

ASYMPTOTIC ESTIMATES FOR ROOTS OF THE CUBOID CHARACTERISTIC EQUATION IN THE LINEAR REGION.

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ABSTRACT. A perfect cuboid is a rectangular parallelepiped whose edges, whose face diagonals, and whose space diagonal are of integer lengths. The second cuboid conjecture specifies a subclass of perfect cuboids described by one Diophantine equation of tenth degree and claims their non-existence within this subclass. This Diophantine equation is called the cuboid characteristic equation. It has two parameters. The linear region is a domain on the coordinate plane of these two parameters given by certain linear inequalities. In the present paper asymptotic expansions and estimates for roots of the characteristic equation are obtained in the case where both parameters tend to infinity staying within the linear region. Their applications to the cuboid problem are discussed.

1. INTRODUCTION.

The cuboid characteristic equation in the case of the second cuboid conjecture is a polynomial Diophantine equation with two parameters p and q :

$$Q_{pq}(t) = 0. \quad (1.1)$$

The polynomial $Q_{pq}(t)$ from (1.1) is given by the explicit formula

$$\begin{aligned} Q_{pq}(t) = & t^{10} + (2q^2 + p^2)(3q^2 - 2p^2)t^8 + (q^8 + 10p^2q^6 + \\ & + 4p^4q^4 - 14p^6q^2 + p^8)t^6 - p^2q^2(q^8 - 14p^2q^6 + 4p^4q^4 + \\ & + 10p^6q^2 + p^8)t^4 - p^6q^6(q^2 + 2p^2)(3p^2 - 2q^2)t^2 - q^{10}p^{10}. \end{aligned} \quad (1.2)$$

The tenth degree polynomial (1.2) is related to the perfect cuboid problem through the following theorem (see Theorem 8.1 in [1] or in [2]).

Theorem 1.1. *A triple of positive integer numbers p , q , and t satisfying the equation (1.1) and such that $p \neq q$ are coprime produces a perfect cuboid if and only if the following inequalities are fulfilled:*

$$t > p^2, \quad t > pq, \quad t > q^2, \quad (p^2 + t)(pq + t) > 2t^2. \quad (1.3)$$

Once a triple of numbers p , q , t obeying Theorem 1.1 is found, there is a definite procedure for producing a perfect cuboid from them. First of all three rational

numbers α , β , and v are produced in one of the two ways: 1) using the formulas

$$\alpha = \frac{p^2}{t}, \quad \beta = \frac{pq}{t}, \quad v = \frac{q^2}{t}, \quad (1.4)$$

or 2) using the other three formulas

$$\alpha = \frac{pq}{t}, \quad \beta = \frac{p^2}{t}, \quad v = \frac{q^2}{t}. \quad (1.5)$$

Both ways (1.4) and (1.5) are acceptable.

Then the rational number z is produced using the following formula for it:

$$z = \frac{(1+v^2)(1-\beta^2)(1+\alpha^2)}{2(1+\beta^2)(1-\alpha^2v^2)}. \quad (1.6)$$

And finally, the numbers α , β , v along with z from (1.6) are used in the formulas

$$\begin{aligned} \frac{x_1}{L} &= \frac{2v}{1+v^2}, & \frac{d_1}{L} &= \frac{1-v^2}{1+v^2}, \\ \frac{x_2}{L} &= \frac{2z(1-v^2)}{(1+v^2)(1+z^2)}, & \frac{x_3}{L} &= \frac{(1-v^2)(1-z^2)}{(1+v^2)(1+z^2)}, \\ \frac{d_2}{L} &= \frac{(1+v^2)(1+z^2)+2z(1-v^2)}{(1+v^2)(1+z^2)}\beta, & \frac{d_3}{L} &= \frac{2(v^2z^2+1)}{(1+v^2)(1+z^2)}\alpha. \end{aligned} \quad (1.7)$$

They produce six rational numbers in the right hand sides of the formulas (1.7). Then L is chosen as a common denominator for all these six rational numbers. Such a choice assures that $x_1, x_2, x_3, d_1, d_2, d_3$ are integer numbers. They are edges and face diagonals of a perfect cuboid, while L is its space diagonal. In the other words, the integer numbers $x_1, x_2, x_3, d_1, d_2, d_3$, and L satisfy the cuboid equations

$$\begin{aligned} x_1^2 + x_2^2 + x_3^2 &= L^2, & x_2^2 + x_3^2 &= d_1^2, \\ x_3^2 + x_1^2 &= d_2^2, & x_1^2 + x_2^2 &= d_3^2. \end{aligned} \quad (1.8)$$

The formulas (1.7) were derived from (1.8) in [3] for the general case where, instead of (1.2), a twelfth degree polynomial arises. It reduces to the polynomial (1.2) in a special case, which is called the case of the second cuboid conjecture. The second cuboid conjecture itself is formulated as follows (see [4]).

Conjecture 1.1. *For any positive coprime integers $p \neq q$ the polynomial $Q_{pq}(t)$ is irreducible over the ring of integer numbers.*

Conjecture 1.1 means that the equation (1.1) has no integer roots. However, this conjecture is not yet proved nor disproved. It is just a conjecture. Therefore in the present paper we consider the roots of the equation (1.1) and study their dependence on p and q . This research continues the research from [3–7] and [1, 2]. As for the Diophantine equations (1.8), they are being studied for almost 300 years. For the history and various approaches to them the reader is referred to [8–50]. The approach of the papers [51–63] is based on so-called multisymmetric polynomials.

It is different from the approach of the present paper. Therefore we do not consider the papers [51–63] below.

The linear region associated with the cuboid characteristic polynomial (1.2) in the case of the second cuboid conjecture was defined in [2]. It is a domain in the positive quadrant of the pq -coordinate plane given by the linear inequalities

$$\frac{q}{59} < p, \quad p < 59q. \quad (1.9)$$

The main goal of the present paper is to obtain asymptotic expansions and estimates for the roots of the characteristic equation (1.1) as $p \rightarrow +\infty$ and $q \rightarrow +\infty$ simultaneously staying within the linear region (1.9).

