A STRATEGY OF NUMERIC SEARCH FOR PERFECT CUBOIDS IN THE CASE OF THE SECOND CUBOID CONJECTURE.

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Abstract. A perfect cuboid is a rectangular parallelepiped whose edges, whose face diagonals, and whose space diagonal are of integer lengths. The problem of finding such cuboids or proving their non-existence is not solved thus far. The second cuboid conjecture specifies a subclass of perfect cuboids described by one Diophantine equation of tenth degree and claims their non-existence within this subclass. Regardless of proving or disproving this conjecture in the present paper the Diophantine equation associated with it is studied and is used in order to build an optimized strategy of computer-assisted search for perfect cuboids within the subclass covered by the second cuboid conjecture.

1. Introduction.

For the history and various approaches to the problem of perfect cuboids the reader is referred to [1–42]. In this paper we resume the research initiated in [43–47]. The papers [48–60] deal with another approach based on so-called multisymmetric polynomials. In this paper we do not touch this approach.

Perfect cuboids are described by six Diophantine equations. These equations are immediate from the Pythagorean theorem:

$$\begin{aligned} &x_1^2+x_2^2+x_3^2-L^2=0, && x_2^2+x_3^2-d_1^2=0,\\ &x_3^2+x_1^2-d_2^2=0, && x_1^2+x_2^2-d_3^2=0. \end{aligned} \tag{1.1}$$

The variables x_1, x_2, x_3 in (1.1) stand for three edges of a cuboid, the variables d_1, d_2, d_3 correspond to its face diagonals, and L represents its space diagonal.

In [43] an algebraic parametrization for the Diophantine equations (1.1) was suggested. It uses four rational variables α, β, υ, and z:

$$\begin{aligned} &\frac{x_1}{L}=\frac{2\,\upsilon}{1+\upsilon^2}, && \frac{d_1}{L}=\frac{1-\upsilon^2}{1+\upsilon^2},\\ &\frac{x_2}{L}=\frac{2\,z\,(1-\upsilon^2)}{(1+\upsilon^2)\,(1+z^2)}, && \frac{x_3}{L}=\frac{(1-\upsilon^2)\,(1-z^2)}{(1+\upsilon^2)\,(1+z^2)},\\ &\frac{d_2}{L}=\frac{(1+\upsilon^2)\,(1+z^2)+2\,z(1-\upsilon^2)}{(1+\upsilon^2)\,(1+z^2)}\,\beta, && \frac{d_3}{L}=\frac{2\,(\upsilon^2\,z^2+1)}{(1+\upsilon^2)\,(1+z^2)}\,\alpha. \end{aligned} \tag{1.2}$$

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The variables α, β, υ in (1.2) are different from the original ones which are used in [43], here we use α and β instead of a and b, and we use υ instead of u.

Only two of the four variables α, β, υ, and z are independent. The variables α and β are taken for independent ones. Then the variable υ is expressed through α and β as a solution of the following algebraic equation:

$$\begin{gathered}
\upsilon^4\,\alpha^4\,\beta^4+(6\,\alpha^4\,\upsilon^2\,\beta^4-2\,\upsilon^4\,\alpha^4\,\beta^2-2\,\upsilon^4\,\alpha^2\,\beta^4)+(4\,\upsilon^2\,\beta^4\,\alpha^2+\\
+4\,\alpha^4\,\upsilon^2\,\beta^2-12\,\upsilon^4\,\alpha^2\,\beta^2+\upsilon^4\,\alpha^4+\upsilon^4\,\beta^4+\alpha^4\,\beta^4)+(6\,\alpha^4\,\upsilon^2+6\,\upsilon^2\,\beta^4-\\
-8\,\alpha^2\,\beta^2\,\upsilon^2-2\,\upsilon^4\,\alpha^2-2\,\upsilon^4\,\beta^2-2\,\alpha^4\,\beta^2-2\,\beta^4\,\alpha^2)+(\upsilon^4+\beta^4+\\
+\alpha^4+4\,\alpha^2\,\upsilon^2+4\,\beta^2\,\upsilon^2-12\,\beta^2\,\alpha^2)+(6\,\upsilon^2-2\,\alpha^2-2\,\beta^2)+1=0.
\end{gathered}\tag{1.3}$$

Once the variable υ is expressed as a function of α and β by solving the equation (1.3), the variable z is given by the formula

$$z=\frac{(1+\upsilon^2)\,(1-\beta^2)\,(1+\alpha^2)}{2\,(1+\beta^2)\,(1-\alpha^2\,\upsilon^2)}.\tag{1.4}$$

The equation (1.3), along with the formula (1.4), produces two algebraic functions

$$\upsilon=\upsilon(\alpha,\beta), \qquad\qquad z=z(\alpha,\beta).\tag{1.5}$$

Substituting (1.5) into (1.2), we get six algebraic functions

$$\begin{aligned}
&x_1=x_1(\alpha,\beta,L), &&d_1=d_1(\alpha,\beta,L),\\
&x_2=x_2(\alpha,\beta,L), &&d_2=d_2(\alpha,\beta,L),\\
&x_3=x_3(\alpha,\beta,L), &&d_3=d_3(\alpha,\beta,L),
\end{aligned}\tag{1.6}$$

which are linear with respect to L. The functions (1.6) satisfy the cuboid equations (1.1) identically with respect to α, β, and L. This fact is presented by the following theorem (see Theorem 5.2 in [43]).

Theorem 1.1. *A perfect cuboid does exist if and only if there are three rational numbers α, β, and υ satisfying the equation (1.3) and obeying four inequalities $0<\alpha<1$, $0<\beta<1$, $0<\upsilon<1$, and $(\alpha+1)\,(\beta+1)>2$.*

The rational numbers α, β, and υ cam be brought to a common denominator:

$$\alpha=\frac{a}{t}, \qquad\qquad \beta=\frac{b}{t}, \qquad\qquad \upsilon=\frac{u}{t}.\tag{1.7}$$

Substituting (1.7) into (1.3), one easily derives the Diophantine equation

$$\begin{gathered}
t^{12}+(6\,u^2\,-2\,a^2\,-2\,b^2)\,t^{10}+(u^4\,+b^4+a^4+4\,a^2\,u^2+\\
+4\,b^2\,u^2-12\,b^2\,a^2)\,t^8+(6\,a^4\,u^2+6\,u^2\,b^4-8\,a^2\,b^2\,u^2-\\
-2\,u^4\,a^2-2\,u^4\,b^2-2\,a^4\,b^2-2\,b^4\,a^2)\,t^6+(4\,u^2\,b^4\,a^2+\\
+4\,a^4\,u^2\,b^2-12\,u^4\,a^2\,b^2+u^4\,a^4+u^4\,b^4+a^4\,b^4)\,t^4+\\
+(6\,a^4\,u^2\,b^4-2\,u^4\,a^4\,b^2-2\,u^4\,a^2\,b^4)\,t^2+u^4\,a^4\,b^4=0.
\end{gathered}\tag{1.8}$$

Theorem 1.1 then is reformulated in the following form (see Theorem 4.1 in [44]).

