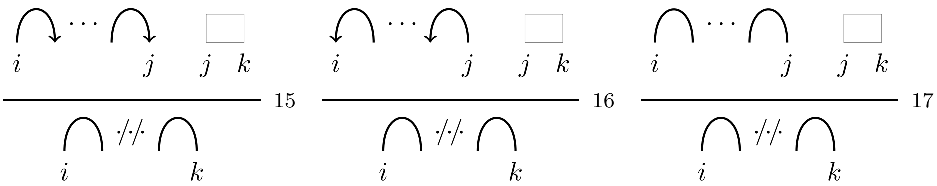
Group 2: The first argument is maxmin-connected



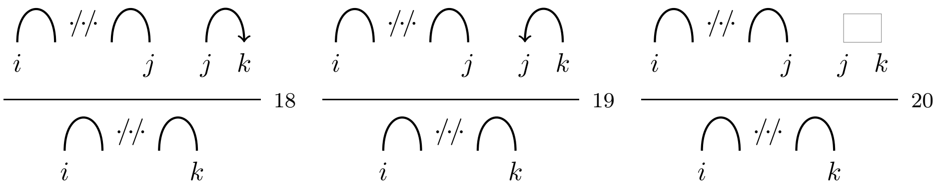
Group 3: The first argument is mix-connected



Concatenate a connected graph and the elementary graph The next rules simulate the concatenation of a connected graph and the elementary graph. The result of such a concatenation is an unconnected graph. There are 3 cases:

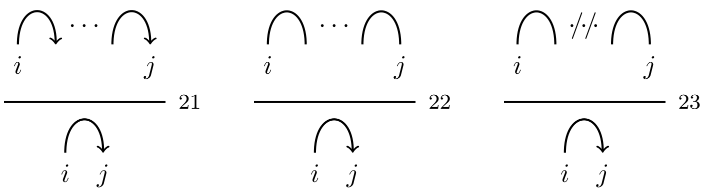


Concatenate to an unconnected graph The next rules simulate the concatenation to an unconnected graph. The result of such a concatenation is another unconnected graph. I consider 3 cases:

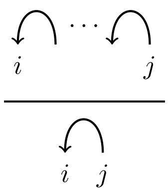


Cover a graph The rules in the final set simulate the cover operations. The result of such an operation is an edge-covered graph. There are 6 cases; I group them based on the direction of the covering edge.

Group 1: The covering edge goes from the minimal vertex to the maximal vertex



Group 2: The covering edge goes from the maximal vertex to the minimal vertex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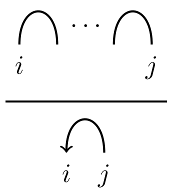
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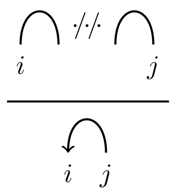
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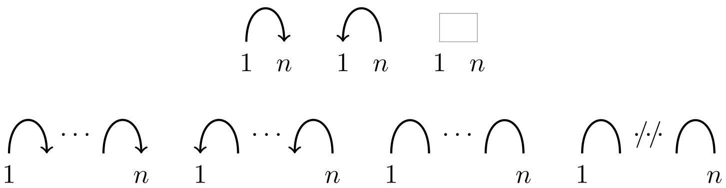
This completes the presentation of the rules.

3.4 Goal Items

In contrast to the deduction systems of Shieber et al. [1995], the proposed tabulation does not have a unique goal item but 7 different goal items, corresponding to the 7 types of noncrossing acyclic digraphs (Section 2.1).



3.5 Properties

While I shall not provide a complete formal analysis of the tabulation, I briefly mention some crucial properties:

- The runtime of the tabulation is in $O(n^3)$ and the space required for it is in $O(n^2)$. This can be seen by counting the number of possible instances of each inference rule and item.
- The deduction system is *sound*, meaning that for each rule, if the statements encoded by the antecedents hold, then the statement encoded by the consequent holds as well. To see this, one can check the soundness of each rule.
- The deduction system is *complete*, meaning that every noncrossing acyclic digraph can be constructed in a way that can be simulated by the inference rules. The completeness argument starts from the observation that the classification given in Section 2.1 is exhaustive, and then checks for each rule that undoing the operation simulated by that rule decomposes a graph represented by the consequent item into the graphs represented by the antecedents.
- Every noncrossing acyclic digraph has a *unique* derivation in the deduction system. This property is useful because it means that we do not need to distinguish between graphs and their derivations. In particular, we

can count graphs by counting their derivations. The uniqueness argument makes use of the observations that the graph types distinguished in Section 2.1 are non-overlapping, and that the backward application of rules is deterministic in the sense that for each graph there is at most one rule in which this graph appears as the consequent item.

4 Derived Tabulations

I conclude this note by noting how tabulation techniques for other classes of noncrossing graphs can be derived from the proposed technique.

4.1 Enforcing Weak Connectivity

To obtain a deduction system for *weakly connected* noncrossing acyclic digraphs one removes rule 20 (concatenate an unconnected graph and the elementary graph) and deletes the item for unconnected graphs from the list of goal items. Along with this change comes a revised intended interpretation for the items for unconnected graphs: 'It is possible to construct a noncrossing acyclic digraph on the vertices $i, \dots, j$ that is not edge-covered and has exactly two weakly connected components.' The integer sequence for weakly connected noncrossing acyclic digraphs is:

1, 2, 18, 242, 3890, 69074, 1306466, 25809826, 526358946, 10997782882, ...

4.2 Unrestricted Noncrossing Digraphs

To obtain a deduction system for *unrestricted* (not necessarily acyclic) noncrossing digraphs, one does away with the items for minmax-connected and maxmin-connected graphs, and deletes all rules that reference them – with the exception of rules 01 and 02 (concatenating two edge-covered graphs with the same directionality of the covering edge), which should be changed to produce mix-connected items. This yields the following integer sequence for unrestricted noncrossing digraphs:

1, 4, 64, 1792, 62464, 2437120, 101859328, 4459528192, 201889939456, ...

4.3 Noncrossing Undirected Graphs

By removing rules 24–26 (or 21–23), one obtains a tabulation of noncrossing *undirected* graphs. This is also known as the class of (undirected) graphs with pagenumber 1 under a fixed ordering of the vertices along the spine. This class is counted by A054726 resp. A007297 (if additionally one requires the graph to be connected) in the OEIS [OEIS Foundation Inc., 2011]:

1, 1, 2, 8, 48, 352, 2880, 25216, 231168, 2190848, ...
1, 4, 23, 156, 1162, 9192, 75819, 644908, 5616182, 49826712, ...

Acknowledgments

The decomposition that is the basis for the tabulation presented in this note was inspired by a counting technique for noncrossing acyclic digraphs proposed by Tirrell [2014]. I benefited greatly from discussions with the participants of the Dagstuhl Seminar 15122 'Formal Models of Graph Transformation in Natural Language Processing'.

Document history

2014-08-12 First version.

2014-08-13 Add a section on derived tabulations (weakly connected noncrossing digraphs, unrestricted noncrossing digraphs).

2015-04-20 Add a mention that the tabulation can be used to count the number of noncrossing undirected graphs.

References

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Stuart M. Shieber, Yves Schabes, and Fernando Pereira. Principles and implementation of deductive parsing. *Journal of Logic Programming*, 24(1–2):3–36, 1995.

Jordan Tirrell. What is the number of noncrossing acyclic digraphs? MathOverflow, 2014. http://mathoverflow.net/q/177008 (version: 2014-08-01).