

# Multiple Landen values and the tribonacci numbers

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Multiple Landen values (MLVs) are defined as iterated integrals on the interval  $x \in [0, 1]$  of the differential forms  $A = d \log(x)$ ,  $B = -d \log(1 - x)$ ,  $F = -d \log(1 - \rho^2 x)$  and  $G = -d \log(1 - \rho x)$ , where  $\rho = (\sqrt{5} - 1)/2$  is the golden section. I conjecture that the dimension of the space of  $\mathbb{Z}$ -linearly independent MLVs of weight  $w$  is a tribonacci number  $T_w$ , generated by  $1/(1 - x - x^2 - x^3) = 1 + \sum_{w>0} T_w x^w$ , and that a basis is provided by all the words in the  $\{A, G\}$  sub-alphabet that neither end in  $A$  nor contain  $A^3$ . For  $w < 9$ , I construct a much more efficient basis, for a MLV datamine, where no prime greater than 11 occurs in the denominators of 3,357,257 coefficients of rational reduction of 49,151 MLVs. Numerical data for 40 primitives then enable fast evaluation of all of these MLVs to 20,000 digits. The datamine provides reductions of Apéry-type sums  $A_w = \sum_{n>0} (-1)^{n+1} n^{-w} / \binom{2n}{n}$  and 6 ladder-combinations of depth-1 polylogarithms  $\text{Li}_w(\rho^p) = \sum_{n>0} \rho^{pn} n^{-w}$  with  $p \in \{1, 2, 3, 4, 6, 8, 10, 12, 20, 24\}$  and coefficients given by Landen, Coxeter and Lewin at  $w = 2$ . I prove that the former evaluate to MLVs and conjecture that the latter do. Comparison is made between the properties of MLVs and multiple polylogarithms at roots of unity, encountered in the quantum field theory of the standard model of particle physics.

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# 1 Introduction

In 1780, John Landen, a land-agent [19, 39] in the English county of Northamptonshire, adjacent to my own county of residence, published a book, *Mathematical memoirs respecting a variety of subjects* [27]. Memoir V, *A new method of obtaining the sums of certain series*, gives reductions of the dilogarithm  $\text{Li}_2(x) = \sum_{n>0} x^n/n^2$  to rational combinations of  $\pi^2$  and squares of logarithms, for the special values  $x = \frac{1}{2}$ ,  $x = \rho$  and  $x = \rho^2 = 1 - \rho$ , where  $\rho = (\sqrt{5} - 1)/2$  is the *golden section*. The case  $x = \frac{1}{2}$  had been studied by Euler and independently by Landen, 20 years earlier [26], but Landen's golden results were new. In the subsequent 235 years, no further result of this type has been found, for  $1 > x > 0$ . Landen also reduced the trilogarithmic combinations  $\text{Li}_3(\frac{1}{2}) - \frac{7}{8}\text{Li}_3(1)$  and  $\text{Li}_3(\rho^2) - \frac{4}{5}\text{Li}_3(1)$  to products of polylogarithms of lesser weight. No other such relation between a pair of polylogarithms has been found [33].

To achieve these feats, Landen exploited the happy fact that polylogarithms are iterated fluents, or iterated integrals, as we now call them. Thus, for example, he proved that

$$x + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \frac{x^4}{4^2} \text{ \&c.} = \text{fl. } \frac{\dot{x}}{x} \cdot \text{fl. } \frac{\dot{x}}{1-x} = \frac{2}{5}a^2 - \text{sq. Log. } x$$

for  $x = (\sqrt{5} - 1)/2$  and  $a$  defined as the length of an arc of a quadrant of a circle of unit radius. In modern terms, that evaluates

$$\text{Li}_2(\rho) \equiv \sum_{n=1}^{\infty} \frac{\rho^n}{n^2} = \int_0^{\rho} \frac{dx_1}{x_1} \int_0^{x_1} \frac{dx_2}{1-x_2} = \frac{2}{5} \left(\frac{\pi}{2}\right)^2 - (\log(\rho))^2. \quad (1)$$

The present work is devoted to the study of multiple *Landen* values (MLVs), which I define as iterated integrals, on the interval  $x \in [0, 1]$ , of the differential forms  $A = d \log(x)$ ,  $B = -d \log(1 - x)$ ,  $F = -d \log(1 - \rho^2 x)$  and the *golden* letter  $G = -d \log(1 - \rho x)$ . By  $Z(W)$ , I denote the map from a word  $W$  in the  $\{A, B, F, G\}$  alphabet to the iterated integral encoded by the letters of  $W$ . The *weight* of a word  $W$  is the number of letters in  $W$  and its *depth* is the number of letters not equal to  $A$ . Thus Landen's result (1) may be re-written as  $Z(AG) = \frac{3}{5}Z(AB) - \frac{1}{2}Z(GG)$ , which yields the integer relation  $6Z(AB) = 10Z(AG) + 5Z(GG)$ , at weight  $w = 2$ . At  $w = 3$ , the

integer relations  $2Z(ABB) = 2Z(AAB) = 10Z(AGG) + 5Z(GGG)$  likewise reduce multiple *zeta* values (MZVs), formed from words in the sub-alphabet  $\{A, B\}$ , to multiple *golden* values (MGVs), formed from words in the golden sub-alphabet  $\{A, G\}$ .

Theorem 1 shows that something quite new happens at  $w = 4$ , where there is an integer relation between MGVs. This is the beginning of a wonderful sequence of relations between MGVs which lead me to claim in Conjecture 1 that the dimension of the space of  $\mathbb{Z}$ -linearly independent finite MGVs of weight  $w$  is a tribonacci number  $T_w$ , generated by  $1/(1 - x - x^2 - x^3) = 1 + \sum_{w>0} T_w x^w$ . Conjecture 2 is even bolder and claims that  $T_w$  is also the dimension of the space of  $\mathbb{Z}$ -linearly independent finite MLVs of weight  $w$  in the *full* alphabet  $\{A, B, F, G\}$ . That implies the existence of precisely 36,783 integer relations between MLVs at  $w = 8$ , all of which are now recorded in a datamine of MLVs, available on request to the author.

In the course of this work, I obtain Theorem 2, on the alternating binomial sums [3]  $A_w = \sum_{n>0} (-1)^{n+1} n^{-w} / \binom{2n}{n}$  and give a novel conjecture for 6 ladder-combinations of depth-1 polylogarithms  $\text{Li}_w(\rho^p) = \sum_{n>0} \rho^{pn} n^{-w}$  with exponents  $p \in \{1, 2, 3, 4, 6, 8, 10, 12, 20, 24\}$  and coefficients given by Landen [27], Coxeter [21] and Lewin [2, 30, 31, 32, 33] at  $w = 2$ .

This paper is organized as follows. Section 2 explains a crucial difference between MZVs and MLVs. Multiple zeta values enjoy two algebras, coming from the shuffles of iterated integrals and the stuffles of nested sums, but for multiple Landen values the algebra of nested sums is not closed. This makes it rather hard to prove many of the fascinating structural relations between MLVs discovered in the course of my work. Section 3 gives a modest number of proofs; Section 4 contains a wealth of empirical results and stringently investigated conjectures. In Section 5, I turn attention to the ladder relations of [33], which end at  $w = 9$ . Yet I conjecture that the 6 combinations of polylogarithms to which these relations refer are always reducible to MGVs. Section 6 compares and contrasts MLVs with iterated integrals encountered in my chosen specialism of quantum field theory, whose practical agenda has done much to enrich the study of iterated integrals. Section 7 offers conclusions.

## 2 Many shuffles but fewer stuffles

The product  $Z(U)Z(V)$  of a pair of MLVs is a sum of MLVs, namely  $\sum_{W \in \mathcal{S}(U,V)} Z(W)$  where  $\mathcal{S}(U,V)$  is the set of all words  $W$  that result from shuffling the words  $U$  and  $V$ . Shuffles preserve the order of letters in  $U$  and the order of letters in  $V$ , but are otherwise unconstrained. Thus, for example, we obtain the shuffle product

$$Z(AB)Z(GF) = Z(A(BGF + GBF + GFB) + G(ABF + AFB + FAB))$$

with a notation in which  $\sum_n Z(c_n W_n) = \sum_n c_n Z(W_n)$ , for real  $c_n$ . Note that each term on the right has weight  $w = 4$  and depth  $d = 3$ , which are the sums of the weights and depths of  $Z(AB)$  and  $Z(GF)$ .

If  $W$  is a word of weight  $w$  and depth  $d$  in the alphabet  $\{A, B, F, G\}$  and  $W$  neither begins with  $B$  nor ends with  $A$ , then  $Z(W)$  is a finite MLV that may be written as a  $d$ -fold sum of the form

$$\text{Li}_{a_1, a_2, \dots, a_d}(z_1, z_2, \dots, z_d) \equiv \sum_{n_1 > n_2 > \dots > n_d > 0} \prod_{j=1}^d \frac{z_j^{n_j}}{n_j^{a_j}}. \quad (2)$$

To determine the arguments, let  $L_j$  be the  $j$ -th letter in  $W$  that is not  $A$ . Then  $a_j - 1$  is the exponent of  $A$  before  $L_j$ . If  $L_j = B$ , set  $p_j = 0$ ; if  $L_j = F$ , set  $p_j = 2$ ; if  $L_j = G$ , set  $p_j = 1$ . Then the exponent of  $\rho$  in  $z_j$  is  $p_j - p_{j-1}$ , with  $p_0 = 0$ . Thus  $Z(AAABAGFAAB) = \text{Li}_{4,2,1,3}(1, \rho, \rho, 1/\rho^2)$ .