Theorem 1.2. *A perfect cuboid does exist if and only if for some positive coprime integer numbers a, b, and u the Diophantine equation* (1.8) *has a positive solution t obeying the inequalities $t > a$, $t > b$, $t > u$, and $(a+t)\,(b+t) > 2\,t^2$.*

In [44] the Diophantine equation was treaded as a polynomial equation for t, while a, b, and u were considered as parameters. As a result in [44] several special cases of the equation (1.8) were specified. They are introduced through the following relationships for the parameters a, b, and u:

$$\begin{array}{lll} 1)\ \ a=b\neq u; & 3)\ \ b\,u=a^2; & 5)\ \ a=u\neq b;\\ 2)\ \ a=b=u; & 4)\ \ a\,u=b^2; & 6)\ \ b=u\neq a. \end{array}\tag{1.9}$$

The cases 2, 5, and 6 are trivial. They produce no perfect cuboids (see [44]). The case 1 corresponds to the first cuboid conjecture (see [44]). It is less trivial, but it produces no perfect cuboids either (see [45]). The cases 2 and 4 correspond to the second cuboid conjecture (see [44] and [46]). The case, where none of the conditions (1.9) is fulfilled, corresponds to the third cuboid conjecture (see [44] and [47]).

In this paper we consider the cases 3 and 4 associated with the second cuboid conjecture. In the case 3 the equality $b\,u = a^2$ is resolved by substituting

$$a = p\,q, \qquad b = p^2, \qquad u = q^2. \tag{1.10}$$

Here $p \neq q$ are two positive coprime integers. Upon substituting (1.10) into the equation (1.8) it reduces to the equation

$$(t-a)\,(t+a)\,Q_{pq}(t) = 0 \tag{1.11}$$

(see [45]), where $Q_{pq}(t)$ is the following polynomial of tenth degree:

$$\begin{aligned} Q_{pq}(t) &= t^{10} + (2\,q^2+p^2)\,(3\,q^2-2\,p^2)\,t^8 + (q^8+10\,p^2\,q^6+\\ &+4\,p^4\,q^4-14\,p^6\,q^2+p^8)\,t^6-p^2\,q^2\,(q^8-14\,p^2\,q^6+4\,p^4\,q^4+\\ &+10\,p^6\,q^2+p^8)\,t^4-p^6\,q^6\,(q^2+2\,p^2)\,(3\,p^2-2\,q^2)\,t^2-q^{10}\,p^{10}. \end{aligned}\tag{1.12}$$

The case 4 is similar. In this case the equality $a\,u = b^2$ is resolved by substituting

$$a = p^2, \qquad b = p\,q, \qquad u = q^2. \tag{1.13}$$

Upon substituting (1.13) into the equation (1.8) it reduces to the equation

$$(t-b)\,(t+b)\,Q_{pq}(t) = 0. \tag{1.14}$$

The roots $t = a$, $t = -a$, $t = b$, and $t = -b$ of the equations (1.11) and (1.14) do not produce perfect cuboids (see Theorem 1.2). Upon splitting off the linear factors from (1.11) and (1.14) we get the equation

$$\begin{aligned} &t^{10} + (2\,q^2+p^2)\,(3\,q^2-2\,p^2)\,t^8 + (q^8+10\,p^2\,q^6+4\,p^4\,q^4-\\ &-14\,p^6\,q^2+p^8)\,t^6-p^2\,q^2\,(q^8-14\,p^2\,q^6+4\,p^4\,q^4+10\,p^6\,q^2+\\ &+p^8)\,t^4-p^6\,q^6\,(q^2+2\,p^2)\,(3\,p^2-2\,q^2)\,t^2-q^{10}\,p^{10}=0. \end{aligned}\tag{1.15}$$

Conjecture 1.1. *For any positive coprime integers $p \neq q$ the polynomial $Q_{pq}(t)$ in (1.12) is irreducible in the ring $\mathbb{Z}[t]$.*

Conjecture 1.1 is known as the second cuboid conjecture. It was formulated in [44]. In particular it claims that the equation (1.15) has no integer roots for any positive coprime integers $p \neq q$. We do not try to prove or disprove Conjecture 1.1 in this paper. Instead, we study real positive roots of the equation (1.15) in the case where q is much larger than p. Using asymptotic expansions for the roots of the equation (1.15) as $q \to +\infty$, below we build an optimized strategy of computer-assisted search for perfect cuboids in the realm of Conjecture 1.1.

2. Asymptotic expansions for roots of the polynomial equation.

Note that the polynomial $Q_{pq}(t)$ in (1.12) is even. Along with each root t it has the opposite root $-t$. We use the condition

$$\begin{cases} t > 0 \ \text{ if } \ t \ \text{ is a real root,} \\ \operatorname{Re}(t) \geqslant 0 \ \text{ and } \ \operatorname{Im}(t) > 0 \ \text{ if } \ t \ \text{ is a complex root,} \end{cases} \tag{2.1}$$

in order to divide the roots of the equation (1.15) into two groups. We denote through t_1, t_2, t_3, t_4, t_5 the roots that obey the conditions (2.1). Then t_6, t_7, t_8, t_9, t_{10} are opposite roots of the equation (1.15):

$$t_6 = -t_1, \qquad t_7 = -t_2, \qquad t_8 = -t_3, \qquad t_9 = -t_4, \qquad t_{10} = -t_5. \tag{2.2}$$

Typically, asymptotic expansions for roots of a polynomial equation look like power series (see [61]). In our case we have the expansions

$$t_i(p, q) = C_i\, q^{\,\alpha_i} \left(1 + \sum_{s=1}^{\infty} \beta_{is}\, q^{-s}\right) \ \text{ as } \ q \to +\infty. \tag{2.3}$$

The coefficient C_i in (2.3) should be nonzero: $C_i \neq 0$.

Let's substitute (2.3) into the equation (1.15). For this purpose we represent the polynomial $Q_{pq}(t)$ from (1.12) formally as the sum

$$Q_{pq}(t) = \sum_{m=0}^{10} \sum_{r=0}^{10} A_{m\,r}(p)\, q^{\,r}\, t^{m}. \tag{2.4}$$

Each nonzero term in (2.4), i.e. a term $A_{m\,r}(p)\, q^{\,r}\, t^{m}$ with the nonzero coefficient

$$A_{m\,r}(p) \neq 0,$$

yields the sum

$$S_{m\,r}(p, q) = A_{m\,r}(p)\ C_i^{\,m}\, q^{\,m\,\alpha_i + r} \ \ + \sum_{s < m\,\alpha_i + r} \gamma_{irms}\, q^{s}. \tag{2.5}$$

Taking into account (2.5), the equation (1.15) is written as

$$\sum_{m=0}^{10} \sum_{r=0}^{10} S_{m\,r}(p, q) = 0. \tag{2.6}$$

The equality (2.6) should be fulfilled identically with respect to the variable $q \to +\infty$. Since $C_i \neq 0$, a necessary condition for that is the coincidence of exponents of at least two summands of the form $S_{m\,r}(q)$ in the leading order with respect to the variable q. This yields the equalities

$$m_1\,\alpha_i + r_1 = m_2\,\alpha_i + r_2 = s_{\max}. \tag{2.7}$$

The maximality of the exponent in (2.7) means that all exponents are not greater than $s_{\max}$, i. e. we have the following inequality:

$$m\,\alpha_i + r \leqslant s_{\max} \quad \text{for all} \quad r \quad \text{and} \quad m \quad \text{such that} \quad A_{m\,r}(p) \neq 0. \tag{2.8}$$

