Counting permutations by runs

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Abstract

In his Ph.D. thesis, Ira Gessel proved a reciprocity formula for noncommutative symmetric functions which enables one to count words and permutations with restrictions on the lengths of their increasing runs. We generalize Gessel's theorem to allow for a much wider variety of restrictions on increasing run lengths, and use it to complete the enumeration of permutations with parity restrictions on peaks and valleys, and to give a systematic method for obtaining generating functions for permutation statistics that are expressible in terms of increasing runs. Our methods can also be used to obtain analogous results for alternating runs in permutations.

Keywords: permutations, increasing runs, peaks, valleys, noncommutative symmetric functions

1. Introduction

Given a set A, let A^* be the set of all finite sequences of elements of A, including the empty sequence. We call A an *alphabet*, the elements of A *letters*, A^* the *free monoid* on A,[1] and the elements of A^* *words*. If we refer to words without specifying an alphabet, then we take the alphabet to be $\mathbb{P}$, the set of positive integers. Suppose that our alphabet A is a totally ordered set, such as $\mathbb{P}$. Then every word in A^* can be uniquely decomposed into a sequence of maximal weakly increasing consecutive subsequences, which we call *increasing runs*. For example, the increasing runs of 2142353 are 2, 14, 235, and 3. The notion of increasing runs clearly extends to permutations as well.

As part of his 1977 Ph.D. thesis, Gessel proved a very general identity involving noncommutative symmetric functions [6, Theorem 5.2] from which we can obtain many generating functions for words and permutations with restrictions on the lengths of their increasing runs. We call this result the "run theorem". In this paper, we present a generalization of Gessel's run theorem that will enable us to count words and permutations with an even wider variety

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[1] Indeed, A^* is a free monoid generated by the letters in A, under the operation of concatenation.

of restrictions on increasing run lengths. Specifically, these restrictions are those which can be encoded by a special type of digraph that we shall call a "run network".

The organization of this paper is as follows. In Section 2, we introduce some preliminary definitions, state Gessel's run theorem, and present our generalization of the run theorem. In Sections 3 and 4, we present two separate applications of the generalized run theorem to permutation enumeration.

Let $\pi = \pi_1\pi_2\cdots\pi_n$ be a permutation in $\mathfrak{S}_n$, the set of permutations of $[n] = \{1, 2, \ldots, n\}$ (or more generally, any sequence of n distinct integers); such permutations are called *n-permutations*. We say that i is a *peak* of π if $\pi_{i-1} < \pi_i > \pi_{i+1}$ and that i is a *valley* of π if $\pi_{i-1} > \pi_i < \pi_{i+1}$. For example, given $\pi = 5736214$, its peaks are 2 and 4, and its valleys are 3 and 6.

In [9], Gessel and Zhuang found the exponential generating function

$$\frac{3\sin\left(\frac{1}{2}x\right) + 3\cosh\left(\frac{1}{2}\sqrt{3}x\right)}{3\cos\left(\frac{1}{2}x\right) - \sqrt{3}\sinh\left(\frac{1}{2}\sqrt{3}x\right)} \tag{1.1}$$

for permutations with all peaks odd and all valleys even. Amazingly, (1.1) can also be expressed as

$$\left(1 - E_1x + E_3\frac{x^3}{3!} - E_4\frac{x^4}{4!} + E_6\frac{x^6}{6!} - E_7\frac{x^7}{7!} + \cdots\right)^{-1} \tag{1.2}$$

where the Euler numbers E_n are defined by the identity $\sum_{n=0}^{\infty} E_n x^n/n! = \sec x + \tan x$, which is reminiscent of David and Barton's [3] generating function

$$\left(1 - x + \frac{x^3}{3!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \frac{x^7}{7!} + \cdots\right)^{-1} \tag{1.3}$$

for permutations with no increasing runs of length 3 or greater. The authors explained the similarity between these two generating functions by applying two different homomorphisms to an identity obtained by the run theorem, which we review in Section 2.2 after introducing the run theorem.

The generating function (1.2) also counts permutations with all peaks even and all valleys odd, because these permutations are in bijection with permutations with all peaks odd and all valleys even; the peaks (respectively, valleys) of $\pi = \pi_1\pi_2\cdots\pi_n$ are precisely the valleys (respectively, peaks) of its complement

$$\pi^c = (n+1-\pi_1)(n+1-\pi_2)\cdots(n+1-\pi_n),$$

and vice versa. In Section 3, we complete the enumeration of permutations with parity restrictions on peaks and valleys using the generalized run theorem; we show that

$$(1+x)\frac{2 + 2\cosh(\sqrt{2}x) + \sqrt{2}x\sinh(\sqrt{2}x)}{2 + 2\cosh(\sqrt{2}x) - \sqrt{2}x\sinh(\sqrt{2}x)}$$

is the exponential generating function for permutations with all peaks and valleys even, and that

$$\frac{2 + 2\cosh(\sqrt{2}x) + \sqrt{2}(2+x)\sinh(\sqrt{2}x)}{2 + 2\cosh(\sqrt{2}x) - \sqrt{2}x\sinh(\sqrt{2}x)}$$

is the exponential generating function for permutations with all peaks and valleys odd. We give several identities involving the terms of these generating functions which are proved combinatorially.

Finally, in Section 4 we use the generalized run theorem to find formulae for bivariate generating functions of the form

$$\sum_{n=0}^{\infty} \sum_{\pi \in \mathfrak{S}_n} t^{\mathrm{st}(\pi)} \frac{x^n}{n!}$$

for many permutation statistics st which are expressible in terms of increasing runs. These statistics include variations of peaks and valleys, "double ascents" and "double descents", and "biruns" and "up-down runs". Although equivalent formulae for the generating functions for some of these statistics have been discovered already using other methods, the generalized run theorem provides a straightforward, systematic method for obtaining these generating functions.

2. The run theorem

2.1. Basic definitions

Here we give the definitions for some of the basic concepts of this paper, including descent compositions and noncommutative symmetric functions.

Given a permutation π in $\mathfrak{S}_n$, we say that $i \in [n-1]$ is a *descent* of π if $\pi_i > \pi_{i+1}$. Then, increasing runs of π can be characterized as maximal consecutive subsequences of π containing no descents. Let us call an increasing run *short* if it has length 1, and *long* if it has length at least 2.

The number of increasing runs of a nonempty permutation is one more than its number of descents; in fact, the lengths of the increasing runs determine the descents, and vice versa. Given a subset $S \subseteq [n-1]$ with elements $s_1 < s_2 < \cdots < s_j$, let $C(S)$ be the composition $(s_1, s_2 - s_1, \ldots, s_j - s_{j-1}, n - s_j)$ of n, and given a composition $L = (L_1, L_2, \ldots, L_k)$, let $D(L) = \{L_1, L_1 + L_2, \ldots, L_1 + \cdots + L_{k-1}\}$ be the corresponding subset of $[n-1]$. Then, C and D are inverse bijections. If π is a permutation with descent set S, we call $C(S)$ the *descent composition* of π, which give the lengths of the increasing runs of π. Applying D to the descent composition of a permutation gives its descent set.

The definitions and properties of descents, increasing runs, and descent compositions extend naturally to words in the free monoid on any totally ordered alphabet such as $[n]$ or $\mathbb{P}$. Note that increasing runs in words are allowed to be weakly increasing, whereas increasing runs in permutations are necessarily strictly increasing since no letters repeat.

Throughout this section, fix a field F of characteristic zero. (We can take F to be $\mathbb{C}$ in subsequent sections.) Then $F\langle\langle X_1, X_2, \ldots \rangle\rangle$ is the F-algebra of formal power series in countably many noncommuting indeterminates $X_1, X_2, \ldots$. Consider the elements

$$\mathbf{h}_n \coloneqq \sum_{i_1 \leq \cdots \leq i_n} X_{i_1} X_{i_2} \cdots X_{i_n}$$

of $F\langle\langle X_1, X_2, \ldots \rangle\rangle$, which are noncommutative versions of the complete symmetric functions h_n. Note that $\mathbf{h}_n$ is the noncommutative generating function for weakly increasing words in

$\mathbb{P}^*$ of length n. For example, the weakly increasing word 13449 is encoded by $X_1 X_3 X_4^2 X_9$, which appears in $\mathbf{h}_5$. For a composition $L = (L_1, \dots, L_k)$, we let $\mathbf{h}_L = \mathbf{h}_{L_1} \cdots \mathbf{h}_{L_k}$. Then the F-algebra generated by the elements $\mathbf{h}_L$ is the algebra $\mathbf{Sym}$ of *noncommutative symmetric functions* with coefficients in F, which is a subalgebra of $F\langle\langle X_1, X_2, \dots \rangle\rangle$.

Next, let $\mathbf{Sym}_n$ be the vector space of noncommutative symmetric functions homogeneous of degree n, so $\mathbf{Sym}_n$ is spanned by $\{\mathbf{h}_L\}_{L \models n}$ where $L \models n$ indicates that L is a composition of n, and $\mathbf{Sym}$ is a graded F-algebra with

$$\mathbf{Sym} = \bigoplus_{n=0}^{\infty} \mathbf{Sym}_n.$$

For any composition $L = (L_1, \dots, L_k)$, we also define

$$\mathbf{r}_L = \sum_{L} X_{i_1} X_{i_2} \cdots X_{i_n}$$

where the sum is over all $(i_1, \dots, i_n)$ satisfying

$$\underbrace{i_1 \leq \cdots \leq i_{L_1}}_{L_1} > \underbrace{i_{L_1+1} \leq \cdots \leq i_{L_1+L_2}}_{L_2} > \cdots > \underbrace{i_{L_1+\cdots+L_{k-1}+1} \leq \cdots \leq i_n}_{L_k}.$$

Then, $\mathbf{r}_L$ is the noncommutative generating function for words in $\mathbb{P}^*$ with descent composition L. These $\mathbf{r}_L$ are noncommutative versions of the ribbon Schur functions r_L.

It can be shown that the noncommutative symmetric functions $\mathbf{h}_L$ and $\mathbf{r}_L$ can be expressed in terms of each other via inclusion-exclusion, and that both $\{\mathbf{h}_L\}_{L \models n}$ and $\{\mathbf{r}_L\}_{L \models n}$ are bases for the vector space $\mathbf{Sym}_n$; see [9, Section 4.1].