2. ASYMPTOTIC EXPANSIONS WITH CONSTANT RATIO.

It is easy to see that the inequalities defining the linear region (1.9) set upper and lower bounds for the ratio of two parameters p and q . They are written as

$$\frac{1}{59} < \frac{p}{q} < 59. \quad (2.1)$$

Due to (2.1) below we consider the case where

$$p \rightarrow +\infty, \quad q \rightarrow +\infty, \quad \frac{p}{q} \rightarrow \theta \neq \infty \quad (2.2)$$

and where θ is a rational number. In this case we can write

$$\theta = \frac{a_{12}}{a_{22}}, \quad (2.3)$$

where a_{11} and a_{12} are two positive coprime integers, i. e. the fraction (2.3) is irreducible. For any two positive coprime integers a_{11} and a_{12} there are two other coprime integers a_{21} and a_{22} such that

$$a_{11} a_{22} - a_{21} a_{12} = 1. \quad (2.4)$$

This fact follows from the Euclidean division algorithm (see [64] or [65]).

Using the numbers a_{11} , a_{12} , a_{21} , a_{22} from (2.3) and (2.4), we define two matrices

$$S = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad T = \begin{vmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{vmatrix}. \quad (2.5)$$

The equality (2.4) means that $\det S = 1$ and $\det T = 1$. Moreover, the matrices (2.5) are inverse to each other. We use them as transition matrices (see [66]) and define the following change of coordinates in the pq -coordinate plane:

$$\begin{vmatrix} \tilde{p} \\ \tilde{q} \end{vmatrix} = \begin{vmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{vmatrix} \cdot \begin{vmatrix} p \\ q \end{vmatrix}, \quad \begin{vmatrix} p \\ q \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} \tilde{p} \\ \tilde{q} \end{vmatrix}. \quad (2.6)$$

The formulas (2.6) can be written in a non-matrix way:

$$\begin{cases} \tilde{p} = a_{22} p - a_{12} q, \\ \tilde{q} = -a_{21} p + a_{11} q, \end{cases} \quad \begin{cases} p = a_{11} \tilde{p} + a_{12} \tilde{q}, \\ q = a_{21} \tilde{p} + a_{22} \tilde{q}. \end{cases} \quad (2.7)$$

Using (2.7), (2.2), (2.3), and (2.4), we derive

$$\frac{\tilde{p}}{q} = a_{22} \frac{p}{q} - a_{12} \rightarrow a_{22} \theta - a_{12} = \frac{a_{22} a_{12} - a_{12} a_{22}}{a_{22}} = 0, \quad (2.8)$$

$$\frac{\tilde{q}}{q} = -a_{21} \frac{p}{q} + a_{11} \rightarrow -a_{21} \theta + a_{11} = \frac{-a_{21} a_{12} + a_{11} a_{22}}{a_{22}} = \frac{1}{a_{22}} \quad (2.9)$$

as $q \rightarrow +\infty$. Relying on (2.8) and (2.9), we use the more restrictive condition

$$\tilde{p} = \text{const} \quad \text{as} \quad \tilde{q} \rightarrow +\infty \quad (2.10)$$

when passing to the new variables \tilde{p} and \tilde{q} . It is worth to note the following lemma describing \tilde{p} and \tilde{q} in (2.7).

Lemma 2.1. *If p and q are coprime then the numbers \tilde{p} and \tilde{q} produced according to the first couple of the formulas (2.7) are also coprime.*

The proof is immediate from the second couple of the formulas (2.7). Indeed, if \tilde{p} and \tilde{q} have some common divisor r , then from the second couple of the formulas (2.7) we derive that r is a common divisor p and q , which contradicts their coprimality.

Using (2.7), let's substitute $p = a_{11} \tilde{p} + a_{12} \tilde{q}$ and $q = a_{21} \tilde{p} + a_{22} \tilde{q}$ into the polynomial (1.2). As a result we get another polynomial $Q_{\tilde{p}\tilde{q}}(t)$. This polynomial is given by an explicit formula. However, the formula for the polynomial $Q_{\tilde{p}\tilde{q}}(t)$ is rather huge. It is placed to the ancillary file `strategy_formulas_03.txt` in a machine-readable form.

Using the polynomial $Q_{\tilde{p}\tilde{q}}(t)$, we replace the equation (1.1) by the equation

$$Q_{\tilde{p}\tilde{q}}(t) = 0. \quad (2.11)$$

It is a tenth degree equation with respect to the variable t . Like $Q_{pq}(t)$, the polynomial $Q_{\tilde{p}\tilde{q}}(t)$ in (2.11) is even with respect to t . Along with each root t it has the opposite root $-t$. Therefore we use the condition

$$\begin{cases} t > 0 & \text{if } t \text{ is a real root,} \\ \text{Im}(t) > 0 & \text{if } t \text{ is a complex root,} \end{cases} \quad (2.12)$$

in order to divide the roots of the equation (2.11) into two groups. We denote through t_1, t_2, t_3, t_4, t_5 those roots that obey the conditions (2.12). Then $t_6, t_7, t_8, t_9, t_{10}$ are opposite roots of the equation (2.11):

$$t_6 = -t_1, \quad t_7 = -t_2, \quad t_8 = -t_3, \quad t_9 = -t_4, \quad t_{10} = -t_5. \quad (2.13)$$

Typically, asymptotic expansions for roots of a polynomial equation look like power series (see [67]). By analogy to (2.3) in [1] and according to (2.10) we write

$$t_i(\tilde{p}, \tilde{q}) = C_i \tilde{q}^{\alpha_i} \left(1 + \sum_{s=1}^{\infty} \beta_{is} \tilde{q}^{-s} \right) \quad \text{as } \tilde{q} \rightarrow +\infty. \quad (2.14)$$

The coefficients C_i in (2.14) should be nonzero: $C_i \neq 0$.

