COUNTING THE IDEALS OF GIVEN CODIMENSION OF THE ALGEBRA OF LAURENT POLYNOMIALS IN TWO VARIABLES

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Abstract. We establish an explicit formula for the number $C_n(q)$ of ideals of codimension n of the algebra $\mathbb{F}_q[x,y,x^{-1},y^{-1}]$ of Laurent polynomials in two variables over a finite field $\mathbb{F}_q$ of cardinality q. This number is a palindromic polynomial of degree $2n$ in q. Moreover, $C_n(q) = (q-1)^2 P_n(q)$, where $P_n(q)$ is another palindromic polynomial; the latter is a q-analogue of the sum of divisors of n, which happens to be the number of subgroups of $\mathbb{Z}^2$ of index n.

1. Introduction

Let $\mathbb{F}_q$ be a finite field of cardinality q and $\mathbb{F}_q[x,y,x^{-1},y^{-1}]$ be the algebra of Laurent polynomials in two variables with coefficients in $\mathbb{F}_q$.

Our main aim is to give a formula for the number $C_n(q)$ of ideals of codimension n of $\mathbb{F}_q[x,y,x^{-1},y^{-1}]$. Our main result is the following.

Theorem 1.1. *For each integer $n \geqslant 1$ we have*

$$C_n(q) = \sum_{\lambda \vdash n} (q-1)^{2\nu(\lambda)} \, q^{n-\ell(\lambda)} \prod_{\substack{i=1,\dots,t \\ d_i \geqslant 1}} \frac{q^{2d_i}-1}{q^2-1},$$

where the sum runs over all partitions λ of n. The expression $C_n(q)$ is a monic polynomial of degree $2n$ in the variable q with integer coefficients. Moreover, the polynomial $C_n(q)$ is divisible by $(q-1)^2$.

The notation $\ell(\lambda)$, $\nu(\lambda)$, d_i appearing in the formula will be explained in Section 3.1. The proof of the theorem will be given in Section 5.3; it relies on a parametrization by Conca and Valla [6] of the affine cells in the Ellingsrud–Strømme decomposition of the Hilbert scheme of n points on the affine plane.

Note that since $C_n(q)$ is divisible by $(q-1)^2$, we may define for each $n \geqslant 1$ a unique polynomial $P_n(q)$ by

$$C_n(q) = (q-1)^2 P_n(q), \tag{1.1}$$

which clearly implies $C_n(1) = 0$ for all $n \geqslant 1$. Table 1 (resp. Table 2) at the end of the paper displays the polynomials $C_n(q)$ (resp. the polynomials $P_n(q)$) for $n \leqslant 12$.

Theorem 1.1 has two interesting consequences. The first one concerns the polynomials $P_n(q)$. Let us state it.

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Corollary 1.2. *For each $n \geqslant 1$ the polynomial $P_n(q)$ is a monic polynomial of degree $2n-2$ with integer coefficients and we have*

$$P_n(1) = \sigma(n) = \sum_{d|n\,;\,d\geqslant 1} d.$$

As is well known, the sum $\sigma(n)$ of positive divisors of n is equal to the number of subgroups of index n of the free abelian group $\mathbb{Z}^2$ of rank two. Thus Theorem 1.1 and Corollary 1.2 imply that the number of ideals of codimension n of the Laurent polynomial algebra $\mathbb{F}_q[x,y,x^{-1},y^{-1}]$, i.e. of the algebra of the group $\mathbb{Z}^2$, is, up to the factor $(q-1)^2$, a q-analogue[1] of the number of subgroups of index n of $\mathbb{Z}^2$.

A similar phenomenon had been observed by Bacher and the second-named author in [3]: up to a power of $q-1$, the number of right ideals of codimension n of the algebra $\mathbb{F}_q[F_2]$ of the rank two free group F_2 is a q-analogue of the number of subgroups of index n of F_2. Actually it was this observation that prompted us to compute the number of ideals of codimension n of the algebra $\mathbb{F}_q[\mathbb{Z}^2]$ of the free abelian group $\mathbb{Z}^2$, i.e. of $\mathbb{F}_q[x,y,x^{-1},y^{-1}]$.

In a similar context, the following holds.

