Intersection cohomology of the symmetric reciprocal plane

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Abstract. We compute the Kazhdan-Lusztig polynomial of the uniform matroid of rank $n-1$ on n elements by proving that the coefficient of t^i is equal to the number of ways to choose i non-intersecting chords in an $(n-i+1)$-gon. We also show that the corresponding intersection cohomology group is isomorphic to the irreducible representation of S_n associated to the partition $[n-2i, 2, \ldots, 2]$.

1 Introduction

Fix an integer $n \geq 2$. Let

$$U_n := \left\{z \in (\mathbb{C}^\times)^n \mid \tfrac{1}{z_1} + \ldots + \tfrac{1}{z_n} = 0\right\},$$

and let X_n be the closure of U_n in $\mathbb{C}^n$. Equivalently, X_n is the hypersurface defined by the $(n-1)^{\text{st}}$ elementary symmetric polynomial. Since X_n is isomorphic to the "reciprocal plane" [PS06, SSV13, DGT14] associated to a symmetric configuration of n vectors spanning a vector space of dimension $n-1$, we will refer to it as the **symmetric reciprocal plane**. The intersection cohomology of X_n (with complex coefficients) vanishes in odd degrees [EPW, 3.12], and we let $c_{n,i} := \dim \mathrm{IH}^{2i}(X_n)$. Let

$$P_n(t) := \sum_i c_{n,i}\, t^i \qquad \text{and} \qquad \Phi(t,u) := \sum_{n=2}^{\infty} P_n(t) u^{n-1}.$$

The polynomial $P_n(t)$ is called the **Kazhdan-Lusztig polynomial** of the uniform matroid of rank $n-1$ on a set of n elements. It is clear that the collection of numbers $c_{n,i}$, the collection polynomials $P_n(t)$, and the single power series $\Phi(t,u)$ all encode the same data. The following theorem gives three equivalent versions of a recursive procedure to compute these data [EPW, 2.2, 2.19, 2.21, 3.12].[3]

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[3]In [EPW], $c_{n,i}$ was denoted by $c^i_{1,n-1}$ and $P_n(t)$ was denoted by $P_{1,n-1}(t)$.

Theorem 1.1. *We have $c_{n,i} = 0$ for all $i \geq \frac{n-1}{2}$, and the following (equivalent) equations hold.*

(1) *For all n and i,* $c_{n,i} = (-1)^i \binom{n}{i} + \sum_{j=0}^{i-1} \sum_{k=2j+2}^{i+j+1} (-1)^{i+j+k+1} \binom{n}{k, i+j-k+1, n-i-j-1} c_{k,j}.$

(2) *For all n,* $t^{n-1} P_n(t^{-1}) = \sum_{j=0}^{n-1} (-1)^j \binom{n}{j} \left(t^{n-i-1} - 1\right) + \sum_{k=2}^{n} \binom{n}{k} (t-1)^{n-k} P_k(t).$

(3) $\Phi(t^{-1}, tu) = \dfrac{tu - u}{(1 - tu + u)(1 + u)} + \dfrac{1}{(1 - tu + u)^2} \, \Phi\left(t, \dfrac{u}{1 - tu + u}\right).$

The purpose of this paper is to give an explicit formula for $c_{n,i}$. In fact, we do even better: the intersection cohomology group $\mathrm{IH}^{2i}(X_n)$ is not just a vector space, but also a representation of the symmetric group S_n, which acts on X_n by permuting the coordinates. We categorify the recursion in Theorem 1.1(1) to obtain a recursion in the virtual representation ring of the symmetric group, which we then solve as follows. For any partition λ of n, let V_λ be the corresponding irreducible representation of S_n.

Theorem 1.2. *For all $n \geq 2$ and $i \geq 0$, the following identities hold.*

(1) $c_{n,i} = \dfrac{1}{i+1} \dbinom{n-i-2}{i} \dbinom{n}{i}.$

(2) $\mathrm{IH}^{2i}(X_n) \cong V_{[n-2i,2,\ldots,2]} = V_{[n-2i,2^i]}.$

Remark 1.3. Cayley [Cay90] proved that the quantity $\frac{1}{i+1}\binom{n-i-2}{i}\binom{n}{i}$ is equal to the number of ways to choose i non-intersecting chords in an $(n-i+1)$-gon. This combinatorial interpretation will actually play a role in our proof of Theorem 1.2(1).