The nested sums (2) are endowed with a stuffle algebra that preserves weight but not depth. For example, the stuffle product  $\text{Li}_a(x)\text{Li}_b(y) = \text{Li}_{a,b}(x, y) + \text{Li}_{b,a}(y, x) + \text{Li}_{a+b}(xy)$  contains a depth-1 term, coming from coincidence of indices of summation. Unfortunately this is often useless for constraining MLVs. None of the three terms in  $Z(AF)Z(AG) = \text{Li}_{2,2}(\rho^2, \rho) + \text{Li}_{2,2}(\rho, \rho^2) + \text{Li}_4(\rho^3)$  is a MLV. The stuffle product of a MZV and a MLV gives MLVs, as does the stuffle product of a MGW with a MGW. Thus the stuffle relations

$$Z(AB)Z(AF) = Z(ABAF + AF AF) + Z(AAAF)$$

$$Z(AG)Z(GG) = Z(AGFF + GAFF + GGAF) + Z(AAFF + GAAF)$$

add new information to the depth-conserving shuffles. However, none of the information gained from the many shuffles and the fewer stuffles uses the golden relation  $\rho^2 = 1 - \rho$ , to which I now turn.

### 3 A modicum of proof

I present a slender, yet seminal, body of proof, before presenting empirical findings. A pragmatic reader may skip to Subsection 3.4.

#### 3.1 An integer relation at weight 4

Let  $W$  be a word in a binary alphabet  $\{A, B\}$  with  $A = d \log(x)$  and  $B = -d \log(1 - x)$ . If  $W$  does not end in  $A$ , let  $L(W, y)$  be the iterated integral from  $x = 0$  to  $x = y$  of the sequence of differential forms encoded by  $W$ . Hence  $yL'(AW, y) = (1 - y)L'(BW, y) = L(W, y)$ . We declare that  $L(1, y) = 1$ , with unity denoting the empty word, and linearly extend by  $\sum_n L(c_n W_n, y) = \sum_n c_n L(W_n, y)$ , for real  $c_n$ . Then  $L(W(A, B), y) = Z(W(A, Y))$  with  $Y = -d \log(1 - xy)$  replacing  $B = -d \log(1 - x)$ .

For any  $y \in [0, 1]$  and  $W$  not beginning with  $B$ , we have a MZV evaluation

$$Z(W) \equiv L(W, 1) = \sum_{W=UV} L(\tilde{U}, 1 - y)L(V, y) = Z(\tilde{W}) \quad (3)$$

where the sum is over all deconcatenations [7] of  $W$  into a first part,  $U$ , and a second part,  $V$ , and  $\tilde{U}$  is the dual of  $U$ , obtained by reversing the order of letters and exchanging  $A$  and  $B$ . Thus the dual of  $AAB$  is  $ABB$ . For a MZV of weight  $w$  there are  $w + 1$  deconcatenations in (3), corresponding to the places that  $y$  may sit inside the inequalities  $1 > x_1 > \dots > x_w > 0$  for the integration variables. The integrations with  $y > x_i$  yield  $L(V, y)$  and those with  $x_j > y$  yield  $L(\tilde{U}, 1 - y)$ , after transforming  $x_j \rightarrow 1 - x_j$ .

Now let us, *pro tempore*, discard products, denoting their neglect by  $\simeq$ . Then by setting  $y = \rho^2$  in (3) we obtain

$$Z(W) \simeq L(\tilde{W}, \rho) + L(W, \rho^2) \quad (4)$$

where the first term is a MGW in the  $\{A, G\}$  sub-alphabet and the second is a MLV in the  $\{A, F\}$  sub-alphabet.

**Lemma 1:** Every MLV in the  $\{A, F\}$  sub-alphabet is a  $\mathbb{Q}$ -linear combination of MZVs, MGWs and products of these two types of term.

**Proof:** For weight  $w > 1$ , suppose that this is true for smaller weights. Let  $W$  be a Lyndon word in the  $\{A, B\}$  sub-alphabet, namely a word for

which all deconcatenations  $W = UV$  have  $U$  preceding  $V$ , in lexicographic ordering. Then we may use (4) at weight  $w$ , since omitted products are, by assumption, of the required form. Hence the MLV obtained by replacing  $B$  by  $F$  in the Lyndon word  $W$  is also of the required form. Thus, at weight  $w$ , all MLVs in the  $\{A, F\}$  sub-alphabet are of the required form, since the shuffle algebra gives them in terms of Lyndon words and products. The observation that  $2Z(F) = Z(G)$  completes the proof by induction. ■

Relation (3) acquires more power when we combine it with

$$L(W, -\rho) = L(\overline{W}, \rho^2) \quad (5)$$

with a conjugate defined by  $\overline{W}(A, B) \equiv W(A + B, -B)$ . The operations of taking a conjugate (denoted by bar) or a dual (denoted by tilde) are self-inverse and commute. Thus if  $W = \overline{U} = \widetilde{V}$ , then  $U = \overline{W}$ ,  $V = \widetilde{W}$  and  $\widetilde{U} = \overline{V}$ . Relation (5) results from transformation of variables of integration by  $x_j \rightarrow -x_j/(1 - x_j)$ , combined with the identity  $\rho/(1 + \rho) = \rho^2$ .

Next consider the depth-1 word  $W = A^{w-1}B$ , which yields the polylogarithm  $L(A^{w-1}B, y) = \text{Li}_w(y)$  for which the transformation of integration variables  $x_j \rightarrow x_j^2$  gives  $\text{Li}_w(y) = 2^{1-w}\text{Li}(y^2) - \text{Li}_w(-y)$ . Setting  $y = \rho$  and using (5), we obtain

$$Z(A^{w-1}G) = 2^{1-w}Z(A^{w-1}F) + Z((A + F)^{w-1}F) \quad (6)$$

with a final term that yields  $2^{w-1}$  distinct MLVs, with depths up to  $w$ , when we expand  $(A + F)^{w-1}$ , remembering that  $A$  and  $F$  do not commute.

Then we expect to be able to reduce all MZVs with weight  $w < 8$  to MGVs, by diligent use of (3) and (6), and hence by Lemma 1 to reduce all MLVs with  $w < 8$  in the  $\{A, F\}$  sub-alphabet to words in the golden sub-alphabet. To show that we may do so, it is necessary to prove that  $\zeta(2)$ ,  $\zeta(3)$ ,  $\zeta(5)$  and  $\zeta(7)$  are reducible to MGVs, which I shall do.

At  $w = 2, 3$ , we obtain intriguingly similar reductions of MZVs to MGVs

$$6\zeta(2) = 5Z((2A + G)G) \quad (7)$$

$$2\zeta(3) = 5Z((2A + G)G^2) \quad (8)$$

with (7) being Landen's result  $\pi^2 = 10 \text{Li}_2(\rho) + 10(\log(\rho))^2$ , obtained by simple algebra as follows. First use (6) to show that  $Z(G) = 2Z(F)$  and that  $Z(AG) = \frac{3}{2}Z(AF) + Z(FF)$ . The shuffle algebra gives  $Z(G^w) =$

$(Z(G))^w/w! = Z(F^w)/2^w$ . Hence we reduce  $Z(AF) = \frac{2}{3}Z(AG) - \frac{1}{6}Z(GG)$  to MGVs. Then we obtain  $\zeta(2) = Z(AF) + Z(F)Z(G) + Z(AG)$  from (3) and lift the product  $Z(F)Z(G)$  to  $Z(GG)$ , obtaining (7). Similar (but more tedious) elimination and lifting at  $w = 3$  give the neat result (8), from which the depth-1 sum  $Z(AAG) = \text{Li}_3(\rho)$  is notably absent.

We now know, from Lemma 1, that all MLVs in the  $\{A, F\}$  alphabet with  $w < 5$  are reducible to MGVs, since there is no primitive MZV at  $w = 4$ .

**Theorem 1:** There is an integer relation between MGVs of weight 4.

**Proof:** To prove existence we do not need to retain products. From (6) we obtain  $Z(AAAG) \simeq \frac{9}{8}Z(AAAF) + Z(U) + Z(V)$  where  $U = FAAF + AF AF + AAF F$  and  $V = FF AF + F AF F + AF FF$ . The shuffle products  $Z(F)Z(AAF) = Z(FAAF + AF AF + 2AAF F)$  and  $Z(F)Z(FAF) = 2Z(FF AF + F AF F)$  give  $Z(U) \simeq -Z(AAF F)$  and  $Z(V) \simeq Z(AF FF)$ . Next we obtain  $Z(AF FF + AAAG) \simeq Z(AAF F + AAGG) \simeq Z(AAAF + AGGG) \simeq 0$  from (4), since all MZVs of weight 4 are rational multiples of  $\pi^4$  and hence products of MGVs. Collecting terms we obtain

$$8Z(A^2G^2) \simeq 16Z(A^3G) + 9Z(AG^3) \quad (9)$$

with neglect of products that may be lifted to MGVs. This lifting cannot trivialize (9), because it contains only Lyndon words. ■

After some rather heavy lifting duty, we obtain the relation explicitly as

$$48Z((2A + G)AAG) = Z((2A + G)(176AG + 76GA + 123GG)G) \quad (10)$$

with less attractive integers than in the Lyndonized version (9), but now a pleasing ubiquity of  $(2A + G)$ .

