REVERSE ASYMPTOTIC ESTIMATES FOR ROOTS OF THE CUBOID CHARACTERISTIC EQUATION IN THE CASE OF THE SECOND CUBOID CONJECTURE.

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Abstract. A perfect cuboid is a rectangular parallelepiped whose edges, whose face diagonals, and whose space diagonal are of integer lengths. The second cuboid conjecture specifies a subclass of perfect cuboids described by one Diophantine equation of tenth degree and claims their non-existence within this subclass. This Diophantine equation has two parameters. Previously asymptotic expansions and estimates for roots of this equation were obtained in the case where the first parameter is fixed and the other tends to infinity. In the present paper reverse asymptotic expansions and estimates are derived in the case where the second parameter is fixed and the first one tends to infinity. Their application to the perfect cuboid problem is discussed.

1. Introduction.

Perfect cuboids are described by six Diophantine equations. These equations are elementary. They are derived with the use of the Pythagorean theorem:

$$\begin{aligned} &x_1^2+x_2^2+x_3^2=L^2, && x_2^2+x_3^2=d_1^2,\\ &x_3^2+x_1^2=d_2^2, && x_1^2+x_2^2=d_3^2. \end{aligned}\tag{1.1}$$

Here x_1, x_2, x_3 are the lengths of the edges of a cuboid, d_1, d_2, d_3 are the lengths of its face diagonals and L is the length of its space diagonal.

None of the solutions for the equations (1.1) are known. Their non-existence is also not proved. This is an open mathematical problem. For the history and various approaches to the problem of perfect cuboids the reader is referred to [1–43]. In the present paper we continue the research from [44–49]. The series of papers [50–62] is devoted to another approach. This approach is not considered here.

In [44] an algebraic parametrization for the Diophantine equations (1.1) was suggested. It uses four rational variables α, β, υ, and z:

$$\begin{aligned} &\frac{x_1}{L}=\frac{2\,\upsilon}{1+\upsilon^2}, && \frac{d_1}{L}=\frac{1-\upsilon^2}{1+\upsilon^2},\\ &\frac{x_2}{L}=\frac{2\,z\,(1-\upsilon^2)}{(1+\upsilon^2)\,(1+z^2)}, && \frac{x_3}{L}=\frac{(1-\upsilon^2)\,(1-z^2)}{(1+\upsilon^2)\,(1+z^2)},\\ &\frac{d_2}{L}=\frac{(1+\upsilon^2)\,(1+z^2)+2\,z(1-\upsilon^2)}{(1+\upsilon^2)\,(1+z^2)}\,\beta, && \frac{d_3}{L}=\frac{2\,(\upsilon^2\,z^2+1)}{(1+\upsilon^2)\,(1+z^2)}\,\alpha. \end{aligned}\tag{1.2}$$

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The variables α, β, υ, and z in (1.2) are not independent. The variable υ is expressed through α and β as a solution of the following algebraic equation:

$$\begin{gathered}\upsilon^4\,\alpha^4\,\beta^4+(6\,\alpha^4\,\upsilon^2\,\beta^4-2\,\upsilon^4\,\alpha^4\,\beta^2-2\,\upsilon^4\,\alpha^2\,\beta^4)+(4\,\upsilon^2\,\beta^4\,\alpha^2+\\+4\,\alpha^4\,\upsilon^2\,\beta^2-12\,\upsilon^4\,\alpha^2\,\beta^2+\upsilon^4\,\alpha^4+\upsilon^4\,\beta^4+\alpha^4\,\beta^4)+(6\,\alpha^4\,\upsilon^2+6\,\upsilon^2\,\beta^4-\\-8\,\alpha^2\,\beta^2\,\upsilon^2-2\,\upsilon^4\,\alpha^2-2\,\upsilon^4\,\beta^2-2\,\alpha^4\,\beta^2-2\,\beta^4\,\alpha^2)+(\upsilon^4+\beta^4+\\+\alpha^4+4\,\alpha^2\,\upsilon^2+4\,\beta^2\,\upsilon^2-12\,\beta^2\,\alpha^2)+(6\,\upsilon^2-2\,\alpha^2-2\,\beta^2)+1=0.\end{gathered}\tag{1.3}$$

Then the variable z is expressed through α, β, and υ by the formula

$$z=\frac{(1+\upsilon^2)\,(1-\beta^2)\,(1+\alpha^2)}{2\,(1+\beta^2)\,(1-\alpha^2\,\upsilon^2)}.\tag{1.4}$$

The equation (1.3) and the formula (1.4) mean that we have two algebraic functions

$$\upsilon=\upsilon(\alpha,\beta), \qquad\qquad z=z(\alpha,\beta).\tag{1.5}$$

Substituting (1.5) into (1.2), we get six algebraic functions

$$\begin{aligned}&x_1=x_1(\alpha,\beta,L), &&x_2=x_2(\alpha,\beta,L), &&x_3=x_3(\alpha,\beta,L),\\ &d_1=d_1(\alpha,\beta,L), &&d_2=d_2(\alpha,\beta,L), &&d_3=d_3(\alpha,\beta,L),\end{aligned}\tag{1.6}$$

They are linear with respect to L. The functions (1.6) satisfy the cuboid equations (1.1) identically with respect to α, β, and L (see Theorem 5.2 in [44]).