Lemma 2.1. *The coincidence $m_1 = m_2$ in the formula (2.7) is impossible.*

Proof. Indeed, due to (2.8) the coincidence $m_1 = m_2$ would mean $r_1 = r_2$. But the sum (2.6) has no two summands with simultaneously coinciding indices r and m. Lemma 2.1 is proved. □

Let's treat m and r as coordinates of a point on the coordinate plane. Since m and r are integer, such a point belongs to the integer grid, being its node. The

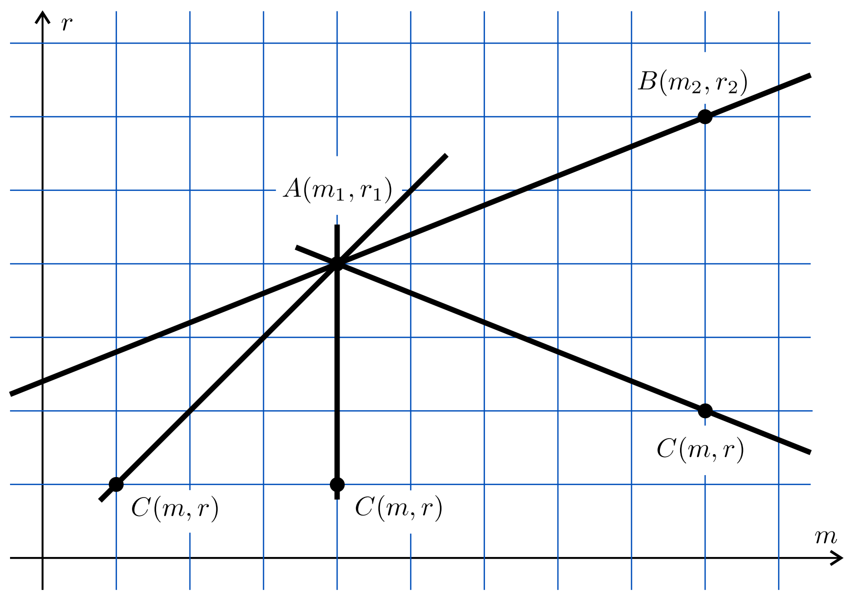

Fig. 2.1

numbers m_1, r_1 and m_2, r_2 from (2.7) mark two nodes of this grid. These are the points A and B in Fig. 2.1. Due to Lemma 2.1 from the equality (2.7) we derive the following formula for the exponent α_i:

$$\alpha_i = -\frac{r_2 - r_1}{m_2 - m_1}. \tag{2.9}$$

The right hand side of the formula (2.9) up to the sign coincides with the slope of the straight line connecting the nodes A and B in Fig. 2.1:

$$\alpha_i = -k_{AB}. \tag{2.10}$$

The nodes A and B correspond to some nonzero summands in the sum (2.4) being a formal presentation of the polynomial (1.12). They are selected by the maximality condition for the parameter $s = m\,\alpha_i + r$. The maximum is taken over all summands in the sum (2.4) for a fixed value of α_i.

Lemma 2.2. *The exponent α_i in the asymptotic expansion (2.3) is determined by the slope of a straight line connecting some two nodes of the integer grid associated with some two nonzero terms in the polynomial (2.4).*

Let C be some node of the integer grid in Fig. 2.1 associated with some nonzero summand of the sum (2.4) and different from the nodes A and B. Its coordinates m and r satisfy the inequality (2.8). From (2.7) and (2.8) one derives the inequality

$$m\,\alpha_i + r \leqslant m_1\,\alpha_i + r_1.$$

Let's write this inequality as follows:

$$\alpha_i\,(m - m_1) \leqslant -(r - r_1). \tag{2.11}$$

In Fig. 3.1 three positions of the node C relative to the node A are shown. The node C can be located to the left of the node A, to the right of the node A, or on the same vertical line with the node A. In the first case $m < m_1$. In the second case $m > m_1$. And finally, in the third case $m = m_1$.

In the first case, i. e. if $m < m_1$, from (2.11) we derive

$$\alpha_i \geqslant -\frac{r - r_1}{m - m_1}. \tag{2.12}$$

The right hand side of the inequality (2.12) up to the sign coincides with the slope of the line AC. Applying (2.10), we get the inequality $-k_{AB} \geqslant -k_{AC}$. Inverting signs, we write this inequality in the following form:

$$k_{AC} \geqslant k_{AB}. \tag{2.13}$$

In the second case, i. e. if $m > m_1$, from (2.11) we derive

$$\alpha_i \leqslant -\frac{r - r_1}{m - m_1}. \tag{2.14}$$

By analogy with (2.13) the inequality (2.14) is transformed to

$$k_{AC} \leqslant k_{AB}. \tag{2.15}$$

And finally, in the third case, i. e. if $m = m_1$, from the inequality (2.11) we derive

$$0 \leqslant -(r - r_1). \tag{2.16}$$

The inequality (2.16) is equivalent to the following inequality:

$$r \leqslant r_1. \tag{2.17}$$

Each of the inequalities (2.13), (2.15), and (2.17) in its case means that the point C is located not above the line AB. This fact is formulated as a lemma.

Lemma 2.3. *All nodes (m, r) of the integer grid associated with nonzero summands in the polynomial (2.4) are located not above the line AB on which the nodes implementing the maximum of the parameter $s = m\,\alpha_i + r$ are located.*

In order to apply Lemmas 2.1, 2.2, and 2.3 let's mark all of the nodes associated with the polynomial (1.12) on the coordinate plane.