### 3.2 A theorem for Apéry's alternating binomial sums

Now consider the Apéry-type alternating binomial sums [3, 8]

$$A_w \equiv \sum_{n>0} \frac{(-1)^{n+1}}{n^w \binom{2n}{n}}. \quad (11)$$

**Theorem 2:** For  $w > 1$ ,  $A_w$  is a  $\mathbb{Z}$ -linear combination of MGVs.

**Proof:** Multiply the summand in (11) by

$$2n^w \int_0^1 \left( \frac{(-2 \log(y))^{w-1}}{(w-1)!} \right) y^{2n-1} dy = 1$$

and exchange the order of summation and integration to obtain

$$A_w = \int_0^1 \left( \frac{(-2 \log(y))^{w-1}}{(w-1)!} \right) g(y) dy$$

where  $g(y) = f'(y)$  is the derivative of the summable series [8, 1]

$$f(y) \equiv - \sum_{n>0} \frac{(-y^2)^n}{n \binom{2n}{n}} = \frac{y \log \left( \sqrt{1 + y^2/4} + y/2 \right)}{\sqrt{1 + y^2/4}}.$$

Hence  $A_1 = f(1) = -2 \log(\rho)/\sqrt{5} = Z(G)/\sqrt{5}$  and for  $w > 1$  we obtain

$$A_w = 2 \int_0^1 \left( \frac{(-2 \log(y))^{w-2}}{(w-2)!} \right) \frac{\log \left( \sqrt{1 + y^2/4} + y/2 \right)}{\sqrt{1 + y^2/4}} dy$$

using integration by parts. Now make the substitution  $y = \rho x / \sqrt{1 - \rho x}$ , which maps  $y = 1$  to  $x = 1$  and gives  $A_w = I_{w-2,2}$ , where

$$I_{a,b} \equiv \int_0^1 \frac{(\log(1 - \rho x) - 2 \log(\rho) - 2 \log(x))^a (-\log(1 - \rho x))^{b-1} \rho dx}{a!(b-1)!(1 - \rho x)} \quad (12)$$

and a trinomial expansion gives

$$I_{a,b} = \sum_{i+j+k=a} (-1)^k \binom{k+b-1}{b-1} Z(G^i) Z((2A)^j G^{k+b}). \quad (13)$$

Then the shuffle algebra lifts this to a  $\mathbb{Z}$ -linear combination of MGVs . ■

The lift of (13) is  $I_{a,b} = Z((2A + G)^a G^b)$ , which delivers  $A_2 = I_{0,2} = Z(G^2) = 2(\log(\rho))^2$  and, on use of (8),  $A_3 = I_{1,2} = Z((2A + G)G^2) = \frac{2}{5}\zeta(3)$ , from which Apéry [3], with great ingenuity [38], proved the irrationality of  $\zeta(3)$ . Thereafter, expansion (13) is more revealing, since it tells us the primitive part of  $A_w$  directly, as a sum of  $(w-2)$  Lyndon words:

$$A_w = Z((2A + G)^{w-2} G^2) \simeq \sum_{k=0}^{w-3} (-1)^k (k+1) Z((2A)^{w-2-k} G^{k+2}). \quad (14)$$



Hence  $A_4 \simeq 4Z(A^2G^2 - AG^3)$ , from which we eliminate the depth-2 term, using (9), and transform the depth-3 term, using  $Z(AG^3 + A^3F) \simeq 0$ , to obtain the primitive part of

$$A_4 = Z((2A + G)^2G^2) \simeq \frac{1}{2}Z(A^3(16G - F)) = \sum_{n>0} \frac{8\rho^{2n-1}}{(2n-1)^4} \quad (15)$$

as a depth-1 sum over *odd* powers of the golden section.

I now prove a result conjectured in [8], namely that the primitive part of

$$A_5 = Z((2A + G)^3G^2) \simeq \frac{1}{2}Z(A^4(5F - 4B)) = \sum_{n>0} \frac{5\rho^{2n} - 4}{2n^5} \quad (16)$$

is a depth-1 sum with only *even* powers of  $\rho$  appearing. To do this, use the shuffle algebra to remove products from (6), in the manner that we did in (14), obtaining

$$Z(A^{w-1}(G - (2^{1-w} + 1)F)) \simeq F_w \equiv \sum_{k=1}^{w-2} (-1)^k Z(A^{w-1-k}F^{k+1}). \quad (17)$$

Then (4) shows that  $F_5 \simeq Z(A^2G^3 - A^3G^2) + Z(A^4G) - \zeta(5)$ , thanks to the duality  $Z(A^2B^3) = Z(A^3B^2)$  of the MZVs with depth  $d > 1$ . Thus  $Z(A^4G)$  cancels in (17), to give  $Z(A^2G^3 - A^3G^2) \simeq \zeta(5) - \frac{17}{16}Z(A^4F)$ , and (14) gives  $A_5 \simeq Z(8A^3G^2 - 8A^2G^3 + 6AG^4)$ . Using (4) to eliminate  $Z(AG^4) \simeq \zeta(5) - Z(A^4F)$ , we prove the claimed result (16). Then shuffle algebra restores the temporarily neglected products and lifts (16) to give

$$\begin{aligned} \sum_{n>0} \frac{(-1)^{n+1}}{n^5 \binom{2n}{n}} &= \frac{5 \operatorname{Li}_5(\rho^2) - 4 \zeta(5)}{2} - 5 \operatorname{Li}_4(\rho^2) \log(\rho) + 4 \zeta(3) (\log(\rho))^2 \\ &\quad + \left(\frac{2\pi}{3}\right)^2 (\log(\rho))^3 - \frac{4}{3} (\log(\rho))^5. \end{aligned} \quad (18)$$

### 3.3 Theorems at higher weight

**Theorem 3:** For weights  $w < 8$ , every MZV and every MLV in the  $\{A, F\}$  sub-alphabet is a  $\mathbb{Q}$ -linear combination of MGVs.

**Proof:** For odd weights, the duality relation  $Z(A^aB^b) = Z(A^bB^a)$  ensures the cancellation of MZVs with depth  $d > 1$  in the reduction of (17) to

MGVs and MZVs, giving

$$\zeta(2n+1) \simeq 4^n \sum_{k=2}^{2n} (-1)^k Z(A^{2n+1-k}G^k) + Z(AG^{2n}) \quad (19)$$

from which  $Z(A^{2n}G) = \text{Li}_{2n+1}(\rho)$  is absent. All MZVs with  $w < 8$  are reducible to  $\zeta(2)$ ,  $\zeta(3)$ ,  $\zeta(5)$ ,  $\zeta(7)$  and their products. Thus (19) reduces them to MGVs and then Lemma 1 shows that MLVs of the  $\{A, F\}$  sub-alphabet reduce to MGVs, for  $w < 8$ . ■

**Theorem 4:** There is at least one integer relation between weight-6 MGVs and at least one between weight-8 MGVs.

**Proof:** For even weight,  $w = 2n$ , we obtain from (17)

$$\sum_{k=2}^{2n-2} (-1)^k Z(A^{2n-k}G^k) \simeq 2Z(A^{2n-1}G) + (2^{1-2n} + 1)Z(AG^{2n-1}) + Z_{2n} \quad (20)$$

where MZVs of depth  $d > 1$  contribute to the summands of

$$Z_w \equiv \sum_{k=2}^{w-2} (-1)^k Z(A^{w-k}B^k). \quad (21)$$

However  $Z_6 \simeq 0$ , since there is no primitive MZV of weight 6, and the MZV datamine [6] proves that  $Z_8$  is a rational multiple of  $\pi^8$ . ■

In fact it is possible to prove that  $Z_{2n} = 4^{1-n}\zeta(2n)$ , but that is not the issue. The important point is that  $Z_8$  does *not* contain the depth-2 primitive  $\zeta(5, 3) \equiv \text{Li}_{5,3}(1, 1)$ . Hence we are stuck: we cannot extend Theorem 3 to higher weights using the limited methods of this section. Nor can we easily extend Theorem 4, since the argument by induction, for lifting all neglected products to MGVs, cannot be relied on for  $w > 8$ .

### 3.4 Seed corn

I have refrained from giving any experimental result in this section. Hence we have learnt only a few things, thus far. The following are seminal.

1. MGVs of the golden sub-alphabet  $\{A, G\}$  have at least one integer relation at each of the even weights 4, 6 and 8.

2. Apéry's alternating binomial sums reduce to  $\mathbb{Z}$ -linear sums of MGVs at all weights  $w > 1$ . For  $w < 6$  they have proven reductions to the depth-1 sums  $\text{Li}_w(\rho^p)$ , with  $p \in \{0, 1, 2\}$ , and their products.
3. With the possible exception of  $\zeta(5, 3)$ , we may reduce all MZVs with weight  $w < 9$  to MGVs.
4. We may reduce all MLVs of the sub-alphabet  $\{A, F\}$  with  $w < 9$  to MGVs and, if necessary,  $\zeta(5, 3)$ .

## 4 A cornucopia of computation and conjecture

Unless otherwise indicated, all further results are empirical and all further claims are conjectural. Hence I am now allowed to say that  $\zeta(5, 3)$  has an empirical reduction to MGVs.

### 4.1 The golden sub-alphabet for weights $w < 9$

Let  $D_w$  be the number of  $\mathbb{Z}$ -linearly independent MLVs of weight  $w$  in the full alphabet  $\{A, B, F, G\}$ . We shall not be able to determine  $D_6$  until someone figures out how to prove the irrationality of  $(\zeta(3))^2/\pi^6$  (and much else). However a plausible lower bound is provided by numerical study of a sub-alphabet, using, for example, the `lindep` procedure of `Pari-GP` [36] to find integer relations that are supported by many more digits of evidence than were required to discover them. So the first question is clear: how many relations are there between weight-5 Lyndon words in the golden sub-alphabet? I found precisely two

$$10Z(AAGAG) \simeq Z(8A^4G - 138A^3G^2 + 111A^2G^3 - 114AG^4) \quad (22)$$

$$10Z(AGAGG) \simeq Z(-16A^4G + 8A^3G^2 - 24A^2G^3 - 3AG^4) \quad (23)$$

written here modulo 9 products of Lyndon words of lesser weight, which may be restored by using the MLV datamine. These relations hold at 3000 digit precision. Moreover, there is no further relation for  $w < 6$  with integer coefficients of less than 200 digits.