The rational numbers α, β, and υ cam be brought to a common denominator:

$$\alpha=\frac{a}{t}, \qquad\qquad \beta=\frac{b}{t}, \qquad\qquad \upsilon=\frac{u}{t}.\tag{1.7}$$

Substituting (1.7) into (1.3), one easily derives the Diophantine equation

$$\begin{gathered}t^{12}+(6\,u^2\,-2\,a^2\,-2\,b^2)\,t^{10}+(u^4\,+b^4+a^4+4\,a^2\,u^2+\\+4\,b^2\,u^2-12\,b^2\,a^2)\,t^8+(6\,a^4\,u^2+6\,u^2\,b^4-8\,a^2\,b^2\,u^2-\\-2\,u^4\,a^2-2\,u^4\,b^2-2\,a^4\,b^2-2\,b^4\,a^2)\,t^6+(4\,u^2\,b^4\,a^2+\\+4\,a^4\,u^2\,b^2-12\,u^4\,a^2\,b^2+u^4\,a^4+u^4\,b^4+a^4\,b^4)\,t^4+\\+(6\,a^4\,u^2\,b^4-2\,u^4\,a^4\,b^2-2\,u^4\,a^2\,b^4)\,t^2+u^4\,a^4\,b^4=0.\end{gathered}\tag{1.8}$$

The following theorem claims exact relation of the equation (1.8) to the cuboid equations (1.1) (see Theorem 5.2 in [44] and Theorem 4.1 in [45]).

Theorem 1.1. *A perfect cuboid does exist if and only if for some positive coprime integer numbers a, b, and u the Diophantine equation* (1.8) *has a positive solution t obeying the inequalities $t>a$, $t>b$, $t>u$, and $(a+t)\,(b+t)>2\,t^2$.*

Due to Theorem 1.1 it is natural to call (1.8) the cuboid characteristic equation, though this term was not previously used for this equation.

In the case of the second cuboid conjecture the parameters a, b, and u are related to each other according to one of the two formulas:

$$b\,u = a^2, \qquad\qquad a\,u = b^2. \tag{1.9}$$

The first equality (1.9) is resolved by means of the formulas

$$a = p\,q, \qquad\qquad b = p^2, \qquad\qquad u = q^2. \tag{1.10}$$

Here $p \neq q$ are two positive coprime integers. Upon substituting (1.10) into the equation (1.8) it reduces to the equation

$$(t-a)\,(t+a)\,Q_{pq}(t) = 0 \tag{1.11}$$

(see [45] or [47]), where $Q_{pq}(t)$ is the following polynomial of tenth degree:

$$\begin{aligned}Q_{pq}(t) = t^{10} + (2\,q^2+p^2)\,(3\,q^2-2\,p^2)\,t^8 + (q^8+10\,p^2\,q^6+\\ +4\,p^4\,q^4-14\,p^6\,q^2+p^8)\,t^6-p^2\,q^2\,(q^8-14\,p^2\,q^6+4\,p^4\,q^4+\\ +10\,p^6\,q^2+p^8)\,t^4-p^6\,q^6\,(q^2+2\,p^2)\,(3\,p^2-2\,q^2)\,t^2-q^{10}\,p^{10}.\end{aligned} \tag{1.12}$$

The second equality (1.9) is similar yo the first one. It is resolved by means of the following formulas similar to (1.10):

$$a = p^2, \qquad\qquad b = p\,q, \qquad\qquad u = q^2. \tag{1.13}$$

Upon substituting (1.13) into the equation (1.8) it reduces to the equation

$$(t-b)\,(t+b)\,Q_{pq}(t) = 0. \tag{1.14}$$

The roots $t = a$, $t = -a$, $t = b$, and $t = -b$ of the equations (1.11) and (1.14) do not produce perfect cuboids (see Theorem 1.1). Upon splitting off the inessential linear factors from (1.11) and (1.14) we get the equation

$$\begin{aligned}t^{10} + (2\,q^2+p^2)\,(3\,q^2-2\,p^2)\,t^8 + (q^8+10\,p^2\,q^6+4\,p^4\,q^4-\\ -14\,p^6\,q^2+p^8)\,t^6-p^2\,q^2\,(q^8-14\,p^2\,q^6+4\,p^4\,q^4+10\,p^6\,q^2+\\ +p^8)\,t^4-p^6\,q^6\,(q^2+2\,p^2)\,(3\,p^2-2\,q^2)\,t^2-q^{10}\,p^{10}=0.\end{aligned} \tag{1.15}$$

Conjecture 1.1. *For any positive coprime integers $p \neq q$ the polynomial $Q_{pq}(t)$ in (1.12) is irreducible in the ring $\mathbb{Z}[t]$.*

This conjecture is known as the second cuboid conjecture. It was formulated in [45]. In particular it means that if it is true, the equation (1.15) has no integer roots for any positive coprime integers $p \neq q$. Like in [49], here we shall not try to prove or disprove Conjecture 1.1. Instead, we study real positive roots of the equation (1.15) in the case where p is much larger than q. This case is reverse to that of [49]. Below we find asymptotic expansions and estimates for the roots of the equation (1.15) as $p \to +\infty$. They can be applied to a numeric search for perfect cuboids. In particular they can be applied to further improvements of the numeric algorithm proposed in [49].

2. The parameters reversion formula.

In order to obtain asymptotic expansions for roots of the equation (1.15) as $p\to+\infty$ one can go through the same steps as in [49] for the case where $q\to+\infty$. However, this is too expensive way. In order to avoid this way below we use the reversion formula derived in [47]. Here is this formula:

$$Q_{pq}(t)=-\frac{Q_{qp}(p^2\,q^2/t)\,t^{10}}{p^{10}\,q^{10}}. \tag{2.1}$$

The formula (2.1) relates the polynomial $Q_{pq}(t)$ with the reverse polynomial $Q_{qp}(t)$. Both polynomials are even. Therefore, like in [49], we use the condition

$$\begin{cases} t>0 \ \text{ if } \ t \ \text{ is a real root,}\\ \operatorname{Re}(t)\geqslant 0 \ \text{ and } \ \operatorname{Im}(t)>0 \ \text{ if } \ t \ \text{ is a complex root}\end{cases} \tag{2.2}$$

in order to divide their roots into two groups. Let's denote through $t_1,\ t_2,\ \dots,$ t_{10} the roots of the polynomial $Q_{pq}(t)$ and through $\tilde t_1,\ \tilde t_2,\ \dots,\ \tilde t_{10}$ the roots of the reverse polynomial $Q_{qp}(t)$. Then the roots $t_1,\ t_2,\ \dots,\ t_5$ and $\tilde t_1,\ \tilde t_2,\ \dots,\ \tilde t_5$ are those roots that satisfy the condition (2.2). Other roots are opposite to them:

$$\begin{aligned} &t_6=-t_1, &&t_7=-t_2, &&t_8=-t_3, &&t_9=-t_4, &&t_{10}=-t_5,\\ &\tilde t_6=-\tilde t_1, &&\tilde t_7=-\tilde t_2, &&\tilde t_8=-\tilde t_3, &&\tilde t_9=-\tilde t_4, &&\tilde t_{10}=-\tilde t_5.\end{aligned} \tag{2.3}$$