The exponents α_i and the coefficients C_i in (2.14) are determined using the Newton polygon (see [1]). The Newton polygon associated with the polynomial $Q_{\tilde{p}\tilde{q}}(t)$ in (2.11) is a triangle (see Fig. 2.1 below). Its boundary consists of three

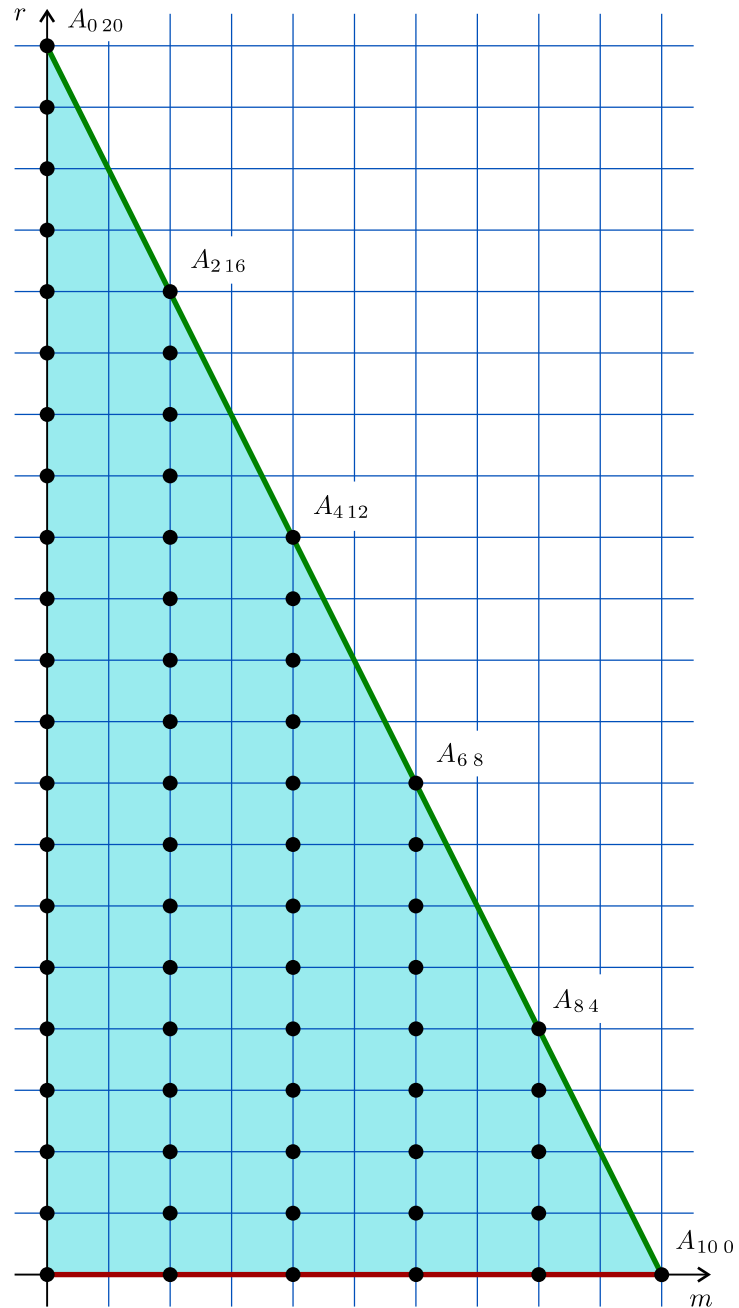


Fig. 2.1

parts — the upper part, the lower part, and the vertical part. The upper part is drawn in green, the lower part is drawn in red. The upper part of the Newton polygon in Fig. 2.1 is a segment of a straight line. It comprises six nodes associated with the equation (2.11). Here are the coefficients of these nodes:

$$\begin{aligned}
A_{10\ 0} &= 1, \\
A_{8\ 4} &= 6 a_{22}^4 - 2 a_{12}^4 - a_{12}^2 a_{22}^2, \\
A_{6\ 8} &= 10 a_{12}^2 a_{22}^6 + a_{12}^8 + a_{22}^8 + 4 a_{12}^4 a_{22}^4 - 14 a_{12}^6 a_{22}^2, \\
A_{4\ 12} &= 14 a_{12}^4 a_{22}^8 - 4 a_{12}^6 a_{22}^6 - a_{12}^2 a_{22}^{10} - 10 a_{12}^8 a_{22}^4 - a_{12}^{10} a_{22}^2, \\
A_{2\ 16} &= a_{12}^8 a_{22}^8 - 6 a_{12}^{10} a_{22}^6 + 2 a_{12}^6 a_{22}^{10}, \\
A_{0\ 20} &= -a_{12}^{10} a_{22}^{10}.
\end{aligned} \tag{2.15}$$

According to [1], the exponent α_i in (2.11) is determined by the slope of the upper boundary of the Newton triangle in Fig. 2.1 by means of the formula $\alpha_i = -k$. In our case $k = -2$, hence for α_i we derive

$$\alpha_i = 2. \tag{2.16}$$

The exponent (2.16) is common for all roots of the equation (2.11). The coefficient C_i in (2.14) is determined by the following equation:

$$\begin{aligned}
A_{10\ 0} C_i^{10} + A_{8\ 4} C_i^8 + A_{6\ 8} C_i^6 + \\
+ A_{4\ 12} C_i^4 + A_{2\ 16} C_i^2 + A_{0\ 20} = 0.
\end{aligned} \tag{2.17}$$

Substituting (2.15) into (2.17), we derive the equation

$$\begin{aligned}
C_i^{10} + (6 a_{22}^4 - 2 a_{12}^4 - a_{12}^2 a_{22}^2) C_i^8 + (10 a_{12}^2 a_{22}^6 + a_{12}^8 + \\
+ a_{22}^8 + 4 a_{12}^4 a_{22}^4 - 14 a_{12}^6 a_{22}^2) C_i^6 + (14 a_{12}^4 a_{22}^8 - \\
- 4 a_{12}^6 a_{22}^6 - a_{12}^2 a_{22}^{10} - 10 a_{12}^8 a_{22}^4 - a_{12}^{10} a_{22}^2) C_i^4 + \\
+ (2 a_{12}^6 a_{22}^{10} - 6 a_{12}^{10} a_{22}^6 + a_{12}^8 a_{22}^8) C_i^2 - a_{12}^{10} a_{22}^{10} = 0.
\end{aligned} \tag{2.18}$$

It is remarkable that the equation (2.18) can be produced from the equation (1.1) by substituting $t = C_i$, $p = a_{12}$ and $q = a_{22}$. Therefore the only known case where the equation (2.18) has a rational solution for C_i is the case $a_{12} = a_{22}$. Relying on the irreducibility of the fraction in (2.3), we set

$$a_{12} = 1, \quad a_{22} = 1. \tag{2.19}$$

Then, in order to satisfy the equality (2.4), we choose

$$a_{11} = 1, \quad a_{21} = 0. \tag{2.20}$$

The choice (2.20) is not unique. But it is the most simple.