I then found, with increasing labour, 4 relations between weight-6 Lyndon words, modulo 19 products, and 8 relations between weight-7 Lyndon

words, modulo 34 products. Thus the number of relations forms the sequence 1, 2, 4, 8 . . . starting at  $w = 4$ .

Then came the *experimentum crucis*, at  $w = 8$ , where I confidently expected to find precisely 15 relations, modulo 66 products, for reasons that will emerge. This required integer relation searches in a space of 82 dimensions, each conducted at 3000-digit precision, in the hope that no integer coefficient might have more than  $\lfloor 3000/82 \rfloor = 36$  digits, and was rewarded by precisely 15 highly credible relations.

**Conjecture 1:** The dimension of the space of  $\mathbb{Z}$ -linearly independent MGVs of weight  $w$  is the tribonacci number  $T_w$  defined by  $T_1 = 1$ ,  $T_2 = 2$ ,  $T_3 = 4$  and  $T_w = T_{w-1} + T_{w-2} + T_{w-3}$  thereafter.

To see how this fits the data, consider the generating function

$$1 + \sum_{w>0} T_w x^w = \frac{1}{1 - x - x^2 - x^3} \quad (24)$$

giving the tribonacci sequence, which for  $w = 1 \dots 14$  is

1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504, 927, 1705, 3136 . . .

Then the conjectured number  $N_w$  of primitives forms the sequence

1, 1, 2, 2, 4, 5, 10, 15, 26, 42, 74, 121, 212, 357 . . .

for  $w = 1 \dots 14$ , determined by the filtration

$$\prod_{w>0} (1 - x^w)^{N_w} = 1 - x - x^2 - x^3. \quad (25)$$

Now define a Lucas-type 3-step sequence,  $S_w$ , with  $S_1 = 1$ ,  $S_2 = 3$ ,  $S_3 = 7$  and  $S_w = S_{w-1} + S_{w-2} + S_{w-3}$  thereafter. Then a Möbius transformation gives  $wN_w = \sum_{d|w} \mu(w/d) S_d$ . The number  $B_w$  of binary Lyndon words (discounting the illegal word  $A$ ) grows much faster, since  $wB_w = \sum_{d|w} \mu(w/d)(2^d - 1)$ . Hence the predicted number of relations between Lyndon words is given by

$$R_w \equiv B_w - N_w = \frac{1}{w} \sum_{d|w} \mu(w/d)(2^d - 1 - S_d) \quad (26)$$

where the sum is over the divisors of  $w$ ,  $\mu(n) = 0$  if  $n$  is divisible by the square of a prime and  $\mu(n) = (-1)^{\omega(n)}$  when  $n$  is a square-free integer with  $\omega(n)$  prime divisors. Then (26) gives, for  $w = 4 \dots 14$ , the sequence

1, 2, 4, 8, 15, 30, 57, 112, 214, 418, 804 . . .

while the number of relations between MGVs,  $2^{w-1} - T_w$ , gives  
1, 3, 8, 20, 47, 107, 238, 520, 1121, 2391, 5056...

which soon far exceeds the number  $R_w$  of relations between Lyndon words.

Note that even at  $w = 5$  there is no unique way of lifting the two relations (22,23) between Lyndon words to the vector space of MGVs, where there are three relations. The third comes from shuffling  $Z(G)$  with relation (10) at  $w = 4$ . At  $w = 8$ , there are  $2^7 - T_8 = 128 - 81 = 47$  relations between MGVs of which only  $R_8 = 15$  are irreducible relations between weight-8 Lyndon words. Hence the task for weight-8 MGVs was to find 15 irreducible relations, each represented by a vector of  $T_8 + 1 = 82$  integers, whose magnitude depends critically on how one has chosen to eliminate MGVs of lower weight. We shall return to this practical issue, in a later subsection. For the present, it is sufficient to note that considerable human intelligence may be needed to complete an investigation of the current scale; if one is not careful, the problem may escalate, alarmingly, as occurred in Erik Panzer's 73-dimensional integer-relation search, needed to determine a 7-loop Feynman period in quantum field theory [11, 37].

## 4.2 The full $\{A, B, F, G\}$ alphabet of MLVs

The harvest of 47 relations between 128 weight-8 MGVs may seem rather modest. I now make the bold claim that these were merely seed corn.

**Conjecture 2:** The dimension of the space of  $\mathbb{Z}$ -linearly independent weight- $w$  MLVs in the alphabet  $\{A, B, F, G\}$  is the tribonacci number  $T_w$ .

If this be true, then there is a glut of 36,783 integer relations between MLVs at  $w = 8$ . I claim to have recorded all of these in the MLV datamine.

## 4.3 Rival bases for vector spaces

**Conjecture 3:** A basis for the vector space of MLVs of weight  $w$  is provided by taking all the words of length  $w$  in the sub-alphabet  $\{A, G\}$  that neither end in  $A$  nor contain  $A^3$ .

This agrees with the MLV datamine. However, the simplistic basis of Conjecture 3 was not the one used to compile the data that confirm the conjecture, for  $w < 9$ , at overwhelming numerical precision.

The golden section  $\rho = (\sqrt{5} - 1)/2$  satisfies  $1 = \rho + \rho^2$ . Conjecture 3 singles out the letter  $G = -d \log(1 - \rho x)$ , which seems a little unfair on  $F = -d \log(1 - \rho^2 x)$ . Let's remedy that.

**Conjecture 4:** A basis for the vector space of MLVs of weight  $w$  is provided by taking all the words of length  $w$  in the sub-alphabet  $\{A, F\}$  that neither end in  $A$  nor contain  $A^3$ .

Again, this agrees with the datamine, which can now act as an adjudicator between rivals, whom I shall personify as  $\rho$  and  $\rho^2$ . Consider the determinant  $|M|$  of the  $81 \times 81$  matrix  $M$  that transforms the weight-8 MGVs of the  $\{A, G\}$  basis of Conjecture 3 to the weight-8 MLVs of the  $\{A, F\}$  basis of Conjecture 4. Every prime that occurs in the *denominator* of  $|M|$  is a bad mark for  $\rho$ , according to  $\rho^2$ , who correctly observes that  $\rho$  must divide by that prime when expressing MLVs in the  $\rho$  basis. But then, of course,  $\rho$  is equally correct in objecting that every prime in the *numerator* of  $|M|$  is a bad mark for  $\rho^2$ , who must divide by that prime to obtain MGVs.

Here are primes that are bad news for  $\rho$ :

2, 7, 37, 41, 643, 198817, 2908441

and here are primes that are bad news for  $\rho^2$ :

3, 113, 1193, 1409, 7793, 2482024049916701

from which it is clear that neither protagonist has cause to rejoice, given that the datamine has no denominator prime greater than 11.

At this point, a third person,  $-\rho$ , speaks up on behalf of the letter  $H = -d \log(1 + \rho x)$  claiming that the  $\{A, H\}$  alphabet has been unfairly excluded from this competition and pleading for it to be enshrined in a further conjecture. We may ignore this request, for the present, thanks to the following lemma.

**Lemma 2:** Let  $f$  be a vector formed by the weight- $w$  MLVs defined in Conjecture 4 and let  $h$  be the corresponding vector of iterated integrals in which  $F = -d \log(1 - \rho^2 x)$  is replaced by  $H = -d \log(1 + \rho x)$ . Then there is a unimodular integral matrix  $U$  such that  $h = Uf$ .

**Proof:** Let  $W(A, B)$  be any binary word of length  $w$  that neither ends in  $A$  nor contains  $A^3$ . Then  $Z(W(A, H))$  is an element of  $h$  and (5) tells us that  $Z(W(A, H)) = Z(W(A + F, -F))$  is a sum of  $2^{w-d}$  elements of  $f$ , each with coefficient  $(-1)^d$ , where  $d$  is the number of occurrences of  $B$  in  $W(A, B)$ . Hence there is a matrix of integers  $U_{i,j}$  such that  $h = Uf$ . Now

we elect to list the elements of vectors  $f$  and  $h$  by the lexicographic order of  $W(A, B)$ . Then  $U$  is upper triangular with diagonal elements  $\pm 1$ . Thus  $|U| = \pm 1$  and the transformation from  $f$  to  $h$  is unimodular, no matter how we order the elements of  $f$  and  $h$ . ■

From this it is clear that every prime that is bad for  $\rho^2$  is also bad for  $-\rho$  and that these primes are even worse than those which are bad for  $\rho$ .

#### 4.4 Conjectural primitives

Using a vector-space criterion for a putative basis is usually a bad idea, for practical purposes. It makes more sense to try to choose, at each weight  $w > 1$ , a set of primitives of the conjectured cardinality  $N_w$  defined by (25), which enumerates binary Lyndon words  $W(A, B)$  that do not contain  $A^3$ . It also enumerates the Lyndon words that do not contain  $B^3$ . Moreover, we may take the duals of those two choices, since the dual of a Lyndon word is not necessarily a Lyndon word. So now we have  $2 \times 2 = 4$  choices for the set of words.