(a) By [8] (see also Section 3.1 below) the number of ideals of codimension n of the polynomial algebra $\mathbb{F}_q[x,y]$, which is the algebra of the free abelian monoid $\mathbb{N}^2$, is a q-analogue of the number $p(n)$ of partitions of n; as is well known, the latter is equal to the number of ideals of the monoid $\mathbb{N}^2$ whose complement is of cardinality n.

(b) In a non-commutative setting, by [20, 2], the number of right ideals of codimension n of the free algebra $\mathbb{F}_q\langle x,y\rangle$ is a q-analogue of the number of right ideals of the free monoid $\langle x,y\rangle^*$ whose complement is of cardinality n.

(c) It may be shown that the number of right ideals of codimension 2 of the algebra $\mathbb{F}_q[F_3]$ of the rank three free group F_3 is equal to

$$q^2(q-1)^5\left((q+1)^3-1\right).$$

The last factor is obviously a q-analogue of $2^3-1=7$, which is the number of subgroups of index 2 of F_3.

We conjecture the number of right ideals of codimension 2 of the algebra $\mathbb{F}_q[F_r]$ of the free group F_r with r generators to be of the form $q^i(q-1)^j\left((q+1)^r-1\right)$ for some non-negative integers i, j; the last factor is then a q-analogue of the number 2^r-1 of subgroups of index 2 of F_r. More generally, we expect the number of right ideals of codimension n of $\mathbb{F}_q[F_r]$, up to a power of $q-1$, to be a q-analogue of the number of subgroups of index n of F_r (see also the conclusion of [3]).

Remark 1.3. The commutative algebra $L_r = \mathbb{F}_q[x_1, x_1^{-1}, \ldots, x_r, x_r^{-1}]$ of Laurent polynomials in r variables ($r \geqslant 3$) provides a distinct contrast with the cases discussed above. We can show that the number of right ideals of codimension 2 of L_r, which is the algebra of the free abelian group $\mathbb{Z}^r$, is equal to $(q-1)^r R_r(q)$, where

$$R_r(q) = \frac{1}{2}\left((q+1)^r + (q-1)^r\right) + \frac{q^r-1}{q-1} - 1.$$

The latter is a q-analogue of $R_r(1) = 2^{r-1}+r-1$. Now the number of subgroups of index 2 of $\mathbb{Z}^r$ is equal to 2^r-1, which is different from $R_r(1)$ when $r \geqslant 3$.

[1] By a q-analogue of an integer r we mean a polynomial $P(q)$ in the variable q such that $P(1) = r$.

The second consequence of Theorem 1.1 expresses the generating function of the polynomials $C_n(q)$ as a nice infinite product.

Corollary 1.4. *(a) We have*

$$1 + \sum_{n\geqslant 1} \frac{C_n(q)}{q^n}\, t^n = \prod_{i\geqslant 1} \frac{(1-t^i)^2}{1-(q+q^{-1})t^i + t^{2i}}.$$

(b) The polynomials $C_n(q)$ and $P_n(q)$ are palindromic.

The previous infinite product shows up in [9, p. 10] (see for instance Equations (9.2) and (10.1)) and probably in other papers on basic hypergeometric series; in an algebraic geometry context it appears in [16, Th. 4.1.3], where it is equal to the generating function of the E-polynomials of the punctual Hilbert schemes of the complex two-dimensional torus (see details in Section 6.3 below).

Using Corollary 1.4, we gave explicit expressions for the coefficients of the polynomials $C_n(q)$ and $P_n(q)$ in the companion paper [18] (see Theorems 1.1 and 1.2 in *loc. cit.*). We obtained a rather striking positivity result, namely the coefficients of $P_n(q)$ are all *non-negative* integers. For the sake of completeness we recall our formulas for the coefficients of the polynomials $C_n(q)$ and $P_n(q)$ in Appendix A.

The paper is organized as follows. Section 2 is devoted to some preliminaries: we first recall the one-to-one correspondence between the ideals of the localization $S^{-1}A$ of an algebra A and certain ideals of A; we also count tuples of polynomials subject to certain constraints over a finite field.

In Section 3 we recall Conca and Valla's parametrization of the affine cells in a decomposition of the Hilbert scheme of n points in the plane; these cells are indexed by the partitions of n. We show how to deduce a parametrization of the cells in the induced decomposition of the Hilbert scheme of n points in a Zariski open subset of the plane.

In Section 4 we apply the techniques of the preceding section to compute the number of ideals of codimension n of $\mathbb{F}_q[x, y, y^{-1}]$. In passing we give a criterion (Proposition 4.1) which will also be used in the proof of Theorem 1.1.