For more on the basic theory of noncommutative symmetric functions, see [5].

2.2. The original run theorem

For the remainder of this section, let A be a unital F-algebra of characteristic zero. We are now ready to state the run theorem, a reciprocity formula involving the noncommutative symmetric functions $\mathbf{h}_n$ and $\mathbf{r}_L$ defined in the previous subsection.

Theorem 1 (Run Theorem)**.** *Let $L = (L_1, \dots, L_k)$ be a composition, $\{w_1, w_2, \dots\}$ a set of weights from A, and write $w_L = w_{L_1} w_{L_2} \cdots w_{L_k}$. Then, the noncommutative generating function for words in $\mathbb{P}^*$, in which each word with descent composition L is weighted w_L, is*

$$\sum_L w_L \mathbf{r}_L = \left(\sum_{n=0}^{\infty} v_n \mathbf{h}_n \right)^{-1}$$

where the sum on the left is over all compositions L and the v_n are defined by

$$\sum_{n=0}^{\infty} v_n x^n = \left(\sum_{k=0}^{\infty} w_k x^k \right)^{-1}$$

with $w_0 = 1$ (the unity element of A).

This theorem appeared in its original form as Theorem 5.2 of Gessel [6], and is similar to Theorem 4.1 of Jackson and Aleliunas [11] and Theorem 4.2.3 of Goulden and Jackson [10]. Gessel's statement of the theorem does not explicitly use noncommutative symmetric functions, which were not formally defined until 1995 in the seminal paper [5] of Gelfand et al. However, Gessel and Zhuang [9, Theorem 11] restated the run theorem using noncommutative symmetric functions and gave a different proof of the result.

Both previous versions of the run theorem—Theorem 5.2 of Gessel [6] and Theorem 11 of Gessel and Zhuang [9]—stated that the weights are to commute with each other, but the proof in [9] does not actually use this condition. Hence, we allow our algebra A to be commutative or noncommutative.[2] Although we can simply set $A = \mathbb{C}$ in most applications, the fact that we can take A to be noncommutative is pivotal to our proof of the generalized run theorem in the next subsection.

We call this result the run theorem because it can be used to obtain many generating functions which count words and permutations with various restrictions on the lengths of runs (see [6, Section 5.2] and [9, Section 4.3]). In fact, Gessel and Zhuang [9] used this result to explain the similarity between David and Barton's result (1.3) and the generating function (1.2) for permutations with all peaks odd and all valleys even. We briefly summarize the argument below.

By taking $w_i = 1$ for $i < m$ and $w_i = 0$ for $i \geq m$, the run theorem gives

$$\sum_L \mathbf{r}_L = \left(\sum_{n=0}^{\infty} (\mathbf{h}_{mn} - \mathbf{h}_{mn+1}) \right)^{-1} \tag{2.1}$$

where the sum on the left is over all compositions L with all parts less than m. Hence, (2.1) counts words with every increasing run having length less than m.

Next, define the homomorphism $\Phi : \mathbf{Sym} \to F[[x]]$ by $\Phi(\mathbf{h}_n) = x^n/n!$. Then, Gessel and Zhuang [9, Lemma 9] showed that $\Phi(\mathbf{r}_L) = \beta(L)x^n/n!$, where $\beta(L)$ is the number of permutations with descent composition L. Applying Φ to (2.1) then gives

$$\left(1 - x + \frac{x^m}{m!} - \frac{x^{m+1}}{(2m+1)!} + \frac{x^{2m}}{(2m)!} - \frac{x^{2m+1}}{(2m+1)!} + \cdots \right)^{-1}$$

as the exponential generating function for permutations with every increasing run having length less than m. Setting $m = 3$ gives (1.3).

To obtain (1.2), we define a variant of descents and increasing runs as follows. Given a permutation π in $\mathfrak{S}_n$, we say that $i \in [n-1]$ is an *alternating descent* of π if i is odd and $\pi_i > \pi_{i+1}$ or if i is even and $\pi_i < \pi_{i+1}$. Similarly, an *alternating run* of π is a maximal consecutive subsequence of π containing no alternating descents, and we can also define alternating descent sets and alternating descent compositions analogously.

We say that $\pi = \pi_1\pi_2\cdots\pi_n$ is an *alternating permutation* if $\pi_1 > \pi_2 < \pi_3 < \pi_4 < \cdots$ and that π is *reverse-alternating* if $\pi_1 < \pi_2 > \pi_3 < \pi_4 > \cdots$. Alternating permutations are in bijection with reverse-alternating permutations by complementation, and it is well known that the number of alternating n-permutations is the Euler number E_n. Incidentally, an

[2] We do, however, require that the weights commute with noncommutative symmetric functions. Formally, this means that we are working in the tensor product algebra $A \otimes_F \mathbf{Sym}$.

alternating run starting at an even position is an alternating permutation and an alternating run starting at an odd position is a reverse-alternating permutation, which hints at a close connection between these alternating analogues of descents and runs with the Euler numbers appearing in the generating function (1.2).

Define another homomorphism $\hat{\Phi} : \mathbf{Sym} \to F[[x]]$ by $\hat{\Phi}(\mathbf{h}_n) = E_n x^n/n!$. Then, $\hat{\Phi}(\mathbf{r}_L) = \hat{\beta}(L)x^n/n!$ [9, Lemma 10], where $\hat{\beta}(L)$ is the number of permutations with alternating descent composition L. Applying $\hat{\Phi}$ to (2.1) gives (1.2) as the exponential generating function for permutations with every alternating run having length less than 3, and it is easy to show that these are precisely the permutations with all peaks odd and all valleys even.

In general, if applying Φ to an identity obtained by assigning weights to the run theorem yields a generating function counting permutations with certain restrictions on increasing runs, then applying $\hat{\Phi}$ to the same identity yields a generating function counting permutations with the same restrictions on alternating runs.

2.3. The generalized run theorem

Suppose that G is a digraph on the vertex set $[m]$, where each arc (i,j) is assigned a nonempty subset $P_{i,j}$ of $\mathbb{P}$, and let P be the set of all pairs (a,e) where $e=(i,j)$ is an arc of G and $a \in P_{i,j}$. In addition, let $\overrightarrow{P^*} \subseteq P^*$ be the subset of all sequences $\alpha = (a_1,e_1)(a_2,e_2)\cdots(a_n,e_n)$ where $e_1e_2\cdots e_n$ is a walk in G. Given $\alpha = (a_1,e_1)(a_2,e_2)\cdots(a_n,e_n)$ in $\overrightarrow{P^*}$, let $\rho(\alpha) = (a_1,a_2,\ldots,a_n)$, and let $E(\alpha) = (i,j)$ where i and j are the initial and terminal vertices, respectively, of the walk $e_1e_2\cdots e_n$.

We call this construction (G,P) a *run network* if for all nonempty $\alpha,\beta \in \overrightarrow{P^*}$, if $\rho(\alpha) = \rho(\beta)$ and $E(\alpha) = E(\beta)$ then $\alpha = \beta$. That is, the same tuple $(a_1,a_2,\ldots,a_n)$ cannot be obtained by traversing two different walks with the same initial and terminal vertices. Given a run network (G,P), suppose that we want to count words in $\mathbb{P}^*$ with descent composition L whose parts are given by a walk in G, with various weights attached. This can be done using the following generalization of the run theorem.

Theorem 2 (Generalized Run Theorem)**.** *Suppose that G is a digraph on $[m]$ and that (G,P) is a run network on $\mathbb{P}^*$. Let $\{w_a^{(i.j)} : (a,(i,j)) \in P\}$ be a set of weights from A, with $w_a^{(i,j)} = 0$ if $(a,(i,j)) \notin P$. Given a composition L and $1 \le i,j \le m$, let $w^{(i,j)}(L) = w_{L_1}^{e_1}\cdots w_{L_k}^{e_k}$ if there exists $\alpha = (L_1,e_1)\cdots(L_k,e_k) \in \overrightarrow{P^*}$ such that $E(\alpha) = (i,j)$ and $L = \rho(\alpha)$, and let $w^{(i,j)}(L) = 0$ otherwise. Then,*

$$\begin{bmatrix} \sum_L w^{(1,1)}(L)\mathbf{r}_L & \cdots & \sum_L w^{(1,m)}(L)\mathbf{r}_L \\ \vdots & \ddots & \vdots \\ \sum_L w^{(m,1)}(L)\mathbf{r}_L & \cdots & \sum_L w^{(m,m)}(L)\mathbf{r}_L \end{bmatrix} = \begin{bmatrix} \sum_{n=0}^\infty v_n^{(1,1)}\mathbf{h}_n & \cdots & \sum_{n=0}^\infty v_n^{(1,m)}\mathbf{h}_n \\ \vdots & \ddots & \vdots \\ \sum_{n=0}^\infty v_n^{(m,1)}\mathbf{h}_n & \cdots & \sum_{n=0}^\infty v_n^{(m,m)}\mathbf{h}_n \end{bmatrix}^{-1}$$

where each sum in the matrix on the left-hand side is over all compositions L and the $v_n^{(i,j)}$

are given by

$$\begin{bmatrix} \sum_{n=0}^{\infty} v_n^{(1,1)} x^n & \cdots & \sum_{n=0}^{\infty} v_n^{(1,m)} x^n \\ \vdots & \ddots & \vdots \\ \sum_{n=0}^{\infty} v_n^{(m,1)} x^n & \cdots & \sum_{n=0}^{\infty} v_n^{(m,m)} x^n \end{bmatrix} = \left(I_m + \begin{bmatrix} \sum_{k=1}^{\infty} w_k^{(1,1)} x^k & \cdots & \sum_{k=1}^{\infty} w_k^{(1,m)} x^k \\ \vdots & \ddots & \vdots \\ \sum_{k=1}^{\infty} w_k^{(m,1)} x^k & \cdots & \sum_{k=1}^{\infty} w_k^{(m,m)} x^k \end{bmatrix} \right)^{-1}.$$

Proof. We apply the original run theorem with weights coming from the matrix algebra $\mathrm{Mat}_m(A)$. Set