For each of those 4 choices of words, we have 3 choices,  $F$ ,  $G$  and  $H$ , for what to substitute for the letter  $B$ , which here stands merely as a binary place-holder. Note that the choice of  $H = -d \log(1 + \rho x)$  is now not trivially disposed of by Lemma 2, which was concerned with a transformation between vector spaces, not between primitives. So we have (at least)  $4 \times 3 = 12$  choices for sets of putative primitives.

**Conjecture 5:** For each of these 12 choices, the specified words are primitive and, with their products, provide a basis for all MLVs.

This agrees with the MLV datamine, for  $w < 9$ .

#### 4.5 Depth-filtered primitives

Being, by upbringing, an empiricist, I wished to reduce all 49,151 MLVs with  $w < 9$  to a set of primitives and their products. That involved, *inter alia*, determination of 36,783 integer relations at  $w = 8$  in a search space of dimension  $T_8 + 1 = 82$ , where even a good choice of primitives might be expected to yield integers with more than 15 digits. Inflation to more than 35 digits, by products of denominator-primes, such as  $37 \times 41 \times 643 \times$

198817  $\times$  2908441, could not be tolerated; some better idea was needed.

Having seen that the strong Conjecture 2, on the number of primitives for the *full* alphabet, was viable at low weights, I resolved not to restrict the primitives to any sub-alphabet. Hence my Aufbau was based on ordering primitive MLVs first by weight,  $w$ , then by depth,  $d$ , and finally, for each  $w$  and  $d$ , by lexicographic order in the  $\{A, B, F, G\}$  alphabet. Thus a systematic choice of primitive words is given by

$$\{F\}, \quad \{AB\}, \quad \{A^2B, A^2G\}, \quad \{A^3F, A^3G\}, \quad \{A^4B, A^4F, A^4G, A^3BF\}$$

for  $w = 1$  to 5, respectively. Note that Landen's result of 1780, for the reduction of  $\text{Li}_3(\rho^2) - \frac{4}{5}\text{Li}_3(1)$ , means that one must skip over  $A^2F$ . The reducibility of  $\zeta(4)$  requires the omission of  $A^3B$ . At  $w = 5$ , the first depth-2 word,  $A^3BF$ , is primitive. Then at  $w = 6$  the set

$$\{A^5F, A^5G, A^4FB, A^4GB, A^4GF\}$$

suffices, with the predicted cardinality  $N_6 = 5$ . At  $w = 7$ , with  $N_7 = 10$ ,

$$\{A^6B, A^6F, A^6G, A^5BF, A^5BG, A^5FB, A^5GB, A^5GF, A^4FAB, A^4FAG\}$$

likewise contains no depth-3 word.

So far, so good: no denominator-prime greater than 11 had appeared in any reduction of a MLV with weight  $w < 8$ . The products of these 25 primitives then supplied  $T_8 - N_8 = 81 - 15 = 66$  elements of the conjectured vector space at  $w = 8$ . Then I worked my way through the 48 weight-8 MLVs that begin with  $A^5$  and do not end in  $A$ , finding at high precision, that

$$\{A^7F, A^7G, A^6FB, A^6GB, A^6GF, A^5BAB, A^5BAF, A^5BAG, \\ A^5FAB, A^5FAG, A^5GAB, A^5GAF, A^5BBF, A^5BBG, A^5BFG\}$$

are independent primitives, with 3 of the 15 appearing at depth  $d = 3$ .

This method is iterative: begin with a 67-dimensional integer relation search, to determine the first primitive; when one is found increment the partial basis; continue until a putative basis of dimension 81 is achieved; the remaining checks are then in 82 dimensions. With only 48 preliminary searches to perform, I could afford the luxury of working at 3000 digits.



## 4.6 Testing the conjectures at $w < 9$

Now comes the big question. Do *all* weight-8 MLVs reduce to a 81-dimensional vector space? The datamine, obtained by the judicious choice of the previous subsection, attests that this is the case. It contains no denominator-prime greater than 11. All the MLVs were computed to 1250 digits but `lindep` was told only 1200 digits. It managed to find credible 82-dimensional integer relations in all 36,783 cases with  $w = 8$ . Then I checked that these reproduce the extra 50 digits that had been hidden from `lindep`. So the probability of a spurious reduction is less than  $1/10^{45}$ .

The largest prime found in a numerator of a coefficient of reduction was 158575062799, with 12 digits, in the coefficient of  $\pi^8$  for the reduction of  $Z(AFGBGAFG)$ . The largest integer found by `lindep` was  $7 \times 15^3 \times 7908074791 = 186828266937375$ , with 15 digits, in the integer relation for  $Z(GAFBGAFB)$ . Had I used the basis of Conjecture 3, the sizes of integers might have been inflated to 35 digits.

The datamine was compiled to test Conjecture 2. Testing of the other conjectures, for  $w < 9$ , is now simply a matter of matrix algebra. Conjecture 1 asserts that the golden sub-alphabet,  $\{A, G\}$ , gives vector spaces no smaller than those for the full alphabet. To confirm this at weight  $w < 9$ , it suffices to find a set, with tribonacci cardinality,  $T_w$ , of weight- $w$  MGVs that are linearly independent, according to the rational vectors of reduction in the datamine. This was easily done. Conjectures 3, 4 and 5 assert the validity of 14 concrete bases, confirmed by the datamine, for  $w < 9$ , by computing rational determinants and showing that none vanishes. I have included these 14 naive choices in the hope that someone might achieve, for one or more of them, a proof comparable to that of Francis Brown, who showed [16] that Michael Hoffman's similarly naive conjectural basis [34] for MZVs is indeed functional at all weights, though it lies beyond the wit of humankind to prove that there are no further rational relations.

## 5 Lewin's golden ladder combinations live for ever

*...glancing through the pages of Edwards' Calculus – a fascinating book if ever there was – when my eye caught a paragraph at the foot of the*

*page recounting some formulae of Landen's*, wrote the electrical engineer, Leonard Lewin, reminiscing [28] about his schooldays in the 1930s.

I learnt about polylogarithms, more than 45 years ago, from Lewin's book, *Dilogarithms and associated functions*, published [28] in 1958, which also contained a good deal of information about polylogarithms and was much in demand in my university's library, by physicists calculating Feynman integrals. I studied it there, again and again. Yet at that time, in the late 1960s, few mathematicians seemed to take interest in this wonderful book. It was something that harked back to seemingly miscellaneous results by Euler, Landen [26, 27], Spence, Abel, Hill [24, 25], Kummer, *et alia*. It was very useful for technical purposes, like mine, but it seemed to be distinctly *passé* in the upper echelons of courtiers of the queen of the sciences – pure mathematics – as I perceived her, at that time.

A second edition, entitled *Polylogarithms and associated functions*, was published [29] in 1981, with little change of content, but now an encouraging preface by Alf van der Poorten. Richard Askey remarked [4] that anyone who appreciates beautiful formulas should become familiar with this book.

It was a source of satisfaction to me that modern mathematicians eventually caught up with this fine scholar and engineer, and his enthusiastic readers working in particle physics, by honouring Lewin with erudite contributions to an American Mathematical Society volume [33] entitled *Structural properties of polylogarithms*, edited by him in 1991. Here he returned to Landen's 1780 formulas for  $\text{Li}_2(\rho)$ ,  $\text{Li}_2(\rho^2)$  and  $\text{Li}_3(\rho^2)$ , remarking that with  $p \in \{1, 2, 3, 4, 6, 8, 10, 12, 20, 24\}$  there are 6 combinations of  $\text{Li}_w(\rho^p) = \sum_{n>0} \rho^{pn} n^{-w}$  that are reducible to  $\pi^2$  and  $(\log(\rho))^2$  at  $w = 2$  and proceeding to find further relations between them, up to  $w = 9$ .

## 5.1 Six characters in search of an afterlife

In addition to two dilogarithmic combinations from Landen [27] in 1780, Lewin remarked on three from Coxeter [21], in 1935, and added a sixth finding [33] of his own. The extension to weight  $w$  is then as follows. Let  $\mathcal{S}_w$  be the set of the following 6 combinations of polylogarithms

$$L_{1,w} \equiv \text{Li}_w(\rho) \tag{27}$$

$$L_{2,w} \equiv \text{Li}_w(\rho^2) \tag{28}$$

$$L_{6,w} \equiv \text{Li}_w(\rho^6) - 2^w \text{Li}_w(\rho^3) \tag{29}$$

$$L_{12,w} \equiv \text{Li}_w(\rho^{12}) - 2^{w-2} 3 \text{Li}_w(\rho^6) - 3^{w-1} \text{Li}_w(\rho^4) \tag{30}$$

$$L_{20,w} \equiv \text{Li}_w(\rho^{20}) - 2^{w-1} \text{Li}_w(\rho^{10}) - 5^{w-1} 3 \text{Li}_w(\rho^4) \tag{31}$$

$$L_{24,w} \equiv \text{Li}_w(\rho^{24}) + 2^{w-1} \text{Li}_w(\rho^{12}) - 3^{w-1} 2 \text{Li}_w(\rho^8) - 2^{2w-3} 7 \text{Li}_w(\rho^6) \tag{32}$$

whose rule of construction is best appreciated at  $w = 1$ , where mere algebra proves that  $-2L_{n,1}/\log(\rho) = 4, 2, -2, -1, -2, -1$ , for  $n = 1, 2, 6, 12, 20, 24$ , respectively. Thus, for example, the terms with  $\rho$  raised to powers  $p < 24$  in the sixth case (32), with  $n = 24$ , are included so as to exploit the identity

$$\rho^2(1 - \rho^{24})^2(1 - \rho^{12})^2 = (1 - \rho^8)^4(1 - \rho^6)^7 \tag{33}$$

which is an algebraic consequence of the defining property  $\rho^2 = 1 - \rho$  of the golden section. The generalization to  $w > 1$  is given by the simple device of multiplying the required coefficient of  $\text{Li}_1(\rho^p)$  in  $L_{n,1}$  by  $(n/p)^{w-1}$ . Since all six elements of  $\mathcal{S}_w$  have the required property at  $w = 1$ , so does any  $\mathbb{Q}$ -linear combination of them. I have chosen to include as few terms as possible in the definitions, omitting divisors  $p|n$  where  $p$  is small enough to be covered by a previous definition.