Remark 1.4. Note that, if $i \geq \frac{n-1}{2}$, then the first binomial coefficient in Theorem 1.2(1) is zero and the "partition" $[n-2i, 2, \ldots, 2]$ in Theorem 1.2(2) is not a partition. Thus the vanishing of these coefficients is built into the statement of the theorem.

Remark 1.5. Theorem 1.2(1) follows immediately from Theorem 1.2(2) via the hook-length formula for the dimension of V_λ. However, we will need to prove Theorem 1.2(1) first in order to prove Theorem 1.2(2).

Remark 1.6. The most surprising aspect of Theorem 1.2(2) is that the representation $\mathrm{IH}^{2i}(X_n)$ is irreducible; we see no *a priori* reason for this to be the case. One way to interpret Theorem 1.2(2) would be to regard it as a new geometric construction of a certain class of irreducible representations of the symmetric group.

Remark 1.7. In a future paper, we will compute the Kazhdan-Lusztig polynomial of a uniform matroid of arbitrary rank on a set of n elements. The reason that we only treat the rank $n-1$ case in this paper is that only this case can be represented by an arrangement that carries a natural action of the symmetric group.

We conclude the introduction with a discussion of structure of the paper. In Section 2, we use Theorem 1.1(3) along with Beckwith's work on polygonal dissections [Bec98] to prove Theorem 1.2(1). As a corollary, we prove that the coefficients of $P_n(t)$ form a log concave sequence, as conjectured in [EPW, 2.5]. In Section 3, we use mixed Hodge theory to construct a spectral sequence which allows us to lift the recursion in Theorem 1.1(1) to the category of representations of the symmetric group. In Section 4, we use Schubert calculus to solve this recursion, thus proving Theorem 1.2(2).

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2 Numbers

In this section we prove Theorem 1.2(1). For all $n \geq 3$ and $i \geq 0$, let $d_{n,i}$ be the number of sets of i non-intersecting diagonals of an n-gon, and let

$$f(t,u) := \sum_{\substack{i \geq 0 \\ n \geq 3}} d_{n,i}\, t^i u^{n-1}.$$

As noted in Remark 1.3, Cayley [Cay90] gave a formula for $d_{n,i}$ which demonstrates that Theorem 1.2(1) is equivalent to the statement $c_{n,i} = d_{n-i+1,i}$, which is in turn equivalent to the identity $\Phi(t,u) = u^{-1} f(tu,u)$. Thus it will be sufficient to prove that $g(t,u) := u^{-1} f(tu,u)$ satisfies the functional equation in Theorem 1.1(3).

Using the combinatorial interpretation of the numbers $d_{n,i}$, Beckwith [Bec98] shows that

$$f(t,u) = \frac{2\left((2t+1)u + \sqrt{1-2\,(2t+1)u+u^2} - 1\right)}{1-(2t+1)^2},$$

and therefore

$$g(t,u) = \frac{2}{u} \cdot \frac{(2tu+1)u + \sqrt{1-2\,(2tu+1)u+u^2} - 1}{1-(2tu+1)^2}.$$

It is straightforward to show (or to check on a computer) that

$$\begin{aligned} g(t^{-1}, tu) &= \frac{1 - tu - 2tu^2 - \sqrt{1 - 2tu - 4tu^2 + t^2u^2}}{2tu^2(1+u)} \\ &= \frac{1 - tu - 2tu^2 - (1 - tu + u)\sqrt{\frac{1-2tu-4tu^2+t^2u^2}{(1-tu+u)^2}}}{2tu^2(1+u)} \\ &= \frac{tu - u}{(1 - tu + u)(1+u)} + \frac{1}{(1 - tu + u)^2}\, g\left(t, \frac{u}{1 - tu + u}\right). \end{aligned}$$

This completes the proof of Theorem 1.2(1). □

In a previous paper, Elias and the first two authors conjectured that the coefficients of the Kazhdan-Lusztig polynomial of any matroid form a log concave sequence [EPW, 2.5]. Theorem 1.2(1) allows us to prove this conjecture for the uniform matroid of rank $n-1$ on n elements.