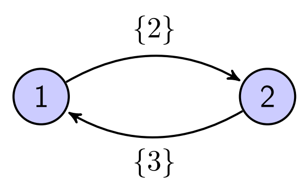
$$w_k = \begin{bmatrix} w_k^{(1,1)} & \cdots & w_k^{(1,m)} \\ \vdots & \ddots & \vdots \\ w_k^{(m,1)} & \cdots & w_k^{(m,m)} \end{bmatrix} \quad \text{and} \quad w_L = \begin{bmatrix} w^{(1,1)}(L) & \cdots & w^{(1,m)}(L) \\ \vdots & \ddots & \vdots \\ w^{(m,1)}(L) & \cdots & w^{(m,m)}(L) \end{bmatrix}.$$

It suffices to verify that if $L = (L_1, \dots, L_k)$ is a composition, then $w_L = w_{L_1} w_{L_2} \cdots w_{L_k}$. Indeed, the (i,j)th entry of $w_{L_1} w_{L_2} \cdots w_{L_k}$ is

$$\sum_{\substack{1 \le p_1, \dots, p_{k+1} \le m \\ p_1 = i,\, p_{k+1} = j}} w_{L_1}^{(p_1,p_2)} w_{L_2}^{(p_2,p_3)} \cdots w_{L_k}^{(p_k,p_{k+1})},$$

but at most one of these summands is nonzero because a run network is defined so that the same descent composition cannot be obtained by traversing two different walks with the same initial and terminal vertices. This precisely gives us $w^{(i,j)}(L)$, the (i,j)th entry of w_L, and thus the theorem is proven. □

For example, suppose that we want to count words having descent compositions of the form $(2,3,2,3\dots,2,3)$. Then, consider the following run network:



The words that we want to count have descent compositions corresponding to walks in this digraph beginning and ending at vertex 1. By taking all nonzero weights to be 1 and applying Theorem 2, it follows that the desired generating function is the $(1,1)$ entry of the matrix

$$\begin{bmatrix} \sum_{n=0}^{\infty} v_n^{(1,1)} \mathbf{h}_n & \sum_{n=0}^{\infty} v_n^{(1,2)} \mathbf{h}_n \\ \sum_{n=0}^{\infty} v_n^{(2,1)} \mathbf{h}_n & \sum_{n=0}^{\infty} v_n^{(2,2)} \mathbf{h}_n \end{bmatrix}^{-1}$$

where the $v_n^{(i,j)}$ are given by

$$\begin{bmatrix} \sum_{n=0}^{\infty} v_n^{(1,1)} x^n & \sum_{n=0}^{\infty} v_n^{(1,2)} x^n \\ \sum_{n=0}^{\infty} v_n^{(2,1)} x^n & \sum_{n=0}^{\infty} v_n^{(2,2)} x^n \end{bmatrix} = \left(I_2 + \begin{bmatrix} 0 & x^2 \\ x^3 & 0 \end{bmatrix} \right)^{-1}$$

$$= \begin{bmatrix} \frac{1}{1-x^5} & -\frac{x^2}{1-x^5} \\ -\frac{x^3}{1-x^5} & \frac{1}{1-x^5} \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{n=0}^{\infty} x^5 & -\sum_{n=0}^{\infty} x^{5n+2} \\ -\sum_{n=0}^{\infty} x^{5n+3} & \sum_{n=0}^{\infty} x^5 \end{bmatrix}. \tag{2.2}$$

Therefore, our desired matrix is

$$\begin{bmatrix} \sum_{n=0}^{\infty} v_n^{(1,1)} \mathbf{h}_n & \sum_{n=0}^{\infty} v_n^{(1,2)} \mathbf{h}_n \\ \sum_{n=0}^{\infty} v_n^{(2,1)} \mathbf{h}_n & \sum_{n=0}^{\infty} v_n^{(2,2)} \mathbf{h}_n \end{bmatrix}^{-1} = \begin{bmatrix} \sum_{n=0}^{\infty} \mathbf{h}_{5n} & -\sum_{n=0}^{\infty} \mathbf{h}_{5n+2} \\ -\sum_{n=0}^{\infty} \mathbf{h}_{5n+3} & \sum_{n=0}^{\infty} \mathbf{h}_{5n} \end{bmatrix}^{-1}. \tag{2.3}$$

The homomorphism Φ defined in the previous subsection induces a homomorphism on the corresponding matrix algebras, which we also call Φ by a slight abuse of notation. Thus, by applying Φ to (2.3), we obtain the matrix

$$\begin{bmatrix} \sum_{n=0}^{\infty} \frac{x^{5n}}{(5n)!} & -\sum_{n=0}^{\infty} \frac{x^{5n+2}}{(5n+2)!} \\ -\sum_{n=0}^{\infty} \frac{x^{5n+3}}{(5n+3)!} & \sum_{n=0}^{\infty} \frac{x^{5n}}{(5n)!} \end{bmatrix}^{-1}, \tag{2.4}$$

whose $(1,1)$ entry is the exponential generating function for permutations having descent composition of the form $(2,3,2,3\ldots,2,3)$. Observe that (2.4) is the inverse of the matrix obtained by taking (2.2) and converting the ordinary generating functions to exponential generating functions.

Similarly, we can apply $\hat{\Phi}$ to (2.3) and obtain the analogous result for alternating runs by taking the $(1,1)$ entry of

$$\begin{bmatrix} \sum_{n=0}^{\infty} E_n \frac{x^{5n}}{(5n)!} & -\sum_{n=0}^{\infty} E_{n+2} \frac{x^{5n+2}}{(5n+2)!} \\ -\sum_{n=0}^{\infty} E_{n+3} \frac{x^{5n+3}}{(5n+3)!} & \sum_{n=0}^{\infty} E_n \frac{x^{5n}}{(5n)!} \end{bmatrix}^{-1}.$$

Finally, we note that the original run theorem can be retrieved from the generalized run theorem by using the run network with one vertex and a loop to which the entire set $\mathbb{P}$ is assigned. Hence, Theorem 2 is indeed a generalization of Theorem 1.

3. Permutations with parity restrictions on peaks and valleys

3.1. The exponential generating functions

For completeness, we restate Gessel and Zhuang's result for the exponential generating function for permutations with all peaks odd and all valleys even.

Theorem 3 (Gessel, Zhuang 2014)**.** *Let a_n be the number of n-permutations with all peaks odd and all valleys even. Then the exponential generating function $A(x)$ for $\{a_n\}_{n\geq 0}$ is*

$$\begin{aligned} A(x) &= \frac{3\sin\left(\frac{1}{2}x\right) + 3\cosh\left(\frac{1}{2}\sqrt{3}x\right)}{3\cos\left(\frac{1}{2}x\right) - \sqrt{3}\sinh\left(\frac{1}{2}\sqrt{3}x\right)} \\ &= \left(1 - E_1 x + E_3\frac{x^3}{3!} - E_4\frac{x^4}{4!} + E_6\frac{x^6}{6!} - E_7\frac{x^7}{7!} + \cdots\right)^{-1} \end{aligned}$$

where $\sum_{k=0}^{\infty} E_k x^k/k! = \sec x + \tan x$.

Our main result in this section is the following:

Theorem 4. *Let b_n be the number of n-permutations with all peaks and valleys even, and let c_n be the number of n-permutations with all peaks and valleys odd. Then the exponential generating functions $B(x)$ for $\{b_n\}_{n\geq 0}$ and $C(x)$ for $\{c_n\}_{n\geq 0}$ are*

$$B(x) = (1+x)\frac{2 + 2\cosh(\sqrt{2}x) + \sqrt{2}x\sinh(\sqrt{2}x)}{2 + 2\cosh(\sqrt{2}x) - \sqrt{2}x\sinh(\sqrt{2}x)}$$

and

$$C(x) = \frac{2 + 2\cosh(\sqrt{2}x) + \sqrt{2}(2+x)\sinh(\sqrt{2}x)}{2 + 2\cosh(\sqrt{2}x) - \sqrt{2}x\sinh(\sqrt{2}x)}.$$

The first several terms of these sequences are as follows:

n	0	1	2	3	4	5	6	7	8	9	10	11	12
a_n	1	1	2	4	13	50	229	1238	7614	52706	405581	3432022	31684445
b_n	1	1	2	6	8	40	84	588	1632	14688	51040	561440	2340480
c_n	1	1	2	2	8	14	84	204	1632	5104	51040	195040	2340480

The sequence $\{a_n\}_{n\geq 0}$ can be found on the OEIS [15, A246012].

By looking at these numbers, we make the following observations:

1. $c_{2n} = b_{2n}$ for $n \geq 0$;

2. $c_{2n+1} < b_{2n+1}$ for $n \geq 1$;

3. $b_n < a_n$ for $n \geq 4$.

The first observation is immediate from the fact that $2n$-permutations with all peaks and valleys odd are in bijection with $2n$-permutations with all peaks and valleys even via reflection. That is, if $\pi = \pi_1\pi_2 \cdots \pi_{2n} \in \mathfrak{S}_{2n}$ has all peaks and valleys odd, then its reflection

$$\pi^r = \pi_{2n}\pi_{2n-1} \cdots \pi_1$$

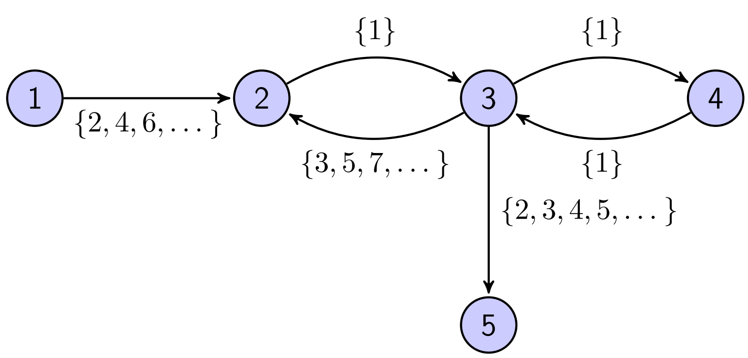
has all peaks and valleys even, and vice versa. We will show that the second and third observations are both true at the end of this section, after computing the generating functions $B(x)$ and $C(x)$.

3.2. Permutations with all peaks and valleys even

We first find the exponential generating function $B(x)$ for permutations with all peaks and valleys even using the generalized run theorem. To construct a suitable run network, we will need to find increasing run patterns for these permutations. This can be done in general, but will depend on whether the permutation begins with an ascent or descent and ends with an ascent or descent.