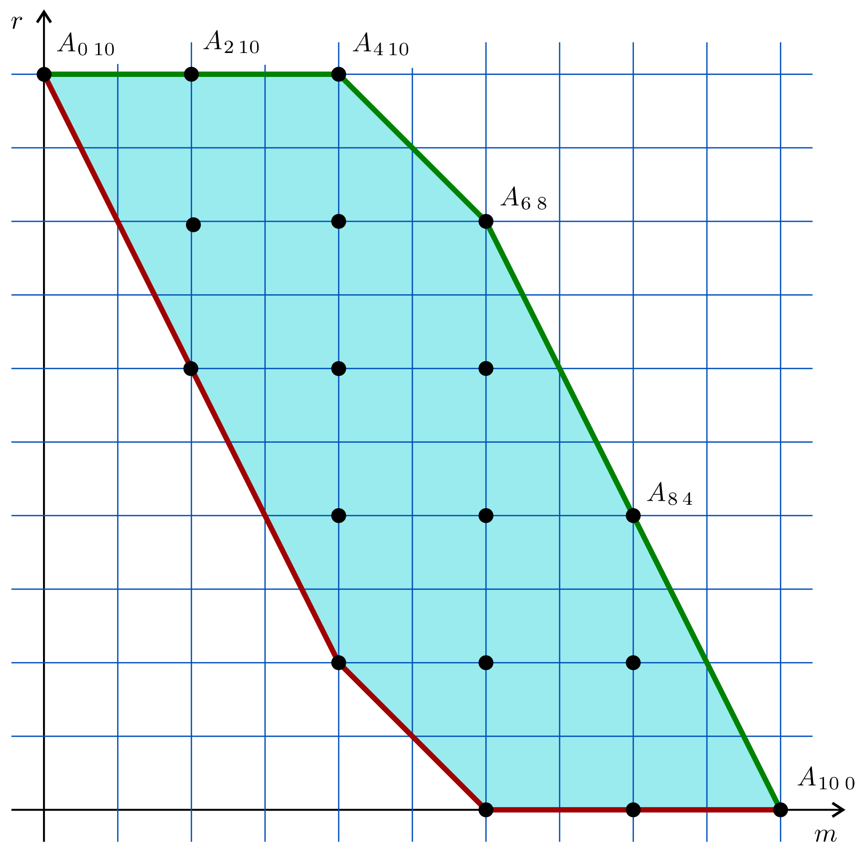

Fig. 2.2

Definition 2.1. For any polynomial of two variables $P(t, q)$ the convex hull of all integer nodes on the coordinate plane associated with monomials of this polynomial is called the Newton polygon of $P(t, q)$.

Remark. Note that in our case the polynomial (1.12) depend on three variables p, q, and t. However, we treat p as a parameter and consider $Q_{pq}(t)$ as a polynomial of two variables when applying Definition 2.1 to it.

The Newton polygon of the polynomial (1.12) is shown in Fig. 2.2. Its boundary consists of two parts — the upper part and the lower part. The upper parts is drawn in green, the lower part is drawn in red. In Fig. 2.2 the nodes on the upper boundary of the Newton polygon are denoted according to the formula (2.4). The coefficients $A_{m\,r}(p)$ in (1.12) associated with these nodes are given by the formulas

$$\begin{aligned} &A_{0\,10}=-p^{10}, && A_{2\,10}=2\,p^{6}, && A_{4\,10}=-p^{2},\\ &A_{6\,8}=1, && A_{8\,4}=6, && A_{10\,0}=1. \end{aligned}\tag{2.18}$$

Theorem 2.1. *The values of exponents α_i in the expansion (2.3) for roots of the equation (1.15) are determined according to the formula $\alpha_i=-k$, where k stands for slopes of segments of the polygonal line being the upper boundary of the Newton polygon in Fig. 2.2.*

Theorem 2.1 is immediate from Lemmas 2.2 and 2.3. The formula $\alpha_i=-k$ in this theorem follows from the formula (2.10). In our particular case we have

$$\alpha_i=0, \qquad\qquad \alpha_i=1, \qquad\qquad \alpha_i=2.\tag{2.19}$$

The options (2.19) are derived from Fig. 2.2 due to the above theorem.

3. Leading terms in asymptotic expansions.

The term $C_i\,q^{\alpha_i}$ obtained upon expanding brackets in (2.3) is called the leading term of the asymptotic expansion (2.3). Three options for the value of α_i are given by the formula (2.19). Let's consider each of these options separately.

The case $\alpha_i=0$. This case corresponds to the horizontal segment on the upper boundary of the Newton polygon in Fig. 2.2. This segment comprises three nodes $A_{4\,10}$, $A_{2\,10}$, and $A_{0\,10}$. Therefore, substituting the expansion (2.3) with $\alpha_i=0$ into the equation (1.15), we get the following equation for C_i:

$$A_{4\,10}\;C_i^{\,4}+A_{2\,10}\;C_i^{\,2}+A_{0\,10}=0.\tag{3.1}$$

Taking into account (2.18), the equation (3.1) is transformed to

$$p^2\,C_i^{\,4}-2\,p^6\,C_i^{\,2}+p^{10}=0.\tag{3.2}$$

The equation (3.2) has two real roots

$$C_i=p^2, \qquad\qquad C_i=-p^2,\tag{3.3}$$

each of which is of multiplicity 2. The condition (2.1) excludes the root $C_i=-1$ from (3.3). The remain is one root of multiplicity 2:

$$C_i=p^2.\tag{3.4}$$

The asymptotic expansion (2.3) corresponding to (3.4) is

$$t_i(p,q)=p^2\biggl(1+\sum_{s=1}^{\infty}\beta_{is}\,q^{-s}\biggr).\tag{3.5}$$

The case $\alpha_i=1$. This case corresponds to the short slant segment in the upper boundary of the Newton polygon in Fig. 2.2. It comprises two nodes $A_{4\,10}$ and

$A_{6\,8}$. Therefore, substituting the expansion (2.3) with $\alpha_i = 1$ into the equation (1.15), we get the following equation for C_i:

$$A_{6\,8}\ {C_i}^6 + A_{4\,10}\ {C_i}^4 = 0. \tag{3.6}$$

The common divisor ${C_i}^4$ can be factored out from the equation (3.6). Since $C_i \neq 0$, we can remove this common divisor. Then the equation takes the form

$$A_{6\,8}\ {C_i}^2 + A_{4\,10} = 0. \tag{3.7}$$

Taking into account (2.18), the equation (3.7) is transformed to

$${C_i}^2 - p^2 = 0. \tag{3.8}$$

The quadratic equation (3.8) has two simple root

$$C_i = p, \qquad\qquad C_i = -p, \tag{3.9}$$

The condition (2.1) excludes the root $C_i = -p$ from (3.9). Therefore as a remain we have only one root, which is of multiplicity 1:

$$C_i = p. \tag{3.10}$$

The asymptotic expansion (2.3) corresponding to (3.10) is

$$t_i(p,q) = p\,q\biggl(1 + \sum_{s=1}^{\infty} \beta_{is}\,q^{-s}\biggr). \tag{3.11}$$

The case $\alpha_i = 2$. This case corresponds to the long slant segment in the upper boundary of the Newton polygon in Fig. 3.2. It comprises three nodes $A_{6\,8}$, $A_{8\,4}$, and $A_{10\,0}$. Therefore, substituting the expansion (2.3) with $\alpha_i = 2$ into the equation (1.15), we get the following equation for C_i:

$$A_{10\,0}\ {C_i}^{10} + A_{8\,4}\ {C_i}^8 + A_{6\,8}\ {C_i}^6 = 0. \tag{3.12}$$

The common divisor ${C_i}^6$ is factored out from the equation (3.12). Since $C_i \neq 0$, we can remove this common divisor. Then the equation takes the form

$$A_{10\,0}\ {C_i}^4 + A_{8\,4}\ {C_i}^2 + A_{6\,8} = 0. \tag{3.13}$$

Taking into account (2.18), the equation (3.13) is transformed to

$${C_i}^4 + 6\ {C_i}^2 + 1 = 0. \tag{3.14}$$

The quartic equation (3.14) has four roots. All of them are complex:

$$C_i = (\sqrt{2} + 1)\,\mathrm{i}, \qquad\qquad C_i = (\sqrt{2} - 1)\,\mathrm{i}, \tag{3.15}$$

$$C_i = -(\sqrt{2} + 1)\,\mathrm{i}, \qquad\qquad C_i = -(\sqrt{2} - 1)\,\mathrm{i}. \tag{3.16}$$

Here $\mathrm{i} = \sqrt{-1}$. The roots (3.16) are excluded by the condition (2.1). The remain is two root (3.15) of multiplicity 1. They yield the following asymptotic expansions:

$$\begin{aligned} t_i(p,q) &= (\sqrt{2} + 1)\,\mathrm{i}\,q^2\biggl(1 + \sum_{s=1}^{\infty} \beta_{is}\,q^{-s}\biggr),\\ t_i(p,q) &= (\sqrt{2} - 1)\,\mathrm{i}\,q^2\biggl(1 + \sum_{s=1}^{\infty} \beta_{is}\,q^{-s}\biggr). \end{aligned} \tag{3.17}$$

The results (3.5), (3.11), (3.17) are summed up in the following theorem.