From this simple game, at  $w = 1$ , we progress to a wonderful result for the dilogarithms at  $w = 2$ , where all 6 elements of  $\mathcal{S}_2$  are rational combinations of  $\pi^2$  and  $(\log(\rho))^2$ . So it is natural to enquire what happens for  $w > 2$ .

## 5.2 Two insignificant departures from the ladder party

Let  $C_1 = 6$  and, for  $w > 1$ , let  $C_w$  be the number of independent  $\mathbb{Z}$ -linear combinations of elements of  $\mathcal{S}_w$  that are reducible to  $\zeta(w)$ , modulo products of polylogarithms of lesser weight. Then we know that  $C_2 = 6$ , because no dilogarithm leaves the ladder party.

At weight  $w = 3$ , we know that  $C_3 < 6$ , because  $\text{Li}_3(\rho) \equiv Z(A^2G)$  and  $\zeta(3) \equiv Z(A^2B)$  are independent primitive MLVs, or at least appear to be so at 3000-digit precision. On the other hand, Landen proved that  $\text{Li}_3(\rho^2) \equiv Z(A^2F) \simeq \frac{4}{5}Z(A^2B)$  remains at Lewin's ladder party. Even

better, there are  $C_3 = 5$  independent combinations of trilogarithms in  $\mathcal{S}_3$  that are reducible to  $\zeta(3)$  and products.

At weight  $w = 4$ , we know that  $C_4 < 5$ , because  $Z(A^3F)$  and  $Z(A^3G)$  are independent primitives of the  $\{A, B, F, G\}$  alphabet and  $Z(A^3B)$  is a multiple of  $\pi^4$  and hence not a primitive. Again it is notable that  $C_4 = 4$  is as large as it could possibly be, given what we know about MLVs.

We may summarize the situation thus far by saying that for weights  $w < 5$  all 6 of the ladder combination evaluate as MLVs. Those polylogarithms that left the ladder were MLVs, by definition. Hence those that remain must, despite their appearance, be combinations of MLVs, for  $w < 5$ , since Lewin found that they are linearly related to those that departed.

### 5.3 A significant departure and arrival at weight 5

By numerical methods, Lewin determined the sequence for  $C_w$  as

6, 6, 5, 4, 3, 2, 2, 1, 1, 0...

for  $w = 1 \dots 10$ , with no ladder relation remaining for weights  $w > 9$ . He charted the departures in Figure 4.1 on page 52 of [33], which shows, in the present notation, that  $L_{6,5}$  leaves his party, because no  $\mathbb{Z}$ -linear combination of

$$L_{6,5} \equiv \text{Li}_5(\rho^6) - 32 \text{Li}_5(\rho^3) = -32 \sum_{n>0} \frac{\rho^{6n-3}}{(2n-1)^5} \quad (34)$$

and the MLV polylogarithms  $\text{Li}_5(\rho^p)$ , with  $p \in \{0, 1, 2\}$ , was found by him to be reducible to products of polylogarithms of lesser weight.

This departee from Lewin's realm of ladder polylogarithms is greeted with joy at the pearly gates of MLV-land, where  $L_{6,5}$  is crowned in glory as a primitive MLV of depth 2. The list of primitives systematically accumulated in Subsection 4.5 offers  $L_{6,5}$  a home as proxy for  $Z(A^3BF) = \text{Li}_{4,1}(1, \rho^2)$ , since numerical investigation quickly confirms that

$$L_{6,5} + \frac{165}{2} \text{Li}_{4,1}(1, \rho^2) \simeq 165 \zeta(5) - 216 \text{Li}_5(\rho) + \frac{27}{4} \text{Li}_5(\rho^2) \quad (35)$$

reduces to  $\text{Li}_5(\rho^p)$  with  $p \in \{0, 1, 2\}$ , modulo products of terms of lesser weight that I here omit but are in the MLV datamine. Metaphor apart, it is quite remarkable that, at the precise point where a depth-2 primitive

MLV is first required, the depth-1 sum in (34) appears as its proxy. This discovery seemed to me to be a fine compensation for the failure of MLVs to close under stuffles and led me to a wider conjecture.

#### 5.4 A conjecture for ladder combinations at all weights

**Conjecture 6:** For every weight  $w$ , the elements (27) to (32) of  $\mathcal{S}_w$  are  $\mathbb{Q}$ -linear combinations of the tribonacci number  $T_w$  of independent MGVs.

Combining Conjectures 2 and 6 with Lewin's list of departures, we predict that the depth-1 sums  $L_{6,6}$  and  $L_{12,6}$  can stand as proxies for two combinations of the weight-6 depth-2 primitive words  $A^4FB$ ,  $A^4GB$  and  $A^4GF$ . Numerical computation confirms this and also shows that  $L_{12,6}$  is absent from  $A_6$ , whose primitive part contains only *odd* powers of  $\rho$ :

$$A_6 = Z((2A + G)^4G^2) \simeq \frac{32}{99} \sum_{n>0} \frac{81\rho^{2n-1} - 4\rho^{6n-3}}{(2n-1)^6}. \quad (36)$$

At  $w = 7$ , there is no new arrival. As predicted,  $L_{6,7}$  and  $L_{12,7}$  stand as proxies for primitive MLVs with  $w = 7$  and  $d = 2$ .

At  $w = 8$ , Lewin observed another departure, leaving only one ladder combination. Here the situation becomes rather interesting, since my method determined that 3 of the 15 primitive weight-8 MLVs have depth  $d = 3$ . The new arrival  $L_{20,8}$  combines with  $L_{12,8}$  and  $L_{6,8}$ , giving proxies for one primitive at  $d = 3$  and two at  $d = 2$ .

Thus Conjecture 6 is neatly confirmed, at 3000-digit precision, for  $w < 9$ , by reductions of all 6 of the ladder combinations, at each of those 8 weights, to a datamine basis of tribonacci dimensionality  $T_w$ .

At  $w = 9$ , there can be no new arrival, since Lewin observed no departure.

At  $w = 10$ , the last departure occurs: no ladder combination survives. Thus Conjecture 6 predicts that  $L_{n,10}$ , with  $n \in \{6, 12, 20, 24\}$ , will serve as proxies for 4 weight-10 primitive MLVs of depth  $d > 1$ . I cannot determine how they distribute themselves by depth, since  $T_{10} = 274$  is too large a basis size for me to handle, with current methods.

## 6 Comparisons with roots of unity in physics

The quantum field theory of the standard model of particle physics leads to Feynman diagrams that define integrals whose evaluation often yields multiple polylogarithms of the type  $\text{Li}_{a_1, a_2, \dots, a_d}(z_1, z_2, \dots, z_d)$  defined in (2).

When there is a single large scale in the problem, set by a large external energy or a large internal mass, the neglect of smaller physical quantities, such as the masses of light quarks, leads to arguments  $z_j$  that are algebraic numbers. These are often roots of unity [12], with  $z_j^N = 1$  for some modest value of  $N$ , with  $N = 1, 2, 6$  being prominent. Hence a great deal of effort has been expended on trying to understand structural properties of iterated integrals defined by words in alphabets with the letter  $A = d \log(x)$  and other letters of the form  $-d \log(1 - z_j x)$  with  $z_j^N = 1$ . These have a shuffle algebra, but not necessarily a stuffle algebra, for which one needs to include all of the  $N$ th roots of unity, while the physics may require only a subset.

There has been fruitful dialogue between highly focused physicists, who need to compute such numbers, algebraic geometers, who are interested in the parametric integrands that produce them, and number theorists, who are interested in the periods that result [15]. Thus it is that I have had the privilege of interaction with people like Spencer Bloch, Francis Brown, Pierre Cartier, Alain Connes, Pierre Deligne, Sasha Goncharov and Don Zagier, who have patiently instructed me in issues of importance to mathematicians and courteously attended to accounts from the frontier of concrete calculation that is an imperative for the standard model of particle physics.

### 6.1 The most puzzling root of unity

From the perspective of iterated integrals, the most puzzling root of unity is unity itself. The case  $N = 1$  gives MZVs and for these the infamous [17] Broadhurst-Kreimer (BK) conjecture [14] provides a bizarre answer to what seems to be a very simple question: how many independently primitive MZVs in the  $\{A, B\}$  alphabet are there at a given weight and depth? The

conjectured answer,  $N_{w,d}$ , is generated by

$$\prod_{w>2} \prod_{d>0} (1 - x^w y^d)^{N_{w,d}} = 1 - \frac{x^3 y}{1 - x^2} + \frac{x^{12} y^2 (1 - y^2)}{(1 - x^4)(1 - x^6)} \quad (37)$$

which Dirk Kreimer and I proposed, nearly 20 years ago, and has been tested, with considerable ferocity, by numerical and exact [6] methods.