In Section 5 we define what we call an invertible Gröbner cell, which is a Zariski open subset of the corresponding affine cell, and compute its cardinality over a finite field. We derive a proof of Theorem 1.1.

The proofs of Corollary 1.4 of and of Corollary 1.2 are given in Section 6.

In Appendix A we briefly recall the results on the coefficients of $C_n(q)$ and $P_n(q)$ we obtained in [18].

2. Preliminaries

We fix a ground field k. By algebra we mean an associative unital k-algebra. In this paper all algebras are assumed to be *commutative*.

2.1. **Ideals in localizations.** Let A be a (commutative) algebra, S a multiplicative submonoid of A not containing 0, and $S^{-1}A$ the corresponding localization of A. We assume that the canonical algebra map $i : A \to S^{-1}A$ is injective (this is the case, for instance, when A is a domain).

Recall the well-known correspondence between the ideals of $S^{-1}A$ and those of A (see [4, Chap. 2, § 2, n$^{\text{o}}$ 4–5], [7, Prop. 2.2]).

(a) For any ideal J of $S^{-1}A$, the set $i^{-1}(J) = J \cap A$ is an ideal of A and we have $J = i^{-1}(J)S^{-1}A$. The map $J \mapsto i^{-1}(J)$ is an injection from the set of ideals of $S^{-1}A$ to the set of ideals of A.

(b) An ideal I of A is of the form $i^{-1}(J)$ for some ideal J of $S^{-1}A$ if and only if for all $s \in S$ the endomorphism of A/I induced by the multiplication by s is injective.

Given an integer $n \geqslant 1$, a *n-codimensional* ideal of A is an ideal such that $\dim_k A/I = n$. For such an ideal, the previous condition (b) is then equivalent to: for all $s \in S$, the endomorphism of A/I induced by the multiplication by s is a linear isomorphism.

We leave the proof of the following lemma to the reader.

Lemma 2.1. *If J is a finite-codimensional ideal of $S^{-1}A$, then the canonical algebra map $i : A \to S^{-1}A$ induces an algebra isomorphism*

$$A/i^{-1}(J) \cong (S^{-1}A)/J.$$

It follows that there is a bijection between the set of n-codimensional ideals of $S^{-1}A$ and the set of n-codimensional ideals I of A such that for all $s \in S$, the endomorphism of A/I induced by the multiplication by s is a linear isomorphism. The latter assertion is equivalent to s being invertible modulo I, that is the image of s in A/I being invertible.

The following criterion will be used in Sections 4 and 5.

Lemma 2.2. *Let A be a commutative algebra. For any $s \in A$, let $p : A \to A/(s)$ be the natural projection onto the quotient algebra of A by the ideal generated by s. If I is an ideal of A, then s is invertible modulo I if and only if $p(I) = A/(s)$.*

Proof. If s is invertible modulo I, then there exists $t \in A$ such that $st-1 \in I$. Hence, $p(1)$ belongs to $p(I)$, which implies $p(I) = A/(s)$. Conversely, if $p(I) = A/(s)$, then $p(1) = p(u)$ for some $u \in I$. Hence $1 - u \in (s)$, which means that there is $t \in A$ such that $1 - u = st$. Thus, $st \equiv 1 \pmod{I}$. □

2.2. **Counting polynomials over a finite field.** In this subsection we assume that $k = \mathbb{F}_q$ is a finite field of cardinality q. We shall need the following in Section 5.

Proposition 2.3. *Let d, h be integers $\geqslant 1$ and $Q_1, \ldots, Q_h \in \mathbb{F}_q[y]$ be coprime polynomials. The number of $(h+1)$-tuples $(P, P_1, \ldots, P_h)$ satisfying the three conditions*

(i) *P is a degree d monic polynomial with $P(0) \neq 0$,*
(ii) *$P_1, \ldots, P_h$ are polynomials of degree $< d$, and*
(iii) *P and $P_1Q_1 + \cdots + P_hQ_h$ are coprime,*

is equal to

$$(q-1)^2\, q^{(h-1)d}\, \frac{q^{2d}-1}{q^2-1}\,.$$

Before giving the proof, we state and prove two auxiliary lemmas.