Applying (2.19) and (2.20) to (2.7), we derive the following formulas:

$$\tilde{p} = p - q, \quad \tilde{q} = q, \quad p = \tilde{p} + \tilde{q}. \tag{2.21}$$

Due to (2.10) the formulas (2.21) mean that we choose the bisectorial direction on the pq -coordinate plane for asymptotic expansions.

3. BISECTORIAL EXPANSIONS.

Let's substitute (2.19) and (2.20) into the equation (2.11). As a result we can write the polynomial $Q_{\tilde{p}\tilde{q}}(t)$ from (2.11) in an explicit form:

$$\begin{aligned}
Q_{\tilde{p}\tilde{q}}(t) = & t^{10} + (3\tilde{q}^4 - 10\tilde{q}^3\tilde{p} - 13\tilde{q}^2\tilde{p}^2 - 8\tilde{p}^3\tilde{q} - 2\tilde{p}^4)t^8 + (\tilde{p}^8 - \\
& - 40\tilde{q}^7\tilde{p} - 136\tilde{q}^4\tilde{p}^4 + 2\tilde{q}^8 + 14\tilde{p}^6\tilde{q}^2 - 148\tilde{q}^6\tilde{p}^2 - 208\tilde{q}^5\tilde{p}^3 + \\
& + 8\tilde{p}^7\tilde{q} - 28\tilde{p}^5\tilde{q}^3)t^6 - (10\tilde{p}^9\tilde{q}^3 + 302\tilde{q}^{10}\tilde{p}^2 + 836\tilde{q}^7\tilde{p}^5 + \\
& + 60\tilde{q}^{11}\tilde{p} + 200\tilde{p}^7\tilde{q}^5 + 704\tilde{q}^9\tilde{p}^3 + \tilde{p}^{10}\tilde{q}^2 + 494\tilde{q}^6\tilde{p}^6 + 2\tilde{q}^{12} + \\
& + 956\tilde{q}^8\tilde{p}^4 + 55\tilde{p}^8\tilde{q}^4)t^4 - (6\tilde{p}^{10}\tilde{q}^6 + 1444\tilde{q}^{11}\tilde{p}^5 + 1230\tilde{q}^{10}\tilde{p}^6 + \\
& + 712\tilde{q}^9\tilde{p}^7 + 40\tilde{q}^{15}\tilde{p} + 1160\tilde{q}^{12}\tilde{p}^4 + 624\tilde{q}^{13}\tilde{p}^3 + 269\tilde{q}^8\tilde{p}^8 + \\
& + 3\tilde{q}^{16} + 60\tilde{p}^9\tilde{q}^7 + 212\tilde{q}^{14}\tilde{p}^2)t^2 - 210\tilde{q}^{16}\tilde{p}^4 - \tilde{q}^{10}\tilde{p}^{10} - \tilde{q}^{20} - \\
& - 210\tilde{q}^{14}\tilde{p}^6 - 120\tilde{q}^{13}\tilde{p}^7 - 45\tilde{q}^{12}\tilde{p}^8 - 10\tilde{q}^{11}\tilde{p}^9 - 10\tilde{q}^{19}\tilde{p} - \\
& - 120\tilde{q}^{17}\tilde{p}^3 - 252\tilde{q}^{15}\tilde{p}^5 - 45\tilde{q}^{18}\tilde{p}^2.
\end{aligned} \tag{3.1}$$

Now let's substitute (2.19) into the equation (2.18). As a result we can factor it:

$$(C_i - 1)(C_i + 1)(C_i^2 + 1)^4 = 0. \tag{3.2}$$

The equation (3.2) has two simple real roots

$$C_i = 1, \quad C_i = -1$$

and two purely imaginary roots

$$C_i = i, \quad C_i = -i$$

of multiplicity four. Here $i = \sqrt{-1}$. The condition (2.12) excludes two roots $C_i = -1$ and $C_i = -i$. Therefore from (2.14) we derive the expansion

$$t_i(\tilde{p}, \tilde{q}) = \tilde{q}^2 \left(1 + \sum_{s=1}^{\infty} \beta_{is} \tilde{q}^{-s} \right) \text{ as } \tilde{q} \rightarrow +\infty \tag{3.3}$$

for real roots of the polynomial (3.1) and the expansion

$$t_i(\tilde{p}, \tilde{q}) = i\tilde{q}^2 \left(1 + \sum_{s=1}^{\infty} \beta_{is} \tilde{q}^{-s} \right) \text{ as } \tilde{q} \rightarrow +\infty \tag{3.4}$$

for complex roots of the polynomial (3.1) in the equation (2.11).