Notice that the permutations counted by $B(x)$ which start and end with ascents are in bijection via complementation with those permutations which start and end with descents. For the same reason, those starting with a descent and ending with an ascent are equinumerous with those starting with an ascent and ending with a descent. Thus, we only need to consider two cases.

First, let us consider permutations with all peaks and valleys even which begin and end with ascents. The descent compositions of these permutations, other than the increasing permutations $12 \cdots n$, are given by walks from vertex 1 to vertex 5 in the following run network, which we call (G_1, P_1):



Indeed, the permutation must begin with an increasing run of even length before reaching a peak, followed by an odd number of short increasing runs before reaching a valley.[3] Then, going from a valley to a peak corresponds to a long increasing run of odd length, and once again followed by an odd number of short increasing runs before reaching another valley.

[3]Recall that an increasing run is called *short* if it has length 1, and it is called *long* if it has length at least 2.

This pattern continues until the permutation reaches its final valley, and then ends with a long increasing run.

Let $B_1(x)$ denote the exponential generating function for the permutations corresponding to walks from 1 to 5 in (G_1, P_1). Applying Theorem 2 with all nonzero weights set equal to 1 and then applying the homomorphism Φ, we know that $B_1(x)$ is the $(1,5)$ entry of

$$\begin{bmatrix} \sum_{n=0}^{\infty} v_n^{(1,1)} \frac{x^n}{n!} & \cdots & \sum_{n=0}^{\infty} v_n^{(1,5)} \frac{x^n}{n!} \\ \vdots & \ddots & \vdots \\ \sum_{n=0}^{\infty} v_n^{(5,1)} \frac{x^n}{n!} & \cdots & \sum_{n=0}^{\infty} v_n^{(5,5)} \frac{x^n}{n!} \end{bmatrix}^{-1},$$

where the $v_n^{(i,j)}$ are given by

$$\begin{bmatrix} \sum_{n=0}^{\infty} v_n^{(1,1)} x^n & \cdots & \sum_{n=0}^{\infty} v_n^{(1,5)} x^n \\ \vdots & \ddots & \vdots \\ \sum_{n=0}^{\infty} v_n^{(5,1)} x^n & \cdots & \sum_{n=0}^{\infty} v_n^{(5,5)} x^n \end{bmatrix} = \left(I_5 + \begin{bmatrix} 0 & \frac{x^2}{1-x^2} & 0 & 0 & 0 \\ 0 & 0 & x & 0 & 0 \\ 0 & \frac{x^3}{1-x^2} & 0 & x & \frac{x^2}{1-x} \\ 0 & 0 & x & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right)^{-1}$$

$$= \begin{bmatrix} 1 & -\frac{(1-x^2)x^2}{1-2x^2} & \frac{x^3}{1-2x^2} & -\frac{x^4}{1-2x^2} & -\frac{x^5}{(1-2x^2)(1-x)} \\ 0 & \frac{1-2x^2+x^4}{1-2x^2} & \frac{(1-x^2)x}{1-2x^2} & \frac{(1-x^2)x^2}{1-2x^2} & \frac{(1+x)x^3}{1-2x^2} \\ 0 & -\frac{x^3}{1-2x^2} & \frac{1-x^2}{1-2x^2} & \frac{(1-x^2)x}{1-2x^2} & -\frac{x^2(1+x)}{1-2x^2} \\ 0 & \frac{x^4}{1-2x^2} & -\frac{(1-x^2)x}{1-2x^2} & -\frac{1-2x^2+x^4}{1-2x^2} & \frac{x^3(1+x)}{1-2x^2} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

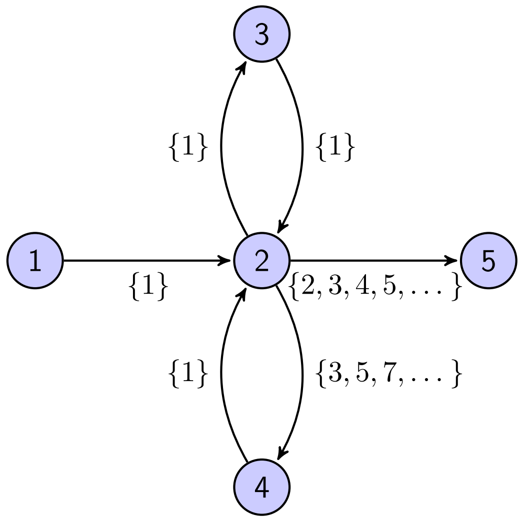
Converting these ordinary generating functions to exponential generating functions yields the matrix

$$\begin{bmatrix} 1 & \frac{1-x^2-\cosh(\sqrt{2}x)}{4} & -\frac{2x-\sqrt{2}\sinh(\sqrt{2}x)}{4} & \frac{1+x^2-\cosh(\sqrt{2}x)}{4} & \frac{3+2x+x^2+\sqrt{2}\sinh(\sqrt{2}x)+\cosh(\sqrt{2}x)-e^x}{4} \\ 0 & \frac{3-x^2+\cosh(\sqrt{2}x)}{4} & -\frac{2x+\sqrt{2}\sinh(\sqrt{2}x)}{4} & -\frac{1-x^2-\cosh(\sqrt{2}x)}{4} & -\frac{1+2x+x^2-\sqrt{2}\sinh(\sqrt{2}x)-\cosh(\sqrt{2}x)}{4} \\ 0 & \frac{2x-\sqrt{2}\sinh(\sqrt{2}x)}{4} & \frac{1+\cosh(\sqrt{2}x)}{2} & -\frac{2x+\sqrt{2}\sinh(\sqrt{2}x)}{4} & \frac{2+2x-\sqrt{2}\sinh(\sqrt{2}x)+2\cosh(\sqrt{2}x)}{4} \\ 0 & -\frac{1+x^2-\cosh(\sqrt{2}x)}{4} & -\frac{2x+\sqrt{2}\sinh(\sqrt{2}x)}{4} & \frac{3+x^2+\cosh(\sqrt{2}x)}{4} & -\frac{1+2x+x^2-\sqrt{2}\sinh(\sqrt{2}x)-\cosh(\sqrt{2}x)}{4} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

whose inverse matrix has $(1,5)$ entry

$$B_1(x) = \frac{4+4x+x^2+(4-x^2-4e^x)\cosh(\sqrt{2}x)+2\sqrt{2}(1+xe^x)\sinh(\sqrt{2}x)-4e^x}{4+4\cosh(\sqrt{2}x)-2\sqrt{2}x\sinh(\sqrt{2}x)}.$$

Next, we consider permutations with all peaks and valleys even which begin with a descent and end with an ascent. The increasing runs of these permutations follow a very similar pattern as before, but it must begin with an odd number of short increasing runs because the first letter is a descent rather than an ascent. Therefore, their descent compositions are given by walks from 1 to 5 in the following run network, which we call (G_2, P_2):



Repeating the same procedure as before, we start by computing

$$\left(I_5 + \begin{bmatrix} 0 & x & 0 & 0 & 0 \\ 0 & 0 & x & \frac{x^3}{1-x^2} & \frac{x^3}{1-x^2} \\ 0 & x & 0 & 0 & 0 \\ 0 & x & x & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\right)^{-1} = \begin{bmatrix} 1 & -\frac{(1-x^2)x}{1-2x^2} & \frac{(1-x^2)x^2}{1-2x^2} & \frac{x^4}{1-2x^2} & \frac{(1+x)x^3}{1-2x^2} \\ 0 & \frac{1-x^2}{1-2x^2} & \frac{(1-x^2)x}{1-2x^2} & -\frac{x^3}{1-2x^2} & -\frac{(1+x)x^2}{1-2x^2} \\ 0 & -\frac{(1-x^2)x}{1-2x^2} & \frac{1-x^2-x^4}{1-2x^2} & \frac{x^4}{1-2x^2} & \frac{(1+x)x^3}{1-2x^2} \\ 0 & -\frac{(1-x^2)x}{1-2x^2} & \frac{(1-x^2)x^2}{1-2x^2} & \frac{(1-x^2)^2}{1-2x^2} & \frac{(1+x)x^3}{1-2x^2} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

and then we convert the ordinary generating functions to exponential generating functions to obtain

$$\begin{bmatrix} 1 & -\frac{2x+\sqrt{2}\sinh(\sqrt{2}x)}{4} & -\frac{1-x^2-\cosh(\sqrt{2}x)}{4} & -\frac{1+x^2-\cosh(\sqrt{2}x)}{4} & -\frac{1+2x+x^2-\sqrt{2}\sinh(\sqrt{2}x)-\cosh(\sqrt{2}x)}{4} \\ 0 & \frac{1+\cosh(\sqrt{2}x)}{2} & -\frac{2x+\sqrt{2}\sinh(\sqrt{2}x)}{4} & \frac{2x-\sqrt{2}\sinh(\sqrt{2}x)}{4} & \frac{2+2x-\sqrt{2}\sinh(\sqrt{2}x)-2\cosh(\sqrt{2}x)}{4} \\ 0 & -\frac{2x+\sqrt{2}\sinh(\sqrt{2}x)}{4} & \frac{3+x^2+\cosh(\sqrt{2}x)}{4} & -\frac{1+x^2-\cosh(\sqrt{2}x)}{4} & -\frac{1+2x+x^2-\sqrt{2}\sinh(\sqrt{2}x)-\cosh(\sqrt{2}x)}{4} \\ 0 & -\frac{2x+\sqrt{2}\sinh(\sqrt{2}x)}{4} & -\frac{1-x^2-\cosh(\sqrt{2}x)}{4} & \frac{3-x^2+\cosh(\sqrt{2}x)}{4} & -\frac{1+2x+x^2-\sqrt{2}\sinh(\sqrt{2}x)-\cosh(\sqrt{2}x)}{4} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The $(1, 5)$ entry of the inverse matrix gives us

$$B_2(x) = \frac{x^2 - x(4+x)\cosh(\sqrt{2}x) + 2\sqrt{2}\sinh(\sqrt{2}x)}{4 + 4\cosh(\sqrt{2}x) - 2\sqrt{2}x\sinh(\sqrt{2}x)}.$$

Now we can obtain $B(x)$ by taking $2B_1(x) + 2B_2(x)$, but will need to add an additional term to account for the increasing and decreasing permutations which were excluded from the above computations. Since e^x is the exponential generating function for increasing permutations and also for decreasing permutations, we add $2e^x$ but also substract $x+1$ because the empty permutation and the length 1 permutation are counted twice by $2e^x$.