Lemma 2.4. *Let R be a finite commutative ring and $a_1, \ldots, a_h \in R$ such that $a_1R + \cdots + a_hR = R$. For any $b \in R$, the number of h-tuples $(x_1, \ldots, x_h) \in R^h$ such that $a_1x_1 + \cdots + a_hx_h = b$ is equal to* $(\mathrm{card}\, R)^{h-1}$.

Proof. The map $(x_1, \dots, x_h) \mapsto a_1 x_1 + \cdots + a_h x_h$ is a homomorphism $R^h \to R$ of additive groups. Since it is surjective, the number of h-tuples satisfying the above condition is equal to the cardinality of its kernel, which is equal to $\operatorname{card} R^h / \operatorname{card} R = (\operatorname{card} R)^{h-1}$. □

Lemma 2.5. *Let $d \geqslant 1$ be an integer. The number of couples $(P, Q) \in \mathbb{F}_q[y]^2$ such that P is a degree d monic polynomial with $P(0) \neq 0$, Q is of degree $< d$, and P and Q are coprime is equal to*

$$c_d = (q-1)^2 \frac{q^{2d} - 1}{q^2 - 1}.$$

Proof. This amounts to counting the number of couples (P, z), where $P \in \mathbb{F}_q[y]$ is a degree d monic polynomial not divisible by y and z is an invertible element of the quotient ring $\mathbb{F}_q[y]/(P)$.

Expanding P into a product of irreducible polynomials and using the Chinese remainder lemma, we have

$$1 + \sum_{d \geqslant 1} c_d t^d = \prod_{\substack{P \text{ irreducible} \\ P \neq y}} \left(1 + \sum_{k \geqslant 1} \operatorname{card}(\mathbb{F}_q[y]/(P))^\times \, t^{k \deg(P)} \right),$$

where the product is taken over all irreducible polynomials of $\mathbb{F}_q[y]$ different from y and where $\deg(P)$ denotes the degree of P. First observe that for any irreducible polynomial $P \in \mathbb{F}_q[y]$ the group $(\mathbb{F}_q[y]/(P))^\times$ of invertible elements of $\mathbb{F}_q[y]/(P)$ is of cardinality $q^{k \deg(P)} - q^{(k-1)\deg(P)}$: indeed, there are $q^{k \deg(P)}$ polynomials of degree $< k \deg(P)$ and $q^{(k-1)\deg(P)}$ of them are divisible by P, hence not invertible in $\mathbb{F}_q[y]/(P)$. Consequently,

$$\begin{aligned} 1 + \sum_{d \geqslant 1} c_d t^d &= \prod_{\substack{P \text{ irreducible} \\ P \neq y}} \left(1 + \left(1 - q^{-\deg(P)} \right) \sum_{k \geqslant 1} (qt)^{k \deg(P)} \right) \\ &= \prod_{\substack{P \text{ irreducible} \\ P \neq y}} \left(1 + \left(1 - q^{-\deg(P)} \right) \frac{(qt)^{\deg(P)}}{1 - (qt)^{\deg(P)}} \right) \\ &= \prod_{\substack{P \text{ irreducible} \\ P \neq y}} \frac{1 - t^{\deg(P)}}{1 - (qt)^{\deg(P)}}. \end{aligned}$$

On one hand the infinite product $\prod_{\substack{P \text{ irreducible} \\ P \neq y}} (1 - t^{\deg(P)})^{-1}$ is equal to the zeta function $Z_{\mathbb{A}^1 \setminus \{0\}}(t)$ of the affine line minus a point. On the other,

$$Z_{\mathbb{A}^1 \setminus \{0\}}(t) = \frac{Z_{\mathbb{A}^1}(t)}{Z_{\{0\}}(t)} = \frac{1 - t}{1 - qt}.$$

Therefore,

$$1 + \sum_{d \geqslant 1} c_d t^d = \frac{1 - qt}{1 - q^2 t} \bigg/ \frac{1 - t}{1 - qt} = \frac{(1 - qt)^2}{(1 - t)(1 - q^2 t)}.$$

Subtracting 1 from both sides, we obtain

$$\sum_{d \geqslant 1} c_d t^d = (q-1)^2 \frac{t}{(1 - t)(1 - q^2 t)},$$

from which it is easy to derive the desired formula for c_d. □

Proof of Proposition 2.3. We have to count the number of those $(h+2)$-tuples $(P, Q, P_1, \dots, P_h)$ such that P is a degree d monic polynomial with $P(0) \neq 0$, Q is a polynomial of degree $< d$ and coprime to P, each polynomial P_i is of degree $< d$, and $\sum_{i=1}^{h} P_i Q_i \equiv Q$ modulo P.