Corollary 2.1. *Fix an integer $n \geq 2$. The sequence $c_{n,0}, c_{n,1}, \ldots, c_{\lfloor n/2 \rfloor - 1}$ is strictly log concave. That is, for all $0 < i < \lfloor n/2 \rfloor - 1$, we have*

$$c_{n,i}^2 > c_{n,i-1}c_{n,i+1}.$$

Proof. Fix such an n and i. By Theorem 1.2(1), we have

$$\frac{c_{n,i-1}c_{n,i+1}}{c_{n,i}^2} = \frac{i}{i+2} \cdot \frac{n-i-1}{n-i+1} \cdot \frac{n-2i-2}{n-2i} \cdot \frac{n-2i-3}{n-2i-1} \cdot \frac{n-i}{n-i-2}.$$

The first four of these factors are clearly each less than 1, while the fifth factor is greater than 1. However, if we combine the fourth and fifth factors, we will find that their product is less than 1. Indeed, for any $0 \leq k < \ell$, we have

$$\frac{k}{k+2} \cdot \frac{\ell+2}{\ell} < 1.$$

Applying this to $k = n - 2i - 3$ and $\ell = n - i - 2$, we see that the product of the fourth and fifth factors is less than 1, thus so is the entire expression. □

3 Deriving the categorified recursion

Our goal in this section is to promote the recursion in Theorem 1.1(1) to the level of virtual representations of symmetric groups, which we will use in the next section to prove Theorem 1.2(2). For any subset $S \subset \{1, \ldots, n\}$, let $Y(S) \subset X_n$ be the locus of points whose vanishing coordinates coincide exactly with S.

Lemma 3.1. *If $|S| > 1$, then there is an open neighborhood of $Y(S)$ in X_n that is isomorphic to an open neighborhood of $Y(S) \cong Y(S) \times \{0\}$ in $Y(S) \times X_{|S|}$. Furthermore, the isomorphism may be chosen to restrict to the identity map on $Y(S)$.*

Proof. Let $W(S) \subset X_n$ be the locus defined by the nonvanishing of z_i for all $i \notin S$. Then $W(S)$ is open in X_n and $Y(S)$ is closed in $W(S)$. Assume without loss of generality that $S = \{1, \ldots, r\}$ for some $r > 1$. Then

$$\mathbb{C}[W(S)] = \mathbb{C}[x_1, \ldots, x_r, x_{r+1}^{\pm}, \ldots, x_n^{\pm}] \Big/ \left\langle \sum_{i=1}^{n} \prod_{j \neq i} x_j \right\rangle$$

and, since $Y(S) \cong (\mathbb{C}^\times)^{n-r}$ [PS06, Proposition 5],

$$\mathbb{C}[Y(S) \times X_{|S|}] \cong \mathbb{C}[x_1, \ldots, x_r, x_{r+1}^{\pm}, \ldots, x_n^{\pm}] \Big/ \left\langle \sum_{i=1}^{r} \prod_{r \geq j \neq i} x_j \right\rangle.$$

In both cases, the subvariety $Y(S)$ is defined by the vanishing of $x_1, \ldots, x_r$. Consider the open subset $V(S) \subset W(S)$ defined by inverting $1 + x_1 \sum_{k>r} x_k^{-1}$, along with the open subset $U(S) \subset Y(S) \times X_{|S|}$ defined by inverting $1 - x_1 \sum_{k>r} x_k^{-1}$. It is clear that both of these open subsets contain $Y(S)$. Now consider the maps

$$\varphi : V(S) \rightleftarrows U(S) : \psi$$

given by the formulas

$$\varphi^*(x_1) := \frac{x_1}{1 + x_1 \sum\limits_{k>r} x_k^{-1}} \qquad \text{and} \qquad \varphi(x_i) = x_i \ \text{ if } i > 1$$

and

$$\psi^*(x_1) := \frac{x_1}{1 - x_1 \sum\limits_{k>r} x_k^{-1}} \qquad \text{and} \qquad \psi(x_i) = x_i \ \text{ if } i > 1.$$

These two maps clearly restrict to the identity on $Y(S)$, and it is straightforward to check that they are mutually inverse. $\square$

Let $Y(p) \subset X_n$ be the union of strata of codimension p. That means that $Y(p) = \bigsqcup_{|S|=p+1} Y(S)$ if $p > 0$ and $Y(0) = Y(\emptyset)$. Let $\iota_p : Y(p) \hookrightarrow X_n$ be the inclusion. The following lemma tells us how to compute the hypercohomology of the shriek pullback to $Y(p)$ of the intersection cohomology sheaf of X_n, which we will need in the proof of Proposition 3.3.