It is an intriguing fact that the expansion of  $x^{12}/((1 - x^4)(1 - x^6))$  enumerates cuspforms of the fundamental modular group. Here I refer the reader to [17, 18] and wish only to observe that setting  $y = 1$  in (37) we obtain the claim that the vector-space dimensions are Padovan numbers, generated by  $1/(1 - x^2 - x^3)$ , in accord with Hoffman's conjectured [34] vector-space basis of finite MZVs with words that contain neither  $A^3$  nor  $B^2$ . It has been proven [16] by Brown, with inspired assistance from Zagier, that Hoffman's conjectural basis is functional; no-one knows how to prove that the Padovan upper bound is tight. Hoffman's basis for MZVs, like that in Conjecture 3 for MLVs, is very inefficient, because of its large denominator primes. The depth-filtered basis of the MZV datamine [6] is far preferable.

Parallels between MZVs and MLVs in the  $\{A, B, F, G\}$  alphabet are clear. There is good support for my conjectures that  $1/(1 - x - x^2 - x^3)$  generates the vector-space dimensions for MLVs and that primitives are provided by Lyndon words in  $\{A, G\}$  that do not contain  $G^3$ . However I have no idea, in general, of how primitives may be filtered by depth. Subsection 4.5 gives the data for  $w < 9$ , where 3 depth-3 primitives appear at  $w = 8$ , of which one may be replaced by depth-1 sums in a ladder combination.

## 6.2 A well behaved root of unity

Minus one is a very well behaved root of unity. The case  $N = 2$ , for the alphabet  $\{A, B, C\}$  with  $C = -d \log(1 + x)$ , gives alternating sums and obviously includes MZVs. Alternating sums were mastered, conjecturally at least, before the BK conjecture, by my claim [9] that the generating function  $1 - xy/(1 - x^2)$  tells us everything about the filtration of primitives by both weight and depth. I proposed to Deligne an empirically viable set of primitives: Lyndon words in  $\{A^2, C\}$ , which he later proved [22] valid at all weights. My study of  $\{A, B, C\}$  began by observing that the Fibonacci

numbers, generated by  $1/(1 - x - x^2)$ , fitted the data on dimensions. It was a question from one my sons, Stephen, then aged about 15, on how to obtain the Fibonacci numbers from Pascal's triangle, that led from an enumeration by weight to filtration by both depth and weight.

There are striking parallels with the results for MLVs, found here. Instead of the Fibonacci sequence, the MLVs appear to follow the tribonacci sequence, generated by  $1/(1 - x - x^2 - x^3)$ . If we strike out  $x$  we enumerate MZVs; if we strike out  $x^3$ , we enumerate alternating sums.

Moreover, the MGVs of the  $\{A, G\}$  sub-alphabet appear to span the vector space of the full  $\{A, B, F, G\}$  alphabet of MLVs, so that we get 4 letters for the price of 2, which is more economical than my old finding that  $\{A, C\}$  spans  $\{A, B, C\}$ , which gave the less enticing offer of 3 for the price of 2.

### 6.3 The primitive sixth root of unity

Let  $D = -\log(1 - \lambda x)$ , where  $\lambda = (1 + \sqrt{-3})/2$  is a primitive sixth root of unity. Then  $1 = \lambda + 1/\lambda$  and setting  $y = 1/\lambda$  in (3) we obtain

$$Z(W) = \sum_{W=UV} L(\tilde{U}, \lambda)L(\tilde{V}, \lambda) = Z(\tilde{W}) \quad (38)$$

since  $L(V, 1/\lambda) = L(\tilde{V}, \lambda)$ . Recalling that  $\tilde{V}(A, B) \equiv V(A + B, -B)$ , we prove that every MZV is a  $\mathbb{Z}$ -linear combination of iterated integrals in the  $\{A, D\}$  alphabet. With  $W = AB$  we obtain  $\zeta(2) = -3Z(DD)$  and with  $W = AAB$  we have two evaluations of  $\zeta(3) = Z(A^2B) = Z(AB^2)$  and hence obtain an integer relation between weight-3 words in  $\{A, D\}$ , namely  $Z((2A^2 + AD + DA + 11D^2)D) = 0$ , with  $\zeta(3) = 3Z((A^2 + 5D^2)D)$ . Note that (38) reduces  $\zeta(5, 3)$  to the  $\{A, D\}$  alphabet, in contrast with the impasse for  $\{A, G\}$  at weight 8, which limited Theorem 3 in Section 3.

Jonathan Borwein, Joel Kamnitzer and I conjectured [8] that the Fibonacci numbers enumerate the vector-space dimensions of the  $\{A, D\}$  alphabet and Deligne later proved [22] that these are upper bounds for the  $\{A, B, D\}$  alphabet, with functional primitives provided by Lyndon words in  $\{A, D\}$  that do not contain  $D^2$ . A better choice for weights  $w < 12$  is given in my datamine [11] for the multiple Deligne values (MDVs) in  $\{A, B, D\}$ .

There is a significant parallel between MDVs, based on  $1 = \lambda + 1/\lambda$ , and MLVs, based on  $1 = \rho + \rho^2$ . In neither case does the stuffle algebra close.



Yet in each we have a very simple set of conjectural primitives: Deligne omits Lyndon words in  $\{A, D\}$  that contain  $D^2$ ; Conjecture 5 asserts, *inter alia*, that in  $\{A, G\}$  we may omit Lyndon words that contain  $G^3$ .

In the Deligne case, the generating function  $1 - x^2y/(1 - x)$  tells us everything about the filtration of primitives by both weight and depth. I am unable to go from  $1/(1 - x - x^2 - x^3)$  to a simple two-variable generating function that accounts for the data on depths in Subsection 4.5, or for its pattern after modification by ladder combinations.

## 6.4 Boring roots of unity

The cases  $N = 3, 4, 8$  were also mastered by Deligne [22]. Here there is little of interest, by way of structure. The enumeration of dimensions is trivial:  $2^w$  for  $N = 3, 4$  and  $3^w$  for  $N = 8$ . Primitives are supplied by Lyndon words in  $\{A, -d \log(1 - \lambda^2 x)\}$  for  $N = 3$ , in  $\{A, -d \log(1 - ix)\}$  for  $N = 4$  and in  $\{A, -d \log(1 - \sqrt{i}x), -d \log(1 + \sqrt{i}x)\}$  for  $N = 8$ , with no known relations between such Lyndon words, for a given value of  $N$ . So far, particle physicists have had no need of these lacklustre cases.

## 6.5 The full 7-letter alphabet for sixth roots of unity

Here the full alphabet is  $\{A, B, C, D, \bar{D}, E, \bar{E}\}$ , where  $E = -d \log(1 - \lambda^2 x)$  and bars denote complex conjugation. Filtration of primitives by weight and depth is generated by  $1 - xy - xy/(1 - x)$  which at  $y = 1$  gives the vector-space dimensions as Fibonacci numbers with even indices:

1, 3, 8, 21, 55, 144, 377, 987, 2584, 6765, 17711, 46368...

for  $w = 1 \dots 12$ . These are generated by  $1/(1 - 3x + x^2)$ . Thus at weight  $w = 6$  we have a Fibonacci dimension  $(\rho^{-12} - \rho^{12})/\sqrt{5} = 144$  that is the same as that for  $w = 11$  in the  $\{A, B, D\}$  sub-alphabet of MDVs. I have given in [12] a conjecture for the primitives, using a well-defined subset of Lyndon words in  $\{A, E, C\}$ , namely those which do not contain  $AC$ .

The full alphabet was studied in [10], where it was needed for evaluating Feynman diagrams dominated by a large mass. The sub-alphabet  $\{A, B\}$  is relevant to Feynman diagrams [13, 14] dominated by a large external energy, as in electron-positron collisions producing hadrons.  $\{A, B, C\}$  is

of the essence [9] for the magnetic moment of the electron, where the same mass appears externally and internally. In a study of a 7-loop diagram [37] that contributes to the running of the self-coupling of the Higgs boson, Panzer encountered the sub-alphabet  $\{A, D, \overline{E}\}$  but found an empirical reduction to  $\{A, D\}$ , with large denominator primes, which I cleaned up in [11], using primitives in  $\{A, B, D\}$  that are more practical than Deligne's.

## 6.6 Other roots of unity

Deligne was silent in [22] on the cases  $N = 5$  and  $N = 7$ . I reported in [12] conjectural enumerations of double sums in those cases, but failed to arrive at overall conclusions. The present study came from the fact that the golden section  $\rho = 2 \sin(\pi/10)$  interacts strongly with the  $N = 5$  problem. For  $w < 6$ , `linddep` gives MLVs as  $\mathbb{Q}$ -linear combinations of iterated integrals in the 6-letter alphabet formed by  $d \log(x)$  and  $-d \log(1 - \exp(2n\pi i/5)x)$  for  $n = 0 \dots 4$ . Yet I am loath to elevate this observation to a conjecture for all weights. Even if it were the case, it would not be of much use. For example, we know that for  $w < 6$  there is only one primitive MLV with depth  $d > 1$  to consider, namely  $Z(A^3BF) \equiv \text{Li}_{4,1}(1, \rho^2)$ , or its proxy,  $L_{6,5} \equiv \text{Li}_5(\rho^6) - 32\text{Li}_5(\rho^3)$ . I determined that the 5th-root alphabet gives vector spaces with empirical dimensions 3, 8, 22, 61 and 168, for  $w = 1 \dots 5$ , and that  $L_{6,5}$  reduces to 5th-root words at  $w = 5$ , where the MLV dimension  $T_5 = 13$  is much smaller. It is hard to see what is gained by embedding a 13-dimensional problem in a 168-dimensional problem.