By Lemma 2.5, the number of couples (P, Q) satisfying these conditions is equal to $(q-1)^2\,(q^{2d}-1)/(q^2-1)$. Since card $\mathbb{F}_q[y]/(P) = q^d$, by Lemma 2.4 we have $q^{d(h-1)}$ choices for the h-tuples $(P_1, \dots, P_h)$. The number we wish to count is the product of the two previous ones. □

3. The Hilbert scheme of points in a Zariski open subset of the plane

Let k be a field. As is well known, the ideals of codimension n of an affine k-algebra A are in bijection with the k-points of the Hilbert scheme parametrizing finite subschemes of colength n of the spectrum of A. For instance the ideals of codimension n of the polynomial algebra $k[x, y]$ are in bijection with the k-points of the Hilbert scheme $\mathrm{Hilb}^n(\mathbb{A}_k^2)$ of n points on the affine plane. Similarly, the ideals of codimension n of the Laurent polynomial algebra $k[x, y, x^{-1}, y^{-1}]$ are in bijection with the k-points of the Hilbert scheme $\mathrm{Hilb}^n((\mathbb{A}_k^1 \backslash \{0\}) \times (\mathbb{A}_k^1 \backslash \{0\}))$ of n points on the two-dimensional torus, which is a Zariski open subset of the plane.

In this paragraph we prove that the Hilbert scheme of n points in a Zariski open subset of the plane is an open subscheme of the Hilbert scheme of n points in the plane, and show how to determine it explicitly.

3.1. **Parametrizing the finite-codimensional ideals of $k[x, y]$.** Computing the homology of Hilbert scheme $\mathrm{Hilb}^n(\mathbb{A}_k^2)$, Ellingsrud and Strømme [8] showed that it has a cellular decomposition indexed by the partitions λ of n, each cell C_λ being an affine space of dimension $n + \ell(\lambda)$, where $\ell(\lambda)$ is the length of λ.

It follows that, in the special case when $k = \mathbb{F}_q$ is a finite field of cardinality q, the number $A_n(q)$ of ideals of $\mathbb{F}_q[x, y]$ of codimension n is finite and given by the polynomial

$$A_n(q) = \sum_{\lambda \vdash n} q^{n+\ell(\lambda)}, \tag{3.1}$$

where the sum runs over all partitions λ of n (we indicate this by the notation $\lambda \vdash n$ or by $|\lambda| = n$). The polynomial $A_n(q)$ clearly has non-negative integer coefficients, its degree is $2n$, and $A_n(1) = p(n)$ is equal to the number of partitions of n (for more on the polynomials $A_n(q)$, see Remark 4.7).

For our purposes we need an explicit description of the affine cells C_λ. We use a parametrization due to Conca and Valla [6]. Let us now recall it.

Given a positive integer n, there is a well-known bijection between the partitions of n and the monomials ideals of codimension n of $k[x, y]$. The correspondence is as follows: to a partition λ of n we associate the sequence

$$0 = m_0 < m_1 \leqslant \cdots \leqslant m_t$$

of integers counting from right to left the boxes in each column of the Ferrers diagram of λ; we have $m_1 + \cdots + m_t = n$. Then the associated monomial ideal I_λ^0 is given by

$$I_\lambda^0 = (x^t, x^{t-1} y^{m_1}, \dots, x y^{m_{t-1}}, y^{m_t}). \tag{3.2}$$

(Note that the generating set in the right-hand side of (3.2) is in general not minimal.) The set $\mathcal{B}_\lambda = \{x^i y^j \mid 0 \leqslant i < t,\ 0 \leqslant j < m_i\}$ induces a linear basis of the n-dimensional quotient algebra $k[x,y]/I^0_\lambda$.

Consider the lexicographic ordering on the monomials $x^i y^j$ given by

$$1 < y < y^2 < \cdots < x < xy < xy^2 < \cdots < x^2 < x^2y < x^2y^2 < \cdots$$

Then the cell C_λ, called *Gröbner cell* in [6], is by definition the set of ideals I of $k[x,y]$ such that the dominating terms (for this ordering) of the elements of I generate the monomial ideal I^0_λ. It was proved in [8] that C_λ is an affine space.