Lemma 3.2. *For any $p > 0$, we have $\mathbb{H}^*(\iota_p^! \operatorname{IC}_{X_n}) \cong \operatorname{H}^*(Y(p)) \otimes \operatorname{IH}^*_c(X_{p+1})$.*

Proof. The cohomology of the complex $\iota_p^! \operatorname{IC}_{X_n}$ is a local system on $Y(p)$ whose fiber at a point is the compactly supported cohomology of the stalk of IC_{X_n} at that point. By Lemma 3.1, this local system is constant with fiber isomorphic to $\operatorname{IH}^*_c(X_{p+1})$. We thus have a spectral sequence with

$$E_2^{j,k} = \operatorname{H}^j(Y(p)) \otimes \operatorname{IH}^k_c(X_{p+1}) \qquad \text{and} \qquad \bigoplus_{j+k=\ell} E_\infty^{j,k} = \mathbb{H}^*(\iota_p^! \operatorname{IC}_{X_n}) \ \text{ for all } \ell.$$

All of these groups carry mixed Hodge structures, and the maps in the spectral sequence are strictly compatible with the weight filtrations. The group $H^j(Y(p))$ is pure of weight $2j$ [Sha93], and $\mathrm{IH}^k_c(X_{p+1})$ is pure of weight k [EPW, 3.9]. Thus all maps vanish and the result follows. □

Let $\rho_n \cong V_{[n]} \oplus V_{[n-1,1]}$ be the permutation representation of S_n.

Proposition 3.3. *Fix integers $n \geq 2$ and $i < \frac{n-1}{2}$. There exists a first quadrant cohomological spectral sequence E in the category of S_n-representations with*

$$E_1^{p,q} = \begin{cases} \mathrm{Ind}_{S_{n-p-1}\times S_{p+1}}^{S_n}\left(\wedge^{2i-p-q}\rho_{n-p-1} \boxtimes \mathrm{IH}^{2(i-q)}(X_{p+1})\right) & \text{if } 0 < p < n-1, \\ \wedge^i \rho_n & \text{if } p = 0 \text{ and } q = i, \\ 0 & \text{otherwise} \end{cases}$$

converging to

$$\bigoplus_{p+q=2i} E_\infty^{p,q} = \mathrm{IH}^{2i}(X_n) \qquad \text{and} \qquad \bigoplus_{p+q\neq 2i} E_\infty^{p,q} = 0.$$

Proof. There is a first quadrant cohomological spectral sequence $\tilde{E}$ with

$$\tilde{E}_1^{p,q} = \mathbb{H}^{p+q}(\iota_p^! \mathrm{IC}_{X_n}) \qquad \text{and} \qquad \bigoplus_{p+q=\ell} \tilde{E}_\infty^{p,q} = \mathrm{IH}^\ell(X_n) \text{ for all } \ell \text{ [BGS96, §3.4]}.$$

Lemma 3.2 tells us that

$$\tilde{E}_1^{p,q} \cong \bigoplus_{j+2k=p+q} \mathrm{H}^j(Y(p)) \otimes \mathrm{IH}^{2k}_c(X_{p+1}) \quad \text{if } p > 0.$$

If $p = 0$, then ι_p is an open inclusion inside of the smooth locus of X_n, so $\tilde{E}_1^{0,q} \cong \mathrm{H}^q(Y(0))$.

As in the proof of Lemma 3.2, our groups carry mixed Hodge structures, and the maps are strictly compatible with weight filtrations. As noted above, $H^j(Y(p))$ is pure of weight $2j$, while $\mathrm{IH}^{2k}(X_n)$ and $\mathrm{IH}^{2k}_c(X_{p+1})$ are both pure of weight $2k$. Let E be the maximal subquotient of $\tilde{E}$ that is pure of weight $2i$. Then

$$E_1^{0,i} = \mathrm{H}^i(Y(0)) \qquad \text{and} \qquad E_1^{0,q} = 0 \quad \text{if } q \neq i,$$

and

$$\bigoplus_{p+q=2i} E_\infty^{p,q} = \mathrm{IH}^{2i}(X_n) \qquad \text{and} \qquad \bigoplus_{p+q\neq 2i} E_\infty^{p,q} = 0.$$