Since $q^{10}\,p^{10}\neq 0$ in (1.12), the roots of both polynomials in (2.3) are nonzero. Hence the formula (2.1) is applicable to all of them.

3. Roots of the reverse polynomial.

Note that $p\to+\infty$ for the reverse polynomial $Q_{qp}(t)$ is equivalent to $q\to+\infty$ for the original polynomial $Q_{pq}(t)$. Therefore we can use the results of [49] and reformulate them as the results for the roots $\tilde t_1,\ \dots,\ \tilde t_{10}$ of the reverse polynomial $Q_{qp}(t)$ as $p\to+\infty$. These results are formulated using five asymptotic intervals. Three of these five intervals are on the real axis:

$$q^2-\frac{5\,q^3}{p}<\tilde t<q^2, \tag{3.1}$$

$$q^2<\tilde t<q^2+\frac{5\,q^3}{p} \tag{3.2}$$

$$q\,p-\frac{16\,q^3}{p}-\frac{5\,q^4}{p^2}<\tilde t<q\,p-\frac{16\,q^3}{p}+\frac{5\,q^4}{p^2}. \tag{3.3}$$

The other two intervals are on the imaginary axis of the complex plane:

$$(\sqrt{2}+1)\,p^2+(\sqrt{2}-2)\,q^2-\frac{5\,q^3}{p}<\operatorname{Im}\tilde t<(\sqrt{2}+1)\,p^2+(\sqrt{2}-2)\,q^2+\frac{5\,q^3}{p}. \tag{3.4}$$

$$(\sqrt{2}-1)\,p^2+(\sqrt{2}+2)\,q^2-\frac{5\,q^3}{p}<\operatorname{Im}\tilde t<(\sqrt{2}-1)\,p^2+(\sqrt{2}+2)\,q^2+\frac{5\,q^3}{p}. \tag{3.5}$$

Now Theorem 6.1 from [49] is formulated as follows.

Theorem 3.1. *For $p \geqslant 59\,q$ five roots $\tilde{t}_1$, $\tilde{t}_2$, $\tilde{t}_3$, $\tilde{t}_4$, $\tilde{t}_5$ of the reverse polynomial $Q_{qp}(t)$ obeying the condition (2.2) are simple. They are located within five disjoint intervals (3.1), (3.2), (3.3), (3.4), (3.5), one per each interval.*

Theorem 3.1 means that three roots $\tilde{t}_1$, $\tilde{t}_2$, $\tilde{t}_3$ of the reverse polynomial $Q_{qp}(t)$ are real. They are arranged as follows:

$$\tilde{t}_1 < \tilde{t}_2 < \tilde{t}_3 \tag{3.6}$$

The other two roots $\tilde{t}_4$ and $\tilde{t}_5$ are imaginary. They are arranged so that

$$\operatorname{Im}\tilde{t}_4 > \operatorname{Im}\tilde{t}_5. \tag{3.7}$$

We are going to preserve the same arrangement (3.6) and (3.7) for the roots of the original polynomial $Q_{pq}(t)$, i.e. we write

$$t_1 < t_2 < t_3, \qquad\qquad \operatorname{Im} t_4 > \operatorname{Im} t_5. \tag{3.8}$$

Then, applying (2.3) and the reversion formula (2.1), from (3.6), (3.7), and (3.8) we derive the following correspondence for the roots of $Q_{pq}(t)$ and $Q_{qp}(t)$:

$$t_1 = \frac{p^2\,q^2}{\tilde{t}_3}, \quad t_2 = \frac{p^2\,q^2}{\tilde{t}_2}, \quad t_3 = \frac{p^2\,q^2}{\tilde{t}_1}, \quad t_4 = -\frac{p^2\,q^2}{\tilde{t}_5}, \quad t_5 = -\frac{p^2\,q^2}{\tilde{t}_4}. \tag{3.9}$$

Below we use the formulas (3.9) in order to derive asymptotic expansions and estimates for the roots of the original polynomial $Q_{pq}(t)$ as $p \to +\infty$.

4. Asymptotics of the real roots.

According to Theorem 3.1, for $p \geqslant 59\,q$ the root $\tilde{t}_3$ of the reverse polynomial $Q_{qp}(t)$ belongs to the third asymptotic interval (3.3). Applying the first formula (3.9), we derive the following interval for the root t_1 of the polynomial $Q_{pq}(t)$:

$$\frac{p^4\,q}{p^3 - 16\,q^2\,p + 5\,q^3} < t < \frac{p^4\,q}{p^3 - 16\,q^2\,p - 5\,q^3}. \tag{4.1}$$

The left and right sides of the inequalities (4.1) have the asymptotic expansions

$$\begin{aligned}
\frac{p^4\,q}{p^3 - 16\,q^2\,p + 5\,q^3} &= p\,q + \frac{16\,q^3}{p} - \frac{5\,q^4}{p^2} + \frac{256\,q^5}{p^3} - \frac{160\,q^6}{p^4} + \ldots\,,\\
\frac{p^4\,q}{p^3 - 16\,q^2\,p - 5\,q^3} &= p\,q + \frac{16\,q^3}{p} + \frac{5\,q^4}{p^2} + \frac{256\,q^5}{p^3} + \frac{160\,q^6}{p^4} + \ldots
\end{aligned} \tag{4.2}$$

as $p \to +\infty$. From (4.1) and (4.2) one can derive an asymptotic expansion for t_1:

$$t_1 = p\,q + \frac{16\,q^3}{p} + R_1(p,q) \quad \text{as} \quad p \to +\infty. \tag{4.3}$$