4. ASYMPTOTIC ESTIMATES FOR REAL ROOTS.

According to the formulas (3.3) and (3.4), both real and complex roots of the polynomial (3.1) are growing as $\tilde{q} \rightarrow +\infty$. Therefore we need to specify the expan-

sions (3.3) and (3.4) up to non-growing terms. For a real root we have

$$t_1(\tilde{p}, \tilde{q}) = \tilde{q}^2 + 5\tilde{p}\tilde{q} + 10\tilde{p}^2 + R_1(\tilde{p}, \tilde{q}) \text{ as } \tilde{q} \rightarrow +\infty. \quad (4.1)$$

The formula (4.1) is in agreement with the formula (3.3). It means that $\beta_{11} = 5\tilde{p}$ and $\beta_{12} = 10\tilde{p}^2$. Like in [1] and [2], we have to obtain an estimate of the form

$$|R_1(\tilde{p}, \tilde{q})| < \frac{C(\tilde{p})}{\tilde{q}} \quad (4.2)$$

for the remainder term $R_1(\tilde{p}, \tilde{q})$ in (4.1). For this purpose we substitute

$$t = \tilde{q}^2 + 5\tilde{p}\tilde{q} + 10\tilde{p}^2 + \frac{c}{\tilde{q}} \quad (4.3)$$

into the polynomial (3.1). Then we replace \tilde{q} with the new variable z :

$$z = \frac{1}{\tilde{q}}. \quad (4.4)$$

As a result of two substitutions (4.3) and (4.4) and upon removing denominators the equation (2.11) with the polynomial (3.1) turns to a polynomial equation in the new variables c and z . This equation can be written as

$$1216\tilde{p}^3 + f(\tilde{p}, c, z) = -32c. \quad (4.5)$$

Here $f(\tilde{p}, c, z)$ is a polynomial of three variables given by an explicit formula. The formula for $f(\tilde{p}, c, z)$ is rather huge. Therefore it is placed to the ancillary file **strategy_formulas_03.txt** in a machine-readable form.

Let $\tilde{q} \geq 97|\tilde{p}|$ and let c belong to one of the following two intervals:

$$\begin{aligned} -74|\tilde{p}|^3 < c < 0 & \text{ if } \tilde{p} > 0, \\ 0 < c < 74|\tilde{p}|^3 & \text{ if } \tilde{p} < 0. \end{aligned} \quad (4.6)$$

From $\tilde{q} \geq 97|\tilde{p}|$ and from (4.4) we derive the estimate $|z| \leq 1/97|\tilde{p}|^{-1}$. Using this estimate and using the inequalities (4.6), by means of direct calculations one can derive the following estimate for the modulus of the function $f(\tilde{p}, c, z)$:

$$|f(\tilde{p}, c, z)| < 1142|\tilde{p}|^3. \quad (4.7)$$

For fixed \tilde{p} and z the estimate (4.7) means that the left hand side of the equation (4.5) is a continuous function of c whose values obey the inequalities

$$\begin{aligned} 74|\tilde{p}|^3 &\leq 1216\tilde{p}^3 + f(\tilde{p}, c, z) \leq 2358|\tilde{p}|^3 \text{ if } \tilde{p} > 0, \\ -2358|\tilde{p}|^3 &\leq 1216\tilde{p}^3 + f(\tilde{p}, c, z) \leq -74|\tilde{p}|^3 \text{ if } \tilde{p} < 0 \end{aligned} \quad (4.8)$$

while c runs over the corresponding interval (4.6). The right hand side of the equation (4.5) is also a continuous function of the variable c . Moreover, it is monotonic.

Multiplying the inequalities (4.6) by -32 , we find that the values of the right hand side of the equation (4.5) fill one of the two intervals

$$\begin{aligned} 0 < -32c < 2368|\tilde{p}|^3 & \text{ if } \tilde{p} > 0, \\ -2368|\tilde{p}|^3 < -32c < 0 & \text{ if } \tilde{p} < 0 \end{aligned} \quad (4.9)$$

while c runs over the corresponding interval (4.6). Comparing (4.8) with (4.9), we see that there is at least one root of the polynomial equation (4.5) somewhere in one of the two intervals (4.6).

The case $\tilde{p} = 0$ is exceptional. In this case $f(\tilde{p}, c, z) = 0$. Hence $c = 0$ is a root of the equation (4.5) for this case.

The variable c is related to the variable t by means of the formula (4.4). Therefore the inequalities (4.6) for c imply the following inequalities for t :

$$\begin{aligned} \tilde{q}^2 + 5\tilde{p}\tilde{q} + 10\tilde{p}^2 - \frac{74|\tilde{p}|^3}{\tilde{q}} < t < \tilde{q}^2 + 5\tilde{p}\tilde{q} + 10\tilde{p}^2 & \text{ if } \tilde{p} > 0, \\ \tilde{q}^2 + 5\tilde{p}\tilde{q} + 10\tilde{p}^2 < t < \tilde{q}^2 + 5\tilde{p}\tilde{q} + 10\tilde{p}^2 + \frac{74|\tilde{p}|^3}{\tilde{q}} & \text{ if } \tilde{p} < 0. \end{aligned} \quad (4.10)$$

The case $\tilde{p} = 0$ is exceptional. In this case we get

$$t = \tilde{q}^2 \text{ if } \tilde{p} = 0. \quad (4.11)$$

The result obtained is formulated as a theorem.

Theorem 4.1. *For each $\tilde{q} \geq 97|\tilde{p}|$ there is at least one real root of the polynomial (3.1) satisfying one of the conditions in (4.10) and (4.11).*

Theorem 4.1 proves the asymptotic expansion (4.1) and provides the estimate of the form (4.2) for the remainder term in it.

5. ASYMPTOTICS FOR COMPLEX ROOTS.

For complex roots of the polynomial (3.1) we have the formula specifying (3.4):

$$t_i(\tilde{p}, \tilde{q}) = \mathbf{i}\tilde{q}^2 + u_i\tilde{p}\tilde{q} - \frac{u_i + \mathbf{i}u_i^2}{2}\tilde{p}^2 + R_i(\tilde{p}, \tilde{q}) \text{ as } \tilde{q} \rightarrow +\infty \quad (5.1)$$

and $i = 2, \dots, 5$. Here $\mathbf{i} = \sqrt{-1}$ and u_i are roots of the following quartic equation:

$$u^4 + 8u^2 - 12\mathbf{i}u - 4 = 0. \quad (5.2)$$

The equation (5.2) is irreducible. All of its roots are irrational. Two of them are purely imaginary. Here are approximate values for these two roots:

$$u_2 \approx 0.4863801704\mathbf{i}, \quad u_3 \approx -3.439109107\mathbf{i}. \quad (5.3)$$