## 7 Conclusions

This investigation turned out better than I had dared to hope.

1. Empirically, MLVs enumerate very simply by weight, following the tribonacci sequence generated by  $1/(1 - x - x^2 - x^3)$ . Striking out  $x$ , one has the Padovan enumeration of that subset of MLVs that are MZVs, well known to physicists. Striking out  $x^3$  one gets the Fibonacci numbers that appear in 3 other enumerations: for alternating sums, for MDVs, and (with even indices) for the full alphabet of sixth roots of unity, all of which figure in physics applications.

2. A credible filtration of primitive MLVs by both depth and weight seems to be much harder to guess. One has only to look at the BK conjecture (37) for MZVs to imagine how long it might take to accumulate sufficient MLV data to tackle this problem empirically.
3. A datamine is available, on request to the author, with reductions of all MLVs with weight  $w < 9$  to a systematically constructed basis that results in 3,357,257 coefficients of rational reduction, without dividing by any prime greater than 11. Then numerical data for merely 40 primitives enable very fast evaluation of all of these 49,151 MLVs to 20,000 digits. A Hoffman-type basis that naively omits words containing  $A^3$  would have introduced large denominator-primes.
4. Apéry's sums,  $A_w = \sum_{n>0} (-1)^{n+1} n^{-w} / \binom{2n}{n}$ , with  $w > 1$ , have proven reductions to MLVs. For  $w < 6$  they reduce to depth-1 sums and their products;  $A_6$  is conjectured to do so, in (36).
5. The available evidence agrees with the conjecture that Lewin's ladder combinations remain MLVs for ever. When they leave his ladder, by failing to reduce to depth-1 MLVs, they provide proxies for primitive MLVs of greater depth. This compensates for the failure of the stuffle algebra to close in the case of MLVs.
6. It is not clear whether there are corresponding phenomena in other algebraic number fields, such the Lehmer field [20] that sets the record [5] for the longest known ladder, reaching up to  $\zeta(17)$ .
7. It would be good to have a proof of the reduction (35), which may be accessible by combining stuffles that leave the MLV alphabet with generalized doubling relations of the type used in [6].

**Note added:** On reading this paper, Pierre Deligne sent me an interesting letter [23], suggesting a possible geometric origin of relations between MLVs, based on 5 points  $\{\infty, 0, 1, \rho^{-1}, \rho^{-2}\} \subset \mathbb{P}^1$ . His intuition leads to the symmetry group of a dodecahedron on a compactified Euclidean plane. To my untrained ear, this resonates with Coxeter's Section 8, *Regular polyhedra inscribed in the hyperbolic absolute*, in the work [21], which fuelled Lewin's interest [30] in dilogarithms  $\text{Li}_2(\rho^p)$ , with  $p > 2$ , and hence mine.

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## References

- [1] J. Ablinger, J. Blümlein, C.G. Raab, C. Schneider, *Iterated binomial sums and their associated iterated integrals*, J. Math. Phys., **55** (2014) 112301, [arXiv:1407.1822](#) .
- [2] M. Abouzahra, L. Lewin, *The polylogarithm in algebraic number fields*, J. Number Theory, **21** (1985) 214–244.
- [3] R. Apéry, *Irrationalité de  $\zeta(2)$  et  $\zeta(3)$* , Astérisque, **61** (1979) 11–13.
- [4] R. Askey, *Polylogarithms and associated functions*, by Leonard Lewin, Bull. Amer. Math. Soc., **6** (1982) 248–251.
- [5] D.H. Bailey, D.J. Broadhurst, *A seventeenth-order polylogarithm ladder*, [arXiv:math/9906134](#) .
- [6] J. Blümlein, D.J. Broadhurst, J.A.M. Vermaseren, *The multiple zeta value data mine*, Comput. Phys. Commun., **181** (2010) 582–625, [arXiv:0907.2557](#) .
- [7] J.M. Borwein, D.M. Bradley, D.J. Broadhurst, P. Lisonek, *Special values of multiple polylogarithms*, Trans. Amer. Math. Soc., **353** (2001) 907–941, [arXiv:math/9910045](#) .
- [8] J.M. Borwein, D.J. Broadhurst, J. Kamnitzer *Central binomial sums, multiple Clausen values and zeta values*, Exper. Math., **10** (2001) 25–34, [arXiv:hep-th/0004153](#) .

- [9] D.J. Broadhurst, *On the enumeration of irreducible  $k$ -fold Euler sums and their roles in knot theory and field theory*, arXiv:hep-th/9604128 .
- [10] D.J. Broadhurst, *Massive 3-loop Feynman diagrams reducible to  $SC^*$  primitives of algebras of the sixth root of unity*, Eur. Phys. J., **C8** (1999) 311-333, arXiv:hep-th/9803091 .
- [11] D. Broadhurst, *Multiple Deligne values: a data mine with empirically tamed denominators*, arXiv:1409.7204 .
- [12] D. Broadhurst, *Polylogs of roots of unity: the good, the bad and the ugly*, talk at meeting on *Mathematical physics, number theory and non-commutative geometry*, Vienna, 12 March 2015, <http://www.noncommutativegeometry.nl/esi2015/slides/> .
- [13] D.J. Broadhurst, D. Kreimer, *Knots and numbers in  $\phi^4$  theory to 7 loops and beyond*, Int. J. Mod. Phys., **C6** (1995) 519–524, arXiv:hep-ph/9504352 .
- [14] D.J. Broadhurst, D. Kreimer, *Association of multiple zeta values with positive knots via Feynman diagrams up to 9 loops*, Phys. Lett., **B393** (1997) 403–412, arXiv:hep-th/9609128 .
- [15] D. Broadhurst, O. Schnetz, *Algebraic geometry informs perturbative quantum field theory*, Proc. Sci., **211** (2014) 078, arXiv:1409.5570 .
- [16] F. Brown, *Mixed Tate motives over  $\mathbb{Z}$* , Annals of Mathematics, **175** (2012) 949–976, arXiv:1102.1312 .
- [17] F. Brown, *Depth-graded motivic multiple zeta values*, arXiv:1301.3053 .
- [18] F. Brown, *Zeta elements in depth 3 and the fundamental Lie algebra of a punctured elliptic curve*, arXiv:1504.04737 .
- [19] A.M. Clerke, *Landen, John (1719–1790)*, entry in *Oxford Dictionary of National Biography*, Oxford University Press, 2004, <http://www.oxforddnb.com/view/article/15973> .
- [20] H. Cohen, L. Lewin, D. Zagier, *A sixteenth-order polylogarithm ladder*, Exper. Math., **1** (1992) 25–34.

- [21] H.S.M. Coxeter, *The functions of Schläfli and Lobatschewsky*, Quart. J. Math., Oxford, **6** (1935) 13–29.
- [22] P. Deligne, *Le groupe fondamental unipotent motivique de  $\mathbf{G}_m - \mu_N$  pour  $N = 2, 3, 4, 6$  ou  $8$* , Publications Mathématiques de l’IHÉS, **112** (2010) 101–141.
- [23] P Deligne, handwritten letter to the author on 22 April 2015, with figure, `Broadhurst_22_avril_2015.pdf` at <http://physics.open.ac.uk/~dbroadhu/cert/> .
- [24] C.J. Hill, *Über die Integration logarithmisch-rationaler Differentiale*, Journal für die reine und angewandte Mathematik, **3** (1828) 101-159.
- [25] C.J. Hill, *Specimen exercitii analytici, functionum integralum  $\int_0^x \frac{dx}{x} L(1 + 2xC\alpha + x^2) = D^\alpha x$  tum quoad amplitudinem, tum quoad modulum comparandi modum exhibentis*, Academia Carolina, Lund, 1830.
- [26] J. Landen, *A new method of computing the sums of certain series*, Phil. Trans. Roy. Soc., **51** (1759) 553–565.
- [27] J. Landen, *Mathematical memoirs respecting a variety of subjects*, Volume 1, Nourse, London, 1780.
- [28] L. Lewin, *Dilogarithms and associated functions*, Macdonald, London, 1958.
- [29] L. Lewin, *Polylogarithms and associated functions*, Elsevier, New York, 1981.
- [30] L. Lewin, *The dilogarithm in algebraic fields*, J. Austral. Math. Soc., **A33** (1982) 302–330.
- [31] L. Lewin, *The inner structure of the dilogarithm in algebraic fields*, J. Number Theory, **19** (1984) 345–373.
- [32] L. Lewin, *The order-independence of the polylogarithmic ladder structure – implications for a new category of functional equations*, *Æquationes Mathematicæ*, **30** (1986) 1–20.
- [33] L. Lewin, *Structural properties of polylogarithms*, American Mathematical Society, Providence, Rhode Island, 1991.

- [34] M.E. Hoffman, *The algebra of multiple harmonic series*, J. Algebra, **194** (1997) 477–495.
- [35] OEIS Foundation Inc. (2011), *On-Line encyclopedia of integer sequences*, <http://oeis.org> .
- [36] PARI Group, PARI/GP version 2.5.0, Bordeaux, 2011, <http://pari.math.u-bordeaux.fr/> .
- [37] E. Panzer, *Feynman integrals via hyperlogarithms*, <http://arxiv.org/abs/1407.0074> .
- [38] A. van der Poorten, *A proof that Euler missed... Apéry's proof of the irrationality of  $\zeta(3)$* , Math. Intelligencer, **1** (1979) 195–203.
- [39] G.N. Watson, *The marquis and the land-agent; a tale of the eighteenth century*, Mathematical Gazette, **17** (1933) 5–17, <http://www.jstor.org/discover/10.2307/3607944> .