Like in [49], our goal here is to obtain an estimate of the form

$$|R_1(p,q)| < \frac{C(q)}{p^2}. \tag{4.4}$$

In order to get such an estimate we substitute

$$t = p\,q + \frac{16\,q^3}{p} + \frac{c}{p^2} \tag{4.5}$$

into the equation (1.15). Then we perform another substitution into the equation obtained as a result of substituting (4.5) into (1.15):

$$p = \frac{1}{z}. \tag{4.6}$$

Upon two substitutions (4.5) and (4.6) and upon removing denominators the equation (1.15) is presented as a polynomial equation in the new variables c and z. It is a peculiarity of this equation that it can be written as

$$f(c,q,z) = -2\,q^5\,c. \tag{4.7}$$

Here $f(c,q,z)$ is a polynomial of three variables given by an explicit formula. However, the formula for $f(c,q,z)$ is rather huge. Therefore it is placed to the ancillary file `strategy_formulas_02.txt` in a machine-readable form.

Let $p \geqslant 59\,q$ and let the parameter c run over the interval from $-5\,q^4$ to $5\,q^4$:

$$-5\,q^4 < c < 5\,q^4. \tag{4.8}$$

From $p \geqslant 59\,q$ and from (4.6) we derive the estimate $|z| \leqslant 1/59\,q^{-1}$. Using this estimate and using the inequalities (4.8), by means of direct calculations one can derive the following estimate for the modulus of the function $f(c,q,z)$:

$$|f(c,q,z)| < 3\,q^9. \tag{4.9}$$

For fixed q and z the estimate (4.9) means that the left hand side of the equation (4.7) is a continuous function of c whose values are within the range from $-3\,q^9$ to $3\,q^9$ while c runs over the interval (4.8). The right hand side of the equation (4.7) is also a continuous function of c. It decreases from $10\,q^9$ to $-10\,q^9$ in the interval (4.8). Therefore somewhere in the interval (4.8) there is at least one root of the polynomial equation (4.7).

The parameter c is related to the variable t by means of the formula (4.5). Therefore the inequalities (4.8) for c imply the following inequalities for t:

$$p\,q + \frac{16\,q^3}{p} - \frac{5\,q^4}{p^2} < t < p\,q + \frac{16\,q^3}{p} + \frac{5\,q^4}{p^2}. \tag{4.10}$$

This result is worth enough, it is formulated as a theorem.

Theorem 4.1. *For each $p \geqslant 59\,q$ there is at least one real root of the equation (1.15) satisfying the inequalities (4.10).*

Note that the inequalities (4.10) are not derived from (4.1) and (4.2). They follow from (4.7) and (4.9) as described above. These inequalities provide an estimate of the form (4.4) for the remainder term in the asymptotic expansion (4.3).

Let's proceed to the roots t_2 and t_3. According to Theorem 3.1 for $p \geqslant 59\,q$ the roots $\tilde{t}_2$ and $\tilde{t}_1$ of the reverse polynomial $Q_{qp}(t)$ belongs to the intervals (3.2) and (3.1) respectively. Applying the second and the third formulas (3.9), we derive the following two intervals for the roots t_2 and t_3 of the original polynomial $Q_{pq}(t)$:

$$\frac{p^3}{p+5\,q} < t < p^2 \tag{4.11}$$

$$p^2 < t < \frac{p^3}{p-5\,q}. \tag{4.12}$$

The left and right hand sides of the above inequalities (4.11) and (4.12) have the following asymptotic expansions as $p \to +\infty$:

$$\begin{aligned} \frac{p^3}{p+5\,q} &= p^2 - 5\,q\,p + 25\,q^2 - \frac{125\,q^3}{p} + \ldots\,, \\ \frac{p^3}{p-5\,q} &= p^2 + 5\,q\,p + 25\,q^2 + \frac{125\,q^3}{p} + \ldots\,. \end{aligned} \tag{4.13}$$

From (4.11), (4.12), and (4.13) one can derive asymptotic formulas for t_2 and t_3:

$$t_2 \sim p^2, \qquad\qquad t_3 \sim p^2 \tag{4.14}$$

as $p \to +\infty$. Unfortunately the asymptotic formulas (4.14) are too rough. They need to be refined. The refinement of the first formula (4.14) looks like

$$t_2 = p^2 - 2\,q\,p - 2\,q^2 + R_2(p,q) \quad \text{as} \quad p \to +\infty. \tag{4.15}$$

Like in (4.4), we need to derive some estimate of the form

$$|R_2(p,q)| < \frac{C(q)}{p} \tag{4.16}$$

for the remainder term $R_2(p,q)$ in (4.15). For this purpose we substitute

$$t = p^2 - 2\,q\,p - 2\,q^2 + \frac{c}{p} \tag{4.17}$$

into the equations (1.15). Then we replace p with the new variable z using the substitution (4.6). As a result of two substitutions (4.17) and (4.6) upon removing denominators we get a polynomial equation in the new variables c and z. As it turns out, this polynomial equation can be written in the following way:

$$80\,q^4 + \varphi(c,q,z) = -16\,q\,c. \tag{4.18}$$

Here $\varphi(c,q,z)$ is a polynomial of three variables given by an explicit formula. The formula for $\varphi(c,q,z)$ is rather huge. Therefore it is placed to the ancillary file `strategy_formulas_02.txt` in a machine-readable form.