The other two roots are complex. Their approximate values are

$$\begin{aligned} u_4 & \approx 0.4600767354 + 1.476364468\mathbf{i}, \\ u_5 & \approx -0.4600767354 + 1.476364468\mathbf{i}. \end{aligned} \quad (5.4)$$

Like in the previous section, below we derive an estimate of the form

$$|R_i(\tilde{p}, \tilde{q})| < \frac{C_i(\tilde{p})}{\tilde{q}} \quad (5.5)$$

for the remainder term in the formula (5.1). For this purpose we substitute

$$t = i\tilde{q}^2 + u\tilde{p}\tilde{q} - \frac{u + iu^2}{2}\tilde{p}^2 + \frac{c}{\tilde{q}}. \quad (5.6)$$

into the polynomial (3.1). Then we replace \tilde{q} with the new variable z using (4.4). As a result of two substitutions (5.6) and (4.4) and upon removing denominators the equation (2.11) with the polynomial (3.1) turns to a polynomial equation in the new variables c and z . This equation can be written as

$$\begin{aligned} 212992 i \tilde{p}^6 - 598016 \tilde{p}^6 u - 446464 i \tilde{p}^6 u^2 + \\ + 110592 \tilde{p}^6 u^3 + \varphi(u, \tilde{p}, c, z) = 352256 \tilde{p}^3 c. \end{aligned} \quad (5.7)$$

Here $\varphi(u, \tilde{p}, c, z)$ is a polynomial of four variables given by an explicit formula. The formula for $\varphi(u, \tilde{p}, c, z)$ is rather huge. Therefore it is placed to the ancillary file `strategy_formulas_03.txt` in a machine-readable form.

Let $\tilde{q} \geq 97 |\tilde{p}|$. Now c is a complex variable. Assume that it runs over the open disk on the complex plane given by the inequality

$$|c| < 51 |\tilde{p}|^3. \quad (5.8)$$

From $\tilde{q} \geq 97 |\tilde{p}|$ and from (4.4) we derive the estimate $|z| \leq 1/97 |\tilde{p}|^{-1}$. Using this estimate and using (5.8), upon substituting the value $u = u_2$ from (5.3) one can derive the following estimate for the modulus of the function $\varphi(u, \tilde{p}, c, z)$:

$$|\varphi(u_2, \tilde{p}, c, z)| \leq 1174818 |\tilde{p}|^6. \quad (5.9)$$

For fixed \tilde{p} and z the estimate (5.9) means that the left hand side of the equation (5.7) is a function of c whose values within the disc (5.8) obey the estimate

$$\begin{aligned} |212992 i \tilde{p}^6 - 598016 \tilde{p}^6 u_2 - 446464 i \tilde{p}^6 u_2^2 + \\ + 110592 \tilde{p}^6 u_2^3 + \varphi(u_2, \tilde{p}, c, z)| \leq 1189840 |\tilde{p}|^6. \end{aligned} \quad (5.10)$$

Note that it is a holomorphic function which is continuous up to the boundary of the disc (5.8). Therefore the estimate (5.10) holds on the boundary of the disc.

The right hand side of the equation (5.7) is also a holomorphic function of c . It has exactly one zero at the origin within the disc (5.8) and its modulus is constant on the boundary of this disc. Indeed, we have

$$|352256 \tilde{p}^3 c| = 17965056 |\tilde{p}|^6 \quad \text{if } |c| = 74 |\tilde{p}|^3. \quad (5.11)$$

Comparing the numbers $1189839 < 17965056$ from (5.10) and (5.11) and applying the Rouché theorem (see [68] or [69]), we conclude that the equation (5.7) with $u = u_2$ has exactly one solution within the disc (5.8).

The same conclusion is valid for the other three roots $u = u_3$, $u = u_4$, and $u = u_5$ of the equation (5.2) from (5.3) and (5.4). However, the estimate (5.10) for them is replaced by the following three estimates:

$$\begin{aligned} & |212992 i \tilde{p}^6 - 598016 \tilde{p}^6 u_3 - 446464 i \tilde{p}^6 u_3^2 + \\ & + 110592 \tilde{p}^6 u_3^3 + \varphi(u_3, \tilde{p}, c, z)| \leq 16504669 |\tilde{p}|^6, \end{aligned} \quad (5.12)$$

$$\begin{aligned} & |212992 i \tilde{p}^6 - 598016 \tilde{p}^6 u_4 - 446464 i \tilde{p}^6 u_4^2 + \\ & + 110592 \tilde{p}^6 u_4^3 + \varphi(u_4, \tilde{p}, c, z)| \leq 2513770 |\tilde{p}|^6, \end{aligned} \quad (5.13)$$

$$\begin{aligned} & |212992 i \tilde{p}^6 - 598016 \tilde{p}^6 u_5 - 446464 i \tilde{p}^6 u_5^2 + \\ & + 110592 \tilde{p}^6 u_5^3 + \varphi(u_5, \tilde{p}, c, z)| \leq 2513770 |\tilde{p}|^6. \end{aligned} \quad (5.14)$$

The numbers 16504669 and 2513770 in the right hand sides of (5.12), (5.13), and (5.14) are smaller than the number 17965056 in (5.11), which is the reason for applying the Rouché theorem.

The variable c is related to the variable t by means of the formula (5.6). Therefore the inequality (5.8) for c implies the following inequalities for t :

$$\left| t - i \tilde{q}^2 - u_2 \tilde{p} \tilde{q} + \frac{u_2 + i u_2^2}{2} \tilde{p}^2 \right| < \frac{51 |\tilde{p}|^3}{\tilde{q}} \quad \text{if } \tilde{p} \neq 0, \quad (5.15)$$

$$\left| t - i \tilde{q}^2 - u_3 \tilde{p} \tilde{q} + \frac{u_3 + i u_3^2}{2} \tilde{p}^2 \right| < \frac{51 |\tilde{p}|^3}{\tilde{q}} \quad \text{if } \tilde{p} \neq 0, \quad (5.16)$$

$$\left| t - i \tilde{q}^2 - u_4 \tilde{p} \tilde{q} + \frac{u_4 + i u_4^2}{2} \tilde{p}^2 \right| < \frac{51 |\tilde{p}|^3}{\tilde{q}} \quad \text{if } \tilde{p} \neq 0, \quad (5.17)$$

$$\left| t - i \tilde{q}^2 - u_5 \tilde{p} \tilde{q} + \frac{u_5 + i u_5^2}{2} \tilde{p}^2 \right| < \frac{51 |\tilde{p}|^3}{\tilde{q}} \quad \text{if } \tilde{p} \neq 0. \quad (5.18)$$

The inequalities (5.15), (5.16), (5.17), and (5.18) define four disks which play the same role as the intervals (4.10) in the previous section.