Theorem 3.1. *For sufficiently large positive values of the parameter q, i. e. for $q > q_{\min}$, the tenth-degree equation (1.15) has five roots of multiplicity 1 satisfying the condition (2.1). Three of them t_1, t_2, and t_3 are real roots. Their asymptotics as $q \to +\infty$ are given by the formulas*

$$t_1 \sim p^2, \qquad t_2 \sim p^2, \qquad t_3 \sim p\,q. \tag{3.18}$$

The rest two roots t_4 and t_5 of the equation (1.15) are complex. Their asymptotics as $q \to +\infty$ are given by the formulas

$$t_4 \sim (\sqrt{2} + 1)\,\mathrm{i}\,q^2, \qquad t_5 \sim (\sqrt{2} - 1)\,\mathrm{i}\,q^2. \tag{3.19}$$

The complex roots (3.19) do not provide perfect cuboids. However, below they are important for determining the exact number of real roots.

4. Asymptotic estimates for real roots.

According to the formula (3.18) the roots t_1 and t_2 are not growing as $q \to +\infty$. For this reason we do not need to calculate β_{is} in (3.5) for them. But we need to find estimates for remainder terms R_1 and R_2 in the formulas

$$t_1 = p^2 + R_1(p, q), \qquad t_2 = p^2 + R_2(p, q) \tag{4.1}$$

as $q \to +\infty$. Our goal is to obtain estimates of the form

$$|R_i(p, q)| < \frac{C(p)}{q}, \ \text{ where } \ i = 1,\, 2. \tag{4.2}$$

In order to get such estimates we substitute

$$t = p^2 + \frac{c}{q} \tag{4.3}$$

into the equation (1.15). Then we perform another substitution into the equation obtained as a result of substituting (4.3) into (1.15):

$$q = \frac{1}{z}. \tag{4.4}$$

Upon two substitutions (4.3) and (4.4) and upon removing denominators the equation (1.15) is written as a polynomial equation in the new variables c and z. It is a peculiarity of this equation that it can be written as

$$16\,p^{12} + f(c, p, z) = 4\,p^6\,c^2. \tag{4.5}$$

Here $f(c, p, z)$ is a polynomial given by an explicit formula. The formula for $f(c, p, z)$ is rather huge. Therefore it is placed to the ancillary file `strategy_formulas.txt` in a machine-readable form.

Let $q \geqslant 59\,p$ and let the parameter c run over the interval from $-5\,p^3$ to 0:

$$-5\,p^3 < c < 0. \tag{4.6}$$

From $q \geqslant 59\,p$ and from (4.4) we derive the estimate $|z| \leqslant 1/59\,p^{-1}$. Using this estimate and using the inequalities (4.6), by means of direct calculations one can derive the following estimate for the modulus of the function $f(c, p, z)$:

$$|f(c, p, z)| < 15\,p^{12}. \tag{4.7}$$

For fixed p and z the estimate (4.7) means that the left hand side of the equation (4.5) is a continuous function of c taking the values within the range from p^{12} to $31\,p^{12}$ while c is in the interval (4.6). The right hand side of (4.5) is also a continuous function of c. It decreases from $100\,p^{12}$ to 0 in the interval (4.6). Therefore somewhere in the interval (4.6) there is at least one root of the equation (4.5).

The parameter c is related to the initial variable t by means of the formula (4.3). The inequalities (4.5) for c imply the following inequalities for t:

$$p^2 - \frac{5\,p^3}{q} < t < p^2. \tag{4.8}$$

The inequalities (4.8) and the above considerations prove the following theorem.

Theorem 4.1. *For each $q \geqslant 59\,p$ there is at least one real root of the equation (1.15) satisfying the inequalities (4.8).*

The above considerations can be repeated for the case where the parameter c runs over the interval from 0 to $5\,p^3$. In this case due to (4.3) from

$$0 < c < 5\,p^3$$

we derive the inequalities

$$p^2 < t < p^2 + \frac{5\,p^3}{q} \tag{4.9}$$

for the variable t and hence we obtain the following theorem.

Theorem 4.2. *For each $q \geqslant 59\,p$ there is at least one real root of the equation (1.15) satisfying the inequalities (4.9).*

Now let's proceed to the growing root t_3 of the equation (1.15) (see Theorem 3.1). Upon refining the asymptotic formula (3.18) for t_3 looks like

$$t_3 = p\,q - \frac{16\,p^3}{q} + R_3(p, q). \tag{4.10}$$

The formula (4.10) is in agreement with the expansion (3.11). It means that

$$\beta_{31} = 0, \qquad\qquad \beta_{32} = -16\,p^2.$$

Like in (4.2), our goal here is to obtain estimates of the form

$$|R_3(q)| < \frac{C(p)}{q^2}. \tag{4.11}$$

In order to get such estimates we substitute

$$t = p\,q - \frac{16\,p^3}{q} + \frac{c}{q^2} \tag{4.12}$$

into the equation (1.15). Immediately after that we perform the substitution (4.4) into the equation obtained by substituting (4.12) into (1.15). As a result of two substitutions (4.12) and (4.4) upon eliminating denominators the equation (1.15) is written as a polynomial equation in the new variables c and z. It looks like

$$\varphi(c, p, z) = -2\,p^5\,c. \tag{4.13}$$

Here $\varphi(c, p, z)$ is a polynomial of three variables. The explicit formula for $\varphi(c, p, z)$ is rather huge. Therefore it is placed to the ancillary file `strategy_formulas.txt` in a machine-readable form.

Let $q \geqslant 59\,p$ and let the parameter c run over the interval from $-5\,p^4$ to $5\,p^4$:

$$-5\,p^4 < c < 5\,p^4. \tag{4.14}$$

From $q \geqslant 59\,p$ and from (4.4) we derive the estimate $|z| \leqslant 1/59\,p^{-1}$. Using this estimate and using the inequalities (4.14), by means of direct calculations one can derive the following estimate for the modulus of the function $\varphi(c, p, z)$:

$$|\varphi(c, z)| < 7\,p^9. \tag{4.15}$$

For fixed p and z the estimate (4.15) means that the left hand side of the equation (4.13) is a continuous function of c taking the values within the range from $-7\,p^9$ to $7\,p^9$ while c is in the interval (4.14). The right hand side of the equation (4.13) is also a continuous function of c. It decreases from $10\,p^9$ to $-10\,p^9$ in the interval (4.14). Therefore somewhere in the interval (4.14) there is at least one root of the polynomial equation (4.13).