Let $p \geqslant 59\,q$ and let the parameter c run over the interval from $-9\,q^3$ to 0:

$$-9\,q^3 < c < 0. \tag{4.19}$$

From $p \geqslant 59\,q$ and from (4.6) we derive the estimate $|z| \leqslant 1/59\,q^{-1}$. Using this estimate and using the inequalities (4.19), by means of direct calculations one can derive the following estimate for the modulus of the function $\varphi(c,q,z)$:

$$|\varphi(c,q,z)| < 52\,q^4. \tag{4.20}$$

For fixed q and z the estimate (4.20) means that the left hand side of the equation (4.18) is a continuous function of c taking its values within the range from $28\,q^4$ to $132\,q^4$ while c runs over the interval (4.19). The right hand side of the equation (4.18) is also a continuous function of c. It decreases from $144\,q^4$ to 0 in the interval (4.19). Therefore somewhere in the interval (4.19) there is at least one root of the polynomial equation (4.18).

The parameter c in (4.18) is related to t by means of the formula (4.17). Therefore the inequalities (4.19) for c imply the following inequalities for t:

$$p^2 - 2\,q\,p - 2\,q^2 - \frac{9\,q^3}{p} < t < p^2 - 2\,q\,p - 2\,q^2. \tag{4.21}$$

This result is formulated as the following theorem.

Theorem 4.2. *For each $p \geqslant 59\,q$ there is at least one real root of the equation* (1.15) *satisfying the inequalities* (4.21).

Note that the inequalities (4.21) prove the asymptotic expansion (4.15) and provide an estimate of the form (4.16) for the remainder term $R_2(p,q)$ in it.

The root t_3 of the equation (1.15) is handled in a similar way. The refinement of the second asymptotic formula (4.14) for this root looks like

$$t_3 = p^2 + 2\,q\,p - 2\,q^2 + R_3(p,q) \ \text{ as } \ p \to +\infty. \tag{4.22}$$

Its remainder term $R_3(p,q)$ obeys the estimate of the form

$$|R_3(p,q)| < \frac{C(q)}{p}. \tag{4.23}$$

In order to prove the formulas (4.22) and (4.23) we substitute

$$t = p^2 + 2\,q\,p - 2\,q^2 + \frac{c}{p} \tag{4.24}$$

into the equations (1.15). Then we replace p with the new variable z using the substitution (4.6). As a result of two substitutions (4.24) and (4.6) upon removing denominators we get a polynomial equation in the new variables c and z. As it turns out, this polynomial equation can be written in the following way:

$$80\,q^4 + \psi(c,q,z) = 16\,q\,c. \tag{4.25}$$

Here $\psi(c,q,z)$ is a polynomial of three variables given by an explicit formula. The formula for $\psi(c,q,z)$ is rather huge. Therefore it is placed to the ancillary file `strategy_formulas_02.txt` in a machine-readable form.

Let $p \geqslant 59\,q$ and let the parameter c run over the interval from 0 to $9\,q^3$:

$$0 < c < 9\,q^3. \tag{4.26}$$

From $p \geqslant 59\,q$ and from (4.6) we derive the estimate $|z| \leqslant 1/59\,q^{-1}$. Using this estimate and using the inequalities (4.26), by means of direct calculations one can derive the following estimate for the modulus of the function $\psi(c,q,z)$:

$$|\psi(c,q,z)| < 52\,q^4. \tag{4.27}$$

For fixed q and z the estimate (4.27) means that the left hand side of the equation (4.25) is a continuous function of c taking its values within the range from $28\,q^4$ to $132\,q^4$ while c runs over the interval (4.26). The right hand side of the equation (4.25) is also a continuous function of c. It increases from 0 to $144\,q^4$ in the interval (4.26). Therefore somewhere in the interval (4.26) there is at least one root of the polynomial equation (4.25).

The parameter c is related to the variable t by means of the formula (4.24). Therefore the inequalities (4.26) for c imply the following inequalities for t:

$$p^2 + 2\,q\,p - 2\,q^2 < t < p^2 + 2\,q\,p - 2\,q^2 + \frac{9\,q^3}{p}. \tag{4.28}$$

This result is formulated as the following theorem.

Theorem 4.3. *For each $p \geqslant 59\,q$ there is at least one real root of the equation* (1.15) *satisfying the inequalities* (4.28).

Apart from Theorem 4.3, the inequalities (4.28) prove the asymptotic expansion (4.22) and the estimate (4.23) for the remainder term in it.

5. Asymptotics of the complex roots.

According to Theorem 3.1, for $p \geqslant 59\,q$ the root $\tilde{t}_5$ of the reverse polynomial $Q_{qp}(t)$ belongs to the fifth asymptotic interval (3.5). Applying the fourth formula (3.9), we derive the following interval for the root t_4 of the polynomial $Q_{pq}(t)$:

$$\frac{p^3\,q^2}{(\sqrt{2}-1)\,p^3+(\sqrt{2}+2)\,p\,q^2+5\,q^3} < \operatorname{Im}\,t < \frac{p^3\,q^2}{(\sqrt{2}-1)\,p^3+(\sqrt{2}+2)\,p\,q^2-5\,q^3}. \tag{5.1}$$

Like in (4.2) and (4.13), using asymptotic expansions for both sides of the inequalities (5.1), we derive the following asymptotic expansion for the root t_4:

$$t_4 = (\sqrt{2}+1)\,\mathrm{i}\,q^2 + R_4(p,q) \quad \text{as} \quad p \to +\infty. \tag{5.2}$$

Here $\mathrm{i} = \sqrt{-1}$. The expansion (5.2) is analogous to (4.3), (4.15), and (4.22). Our goal here is to derive an estimate for the remainder term $R_4(p,q)$ of the form

$$|R_4(p,q)| < \frac{C(q)}{p}. \tag{5.3}$$

In order to obtain such an estimate we substitute

$$t=(\sqrt{2}+1)\,\mathrm{i}\,q^2+\frac{\mathrm{i}\,c}{p^2} \tag{5.4}$$

into the equation (1.15). Then we apply the substitution (4.6) to the equation obtained as a result of substituting (5.2) into (1.15). Upon applying two substitutions (5.4) and (4.6) and upon removing denominators the equation (1.15) is written as a polynomial equation in the new variables c and z. It can be represented as

$$\eta(c,q,z)=16\,c. \tag{5.5}$$

Here $\eta(c,q,z)$ is a polynomial of three variables given by an explicit formula. The formula for $\eta(c,q,z)$ is rather huge. Therefore it is placed to the ancillary file `strategy_formulas_02.txt` in a machine-readable form.