The case $\tilde{p} = 0$ is exceptional. In this case the disks (5.15), (5.16), (5.17), and (5.18) collapse to the point $t = i \tilde{q}^2$ thus producing a multiple root:

$$t = i \tilde{q}^2. \quad (5.19)$$

Now we can formulate the result of this section as a theorem.

Theorem 5.1. *For each $\tilde{q} \geq 97 |\tilde{p}|$ there is exactly one root of the polynomial (3.1) in each of the four disks (5.15), (5.16), (5.17), and (5.18) or there is one multiple root given by the formula (5.19).*

Theorem 5.1 proves the asymptotic expansion (5.1) and provides the estimate of the form (5.5) for the remainder terms in it.

6. NON-INTERSECTION OF ASYMPTOTIC SITES.

In the previous two sections we have found five sites where roots of the polynomial (3.1) are located. They are the intervals (4.10) and the disks (5.15), (5.16), (5.17),

(5.18) in the non-degenerate case $\tilde{p} \neq 0$ and the points (4.10) and (5.19) in the degenerate case $\tilde{p} = 0$. The points (4.10) and (5.19) do not coincide since $\tilde{q} = q > 0$ (see (2.21) and Theorem 1.1). In the case $\tilde{p} \neq 0$ we have the following lemma.

Lemma 6.1. *For $\tilde{q} \geq 97 |\tilde{p}| \neq 0$ the asymptotic sites (4.10), (5.15), (5.16), (5.17), and (5.18) do not intersect with each other.*

Proof. In order to prove Lemma 6.1 for discs it is sufficient to calculate the distances between their centers and compare them with the double of their radius, which is the same for all of them. The intervals (4.10) are centered about the point $t = \tilde{q}^2 + 5\tilde{p}\tilde{q} + 10\tilde{p}^2$. They can be enclosed in one disc with the radius $74 |\tilde{p}|^3 / \tilde{q}$ for rough estimates. By means of direct calculations one can derive a lower bound for the distances from the center of this disc to the centers of the other four discs:

$$d_{1i} \geq 1.4 (\tilde{q} - 2.5 |\tilde{p}|)^2 - 21.75 |\tilde{p}|^2, \quad i = 2, \dots, 5. \quad (6.1)$$

Applying $\tilde{q} \geq 97 |\tilde{p}|$ to (6.1), we obtain the inequality

$$d_{1i} \geq 12480 |\tilde{p}|^2. \quad (6.2)$$

For the sum of the radii of two discs from $\tilde{q} \geq 97 |\tilde{p}|$ we derive

$$r_1 + r_i = \frac{74 |\tilde{p}|^3}{\tilde{q}} + \frac{51 |\tilde{p}|^3}{\tilde{q}} \leq \frac{125}{97} |\tilde{p}|^2 < 2 |\tilde{p}|^2. \quad (6.3)$$

Comparing (6.2) and (6.3), we see that $d_{1i} > r_1 + r_i$, i. e. the intervals (4.10) do not intersect with the discs (5.15), (5.16), (5.17), and (5.18).

Similarly, for the mutual distances between centers of the discs (5.15), (5.16), (5.17), and (5.18) one can obtain the following lower bound:

$$d_{ij} \geq 0.98 |\tilde{p}| \tilde{q} - 8 |\tilde{p}|^2, \quad i = 2, \dots, 5 \text{ and } i \neq j. \quad (6.4)$$

Applying $\tilde{q} \geq 97 |\tilde{p}|$ to (6.4), we obtain the inequality

$$d_{ij} \geq 87 |\tilde{p}|^2. \quad (6.5)$$

For the double radius of the discs (5.15), (5.16), (5.17), and (5.18) from the inequality $\tilde{q} \geq 97 |\tilde{p}|$ we derive the following upper bound:

$$r_i + r_j = 2 r_i = \frac{102 |\tilde{p}|^3}{\tilde{q}} \leq \frac{102}{97} |\tilde{p}|^2 < 2 |\tilde{p}|^2. \quad (6.6)$$

Comparing (6.5) and (6.6), we see that $d_{ij} > r_i + r_j$, i. e. the discs (5.15), (5.16), (5.17), and (5.18) do not intersect with each other. Lemma 6.1 is proved. \square

Lemma 6.2. *For $\tilde{q} \geq 97 |\tilde{p}| \neq 0$ the asymptotic discs (5.15), (5.16), (5.17), and (5.18) are enclosed in upper half-plane of the complex plane and do not intersect with the real axis.*

Proof. The proof is based on the following lower bound for the distances from the centers of the discs (5.15), (5.16), (5.17), (5.18) to the real axis:

$$d_i \geq (\tilde{q} - 2 |\tilde{p}|)^2 - 12 |\tilde{p}|^2, \quad i = 2, \dots, 5. \quad (6.7)$$

Applying $\tilde{q} \geq 97 |\tilde{p}|$ to (6.7), we obtain the inequality

$$d_i \geq 9013 |\tilde{p}|^2. \quad (6.8)$$

On the other hand, for the radius of the discs (5.15), (5.16), (5.17), and (5.18) from the inequality $\tilde{q} \geq 97 |\tilde{p}|$ we derive the following upper bound:

$$r_i = \frac{51 |\tilde{p}|^3}{\tilde{q}} \leq \frac{51}{97} |\tilde{p}|^2 < |\tilde{p}|^2. \quad (6.9)$$

Comparing (6.8) and (6.9), we see that Lemma 6.2 is proved. \square

Lemmas 6.1 and 6.2 are summed up in the following theorem.