The parameter c is related to the initial variable t by means of the formula (4.12). The inequalities (4.14) for c imply the following inequalities for t:

$$p\,q - \frac{16\,p^3}{q} - \frac{5\,p^4}{q^2} < t < p\,q - \frac{16\,p^3}{q} + \frac{5\,p^4}{q^2}. \tag{4.16}$$

The inequalities (4.16) and the above considerations prove the following theorem.

Theorem 4.3. *For each $q \geqslant 59\,p$ there is at least one real root of the equation* (1.15) *satisfying the inequalities* (4.16).

Theorems 4.1, 4.2, and 4.3 solve the problem of obtaining estimates of the form (4.2) and (4.11) for the remainder terms in the refined asymptotic expansions (4.1) and (4.10) for $q \geqslant 59\,p$.

5. Asymptotic estimates for complex roots.

Let's proceed to complex roots of the equation (1.15). Upon refining the asymptotic formula (3.19) for the complex root t_4 is written as

$$t_4 = (\sqrt{2} + 1)\,\mathrm{i}\,q^2 + (\sqrt{2} - 2)\,\mathrm{i}\,p^2 + R_4(p, q), \quad \text{where} \quad \mathrm{i} = \sqrt{-1}. \tag{5.1}$$

The formula (5.1) is in agreement with the first expansion (3.17). It means that $\beta_{41} = 0$ and $\beta_{42} = (4 - 3\,\sqrt{2})\,p^2$. Like in the formula (4.2) and in the formula

(4.11), our goal here is to obtain estimates of the form

$$|R_4(p,q)| < \frac{C(p)}{q}. \tag{5.2}$$

In order to get such estimates we substitute

$$t = (\sqrt{2}+1)\,\mathrm{i}\,q^2 + (\sqrt{2}-2)\,\mathrm{i}\,p^2 + \frac{c\,\mathrm{i}}{q} \tag{5.3}$$

into the equation (1.15). Immediately after that we perform the substitution (4.4) into the equation obtained by substituting (5.3) into (1.15). As a result of two substitutions (5.3) and (4.4) upon eliminating denominators the equation (1.15) is written as a polynomial equation in the new variables c and z. It looks like

$$\psi(z,p,c) = 16\,c. \tag{5.4}$$

Here $\psi(c,p,z)$ is a polynomial of three variables with purely real coefficients. The explicit formula for $\psi(c,p,z)$ is rather huge. Therefore it is placed to the ancillary file `strategy_formulas.txt` in a machine-readable form.

Let $q \geqslant 59\,p$ and let the parameter c run over the interval from $-5\,p^3$ to $5\,p^3$:

$$-5\,p^3 < c < 5\,p^3. \tag{5.5}$$

From $q \geqslant 59\,p$ and from (4.4) we derive the estimate $|z| \leqslant 1/59\,p^{-1}$. Using this estimate and using the inequalities (5.5), by means of direct calculations one can derive the following estimate for the modulus of the function $\psi(c,p,z)$:

$$|\psi(c,p,z)| < 15\,p^3. \tag{5.6}$$

For fixed p and z the estimate (5.6) means that the left hand side of the equation (5.4) is a continuous function of c taking its values within the range from $-15\,p^3$ to $15\,p^3$ while c runs over the interval (5.5). The right hand side of the equation (5.4) is also a continuous function of c. It increases from $-80\,p^3$ to $80\,p^3$ in the interval (5.5). Therefore somewhere in the interval (5.5) there is at least one root of the polynomial equation (5.4).

The parameter c is related to the initial variable t by means of the formula (5.3). Therefore the inequalities (5.5) for c imply the following inequalities for t:

$$(\sqrt{2}+1)\,q^2+(\sqrt{2}-2)\,p^2-\frac{5\,p^3}{q} < \operatorname{Im}\,t < (\sqrt{2}+1)\,q^2+(\sqrt{2}-2)\,p^2+\frac{5\,p^3}{q}. \tag{5.7}$$

The inequalities (5.7) and the above considerations prove the following theorem.

Theorem 5.1. *For each $q \geqslant 59\,p$ there is at least one purely imaginary root of the equation* (1.15) *satisfying the inequalities* (5.7).

The complex root t_5 is similar to the root t_4. Upon refining the asymptotic formula (3.19) for the complex root t_5 is written as

$$t_4 = (\sqrt{2}-1)\,\mathrm{i}\,q^2 + (\sqrt{2}+2)\,\mathrm{i}\,p^2 + R_5(p,q), \text{ where } \mathrm{i}=\sqrt{-1}. \tag{5.8}$$

The formula (5.8) is in agreement with the second expansion (3.17). It means that $\beta_{51} = 0$ and $\beta_{52} = (4 + 3\sqrt{2})\,p^2$. Like in the formulas (4.2), (4.11), and (4.2), our goal here is to obtain estimates of the form

$$|R_5(p,q)| < \frac{C(p)}{q}. \tag{5.9}$$

In order to get such estimates we substitute

$$t = (\sqrt{2} - 1)\,\mathrm{i}\,q^2 + (\sqrt{2} + 2)\,\mathrm{i}\,p^2 + \frac{c\,\mathrm{i}}{q} \tag{5.10}$$

into the equation (1.15). Immediately after that we perform the substitution (4.4) into the equation obtained by substituting (5.10) into (1.15). As a result of two substitutions (5.10) and (4.4) upon eliminating denominators the equation (1.15) is written as a polynomial equation in the new variables c and z. It looks like

$$\eta(z, c) = 16\,c. \tag{5.11}$$

Here $\eta(c, p, z)$ is a polynomial of three variables. The explicit formula for $\eta(c, p, z)$ is rather huge. Therefore it is placed to the ancillary file `strategy_formulas.txt` in a machine-readable form.

Let $q \geqslant 59\,p$ and let the parameter c run over the interval from $-5\,p^3$ to $5\,p^3$ (see (5.5)). From $q \geqslant 59\,p$ and from (4.4) we derive the estimate $|z| \leqslant 1/59\,p^{-1}$. Using this estimate and using the inequalities (5.5), by means of direct calculations one can derive the following estimate for the modulus of the function $\eta(c, p, z)$:

$$|\eta(c, p, z)| < 15\,p^3. \tag{5.12}$$

For fixed p and z the estimate (5.12) means that the left hand side of the equation (5.11) is a continuous function of c taking its values within the range from $-15\,p^3$ to $15\,p^3$ while c runs over the interval (5.5). The right hand side of the equation (5.12) is also a continuous function of c. It increases from $-80\,p^3$ to $80\,p^3$ in the interval (5.5). Therefore somewhere in the interval (5.5) there is at least one root of the polynomial equation (5.12).