Therefore,

$$B(x) = 2B_1(x) + 2B_2(x) + 2e^x - x - 1$$

which simplifies to

$$B(x) = (1+x)\frac{2 + 2\cosh(\sqrt{2}x) + \sqrt{2}x\sinh(\sqrt{2}x)}{2 + 2\cosh(\sqrt{2}x) - \sqrt{2}x\sinh(\sqrt{2}x)}.$$

3.3. Permutations with all peaks and valleys odd

Although the exponential generating function $C(x)$ for permutation with all peaks and valleys odd can be obtained in the same way via run networks, it is easier to derive it from a combinatorial identity relating the odd and even terms of the sequence $\{c_n\}_{n\geq 0}$, which we do here.

Notice that the generating function $B(x)$ for permutations with all peaks and valleys even splits nicely into even and odd parts:

$$B_{\text{even}}(x) \coloneqq \sum_{n=0}^{\infty} b_{2n} \frac{x^{2n}}{(2n)!} = \frac{2 + 2\cosh(\sqrt{2}x) + \sqrt{2}x\sinh(\sqrt{2}x)}{2 + 2\cosh(\sqrt{2}x) - \sqrt{2}x\sinh(\sqrt{2}x)}$$

and

$$B_{\text{odd}}(x) \coloneqq \sum_{n=0}^{\infty} b_{2n+1} \frac{x^{2n+1}}{(2n+1)!} = x\frac{2 + 2\cosh(\sqrt{2}x) + \sqrt{2}x\sinh(\sqrt{2}x)}{2 + 2\cosh(\sqrt{2}x) - \sqrt{2}x\sinh(\sqrt{2}x)}.$$

We immediately deduce from $B_{\text{odd}}(x) = xB_{\text{even}}(x)$ an identity relating the even and odd terms of $\{b_n\}_{n\geq 0}$.

Proposition 5. *For all $n \geq 0$, $b_{2n+1} = (2n+1)b_{2n}$.*

The identity can also be seen from a simple bijection.

Proof. Let π be any $2n$-permutation with all peaks and valleys even, and pick any $m \in [2n+1]$. Let π' be the permutation obtained by replacing the letter k with $k+1$ for every $k \geq m$ in π, and attaching m to the end of π. For example, given $\pi = 1432$ and $m = 3$, we have $\pi' = 15423$. Then, π' is a $(2n+1)$-permutation with all peaks and valleys even. To obtain (π, m) from π', simply take $m = \pi'_{2n+1}$ and apply the reduction map to the word formed by the first $2n$ letters of π' to get π. □

Essentially the same bijection gives us an analogous identity for permutations with all peaks and valleys odd.

Proposition 6. *For all $n \geq 1$, $c_{2n} = 2nc_{2n-1}$.*

Proof. Pick any $(2n-1)$-permutation with all peaks and valleys odd, and any $m \in [2n]$. Applying the same procedure in the proof of Proposition 5 yields a $(2n)$-permutation with all peaks and valleys odd, and we reverse the procedure in the same way to obtain π and m. □

Proposition 6, along with the fact that $b_{2n} = c_{2n}$ for all $n \geq 0$, allows us to deduce that

$$\begin{aligned} C_{\text{odd}}(x) &\coloneqq \sum_{n=1}^{\infty} c_{2n-1} \frac{x^{2n-1}}{(2n-1)!} \\ &= \sum_{n=1}^{\infty} \frac{c_{2n}}{2n} \frac{x^{2n-1}}{(2n-1)!} \\ &= \frac{1}{x} \sum_{n=1}^{\infty} b_{2n} \frac{x^{2n}}{(2n)!} \end{aligned}$$

$$= \frac{1}{x}\left(\frac{2+2\cosh(\sqrt{2}x)+\sqrt{2}x\sinh(\sqrt{2}x)}{2+2\cosh(\sqrt{2}x)-\sqrt{2}x\sinh(\sqrt{2}x)} - 1\right)$$
$$= \frac{2\sqrt{2}\sinh(\sqrt{2}x)}{2+2\cosh(\sqrt{2}x)-\sqrt{2}x\sinh(\sqrt{2}x)}.$$

Furthermore,

$$C_{\text{even}}(x) := \sum_{n=0}^{\infty} c_{2n}\frac{x^{2n}}{(2n)!} = B_{\text{even}}(x),$$

so we have

$$C(x) = C_{\text{odd}}(x) + C_{\text{even}}(x)$$
$$= \frac{2+2\cosh(\sqrt{2}x)+\sqrt{2}(2+x)\sinh(\sqrt{2}x)}{2+2\cosh(\sqrt{2}x)-\sqrt{2}x\sinh(\sqrt{2}x)},$$

which completes the proof of Theorem 4.

Permutations with all peaks and valleys odd are closely related to "balanced permutations", which are defined in terms of standard skew Young tableaux called "balanced tableaux". In fact, balanced permutations of odd length are precisely permutations of odd length with all peaks and valleys odd, counted by $\{c_{2n+1}\}_{n\geq 0}$. Gessel and Greene [8] gave the exponential generating function for balanced permutations, and by comparing generating functions, showed that

$$d_{2n+1} = 2^n c_{2n+1} \tag{3.1}$$

for all $n \geq 0$, where d_n is the number of n-permutations with all valleys odd (and with no parity restrictions on peaks) which were previously studied by Gessel [7]. A bijective proof of (3.1) was later given by La Croix [13].

3.4. Two inequalities

We end this section by verifying two observations made at the beginning of this section relating the sequences $\{a_n\}_{n\geq 0}$, $\{b_n\}_{n\geq 0}$, and $\{c_n\}_{n\geq 0}$.

Proposition 7. *For all $n \geq 1$, $c_{2n+1} < b_{2n+1}$.*

Proof. Fix $n \geq 1$. We provide an injection `shift` from the set of $(2n+1)$-permutations with all peaks and valleys odd to the set of $(2n+1)$-permutations with all peaks and valleys even.

Let $\pi = \pi_1\pi_2\cdots\pi_{2n+1}$ have all peaks and valleys odd, and let $\texttt{shift}(\pi) = \pi_2\cdots\pi_{2n+1}\pi_1$. All of the peaks and valleys of π remain peaks and valleys of $\texttt{shift}(\pi)$, but their positions are shifted by 1, and the only possible new valley or peak is given by π_{2n+1}, which has an even position in $\texttt{shift}(\pi)$. Therefore, all of the peaks and valleys of $\texttt{shift}(\pi)$ are even, and `shift` is injective because it is clearly reversible.

However, `shift` is not a surjection. The increasing permutation $12\cdots(2n)(2n+1)$ has all peaks and valleys even, but $(2n+1)12\cdots(2n)$ does not have all peaks and valleys odd. □

Proposition 8. *For all $n \geq 4$, $b_n < a_n$.*

Proof. Fix $n \geq 4$. As before, we give a suitable injection, this time from the set of n-permutations with all peaks and valleys even to the set of n-permutations with all valleys even and all peaks odd.

Let $\pi = \pi_1\pi_2\cdots\pi_n$ have all peaks and valleys even, and let $\texttt{pkshift}(\pi)$ be the n-permutation obtained by the following algorithm. We iterate through the letters of π from left to right, and for each peak $k < n-1$ of π, we switch the letters π_k and π_{k+1}. If $n-1$ is a peak of π, then we switch π_{n-2} and π_{n-1}. For example, $\pi = 287134596$ is mapped to $\texttt{pkshift}(\pi) = 278134956$.

We must show that $\texttt{pkshift}(\pi)$ has all valleys even and all peaks odd. Since all peaks and valleys of π are even, each peak must be at least 4 greater than the previous peak, so we can look at the "local behavior" of $\texttt{pkshift}$ around the peaks of π. Suppose that $i < n-1$ is a peak of π and is thus even. Then,

$$\pi_{i-2} < \pi_{i-1} < \pi_i > \pi_{i+1} > \pi_{i+2}$$

because π has all peaks and valleys even. (It is possible that π_{i-2} does not exist.) Then this segment of π is mapped to

$$\pi_{i-2} < \pi_{i-1} \lessgtr \pi_{i+1} < \pi_i > \pi_{i+2}$$

by $\texttt{pkshift}$ if $i \neq n-1$. If $n-1$ is a peak of π, then

$$\pi_{n-3} < \pi_{n-2} < \pi_{n-1} > \pi_n$$

is mapped to

$$\pi_{n-3} < \pi_{n-1} > \pi_{n-2} \lessgtr \pi_n$$

by $\texttt{pkshift}$.

These segments may only intersect each other at the leftmost or rightmost positions, which are not changed, so they are completely compatible. All of the letters within two positions of the peaks of π are affected in this manner, and none of the other letters are changed.

In particular, every valley of π is unchanged and becomes a valley of $\texttt{pkshift}(\pi)$, every peak $i < n-1$ of π becomes a peak $i+1$ of $\texttt{pkshift}(\pi)$, and if $n-1$ is a peak of π then it becomes a peak $n-2$ of $\texttt{pkshift}(\pi)$. Moreover, given a peak $i < n-1$ of π and if $\pi_{i-1} > \pi_{i+1}$, then $i+1$ becomes a valley i of $\texttt{pkshift}(\pi)$, and if additionally π_{i-2} exists, then $i-1$ becomes a peak of $\texttt{pkshift}(\pi)$. If $n-1$ is a peak of π and $\pi_{n-2} < \pi_n$, then $n-1$ becomes a valley of $\texttt{pkshift}(\pi)$. All of these peaks and valleys are in accordance with the required parity conditions, and since no other peaks and valleys are introduced, it follows that $\texttt{pkshift}$ is a well-defined map into the set of n-permutations with all peaks odd and all valleys even.

This map can easily be reversed. Iterate through the letters of $\texttt{pkshift}(\pi)$ from right to left. If $n-2$ is a peak of $\texttt{pkshift}(\pi)$, then switch π_{n-2} and π_{n-1}. For all other peaks k of $\texttt{pkshift}(\pi)$, we switch π_{k-1} and π_k. This algorithm eliminates all peaks and valleys that were added and restores the peaks that were shifted, so it is evident that π is recovered from $\texttt{pkshift}(\pi)$. Hence, $\texttt{pkshift}$ is an injection.