Let $p\geqslant 59\,q$ and let the parameter c run over the interval from $-5\,q^3$ to $5\,q^3$:

$$-5\,q^3<c<5\,q^3. \tag{5.6}$$

From $p\geqslant 59\,q$ and from (4.6) we derive the estimate $|z|\leqslant 1/59\,q^{-1}$. Using this estimate and using the inequalities (5.6), by means of direct calculations one can derive the following estimate for the modulus of the function $\eta(c,q,z)$:

$$|\eta(c,q,z)|<14\,q^3. \tag{5.7}$$

For fixed q and z the estimate (5.7) means that the left hand side of the equation (5.5) is a continuous function of c taking its values within the range from $-14\,q^3$ to $14\,q^3$ while c runs over the interval (5.6). The right hand side of the equation (5.5) is also a continuous function of c. It increases from $-80\,q^3$ to $80\,q^3$ in the interval (5.6). Therefore somewhere in the interval (5.6) there is at least one root of the polynomial equation (5.5).

The parameter c is related to the variable t by means of the formula (5.4). Therefore the inequalities (5.6) for c imply the following inequalities for t:

$$(\sqrt{2}+1)\,q^2-\frac{5\,q^3}{p^2}<\operatorname{Im}\,t<(\sqrt{2}+1)\,q^2+\frac{5\,q^3}{p^2}. \tag{5.8}$$

This result leads to the following theorem.

Theorem 5.1. *For each $p\geqslant 59\,q$ there is at least one purely imaginary root of the equation (1.15) satisfying the inequalities (5.8).*

The complex root t_5 of the polynomial $Q_{pq}(t)$ is similar to the root t_4. The asymptotic expansion for this root is derived from the inequalities (3.4) with the use of the fifth formula (3.9). This expansion looks like

$$t_5=(\sqrt{2}-1)\,\mathrm{i}\,q^2+R_5(p,q)\quad\text{as}\quad p\to+\infty. \tag{5.9}$$

Here $\mathrm{i}=\sqrt{-1}$. The expansion (5.9) is analogous to (5.2). Our goal here is to derive an estimate for the remainder term $R_5(p,q)$ of the form

$$|R_5(p,q)|<\frac{C(q)}{p}. \tag{5.10}$$

In order to obtain such an estimate we substitute

$$t = (\sqrt{2} - 1)\,\mathrm{i}\,q^2 + \frac{\mathrm{i}\,c}{p^2} \tag{5.11}$$

into the equation (1.15). Then we apply the substitution (4.6) to the equation obtained as a result of substituting (5.11) into (1.15). Upon applying two substitutions (5.11) and (4.6) and upon removing denominators the equation (1.15) is written as a polynomial equation in the variables c and z. It can be represented as

$$\zeta(c, q, z) = 16\,c. \tag{5.12}$$

Here $\zeta(c, q, z)$ is a polynomial of three variables given by an explicit formula. The formula for $\zeta(c, q, z)$ is rather huge. Therefore it is placed to the ancillary file `strategy_formulas_02.txt` in a machine-readable form.

Let $p \geqslant 59\,q$ and let the parameter c run over the interval (5.6). From $p \geqslant 59\,q$ and from (4.6) we derive the estimate $|z| \leqslant 1/59\,q^{-1}$. Using this estimate and using the inequalities (5.6), by means of direct calculations one can derive the following estimate for the modulus of the function $\zeta(c, q, z)$:

$$|\zeta(c, q, z)| < 14\,q^3. \tag{5.13}$$

For fixed q and z the estimate (5.13) means that the left hand side of the equation (5.12) is a continuous function of c taking its values within the range from $-14\,q^3$ to $14\,q^3$ while c runs over the interval (5.6). The right hand side of the equation (5.12) is also a continuous function of c. It increases from $-80\,q^3$ to $80\,q^3$ in the interval (5.6). Therefore somewhere in the interval (5.6) there is at least one root of the polynomial equation (5.12).

The parameter c is related to the variable t by means of the formula (5.11). Therefore the inequalities (5.6) for c imply the following inequalities for t:

$$(\sqrt{2} - 1)\,q^2 - \frac{5\,q^3}{p^2} < \operatorname{Im} t < (\sqrt{2} - 1)\,q^2 + \frac{5\,q^3}{p^2}. \tag{5.14}$$

This result leads to the following theorem.

Theorem 5.2. *For each $p \geqslant 59\,q$ there is at least one purely imaginary root of the equation* (1.15) *satisfying the inequalities* (5.14).

Theorems 5.1 and 5.2 solve the problem of obtaining estimates of the form (5.3) and (5.10) for the remainder terms in the asymptotic expansions (5.2) and (5.9) for $p \geqslant 59\,q$. Along with Theorems 4.1, 4.2, and 4.3, Theorems 5.1 and 5.2 separate the roots t_1, t_2, t_3, t_4, t_5 of the equation (1.15) from each other for sufficiently large p and specify their locations.

6. Non-intersection of asymptotic intervals.

The roots $\tilde{t}_1$, $\tilde{t}_2$, $\tilde{t}_3$, $\tilde{t}_4$, $\tilde{t}_5$ of the reverse polynomial belong to the intervals (3.1), (3.2), (3.3), (3.4), (3.5), one per each interval. However, the asymptotic intervals (4.10), (4.21), (4.28), (5.8), (5.14) do not exactly correspond to them by virtue of the formula (3.9). For this reason we need to prove some non-intersection results concerning the intervals (4.10), (4.21), (4.28), (5.8), and (5.14).