Here is how Conca and Valla explicitly parametrize C_λ. Given a partition λ of n and the associated sequence $0 = m_0 < m_1 \leqslant \cdots \leqslant m_t$, they first define the sequence of integers $d_1, \ldots, d_t$ by

$$d_i = m_i - m_{i-1} \geqslant 0. \tag{3.3}$$

We have $d_1 = m_1 > 0$.

Later we shall also need the integer

$$v(\lambda) = \mathrm{card}\,\{i = 1, \ldots, t \mid d_i \geqslant 1\}\,; \tag{3.4}$$

this integer is equal to the number of distinct values of the sequence $m_1 \leqslant \cdots \leqslant m_t$. Note that $v(\lambda) \geqslant 1$; moreover, $v(\lambda) = 1$ if and only if the partition is "rectangular", i.e. $m_1 = \cdots = m_t\,(> 0)$.

Let T_λ be the set of $(t+1) \times t$-matrices $(p_{i,j})$ with entries in the one-variable polynomial algebra $k[y]$ satisfying the following conditions: $p_{i,j} = 0$ if $i < j$, the degree of $p_{i,j}$ is less than d_j if $i \geqslant j$ and $d_j \geqslant 1$, and $p_{i,j} = 0$ for all i if $d_j = 0$. The set T_λ is an affine space whose dimension is $n + \ell(\lambda)$.

Now consider the $(t+1) \times t$-matrix

$$M_\lambda = \begin{pmatrix}
y^{d_1} + p_1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
p_{2,1} - x & y^{d_2} + p_2 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
p_{3,1} & p_{3,2} - x & y^{d_3} + p_3 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\
p_{i-1,1} & p_{i-1,2} & p_{i-1,3} & \cdots & y^{d_{i-1}} + p_{i-1} & 0 & 0 & \cdots & 0 \\
p_{i,1} & p_{i,2} & p_{i,3} & \cdots & p_{i,i-1} - x & y^{d_i} + p_i & 0 & \cdots & 0 \\
p_{i+1,1} & p_{i+1,2} & p_{i+1,3} & \cdots & p_{i+1,i-1} & p_{i+1,i} - x & y^{d_{i+1}} + p_{i+1} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
p_{t,1} & p_{t,2} & p_{t,3} & \cdots & p_{t,i-1} & p_{t,i} & p_{t,i+1} & \cdots & y^{d_t} + p_t \\
p_{t+1,1} & p_{t+1,2} & p_{t+1,3} & \cdots & p_{t+1,i-1} & p_{t+1,i} & p_{t+1,i+1} & \cdots & p_{t+1,t} - x
\end{pmatrix}, \tag{3.5}$$

where for simplicity we set $p_i = p_{i,i}$.

By [6, Th. 3.3] the map sending the polynomial matrix $(p_{i,j}) \in T_\lambda$ to the ideal I_λ of $k[x,y]$ generated by all t-minors (the maximal minors) of the matrix M_λ is a bijection of T_λ onto C_λ. These minors are polynomial expressions with integer coefficients in the coefficients of the $p_{i,j}$'s.

3.2. **Localizing.** Let S be a multiplicative submonoid of $k[x,y]$ not containing 0. We assume that S has a finite generating set Σ. In the sequel we shall concentrate on two cases: $\Sigma = \{y\}$ (in Section 4) and $\Sigma = \{x,y\}$ (in Section 5).

It follows from Section 2 that the set of n-codimensional ideals of the localization $S^{-1}k[x,y]$ can be identified with the subset of $\mathrm{Hilb}^n(\mathbb{A}^2_k)$ consisting of the n-codimensional ideals I of $k[x,y]$ such that for all $s \in S$, the endomorphism μ_s of $k[x,y]/I$ induced by the multiplication by s is a linear isomorphism. The latter is equivalent to $\det \mu_s \neq 0$ for all $s \in \Sigma$.

By the considerations of Section 3.1, the set of n-codimensional ideals of the algebra $S^{-1}k[x,y]$ is the disjoint union

$$\coprod_{\lambda \vdash n} C_\lambda^\Sigma ,$$

where C_λ^Σ is the Zariski open subset of the affine Gröbner cell C_λ consisting of the points satisfying $\det \mu_s \neq 0$ for all $s \in \Sigma$.