As before, `pkshift` is not a surjection. Take the n-permutation given by concatenating 2143 with $56\cdots n$ (the latter is empty if $n=4$), which has all peaks odd and all valleys even. For $n=4$ and $n>5$, applying the inverse algorithm yields $241356\cdots n$, which has an odd valley. For $n=5$, the inverse algorithm yields 21345, which has all valleys even and no peaks. So, applying `pkshift` to 21345 gives 21345, which was not the original permutation 21435. Thus the result is proven. □

4. Counting permutations by run-based statistics

4.1. Introduction

Let $\operatorname{des}(\pi)$ and $\operatorname{altdes}(\pi)$ be the number of descents and alternating descents of a permutation π, respectively. It was demonstrated in [9] that by setting all weights equal to t in the original version of the run theorem and applying the homomorphism Φ, we obtain the famous exponential generating function

$$1+\sum_{n=1}^{\infty} A_n(t)\frac{x^n}{n!} = \frac{1-t}{1-te^{(1-t)x}}$$

for the Eulerian polynomials

$$A_n(t) = \sum_{\pi\in\mathfrak{S}_n} t^{\operatorname{des}(\pi)+1},$$

counting permutations by descents. If we apply $\hat{\Phi}$ instead, then we get an analogous result

$$1+\sum_{n=1}^{\infty} \hat{A}_n(t)\frac{x^n}{n!} = \frac{1-t}{1-t(\sec((1-t)x)+\tan((1-t)x))}$$

(previously discovered in an equivalent form by Chebikin [2, Theorem 4.2]) for the "alternating Eulerian polynomials"

$$\hat{A}_n(t) = \sum_{\pi\in\mathfrak{S}_n} t^{\operatorname{altdes}(\pi)+1},$$

counting permutations by alternating descents.

In general, given a permutation statistic st, we define the st-polynomials to be

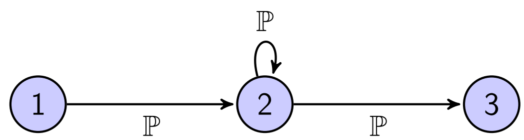
$$P_n^{\mathrm{st}}(t) \coloneqq \sum_{\pi\in\mathfrak{S}_n} t^{\mathrm{st}(\pi)}$$

for $n\geq 0$. So, for instance, the Eulerian polynomials and alternating Eulerian polynomials correspond to the polynomials for the number of increasing runs and the number of alternating runs, respectively. Then the exponential generating functions

$$P^{\mathrm{st}}(t,x) \coloneqq \sum_{n=0}^{\infty} P_n^{\mathrm{st}}(t)\frac{x^n}{n!}$$

are bivariate generating functions counting permutations by length and the statistic st.

In this section, we demonstrate how the generating functions $P^{\mathrm{st}}(t,x)$ for many statistics can be obtained by applying the generalized run theorem on the following run network, which we call (G,P) :



The descent compositions of all non-increasing permutations are given by walks from 1 to 3 in (G,P), but note that we distinguish the initial run and the final run in this run network. Indeed, the permutation statistics that we consider in this section are all determined by the number of non-initial and non-final long increasing runs, as well as whether the permutation begins and ends with a short increasing run or a long increasing run. Hence, by assigning appropriate weights to the letters of P in this run network, the generalized run theorem yields refined results counting permutations by these statistics.

Not every statistic that we consider requires three vertices in our run network. For example, only two vertices are required if we only need to distinguish the initial run or the final run but not both, and only one vertex is required if we do not need to distinguish the initial run or the final run. We will still use the 3-vertex run network (G,P) in the former case, since it eliminates the need for us to define two 2-vertex run networks (one for distinguished initial runs, and one for distinguished final runs) and the computation is no more difficult using a computer algebra system such as Maple. However, the latter case does not require a run network at all, so we will simply apply the original version of the run theorem (Theorem 1).

This general approach can also be used to find multivariate generating functions giving the joint distribution of two or more of these statistics, although we do not do this here.

We note that the result of applying the generalized run theorem to the 3-vertex run network (G,P) is essentially a weighted version of Theorem 6.12 of Gessel [6], which gives formulae for counting words with distinguished initial run and final run, and is similar to results given by Jackson and Aleliunas [11, Sections 10-12]. See also Goulden and Jackson [10, Theorem 4.2.19].

4.2. Counting permutations by peaks and variations

The first two statistics that we consider are the number of peaks and the number of valleys of a permutation. By taking complements, it is immediate that these two statistics are equidistributed on $\mathfrak{S}_n$, so it suffices to find the bivariate generating function for peaks.

Theorem 9. *Let* $\operatorname{pk}(\pi)$ *be the number of peaks of* π. *Then,*

$$P^{\mathrm{pk}}(t,x)=\frac{\sqrt{1-t}\cosh(x\sqrt{1-t})}{\sqrt{1-t}\cosh(x\sqrt{1-t})-\sinh(x\sqrt{1-t})}.$$

Other equivalent formulae have been found, e.g., by Entringer [4] using differential equations, Mendes and Remmel [14] using the "homomorphism method", and Kitaev [12] using the notion of partially ordered permutation patterns.

The first ten pk-polynomials are given below, and their coefficients can be found in the OEIS [15, A008303].

n	$P_n^{\text{pk}}(t)$	n	$P_n^{\text{pk}}(t)$
0	1	5	$16+88t+16t^2$
1	1	6	$32+416t+272t^2$
2	2	7	$64+1824t+2880t^2+272t^3$
3	$4+2t$	8	$128+7680t+24576t^2+7936t^3$
4	$8+16t$	9	$256+31616t+185856t^2+137216t^3+7936t^4$

Proof. Notice that the number of peaks in a permutation is equal to its number of non-final long increasing runs, as every non-final long increasing run necessarily ends with a peak and every peak is at the end of a non-final long increasing run. So, we want to weight every $k \neq 1$ in $P_{1,2}$ and $P_{2,2}$ by t in (G,P). Setting $w_k^{(1,2)} = w_k^{(2,2)} = t$ for all $k \neq 1$ (and setting all other nonzero weights to 1) and then applying Theorem 2, we compute

$$\left(I_3 + \begin{bmatrix} 0 & x+\frac{tx^2}{1-x} & 0 \\ 0 & x+\frac{tx^2}{1-x} & x+\frac{x^2}{1-x} \\ 0 & 0 & 0 \end{bmatrix} \right)^{-1} = \begin{bmatrix} 1 & -\frac{(1-(1-t)x)x}{1-(1-t)x^2} & \frac{(1-(1-t)x)x^2}{(1-(1-t)x^2)(1-x)} \\ 0 & \frac{1-x}{1-(1-t)x^2} & -\frac{x}{1-(1-t)x^2} \\ 0 & 0 & 1 \end{bmatrix},$$

and converting the ordinary generating functions to exponential generating functions gives

$$\begin{bmatrix} 1 & -1+\cosh(x\sqrt{1-t})-\frac{\sinh(x\sqrt{1-t})}{\sqrt{1-t}} & -1-\frac{\sinh(x\sqrt{1-t})}{\sqrt{1-t}}+e^x \\ 0 & \cosh(x\sqrt{1-t})-\frac{\sinh(x\sqrt{1-t})}{\sqrt{1-t}} & -\frac{\sinh(x\sqrt{1-t})}{\sqrt{1-t}} \\ 0 & 0 & 1 \end{bmatrix}. \tag{4.1}$$

Since the increasing permutations have exponential generating function e^x and do not have any peaks, we add e^x to the $(1,3)$ entry of the inverse matrix of (4.1) to obtain our desired generating function

$$P^{\text{pk}}(t,x) = \frac{\sqrt{1-t}\cosh(x\sqrt{1-t})}{\sqrt{1-t}\cosh(x\sqrt{1-t})-\sinh(x\sqrt{1-t})}.$$

□

Let us now introduce several variations of peaks and valleys in permutations. We say that i is a *left peak* of a permutation $\pi = \pi_1\pi_2\cdots\pi_n$ if i is a peak of π or if $i=1$ and $\pi_1 > \pi_2$; we say that i is a *right peak* of π if i is a peak of π or if $i=n$ and $\pi_{n-1} < \pi_n$; and we say that i is a *left-right peak* of π if i is a left peak or a right peak of π, or if $\pi = 1$ and $i=1$. *Left valleys*, *right valleys*, and *left-right valleys* are defined analogously.

By taking reverses and complements, it is easy to see that the number of left peaks, right peaks, left valleys, and right valleys are all equidistributed on $\mathfrak{S}_n$. So, we only need to find the bivariate generating function for one of these statistics, say, right peaks. Taking the complement shows that the number of left-right peaks and left-right valleys are equidistributed as well.