Theorem 6.1. *For $\tilde{q} \geq 97 |\tilde{p}| \neq 0$ five roots t_1, t_2, t_3, t_4, t_5 of the polynomial (3.1) obeying the condition (2.12) are simple. They are located within five disjoint sites (4.10), (5.15), (5.16), (5.17), and (5.18), one per each site.*

Due to (2.13) Theorem 6.1 locates all of the ten roots of the polynomial (3.1). Theorem 6.1 does not cover the degenerate case $\tilde{p} = 0$. However, due to (2.21) the equality $\tilde{p} = 0$ implies $p = q$. Therefore the degenerate case $\tilde{p} = 0$ does not produce perfect cuboids (see Theorem 1.1).

7. INTEGER POINTS OF ASYMPTOTIC SITES.

According to Theorem 6.1, for $\tilde{q} \geq 97 |\tilde{p}| \neq 0$ the equation (2.11) with the polynomial (3.1) has exactly one real positive root t_1 belonging to one of the two asymptotic intervals (4.10). The following theorem is immediate from (4.10).

Theorem 7.1. *If $\tilde{q} \geq 97 |\tilde{p}| \neq 0$ and if $\tilde{q} > 74 |\tilde{p}|^3$, then the asymptotic intervals (4.10) have no integer points.*

8. APPLICATION TO THE CUBOID PROBLEM.

The equation (1.1) is related to the perfect cuboid problem through Theorem 1.1. The equation (2.11) differs from the equation (1.1) by the change of variables (2.21). Let's consider the case $\tilde{p} < 0$ in (4.10). In this case from (4.10) we take

$$t < \tilde{q}^2 + 5 \tilde{p} \tilde{q} + 10 \tilde{p}^2 + \frac{74 |\tilde{p}|^3}{\tilde{q}}. \quad (8.1)$$

Theorem 1.1 provides four additional inequalities (1.3). Since $q = \tilde{q}$ in (2.21), the third of them is written as $t > \tilde{q}^2$. Combining it with (8.1), we get

$$\tilde{q}^2 + 5 \tilde{p} \tilde{q} + 10 \tilde{p}^2 + \frac{74 |\tilde{p}|^3}{\tilde{q}} > \tilde{q}^2. \quad (8.2)$$

Since $\tilde{p} < 0$, the inequality (8.2) turns to the following one:

$$-5 |\tilde{p}| \tilde{q} + 10 |\tilde{p}|^2 + \frac{74 |\tilde{p}|^3}{\tilde{q}} > 0. \quad (8.3)$$

Let's apply the inequality $\tilde{q} \geq 97 |\tilde{p}|$ from Theorem 7.1. Since $\tilde{q} = q > 0$, it yields

$$-5 |\tilde{p}| \tilde{q} \leq -5 \cdot 97 |\tilde{p}|^2, \quad \frac{74 |\tilde{p}|^3}{\tilde{q}} \leq \frac{74 |\tilde{p}|^2}{97}. \quad (8.4)$$

From (8.4) one can easily derive the following inequality:

$$-5 |\tilde{p}| \tilde{q} + 10 |\tilde{p}|^2 + \frac{74 |\tilde{p}|^3}{\tilde{q}} \leq \left(-5 \cdot 97 + 10 + \frac{74}{97}\right) |\tilde{p}|^2 = -\frac{46001}{97} |\tilde{p}|^2 < 0. \quad (8.5)$$

The inequalities (8.3) and (8.5) contradict each other. The contradiction obtained means that Theorem 7.1 can be modified in the following way.

Theorem 8.1. *If $\tilde{q} \geq 97 |\tilde{p}|$ and $\tilde{p} < 0$, then the corresponding asymptotic interval in (4.10) has no integer points producing perfect cuboids.*

Unfortunately Theorem 8.1 cannot be extended to the case $\tilde{p} > 0$. In this case the inequality $\tilde{q} \geq 97 |\tilde{p}|$ does not contradict the cuboid inequalities (1.3). However, Theorem 7.1 is still valid in the case $\tilde{p} > 0$.

Theorem 7.1 provides two inequalities $\tilde{q} \geq 97 |\tilde{p}| \neq 0$ and $\tilde{q} > 74 |\tilde{p}|^3$. In the case $\tilde{p} > 0$, passing to the original variables p and q by means of the formulas (2.21), these two inequalities are written in the following way:

$$p \leq \frac{98}{97} q = q + \frac{q}{97}, \quad p < q + \sqrt[3]{\frac{q}{74}}, \quad (8.6)$$

Due to (2.21) the inequality $\tilde{p} > 0$ itself means $p > q$.

Similarly, Theorem 8.1 provides the inequality $\tilde{q} \geq 97 |\tilde{p}|$ in the case $\tilde{p} < 0$, i. e. when $p < q$. Passing to the original variables p and q by means of (2.21), we can write this inequality in the following way:

$$p \geq \frac{96}{97} q = q - \frac{q}{97}, \quad (8.7)$$

Since the bisector line $p = q$ does not produce perfect cuboids, the inequalities (8.6) and (8.7) can be united and then written as follows:

$$q - \frac{q}{97} \leq p, \quad p \leq q + \min\left(\frac{q}{97}, \sqrt[3]{\frac{q}{74}}\right). \quad (8.8)$$

In this form the above inequalities (8.8) are similar to the linear inequalities (1.9) defining the linear region.

9. CONCLUSIONS.

Theorems 7.1 and 8.1 along with the inequalities (8.8) constitute the main result of this paper. The inequalities (8.8) define a subregion within the linear region (1.9). Theorems 7.1 and 8.1 prove that no cuboids are available within this subregion.

The subregion defined by the inequalities (8.8) is rather small. It looks like a narrow spiky strip surrounding the bisector line $p = q$ within the positive quadrant of the pq -coordinate plane. A substantial part of the linear region (1.9) still remains for numeric search of perfect cuboids.

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