Lemma 6.1. *For $p \geqslant 59\,q$ the asymptotic intervals* (4.10), (4.21), (4.28), (5.8), *and* (5.14) *do not comprise the origin.*

Proof. Indeed, from $p \geqslant 59\,q$ for the left endpoint of the interval (4.10) we derive

$$p\,q + \frac{16\,q^3}{p} - \frac{5\,q^4}{p^2} > p\,q - \frac{5\,q^4}{p^2} \geqslant 59\,q^2 - \frac{5\,q^2}{59^2} > 58\,q^2 > 0. \tag{6.1}$$

For the left endpoint of the interval (4.21) we derive

$$\begin{gathered} p^2 - 2\,q\,p - 2\,q^2 - \frac{9\,q^3}{p} = (p-q)^2 - 3\,q^2 - \frac{9\,q^3}{p} \geqslant \\ \geqslant (58\,q)^2 - 3\,q^2 - \frac{9\,q^2}{59} > 3360\,q^2 > 0. \end{gathered} \tag{6.2}$$

The case of the interval (4.28) is similar. In this case we have

$$p^2 + 2\,q\,p - 2\,q^2 = (p+q)^2 - 3\,q^2 \geqslant (60\,q)^2 - 3\,q^2 = 3597\,q^2 > 0. \tag{6.3}$$

For the bottom endpoints of the intervals (5.8) and (5.14) from $p \geqslant 59\,q$ we derive

$$(\sqrt{2}+1)\,q^2 - \frac{5\,q^3}{p^2} > 2.4\,q^2 > 0, \qquad (\sqrt{2}-1)\,q^2 - \frac{5\,q^3}{p^2} > 0.4\,q^2 > 0. \tag{6.4}$$

The above inequalities (6.1), (6.2), (6.3), and (6.4) prove Lemma 6.1. $\square$

Lemma 6.1 means that for $p \geqslant 59\,q$ the real intervals (4.10), (4.21), and (4.28) do not intersect with the imaginary intervals (5.8) and (5.14). Moreover, the inequalities (6.1), (6.2), (6.3) and (6.4) show that all of these intervals are located within positive half-lines of the real and imaginary axes. Therefore any roots of the equation (1.15) enclosed within these intervals satisfy the condition (2.2).

Lemma 6.2. *For $p \geqslant 59\,q$ the asymptotic intervals* (4.10), (4.21), (4.28), (5.8), *and* (5.14), *do not intersect with each other.*

Proof. Let's compare the left endpoint of the interval (4.21) with the right endpoint of the interval (4.10). For their difference we have the inequalities

$$\begin{gathered} \left(p^2 - 2\,q\,p - 2\,q^2 - \frac{9\,q^3}{p}\right) - \left(p\,q + \frac{16\,q^3}{p} + \frac{5\,q^4}{p^2}\right) = \\ = \left(p - \frac{3\,q}{2}\right)^2 - \frac{17\,q^2}{4} - \frac{25\,q^3}{p} - \frac{5\,q^4}{p^2} \geqslant \\ \geqslant \left(\frac{115\,q}{2}\right)^2 - \frac{17\,q^2}{4} - \frac{25\,q^2}{59} - \frac{5\,q^2}{59^2} > 3301\,q^2 > 0. \end{gathered} \tag{6.5}$$

Similarly, let's compare the left endpoint of the interval (4.28) with the right endpoint of the interval (4.21). For their difference we have the inequalities

$$\bigl(p^2 + 2\,q\,p - 2\,q^2\bigr) - \bigl(p^2 - 2\,q\,p - 2\,q^2\bigr) = 4\,q\,p \geqslant 4\cdot 59\,q^2 > 0. \tag{6.6}$$

In the case of imaginary intervals we compare the bottom endpoint of the interval

(5.8) with the top endpoint of the interval (5.14). For their difference we have

$$\begin{aligned}&\left((\sqrt{2}+1)\,q^2-\frac{5\,q^3}{p^2}\right)-\left((\sqrt{2}-1)\,q^2+\frac{5\,q^3}{p^2}\right)=\\&=\ 2\,q^2-\frac{10\,q^3}{p^2}\geqslant 2\,q^2-\frac{10\,q^2}{59^2}>q^2>0.\end{aligned}\tag{6.7}$$

The above inequalities (6.5), (6.6), and (6.7) prove Lemma 6.2. □

Lemmas 6.1 and 6.2 are summed up in the following theorem.

Theorem 6.1. *For $p\geqslant 59\,q$ five roots t_1, t_2, t_3, t_4, t_5 of the equation* (1.15) *obeying the condition* (2.2) *are simple. They are located within five disjoint intervals* (4.10), (4.21), (4.28), (5.8), (5.14), *one per each interval.*

Due to (2.3) Theorem 6.1 locates all of the ten roots of the equation (1.15).

7. Integer points of asymptotic intervals.

The intervals (4.10), (4.21), (4.28), (5.8), and (5.14) are asymptotically small. Their length decrease as $p\to+\infty$. Like in [49], we use this fact in order to determine the number of integer points within them.

Theorem 7.1. *If $p\geqslant 59\,q$ and $p>9\,q^3$, then the asymptotic intervals* (4.21) *and* (4.28) *have no integer points.*

Theorem 7.2. *If $p\geqslant 59\,q$ and $p^2>10\,q^4$, then the asymptotic interval* (4.10) *has at most one integer point.*

Theorems 7.1 and 7.2 are immediate from the inequalities $9\,q^3/p<1$ and $10\,q^4/p^2<1$ that follow from $p>9\,q^3$ and $p^2>10\,q^4$ respectively. We preserve the inequality $p\geqslant 59\,q$ in Theorems 7.1 and 7.2 in order to emphasize their relation to the roots of the equation (1.15) through Theorem 6.1.