Consequently, the Hilbert scheme $\mathrm{Hilb}^n(\mathrm{Spec}(S^{-1}k[x,y]))$ parametrizing subschemes of colength n in $\mathrm{Spec}(S^{-1}k[x,y])$ is an open subscheme of $\mathrm{Hilb}^n(\mathbb{A}_k^2)$, hence an open subscheme of $\mathrm{Hilb}^n(\mathbb{P}_k^2)$. Since by [10, 12] the latter is smooth and projective, $\mathrm{Hilb}^n(\mathrm{Spec}(S^{-1}k[x,y]))$ is a smooth quasi-projective variety.

The endomorphism μ_x (resp. μ_y) of $k[x,y]/I$ induced by the multiplication by x (resp. by y) can be expressed as a matrix in the basis $\mathcal{B}_\lambda$. Observe that the entries of such a matrix are polynomial expressions with integer coefficients in the coefficients of the $p_{i,j}$'s. Therefore, if any $s \in \Sigma$ is a linear combination with integer coefficients of monomials in the variables x, y, then the Hilbert scheme $\mathrm{Hilb}^n(\mathrm{Spec}(S^{-1}k[x,y]))$ is defined over $\mathbb{Z}$ as a variety.

In particular, the Hilbert schemes $\mathrm{Hilb}^n(\mathbb{A}_k^1 \times (\mathbb{A}_k^1 \backslash \{0\}))$ and $\mathrm{Hilb}^n((\mathbb{A}_k^1 \backslash \{0\})^2)$ are smooth quasi-projective varieties defined over $\mathbb{Z}$.

Example 3.1. Let λ be the unique self-conjugate partition of 3. In this case, $t = 2$, $m_1 = 1$, $m_2 = 2$, hence $d_1 = d_2 = 1$. The corresponding matrix M_λ, as in (3.5), is

$$M_\lambda = \begin{pmatrix} y+a & 0 \\ b-x & y+d \\ c & e-x \end{pmatrix},$$

where a, b, c, d, e are scalars. The associated Gröbner cell C_λ is a 5-dimensional affine space parametrized by these five scalars. The ideal I_λ is generated by the maximal minors of the matrix, namely by $(b-x)(e-x) - c(y+d)$, $(e-x)(y+a)$, and $(y+a)(y+d)$. It follows that modulo I_λ we have the relations

$$x^2 \equiv (b+e)x + cy + (cd - be), \quad xy \equiv -ax + ey + ae, \quad y^2 \equiv -(a+d)y - ad.$$

In the basis $\mathcal{B}_\lambda = \{x, y, 1\}$ the multiplication endomorphisms μ_x and μ_y can be expressed as the matrices

$$\mu_x = \begin{pmatrix} b+e & -a & 1 \\ c & e & 0 \\ cd-be & ae & 0 \end{pmatrix} \quad \text{and} \quad \mu_y = \begin{pmatrix} -a & 0 & 0 \\ e & -(a+d) & 1 \\ ae & -ad & 0 \end{pmatrix}.$$

We have $\det \mu_x = e(ac - cd + be)$ and $\det \mu_y = -ad^2$.

It follows from the above computations that, if for instance $\Sigma = \{x, y\}$, then C_λ^Σ is the complement in the affine space $\mathbb{A}_k^5$ of the union of the three hyperplanes $a = 0$, $d = 0$, $e = 0$ and of the quadric hypersurface $ac - cd + be = 0$.

4. The punctual Hilbert scheme of the complement of a line in an affine plane

In this section we apply the considerations of the previous section to the case $\Sigma = \{y\}$. Here S is the multiplicative submonoid of $k[x,y]$ generated by y and $S^{-1}k[x,y] = k[x,y,y^{-1}] = k[x][y,y^{-1}]$.

By Section 3.2, the Hilbert scheme $\mathrm{Hilb}^n(\mathbb{A}_k^1 \times (\mathbb{A}_k^1 \backslash \{0\}))$, that is the set of n-codimensional ideals of $k[x,y,y^{-1}]$, is the disjoint union over the partitions λ of n

of the sets C_λ^y, where C_λ^y consists of the ideals $I \in C_\lambda$ such that y is invertible in $k[x,y]/I$. We call C_λ^y the *semi-invertible Gröbner cell* associated to the partition λ.

4.1. **A criterion for the invertibility of y.** Let $p_y : k[x,y] \to k[x]$ be the algebra map sending x to itself and y to 0. Then by Lemma 2.2, the set C_λ^y consists of the ideals $I \in C_\lambda$ such that $p_y(I) = k[x]$.