The parameter c is related to the initial variable t by means of the formula (5.10). Therefore the inequalities (5.5) for c imply the following inequalities for t:

$$(\sqrt{2}-1)\,q^2 + (\sqrt{2}+2)\,p^2 - \frac{5\,p^3}{q} < \operatorname{Im}\,t < (\sqrt{2}-1)\,q^2 + (\sqrt{2}+2)\,p^2 + \frac{5\,p^3}{q}. \tag{5.13}$$

The inequalities (5.13) and the above considerations prove the following theorem.

Theorem 5.2. *For each $q \geqslant 59\,p$ there is at least one purely imaginary root of the equation* (1.15) *satisfying the inequalities* (5.13).

Theorems 5.1 and 5.2 solve the problem of obtaining estimates of the form (5.2) and (5.9) for the remainder terms in the refined asymptotic expansions (5.1) and (5.8) for $q \geqslant 59\,p$. Along with Theorems 4.1, 4.2, and 4.3, they separate the roots t_1, t_2, t_3, t_4, t_5 of the equation (1.15) from each other for sufficiently large q and provide rather precise intervals for their location.

6. Non-intersection of asymptotic intervals.

Theorems 4.1, 4.2, 4.3, 5.1, 5.2 define five asymptotic intervals (4.8), (4.9), (4.16), (5.7), and (5.13) for $q \geqslant 59\,p$. It is easy to see that the intervals (4.8) and (4.9) do not intersect. For the other pairs of intervals among (4.8), (4.9), (4.16), (5.7), (5.13) this is not so obvious. Therefore we need some elementary lemmas.

Lemma 6.1. *For $q \geqslant 59\,p$ the asymptotic intervals (4.8), (4.9), (4.16), (5.7), and (5.13) do not comprise the origin.*

Proof. Indeed, from $q \geqslant 59\,p$ for the left endpoint of the interval (4.8) we derive

$$p^2 - \frac{5\,p^3}{q} \geqslant p^2 - \frac{5\,p^2}{59} = \frac{54\,p^2}{59} > 0. \tag{6.1}$$

The left endpoint of the interval (4.9) is obviously positive: $p^2 > 0$. In the case of the interval (4.16) from $q \geqslant 59\,p$ we derive

$$p\,q - \frac{16\,p^3}{q} - \frac{5\,p^4}{q^2} \geqslant \frac{204430\,p^2}{3481} > 58\,p^2 > 0. \tag{6.2}$$

In the case of the imaginary intervals (5.7) and (5.13) from $q \geqslant 59\,p$ we derive

$$\begin{aligned}
&(\sqrt{2}+1)\,q^2 + (\sqrt{2}-2)\,p^2 - \frac{5\,p^3}{q} > 8403\,p^2 > 0,\\
&(\sqrt{2}-1)\,q^2 + (\sqrt{2}+2)\,p^2 - \frac{5\,p^3}{q} > 1445\,p^2 > 0.
\end{aligned} \tag{6.3}$$

The above inequalities (6.1), (6.2), and (6.3) prove Lemma 6.1. $\square$

Lemma 6.1 means that for $q \geqslant 59\,p$ the real intervals (4.8), (4.9), and (4.16) do not intersect with the purely imaginary intervals (5.7) and (5.13). Moreover, the inequalities (6.1), (6.2), and (6.3) show that all of these intervals are located within positive half-lines of the real and imaginary axes. Therefore any roots of the equation (1.15) enclosed within these intervals satisfy the condition (2.1).

Lemma 6.2. *For $q \geqslant 59\,p$ the real asymptotic intervals (4.8), (4.9), and (4.16) do not intersect with each other.*

Proof. The open intervals (4.8) and (4.9) are adjacent. They have one common endpoint $t = p^2$, but this endpoint does not belong to them. Therefore the intervals (4.8) and (4.9) do not intersect with each other.

In order to prove Lemma 6.2 it is sufficient to compare the right endpoint of the interval (4.9) with the left endpoint of the interval (4.16). From $q \geqslant 59\,p$ we derive

$$p^2 + \frac{5\,p^3}{q} \leqslant \frac{64\,p^2}{59} < 2\,p^2. \tag{6.4}$$

Comparing (6.4) with (6.2), we conclude that

$$p^2 + \frac{5\,p^3}{q} < p\,q - \frac{16\,p^3}{q} - \frac{5\,p^4}{q^2} \tag{6.5}$$

for $q \geqslant 59\,p$. The inequality (6.5) completes the proof of Lemma 6.2. $\square$

Lemma 6.3. *For $q \geqslant 59\,p$ the imaginary asymptotic intervals (5.7) and (5.13) do not intersect with each other.*

Proof. In order to prove Lemma 6.3 it is sufficient to compare the bottom endpoint of the interval (5.7) with the top endpoint of the interval (5.13):

$$\operatorname{Im}\bigl(t^{(5.7)}_{\text{bottom}} - t^{(5.13)}_{\text{top}}\bigr) = 2\,q^2 - 4\,p^2 - \frac{10\,p^3}{q}. \tag{6.6}$$

From $q \geqslant 59\,p$ and from (6.6) we derive the inequalities

$$\operatorname{Im}\bigl(t^{(5.7)}_{\text{bottom}} - t^{(5.13)}_{\text{top}}\bigr) \geqslant \frac{410512\,p^2}{59} > 6957\,p^2 > 0. \tag{6.7}$$

The inequalities (6.7) complete the proof of Lemma 6.3. $\square$

Lemmas 6.1, 6.2, and 6.3 are summed up in the following theorem.

Theorem 6.1. *For $q \geqslant 59\,p$ five roots t_1, t_2, t_3, t_4, t_5 of the equation (1.15) obeying the condition (2.1) are simple. They are located within five disjoint intervals (4.8), (4.9), (4.16), (5.7), (5.13), one per each interval.*

Due to (2.2) Theorem 6.1 locates all of the ten roots of the equation (1.15).

7. Integer points of asymptotic intervals.

It is easy to see that the asymptotic intervals (4.8), (4.9), (4.16) become more and more narrow if p is fixed and $q \to +\infty$. Using this observation, one can easily prove the following two theorems.

Theorem 7.1. *If $q \geqslant 59\,p$ and $q > 5\,p^3$, then the asymptotic intervals (4.8) and (4.9) have no integer points.*

Theorem 7.2. *If $q \geqslant 59\,p$ and $q^2 > 10\,p^4$, then the asymptotic interval (4.16) has at most one integer point.*

The next theorem is more complicated.