Theorem 7.3. *If $p\geqslant 59\,q$ and $p\geqslant 16\,q^3+5\,q/16$, then the asymptotic interval* (4.10) *has no integer points.*

Theorem 7.3 is more complicated than Theorems 7.1 and 7.2. But its proof repeats the arguments used in proving Theorem 7.3 in [49]. For this reason we do not provide its proof here.

8. Application to the cuboid problem.

The equation (1.15) is a reduced version of the cuboid characteristic equation (1.8). It is related to the perfect cuboid problem through Theorem 1.1. Substituting either (1.10) or (1.13) into the inequalities $t>a$, $t>b$, and $t>u$ from Theorem 1.1, we get the following result expressed by the inequalities

$$t>p^2,\qquad\qquad t>p\,q,\qquad\qquad t>q^2.\tag{8.1}$$

Similarly, substituting either (1.10) or (1.13) into the inequality $(a+t)\,(b+t)>2\,t^2$ from Theorem 1.1, we get the result expressed by the inequality

$$(p^2+t)\,(p\,q+t)>2\,t^2.\tag{8.2}$$

Theorem 1.1 specified for the case of second cuboid conjecture (see Conjecture 1.1) is formulated in the following way.

Theorem 8.1. *A triple of integer numbers p, q, and t satisfying the equation (1.15) and such that $p \neq q$ are coprime provides a perfect cuboid if and only if the inequalities (8.1) and (8.2) are fulfilled.*

The inequalities (8.1) set lower bounds for t. The inequality (8.2) is different. It sets the upper bound for t in the form of the irrational inequality

$$t < \frac{p^2 + p\,q}{2} + \frac{p\,\sqrt{p^2 + 6\,p\,q + q^2}}{2}.$$

Assume that the inequality $p \geqslant 59\,q$ is fulfilled and assume that t belongs to the second asymptotic interval (4.21). The inequality $t < p^2 - 2\,q\,p - 2\,q^2$ from (4.21) and the inequality $t > p^2$ from (8.1) imply the inequality $p^2 < p^2 - 2\,q\,p - 2\,q^2$. This inequality is contradictory since $2\,q\,p + 2\,q^2 > 0$. The contradiction obtained proves the following theorem.

Theorem 8.2. *If $p \geqslant 59\,q$, then the asymptotic interval (4.21) has no points satisfying the inequalities (8.1).*

Now assume that the inequality $p \geqslant 59\,q$ is fulfilled and assume that t belongs to the first asymptotic interval (4.10). In this case we have the following two inequalities taken from (8.1) and (4.10) respectively:

$$p^2 < t, \qquad\qquad t < p\,q + \frac{16\,q^3}{p} + \frac{5\,q^4}{p^2}. \tag{8.3}$$

The inequalities (8.3) imply an inequality for p and q without t:

$$p^2 < p\,q + \frac{16\,q^3}{p} + \frac{5\,q^4}{p^2}. \tag{8.4}$$

The inequality (8.4) can be transformed in the following way:

$$\Bigl(p - \frac{q}{2}\Bigr)^2 - \frac{q^2}{4} - \frac{16\,q^3}{p} - \frac{5\,q^4}{p^2} < 0. \tag{8.5}$$

On the other hand, applying $p \geqslant 59\,q$ to the left hand side of (8.5), we derive

$$\Bigl(p - \frac{q}{2}\Bigr)^2 - \frac{q^2}{4} - \frac{16\,q^3}{p} - \frac{5\,q^4}{p^2} \geqslant 3421\,q^2 > 0. \tag{8.6}$$

The inequalities (8.5) and (8.6) contradict each other. The contradiction obtained proves the following theorem.

Theorem 8.3. *If $p \geqslant 59\,q$, then the asymptotic interval (4.10) has no points satisfying the inequalities (8.1).*

Theorem 8.3 complements Theorem 7.3. Similarly, Theorem 8.2 complements Theorem 7.2 in the case of the asymptotic interval (4.21). These theorems along with Theorem 8.1 are summarized in the following theorem.

Theorem 8.4. *If $p \geqslant 59\,q$, then the Diophantine equation* (1.15) *has no solutions providing perfect cuboids outside the third asymptotic interval* (4.28).

The third asymptotic interval (4.28) is exceptional in Theorem 8.4. The inequality $p \geqslant 59\,q$ does not contradict the inequalities (8.1) and (8.2) within this interval. However, this interval is cut off by applying Theorem 7.1 to it. This yields the following result.

Theorem 8.5. *If $p \geqslant 59\,q$ and $p > 9\,q^3$, then the Diophantine equation* (1.15) *has no solutions providing perfect cuboids at all.*

9. Conclusions.

Theorems 8.4 and 8.5 constitute the main result of the present paper. They should be complemented with Theorem 8.5 from [49]. It is formulated as follows.

Theorem 9.1. *If $q \geqslant 59\,p$, then the Diophantine equation* (1.15) *has no solutions providing perfect cuboids.*

Theorems 8.4 and 8.5 along with Theorem 9.1 proved in [49] outline three regions in the positive quadrant of the $p\,q$-coordinate plane. These regions are:

1) **linear region** given by the linear inequalities

$$\frac{q}{59} < p, \qquad\qquad p < 59\,q; \tag{9.1}$$

2) **nonlinear region** given by the nonlinear inequalities

$$59\,q \leqslant p, \qquad\qquad p \leqslant 9\,q^3; \tag{9.2}$$

3) **no cuboid region** which is the rest of the positive $p\,q$-quadrant.

The inequalities (9.1) and (9.2), as well as other more special inequalities of this paper, could be used in order to further optimize algorithms for a numeric search of perfect cuboids in the case of the second cuboid conjecture.

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