If $p > 0$, we have

$$\begin{aligned} E_1^{p,q} &\cong \bigoplus_{\substack{j+2k=p+q \\ j+k=i}} \mathrm{H}^j(Y(p)) \otimes \mathrm{IH}^{2k}_c(X_{p+1}) \\ &\cong \mathrm{H}^{2i-p-q}(Y(p)) \otimes \mathrm{IH}^{2(p+q-i)}_c(X_{p+1}) \\ &\cong \mathrm{H}^{2i-p-q}(Y(p)) \otimes \mathrm{IH}^{2(i-q)}(X_{p+1}) \quad \text{by Poincaré duality [Con, 1.4.6.5].} \end{aligned}$$

Note that when $p = n-1$, $Y(p)$ is a point, so $\mathrm{H}^{2i-p-q}(Y(p)) = 0$ unless $q = 2i-n+1$, but then $i - q > \frac{n-1}{2}$, so $\mathrm{IH}^{2(i-q)}(X_{p+1}) = 0$. Thus $E_1^{n-1,q} = 0$ for all q.

Last, we incorporate the action of S_n. For each $p > 0$, S_n acts transitively on the set of components of $Y(p)$ with stabilizer $S_{n-p-1} \times S_{p+1}$, thus

$$E_1^{p,q} \cong \mathrm{Ind}_{S_{n-p-1}\times S_{p+1}}^{S_n}\Big(\mathrm{H}^{2i-p-q}(Y([p])) \otimes \mathrm{IH}^{2(i-q)}(X_{p+1})\Big).$$

Since $p > 0$, we have $Y([p]) \cong (\mathbb{C}^\times)^{n-p-1}$, with S_{n-p-1} permuting the coordinates and S_{p+1} acting trivially, thus $\mathrm{H}^{2i-p-q}(Y([p])) \cong \wedge^{2i-p-q}\rho_{n-p-1}$ as a representation of S_{n-p-1}. The action of $S_{n-p-1} \times S_{p+1}$ factors through S_{p+1}, which is the stabilizer of a single point in $Y([p])$. Finally, since $Y(0)$ is a generic $(n-1)$-dimensional linear slice of $(\mathbb{C}^\times)^n$ and $i < \frac{n-1}{2} < n$, we have $E_1^{0,i} = \mathrm{H}^i(Y(0)) \cong \wedge^i \rho_n$. $\square$

Corollary 3.4. *We have*

$$\mathrm{IH}^{2i}(X_n) \;\cong\; (-1)^i \wedge^i \rho_n \;\oplus \bigoplus_{\substack{0<p<n-1\\ 0\le q\le \min(i,2i-p)}} (-1)^{p+q}\; \mathrm{Ind}_{S_{n-p-1}\times S_{p+1}}^{S_n}\Big(\wedge^{2i-p-q}\rho_{n-p-1} \boxtimes \mathrm{IH}^{2(i-q)}(X_{p+1})\Big)$$

as virtual representations of S_n.

Proof. Given a finite convergent spectral sequence in a semisimple category, one always has the Euler characteristic equation

$$\bigoplus_{p,q} (-1)^{p+q}\, E_\infty^{p,q} \;\cong\; \bigoplus_{p,q} (-1)^{p+q}\, E_1^{p,q}.$$

The result then follows from Proposition 3.3. $\square$

Remark 3.5. Taking dimensions in Corollary 3.4, we obtain the integer equation

$$\begin{aligned} c_{n,i} &= (-1)^i \binom{n}{i} + \sum_{\substack{0<p<n-1\\ 0\le q\le \min(i,2i-p)}} (-1)^{p+q} \binom{n}{n-p-1}\binom{n-p-1}{2i-p-q} c_{p+1,i-q} \\ &= (-1)^i \binom{n}{i} + \sum_{\substack{0<p<n-1\\ 0\le q\le \min(i,2i-p)}} (-1)^{p+q} \binom{n}{p+1,\, 2i-p-q,\, n+q-2i-1} c_{p+1,i-q}. \end{aligned}$$

If we make the substitution $k = p+1$ and $j = i - q$, we obtain the recursion in Theorem 1.1(1). Thus Corollary 3.4 is indeed a categorification of Theorem 1.1.