Theorem 7.3. *If $q \geqslant 59\,p$ and $q \geqslant 16\,p^3 + 5\,p/16$, then the asymptotic interval (4.16) has no integer points.*

Proof. Note that $(1+x)^2 > 1 + 2\,x$ for any positive x. This inequality yields

$$1 + x > \sqrt{1 + 2\,x} \quad \text{for any} \quad x > 0. \tag{7.1}$$

Let's write the inequality $q \geqslant 16\,p^3 + 5\,p/16$ in the following way:

$$q - 8\,p^3 \geqslant 8\,p^3 + \frac{5\,p}{16} = 8\,p^3\left(1 + \frac{5}{128\,p^2}\right). \tag{7.2}$$

Setting $x = 5/(128\,p^2)$ in (7.1) and applying it to (7.2), we get

$$q - 8\,p^3 > 8\,p^3\sqrt{1 + \frac{5}{64\,p^2}}. \tag{7.3}$$

Both sides of the inequality (7.3) are positive. Squaring them, we obtain

$$(q-8\,p^3)^2 > 64\,p^6\Bigl(1+\frac{5}{64\,p^2}\Bigr). \tag{7.4}$$

Expanding both sides of the inequality (7.4), we bring it to

$$q^2-16\,p^3\,q > 5\,p^4. \tag{7.5}$$

And finally, dividing both sides of the inequality (7.5) by q^2, we write it as

$$\frac{16\,p^3}{q}+\frac{5\,p^4}{q^2} < 1. \tag{7.6}$$

Apart from $q \geqslant 16\,p^3+5\,p/16$, we have the inequality $q \geqslant 59\,p$ that yields

$$0 < \frac{5\,p^4}{q^2} \leqslant \frac{5\,p^4}{59\,p\,q} = \frac{5\,p^3}{59\,q} < \frac{16\,p^3}{q}. \tag{7.7}$$

Applying (7.6) and (7.7) to (4.16), we derive the following inequalities:

$$p\,q-1 < t < p\,q. \tag{7.8}$$

Since $p\,q$ is integer, the inequalities (7.8) have no integer solutions for t. This means that the interval (4.16) has no integer points. Theorem 7.3 is proved. $\square$

8. A strategy for numeric search.

The numeric search for perfect cuboids in the case of the second cuboid conjecture (see Conjecture 1.1) is based on the equation (1.15). The equation (1.15) is related to the equation (1.8) through the substitutions (1.10) and (1.13). Substituting either (1.10) or (1.13) into the inequalities $t > a$, $t > b$, and $t > u$ from Theorem 1.2, we get the same result expressed by the inequalities

$$t > p^2, \qquad\qquad t > p\,q, \qquad\qquad t > q^2. \tag{8.1}$$

Similarly, substituting either (1.10) or (1.13) into the inequality $(a+t)\,(b+t) > 2\,t^2$, we get the same result expressed by the inequality

$$(p^2+t)\,(p\,q+t) > 2\,t^2. \tag{8.2}$$

Theorem 1.2 specified for the case of second cuboid conjecture (see Conjecture 1.1) is formulated in the following way.

Theorem 8.1. *A triple of integer numbers p, q, and t satisfying the equation (1.15) and such that $p \neq q$ are coprime provides a perfect cuboid of and only if the inequalities (8.1) and (8.2) are fulfilled.*

Generally speaking, the numeric search based on Theorem 8.1 is a three-parametric search. The inequalities (8.1) set lower bounds for t, but they do not restrict

t to a finite set of values. The inequality (8.2) is different. Since p and q are positive, the inequality (8.2) can be written as follows:

$$t < \frac{p^2 + p\,q}{2} + \frac{p\,\sqrt{p^2 + 6\,p\,q + q^2}}{2}. \tag{8.3}$$

Due to (8.1) and (8.3) for each fixed p and fixed q one should iterate only over a finite set of integer values of t, i. e. the search is two-parametric in effect.

Theorems 7.1 and 7.3 strengthen the restrictions. They say that for each fixed p one should iterate over a finite set of values of q and t, i. e. the search for perfect cuboids becomes effectively one-parametric.

Theorems 7.1 and 7.3 can be further strengthened with the use of the inequalities (8.1). Assume that the condition $q \geqslant 59\,p$ is fulfilled and assume that t belongs to the first asymptotic interval (4.8) (see Theorem 6.1). Then from (8.1) and (4.8) we derive two contradictory inequalities $t > p^2$ and $t < p^2$. Hence we have the following theorem.

Theorem 8.2. *If $q \geqslant 59\,p$, then the asymptotic interval* (4.8) *has no points satisfying the inequalities* (8.1).

Similarly, if $q \geqslant 59\,p$ and if t belongs to the second asymptotic interval (4.9), then from the inequalities (4.9) and (8.1) we derive

$$\begin{aligned} &t > q^2 \geqslant (59\,p)^2 = 3481\,p^2,\\ &t < p^2 + \frac{5\,p^3}{q}. \end{aligned} \tag{8.4}$$

The inequalities (8.4) mean that

$$3481\,p^2 < p^2 + \frac{5\,p^3}{q}. \tag{8.5}$$

From (8.5) one easily derives the inequality $q < 5\,p/3480 = p/696$ that contradicts $q \geqslant 59\,p$. Hence we have the following theorem.

Theorem 8.3. *If $q \geqslant 59\,p$, then the asymptotic interval* (4.9) *has no points satisfying the inequalities* (8.1).

Finally, assume that $q \geqslant 59\,p$ and let t belong to the third asymptotic interval (4.16). We know that $q \geqslant 59\,p$ implies (7.7). From (7.7) and (4.16) we derive

$$t < p\,q - \frac{16\,p^3}{q} + \frac{5\,p^4}{q^2} < p\,q. \tag{8.6}$$

The inequalities (8.6) contradict the inequality $t > p\,q$ from (8.1). This contradiction proves the following theorem.

Theorem 8.4. *If $q \geqslant 59\,p$, then the asymptotic interval* (4.16) *has no points satisfying the inequalities* (8.1).

The fourth and fifth asymptotic intervals (5.7) and (5.13) are purely imaginary. They do not provide perfect cuboids. Therefore Theorems 8.1, 8.2, 8.3, and 8.4 are summarized in the following theorem.

Theorem 8.5. *If $q \geqslant 59\,p$, then the Diophantine equation* (1.15) *has no solutions providing perfect cuboids.*

Theorem 8.5 is a background for an optimized strategy of numeric search for perfect cuboids. It says that only for $q < 59\,p$ perfect cuboids are expected.

Applying the inequality $q < 59\,p$ to the formula (8.3) we can simplify it as

$$t < 61\,p^2. \tag{8.7}$$

The upper bound (8.7) is more simple than (8.3), though it can be computationally less time-efficient for large values of p. Along with (8.1) and the inequality

$$q < 59\,p, \tag{8.8}$$

it provides the following very simple computer code for our optimized strategy:

```
for p from 1 by 1 to +∞ do
 for q from 1 by 1 to 59*p-1 do
  if p<>q and gcd(p,q)=1
   then
    for t from max(p^2,p*q,q^2)+1 by 1 to 61*p^2-1 do
     if Q_pqt=0 and (p^2+t)*(p*q+t)>2*t^2
      then
       Str:=sprintf("Cuboid is found:  p=%a, q=%a, t=%a.",p,q,t):
       writeline(default,Str):
     end if:
    end do:
  end if:
 end do:
end do:
```

Here `gcd(p,q)` stands for the greatest common divisor of p and q, while `Q_pqt` is a computer version of the formula (1.12). In practice the infinity sign $+\infty$ should be replaced by some particular positive integer.

9. Concluding remarks and acknowledgments.

Though they look very simple, Theorem 8.5 and the inequality (8.8) constitute the main result of the present work. As for the above code, it should be further optimized e. g. by some tricky algorithms for fast computing the values of `Q_pqt`. One of such further optimized versions of this code has been run on a desktop PC for p from 1 to 100. It took about 7 hours for that. No perfect cuboids were found.

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