We have the following result:

Theorem 10. *Let* $\text{rpk}(\pi)$ *and* $\text{lrpk}(\pi)$ *be the number of right peaks and left-right peaks of* π*, respectively. Then*

$$P^{\text{rpk}}(t,x) = \frac{\sqrt{1-t}}{\sqrt{1-t}\cosh(x\sqrt{1-t})-\sinh(x\sqrt{1-t})}$$

and

$$P^{\mathrm{lrpk}}(t,x)=\frac{\sqrt{1-t}\cosh(x\sqrt{1-t})-(1-t)\sinh(x\sqrt{1-t})}{\sqrt{1-t}\cosh(x\sqrt{1-t})-\sinh(x\sqrt{1-t})}.$$

The first ten rpk-polynomials are given below, and their coefficients can also be found in the OEIS [15, A008971]. We omit the lrpk-polynomials; as we shall see shortly, they can be easily characterized in terms of the pk-polynomials.

n	$P_n^{\mathrm{rpk}}(t)$	n	$P_n^{\mathrm{rpk}}(t)$
0	1	5	$1+58t+61t^2$
1	1	6	$1+179t+479t^2+61t^3$
2	$1+t$	7	$1+543t+3111t^2+1385t^3$
3	$1+5t$	8	$1+1636t+18270t^2+19028t^3+1385t^4$
4	$1+18t+5t^2$	9	$1+4916t+101166t^2+206276t^3+50521t^4$

Proof. The number of right peaks of a permutation is equal to its total number of long increasing runs, so we now assign a weight t to every such run. Thus, setting $w_k=t$ for all $k\neq 1$ and applying the original run theorem, we have that

$$\left(1+x+\frac{tx^2}{1-x}\right)^{-1}=\frac{1-x}{1-(1-t)x^2},$$

whose coefficients have exponential generating function

$$\cosh(x\sqrt{1-t})-\frac{\sinh(x\sqrt{1-t})}{\sqrt{1-t}}.$$

Finally, taking the reciprocal yields

$$\begin{aligned}P^{\mathrm{rpk}}(t,x)&=\left(\cosh(x\sqrt{1-t})-\frac{\sinh(x\sqrt{1-t})}{\sqrt{1-t}}\right)^{-1}\\&=\frac{\sqrt{1-t}}{\sqrt{1-t}\cosh(x\sqrt{1-t})-\sinh(x\sqrt{1-t})}.\end{aligned}$$

As for left-right peaks, we claim that the number of left-right peaks of a permutation π is equal to one more than the number of valleys of π. The number of valleys of π is equal to its number of non-initial long increasing runs, and every non-initial long increasing run ends with a left-right peak. In fact, all left-right peaks of π can be found at the end of these runs, but also at the end of the first increasing run (regardless of length), which accounts for the difference of 1. Since the number of peaks and number of valleys are equidistributed, we have that

$$\begin{aligned}P^{\mathrm{lrpk}}(t,x)&=tP^{\mathrm{pk}}(t,x)-t+1\\&=\frac{\sqrt{1-t}\cosh(x\sqrt{1-t})-(1-t)\sinh(x\sqrt{1-t})}{\sqrt{1-t}\cosh(x\sqrt{1-t})-\sinh(x\sqrt{1-t})}.\end{aligned}$$

□

By comparing $P^{\text{pk}}(t,x)$ and $P^{\text{rpk}}(t,x)$, we see the following relation:

Corollary 11. *The bivariate generating functions $P^{\text{pk}}(t,x)$ and $P^{\text{rpk}}(t,x)$ for the number of peaks and the number of right peaks, respectively, satisfy*

$$P^{\text{pk}}(t,x) = P^{\text{rpk}}(t,x)\cosh(x\sqrt{1-t}),$$

or equivalently,

$$P^{\text{rpk}}(t,x) = P^{\text{pk}}(t,x)\operatorname{sech}(x\sqrt{1-t}).$$

We do not know of a combinatorial explanation for this fact. Note that some of the coefficients of

$$\cosh(x\sqrt{1-t}) = \sum_{n=0}^{\infty}(1-t)^n\frac{x^{2n}}{(2n)!}$$

are negative, so there may be some sort of inclusion-exclusion phenomenon at play. A combinatorial explanation may also have some connection to alternating permutations, since

$$\operatorname{sech}(x\sqrt{1-t}) = \sum_{n=0}^{\infty}E_{2n}(t-1)^n\frac{x^{2n}}{(2n)!}$$

and E_{2n} is the number of alternating permutations of length $2n$.

4.3. Counting permutations by double ascents and variations

Peaks (respectively, valleys) indicate positions in a permutation where we have a letter that is larger than (respectively, smaller than) the two surrounding letters, so it is natural to consider the case when a letter is larger than one surrounding letter but smaller than the other. Given a permutation π, we say that i is a *double ascent* of π if $\pi_{i-1} < \pi_i < \pi_{i+1}$ and that i is a *double descent* of π if $\pi_{i-1} > \pi_i > \pi_{i+1}$. As with peaks and valleys, the number of double ascents and the number of double descents are equidistributed on $\mathfrak{S}_n$, so we only need to consider double ascents.

Theorem 12. *Let $\operatorname{dasc}(\pi)$ be the number of double ascents of π. Then,*

$$P^{\text{dasc}}(t,x) = \frac{ue^{\frac{1}{2}(1-t)x}}{u\cosh(\frac{1}{2}ux)-(1+t)\sinh(\frac{1}{2}ux)}$$

where $u = \sqrt{(t+3)(t-1)}$.

Below are the first ten dasc-polynomials; see also its OEIS entry [15, A162975].

n	$P_n^{\text{dasc}}(t)$	n	$P_n^{\text{dasc}}(t)$
0	1	5	$70+41t+8t^2+t^3$
1	1	6	$349+274t+86t^2+10t^3+t^4$
2	2	7	$2017+2040t+803t^2+167t^3+12t^4+t^5$
3	$5+t$	8	$13358+16346t+8221t^2+2064t^3+316t^4+14t^5+t^6$
4	$17+6t+t^2$	9	$99377+143571t+86214t^2+28143t^3+4961t^4+597t^5+16t^6+t^7$

Proof. It is clear that short increasing runs contribute no double ascents, and long increasing runs of length $k \geq 2$ contribute $k-2$ double ascents. Thus, we set $w_k = t^{k-2}$ for all $k \neq 1$ and apply the original run theorem to obtain

$$\left(1 + x + \frac{x^2}{1-tx}\right)^{-1} = \frac{1-tx}{1+(1-t)(1+x)x},$$

whose coefficients have exponential generating function

$$e^{-\frac{1}{2}(1-t)x}\left(\cosh\left(\frac{1}{2}ux\right) - \frac{(1+t)}{u}\sinh\left(\frac{1}{2}ux\right)\right)$$

where $u = \sqrt{(t+3)(t-1)}$. Then taking the reciprocal gives us

$$\begin{aligned} P^{\text{dasc}}(t,x) &= \left(e^{-\frac{1}{2}(1-t)x}\left(\cosh\left(\frac{1}{2}ux\right) - \frac{(1+t)}{u}\sinh\left(\frac{1}{2}ux\right)\right)\right)^{-1} \\ &= \frac{ue^{\frac{1}{2}(1-t)x}}{u\cosh(\frac{1}{2}ux) - (1+t)\sinh(\frac{1}{2}ux)}. \end{aligned}$$

□

Now let us consider the variations that we can define for double ascents and double descents as we did for peaks and valleys. We say that i is a *left double ascent* of a permutation $\pi = \pi_1\pi_2\cdots\pi_n$ if i is a double ascent of π or if $i=1$ and $\pi_1 < \pi_2$; we say that i is a *right double ascent* of π if i is a double ascent of π or if $i = n$ and $\pi_{n-1} < \pi_n$; and we say that i is a *left-right double ascent* of π if i is a left double ascent or a right double ascent of π, or if $\pi = 1$ and $i = 1$. *Left double descents*, *right double descents*, and *left-right double descents* are defined analogously. It is evident from taking reverses and complements that we only need to consider right double ascents and left-right double ascents.

Theorem 13. *Let* $\operatorname{rdasc}(\pi)$ *and* $\operatorname{lrdasc}(\pi)$ *be the number of right double ascents and left-right double ascents of* π*, respectively. Then,*

$$P^{\text{rdasc}}(t,x) = \frac{u\cosh(\frac{1}{2}ux) + (1-t)\sinh(\frac{1}{2}ux)}{u\cosh(\frac{1}{2}ux) - (1+t)\sinh(\frac{1}{2}ux)}$$

and

$$P^{\text{lrdasc}}(t,x) = \frac{ue^{-\frac{1}{2}(1-t)x}}{u\cosh(\frac{1}{2}ux) - (1+t)\sinh(\frac{1}{2}ux)}$$

where $u = \sqrt{(t+3)(t-1)}$.

Below are the first ten rdasc- and lrdasc-polynomials. There is an OEIS entry [15, A162976] for the coefficients of the rdasc-polynomials, but there does not seem to be one for the lrdasc-polynomials.

n	$P_n^{\text{rdasc}}(t)$
0	1
1	1
2	$1+t$
3	$3+2t+t^2$
4	$9+11t+3t^2+t^3$
5	$39+48t+28t^2+4t^3+t^4$
6	$189+297t+166t^2+62t^3+5t^4+t^5$
7	$1107+1902t+1419t^2+476t^3+129t^4+6t^5+t^6$
8	$7281+14391t+11637t^2+5507t^3+1235t^4+261t^5+7t^6+t^7$
9	$54351+118044t+111438t^2+56400t^3+19096t^4+3020t^5+522t^6+8t^7+t^8$

n	$P_n^{\text{lrdasc}}(t)$
0	1
1	t
2	$1+t^2$
3	$1+4t+t^3$
4	$6+6t+11t^2+t^4$
5	$19+51t+23t^2+26t^3+t^5$
6	$109+212t+269t^2+72t^3+57t^4+t^6$
7	$588+1571t+1419t^2+1140t^3+201t^4+120t^5+t^7$
8	$4033+10470t+13343t^2+7432t^3+4272t^4+522t^5+247t^6+t^8$
9	$29485+87672t+107853t^2+87552t^3+33683t^4+14841t^5+1291t^6+502t^7+t^9$

Proof. As before, non-final short increasing runs contribute no right double ascents, and non-final long increasing runs of length $k\geq 2$ contribute $k-2$ right double ascents. Moreover, if the final increasing run is of length k, then it contributes $k-1$ right double ascents. So, we take $w_k^{(1,2)}=w_k^{(2,2)}=t^{k-2}$ for all $k\neq 1$ and $w_k^{(2,3)}=t^{k-1}$ for all k in the same run network (G,P) defined earlier, and applying Theorem 2 gives

$$\left(I_3+\begin{bmatrix}0 & x+\frac{x^2}{1-tx} & 0\\ 0 & 1+x+\frac{x^2}{1-tx} & \frac{x}{1-tx}\\ 0 & 0 & 0\end{bmatrix}\right)^{-1}=\begin{bmatrix}1 & -\frac{(1+(1-t)x)x}{1+x(1-t)(1+x)} & \frac{(1+(1-t)x)x^2}{(1+(1-t)x^2)(1-tx)}\\ 0 & \frac{1-tx}{1+x(1-t)(1+x)} & -\frac{x}{1+x(1-t)(1+x)}\\ 0 & 0 & 1\end{bmatrix},$$

and converting to exponential generating functions gives

$$\begin{bmatrix}1 & -1+e^{-\frac{1}{2}(1-t)x}\left(\cosh(\tfrac{1}{2}ux)-\frac{1+t}{u}\sinh(\tfrac{1}{2}ux)\right) & -\frac{2}{u}e^{-\frac{1}{2}(1-t)}\sinh(\tfrac{1}{2}ux)-\frac{1-e^{tx}}{t}\\ 0 & e^{-\frac{1}{2}(1-t)x}\left(\cosh(\tfrac{1}{2}ux)-\frac{1+t}{u}\sinh(\tfrac{1}{2}ux)\right) & -\frac{2}{u}e^{-\frac{1}{2}(1-t)}\sinh(\tfrac{1}{2}ux)\\ 0 & 0 & 1\end{bmatrix}.$$

where $u=\sqrt{(t+3)(t-1)}$. We still need to account for the increasing permutations, and the increasing permutation of length n has $n-1$ right double ascents. So, we take the $(1,3)$ entry to the inverse of the above matrix and add to it $(e^{tx}-1)/t+1$ to obtain

$$P^{\text{rdasc}}(t,x)=\frac{u\cosh(\frac{1}{2}ux)+(1-t)\sinh(\frac{1}{2}ux)}{u\cosh(\frac{1}{2}ux)-(1+t)\sinh(\frac{1}{2}ux)}.$$