Remark 3.6. It is clear how to generalize Corollary 3.4 to arbitrary arrangements. Using the notation of [EPW, §3.1], let $\mathcal{A}$ be an arrangement, let $U_\mathcal{A}$ be the complement of $\mathcal{A}$, and let $X_\mathcal{A}$ denote the reciprocal plane of $\mathcal{A}$. Let L be the lattice of flats; for each $F \in L$, we can define the localization $\mathcal{A}_F$ and the restriction $\mathcal{A}^F$. Suppose that W is a finite group that acts on $\mathcal{A}$, and therefore on L and on $X_{\mathcal{A}}$. For each $F \in L$, let $W(F)$ be its stabilizer, and let $[F]$ be its equivalence class in L/W. Then, in the virtual representation ring of W, we have

$$\mathrm{IH}^{2i}(X_{\mathcal{A}}) \;\cong\; \bigoplus_{\substack{[F]\in L/W\\ j\geq 0}} (-1)^j \;\mathrm{Ind}_{W(F)}^{W}\Big(H^j(U_{\mathcal{A}_F}) \otimes \mathrm{IH}^{2(\operatorname{crk} F - i + j)}(X_{\mathcal{A}^F})\Big).$$

In the case of Corollary 3.4, $W(F)$ decomposes as a product of two groups, one of which acts only on $H^j(U_{\mathcal{A}_F})$ and the other of which acts only on $\mathrm{IH}^{2(\operatorname{crk} F-i+j)}(X_{\mathcal{A}^F})$, which greatly simplifies the formula from a computational standpoint. This phenomenon does not occur in general; for example, it does not occur for the symmetric group acting on the braid arrangement, where flats are indexed by set-theoretic partitions.

4 Solving the categorified recursion

In this section we prove Theorem 1.2(2). We proceed by induction on n. When $n = 2$, we must have $i = 0$, and the theorem says that $\mathrm{IH}^0(X_2)$ is equal to the trivial representation $V_{[2]}$ of S_2. Since intersection cohomology agrees with ordinary cohomology in degree 0, this is obvious. Now let $n > 2$ be given and assume that $\mathrm{IH}^{2i}(X_m) \cong V_{[m-2i,2^i]}$ for all $2 \leq m < n$ and $i < \frac{m-1}{2}$.

Lemma 4.1. *Suppose that $0 \leq i < \frac{n-1}{2}$, $0 < p < n$, and $0 \leq q \leq \min(i, 2i - p)$.*

- (i) $\mathrm{Hom}_{S_n}\Big(V_{[n-2i,2^i]},\ \wedge^i \rho_n\Big) \neq 0$ *if and only if $i = 0$.*
- (ii) $\mathrm{Hom}_{S_n}\Big(V_{[n-2i,2^i]},\ \mathrm{Ind}_{S_{n-p-1}\times S_{p+1}}^{S_n}\Big(\wedge^{2i-p-q}\rho_{n-p-1} \boxtimes \mathrm{IH}^{2(i-q)}(X_{p+1})\Big)\Big) \neq 0$
  if and only if $i > 0$, $p = 2i - 1$, and $q = 1$.

Proof. We begin with part (i). If $i = 0$, then $V_{[n-2i,2^i]} = V_{[n]}$ and $\wedge^i\rho_n$ are both equal to the 1-dimensional trivial representation. If $i > 0$, then

$$\wedge^i \rho_n \cong \wedge^i(V_{[n]} \oplus V_{[n-1,1]}) \cong \bigoplus_{j+k=i} \wedge^j V_{[n]} \otimes \wedge^k V_{[n-1,1]}.$$

But $V_{[n]}$ is trivial, so $\wedge^j V_{[n]}$ is trivial if $j \in \{0, 1\}$ and zero otherwise, and $\wedge^k V_{[n-1,1]} \cong V_{[n-k,1^k]}$ [FH91, Exercise 4.6], thus

$$\wedge^i \rho_n \cong \wedge^i V_{[n-1,1]} \oplus \wedge^{i-1} V_{[n-1,1]} \cong V_{[n-i,1^i]} \oplus V_{[n-i+1,1^{i-1}]}.$$

In particular, it does not contain $V_{[n-2i,2^i]}$ as a summand.