The computation for left-right double ascents is similar, but we have to adjust the weights for the initial increasing run. If the initial increasing run is of length k, then it contributes $k-1$ left-right double ascents; hence, we take $w_k^{(2,2)} = t^{k-2}$ for all $k \neq 1$ and $w_k^{(1,2)} = w_k^{(2,3)} = t^{k-1}$ for all k. Then the computation proceeds in the same way, and we add e^{tx} at the end because the increasing permutation of length n has n left-right double ascents. □

Comparing our expressions for $P^{\text{dasc}}(t,x)$ and $P^{\text{lrdasc}}(t,x)$ gives the following identity:

Corollary 14. *The bivariate generating functions $P^{\text{dasc}}(t,x)$ and $P^{\text{lrdasc}}(t,x)$ for the number of double ascents and the number of left-right double ascents, respectively, satisfy*

$$P^{\text{dasc}}(t,x) = e^{(1-t)x} P^{\text{lrdasc}}(t,x).$$

We do not know of a combinatorial proof.

4.4. Counting permutations by biruns and up-down runs

The number of peaks and valleys in permutations are also closely related to another statistic that we call the number of *biruns*, which we define to be maximal consecutive subsequences of length at least two containing no descents or no ascents, that is, long increasing runs and "long decreasing runs".[4] For example, the biruns of $\pi = 51378624$ are 51, 1378, 862, and 24. We also define an *up-down run* in a permutation to be an initial short run or a birun. So the up-down runs of $\pi = 51378624$ are 5, 51, 1378, 862, and 24.

The notion of biruns has been widely studied in the literature (see, e.g., [1, Chapter 1]), and are closely to related to the Eulerian polynomials and longest alternating subsequences of permutations. A connection with the Eulerian polynomials is exemplified by the identity

$$P_n^{\text{br}}(t) = \left(\frac{1+t}{2}\right)^{n-1} (1+w)^{n+1} A_n\left(\frac{1-w}{1+w}\right), \text{ for } n \geq 2,$$

where $w = \sqrt{\frac{1-t}{1+t}}$ and $\text{br}(\pi)$ is the number of biruns of π [1, p. 30].

We say that a subsequence $\pi_{i_1} \cdots \pi_{i_k}$ of a permutation π is *alternating* if $\pi_{i_1} \cdots \pi_{i_k}$ is an alternating permutation. Let $\text{as}(\pi)$ be the length of the longest alternating subsequence of a permutation π. Then, for example, the n-permutations π with $\text{as}(\pi) = n$ are the length n alternating permutations. The study of longest alternating subsequences of permutations was initiated by Stanley [16], who deduced the bivariate generating function

$$P^{\text{as}}(t,x) = (1-t)\frac{1+v+2te^{vx}+(1-v)e^{2vx}}{1+v-t^2+(1-v-t^2)e^{2vx}}, \tag{4.2}$$

where $v = \sqrt{1-t^2}$, and gave the identity

$$P_n^{\text{as}}(t) = \left(\frac{1+t}{2}\right) P_n^{\text{br}}(t), \text{ for } n \geq 2.$$

[4] Biruns are more commonly called *alternating runs*, but since the term "alternating run" is used for a different concept in this paper, we use the term "birun" which was suggested by Stanley [16, Section 4].

We consider up-down runs because the number of up-down runs in a permutation is equal to the length of its longest alternating subsequence; an alternating subsequence is obtained by taking the last letter of each up-down run, and it is easy to see that this is indeed a longest alternating subsequence. For example, the up-down runs of $\pi = 51378624$ are 5, 51, 1378, 862, and 24, so 51824 is a longest alternating subsequence of π, which has length equal to the number of up-down runs of π.

Thus, the study of up-down runs is essentially the study of longest alternating subsequences through a different lens. In particular, the generating function $P^{\mathrm{udr}}(t,x)$ that we give below is equivalent to (4.2). Other properties of the as statistic can be determined by studying the udr statistic, and vice versa.

Theorem 15. *Let* $\mathrm{br}(\pi)$ *and* $\mathrm{udr}(\pi)$ *be the number of biruns and the number of up-down runs of* π*, respectively. Then,*

$$P^{\mathrm{br}}(t,x) = \frac{v}{(1+t)^2} \cdot \frac{2t + (1+x+t^2(1-x))\cosh(vx) - v(1+x)\sinh(vx)}{v\cosh(vx) - \sinh(vx)}$$

and

$$P^{\mathrm{udr}}(t,x) = \frac{v}{1+t} \cdot \frac{t + \cosh(vx) - v\sinh(vx)}{v\cosh(vx) - \sinh(vx)}$$

where $v = \sqrt{1-t^2}$.

The first ten br- and udr-polynomials are given below. Also see their OEIS entries [15, A059427 and A186370].

n	$P_n^{\mathrm{br}}(t)$
0	1
1	1
2	$2t$
3	$2t + 4t^2$
4	$2t + 12t^2 + 10t^3$
5	$2t + 28t^2 + 58t^3 + 32t^4$
6	$2t + 60t^2 + 236t^3 + 300t^4 + 122t^5$
7	$2t + 124t^2 + 836t^3 + 1852t^4 + 1682t^5 + 544t^6$
8	$2t + 252t^2 + 2766t^3 + 9576t^4 + 14622t^5 + 10332t^6 + 2770t^7$
9	$2t + 508t^2 + 8814t^3 + 45096t^4 + 103326t^5 + 119964t^6 + 69298t^7 + 15872t^8$

n	$P_n^{\mathrm{udr}}(t)$
0	1
1	t
2	$t + t^2$
3	$t + 3t^2 + 2t^3$
4	$t + 7t^2 + 11t^3 + 5t^4$
5	$t + 15t^2 + 43t^3 + 45t^4 + 16t^5$
6	$t + 31t^2 + 148t^3 + 268t^4 + 211t^5 + 61t^6$
7	$t + 63t^2 + 480t^3 + 1344t^4 + 1767t^5 + 1113t^6 + 272t^7$
8	$t + 127t^2 + 1509t^3 + 6171t^4 + 12099t^5 + 12477t^6 + 6551t^7 + 1385t^8$
9	$t + 255t^2 + 4661t^3 + 26955t^4 + 74211t^5 + 111645t^6 + 94631t^7 + 42585t^8 + 7936t^9$

Proof. The number of biruns of a permutation is exactly one more than its total number of peaks and valleys. To see why this is true, observe that every non-final long increasing run ends with a peak and that every non-final long decreasing run ends with a valley, which accounts for every peak, every valley, and all but the final birun. Thus, in finding the bivariate generating function for biruns, we can simply assign weights based on peaks and valleys and make an adjustment at the end.

Recall that the number of peaks in a permutation is equal to its number of non-final long increasing runs and that the number of valleys is equal to its number of non-initial long increasing runs. Hence, using the run network (G, P) as before, we set $w_k^{(1,2)} = w_k^{(2,3)} = t$ and $w_k^{(2,2)} = t^2$ for all $k \neq 1$. Then,

$$\left(I_3 + \begin{bmatrix} 0 & x + \frac{tx^2}{1-x} & 0 \\ 0 & x + \frac{t^2x^2}{1-x} & x + \frac{tx^2}{1-x} \\ 0 & 0 & 0 \end{bmatrix}\right)^{-1} = \begin{bmatrix} 1 & -\frac{(1-(1-t)x)x}{1-(1-t^2)x^2} & \frac{(1-(1-t)x)^2x^2}{(1-(1-t^2)x^2)(1-x)} \\ 0 & \frac{1-x}{1-(1-t^2)x^2} & -\frac{(1-(1-t)x)x}{1-(1-t^2)x^2} \\ 0 & 0 & 1 \end{bmatrix},$$

and converting to exponential generating functions gives

$$\begin{bmatrix} 1 & -\frac{1-\cosh(vx)}{1+t} - \frac{\sinh(vx)}{v} & -1 + \left(\frac{1-t}{1+t}\right)x - \frac{2\sinh(vx)}{(1+t)v} + e^x \\ 0 & \cosh(vx) - \frac{\sinh(vx)}{v} & -\frac{1-\cosh(vx)}{1+t} - \frac{\sinh(vx)}{v} \\ 0 & 0 & 1 \end{bmatrix}$$

where $v = \sqrt{1-t^2}$. Finally, we take the $(1,3)$ entry of the inverse matrix, add e^x to account for the increasing permutations, multiply by t, and then add $-tx - t + x + 1$. The result is

$$P^{\mathrm{br}}(t,x) = \frac{v}{(1+t)^2} \cdot \frac{2t + (1 + x + t^2(1-x))\cosh(vx) - v(1+x)\sinh(vx)}{u\cosh(vx) - \sinh(vx)}.$$

To compute the bivariate generating function $P^{\mathrm{udr}}(t,x)$ for the number of up-down runs, we use the same weights as before but also weight initial short runs. That is, we set $w_k^{(1,2)} = t$ for all k, and set $w_k^{(2,2)} = t^2$ and $w_k^{(2,3)} = t$ for all $k \neq 1$. Then the computation is done in the same way, and at the end we add e^x, multiply by t, and add $-t+1$ to obtain the desired generating function. □

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