We now move on to part (ii). By the argument above, we have

$$\wedge^{2i-p-q}\rho_{n-p-1} \cong V_{[n+q-2i-1,1^{2i-p-q}]} \oplus V_{[n+q-2i,1^{2i-p-q-1}]}$$

as representations of S_{n-p-1} (here we adhere to the convention that if the subscript is not a partition, the corresponding representation is zero). By our inductive hypothesis,

$$\mathrm{IH}^{2(i-q)}(X_{p+1}) \cong V_{[p+2q-2i+1,2^{i-q}]}$$

as representations of S_{p+1}. Thus we are interested in

$$\mathrm{Hom}_{S_n}\Big(V_{[n-2i,2^i]},\ \mathrm{Ind}_{S_{n-p-1}\times S_{p+1}}^{S_n}\left(V_{[n+q-2i-1,1^{2i-p-q}]} \boxtimes V_{[p+2q-2i+1,2^{i-q}]}\right)\Big)$$

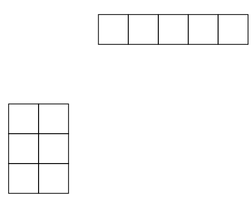
$$\oplus$$

$$\mathrm{Hom}_{S_n}\Big(V_{[n-2i,2^i]},\ \mathrm{Ind}_{S_{n-p-1}\times S_{p+1}}^{S_n}\left(V_{[n+q-2i,1^{2i-p-q-1}]} \boxtimes V_{[p+2q-2i+1,2^{i-q}]}\right)\Big).$$

The dimension $c^{\nu}_{\mu\lambda}$ of

$$\mathrm{Hom}_{S_{|\nu|}}\Big(V_\nu,\ \mathrm{Ind}_{S_{|\mu|}\times S_{|\lambda|}}^{S_\nu}\left(V_\mu \boxtimes V_\lambda\right)\Big)$$

is equal to the number of Littlewood-Richardson tableaux of shape ν/λ and weight μ [Ful97, §5.2]. We will let $\nu = [n-2i, 2^i]$, $\mu = [n+q-2i-1, 1^{2i-p-q}]$, $\mu' = [n+q-2i, 1^{2i-p-q-1}]$, and $\lambda = [p+2q-2i+1, 2^{i-q}]$, and we will compute the coefficients $c^{\nu}_{\mu\lambda}$ and $c^{\nu}_{\mu'\lambda}$. In order for either of these two coefficients to be nonzero, we need the diagram of shape λ to fit inside of the diagram of shape ν, which is the case if and only if $n \geq p+2q+1$. In this case, the skew diagram ν/λ consists of a $1\times(n-p-2q-1)$ component in the top-right and a $q\times 2$ component in the bottom-left. Below is a picture of ν/λ with $n=20$, $i=6$, $p=8$, and $q=3$.



Any Littlewood-Richardson tableau of this shape must consist of all 1's in the upper-right and two ascending sequences of length q in the lower-left, beginning (in the top row) with either 1 or 2. Since μ and μ' are both hooks, it is not possible for a tableau of weight μ or μ' to contain exactly two 2's, thus we may assume that $q \in \{0,1\}$ and the content of our tableau consists of at most one 2 and all remaining entries equal to 1. If the weight is μ, this means that $2i-p-q=0$ or 1, so $\lambda = [q+1, 2^{i-q}]$ or $[q, 2^{i-q}]$. For λ to be a partition and p to be positive, we must have $i>0$, $p = 2i-1$, and $q=1$. Thus we may conclude that $c^{\nu}_{\mu\lambda} = 1$ if $i>0$, $p=2i-1$, and $q=1$, and it is zero otherwise. By similar reasoning, we deduce that $c^{\nu}_{\mu'\lambda}$ is always equal to zero. $\square$

We now complete the proof of Theorem 1.2(2). By Lemma 4.1, there is exactly one term on the right-hand side of the isomorphism in Corollary 3.4 that contains $V_{[n-2i,2^i]}$ as a summand. This implies that $\mathrm{IH}^{2i}(X_n)$ contains $V_{[n-2i,2^i]}$ as a summand. But Theorem 1.2(1), along with the hook-length formula for the dimension of $V_{[n-2i,2^i]}$, tells us that $\dim \mathrm{IH}^{2i}(X_n) = \dim V_{[n-2i,2^i]}$. Thus $\mathrm{IH}^{2i}(X_n)$ must be isomorphic to $V_{[n-2i,2^i]}$. □

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