Recall from Section 3.1 that I_λ is generated by the maximal minors of the matrix M_λ of (3.5), namely by the polynomials $f_0(x,y), \ldots, f_t(x,y)$, where we define $f_i(x,y)$ to be the determinant of the $t \times t$-matrix obtained from M_λ by deleting its $(i+1)$-st row. Then the ideal $p_y(I_\lambda)$ can be identified with the ideal of $k[x]$ generated by the polynomials $f_0(x,0), \ldots, f_t(x,0) \in k[x]$ obtained by setting $y = 0$. We need to determine under what conditions this ideal is equal to the whole algebra $k[x]$.

Recall the entries of the matrix M_λ and particularly the polynomials $p_{i,j}$ and $p_i = p_{i,i} \in k[y]$. Let $a_{i,j} = p_{i,j}(0)$ be the constant term of $p_{i,j}$. As above, we set $a_i = a_{i,i} = p_i(0)$. Note that $a_j = 1$ and $a_{i,j} = 0$ for all $i \neq j$ whenever $d_j = 0$.

Then $f_0(x,0), \ldots, f_t(x,0)$ are the maximal minors of the matrix

$$M_\lambda^y = \begin{pmatrix} a_1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ a_{2,1}-x & a_2 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ a_{3,1} & a_{3,2}-x & a_3 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ a_{i-1,1} & a_{i-1,2} & a_{i-1,3} & \cdots & a_{i-1} & 0 & 0 & \cdots & 0 \\ a_{i,1} & a_{i,2} & a_{i,3} & \cdots & a_{i,i-1}-x & a_i & 0 & \cdots & 0 \\ a_{i+1,1} & a_{i+1,2} & a_{i+1,3} & \cdots & a_{i+1,i-1} & a_{i+1,i}-x & a_{i+1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{t,1} & a_{t,2} & a_{t,3} & \cdots & a_{t,i-1} & a_{t,i} & a_{t,i+1} & \cdots & a_t \\ a_{t+1,1} & a_{t+1,2} & a_{t+1,3} & \cdots & a_{t+1,i-1} & a_{t+1,i} & a_{t+1,i+1} & \cdots & a_{t+1,t}-x \end{pmatrix}.$$

To be precise, $f_i(x,0)$ is the determinant of the square matrix obtained from M_λ^y by deleting its $(i+1)$-st row.

The criterion we need is the following.

Proposition 4.1. *We have $p_y(I_\lambda) = k[x]$ if and only if $a_i \neq 0$ for all $i = 1, \ldots, t$ such that $d_i \geqslant 1$.*

Proof. Since $a_i = 1$ when $d_i = 0$, it is equivalent to prove that $p_y(I_\lambda) = k[x]$ if and only if $a_1 a_2 \cdots a_t \neq 0$.

Set $I_x = p_y(I_\lambda) \subset k[x]$. The condition $a_1 a_2 \cdots a_t \neq 0$ is sufficient. Indeed, the last polynomial, $f_t(x,0)$, is the determinant of a lower triangular matrix whose diagonal entries are the scalars a_i; hence, $f_t(x,0) = a_1 a_2 \cdots a_t$. Thus, if $f_t(x,0)$ is non-zero, then $I_x = k[x]$.

To check the necessity of the condition, we will prove that for each $i = 1, \ldots, t$, the vanishing of the scalar a_i implies that the ideal I_x is contained in a proper ideal generated by a minor of M_λ^y.

If $a_1 = 0$, then $f_1(x,0) = \cdots = f_t(x,0) = 0$ since these are determinants of matrices whose first row is zero. It follows that I_x is the principal ideal generated by the characteristic polynomial $f_0(x,0)$, which is of degree $t \geqslant 1$. Hence, I_x is a proper ideal of $k[x]$.

Let now $i \geqslant 2$. If for $k \geqslant i$, we delete the $(k+1)$-st row of M^y_λ, we obtain a lower block-triangular matrix of the form

$$\begin{pmatrix} M_1 & 0 \\ * & M_2^{(k)} \end{pmatrix},$$

where M_1 is the square submatrix of M^y_λ corresponding to the rows $1, \dots, i$ and to the columns $1, \dots, i$; this is a lower triangular matrix whose diagonal entries are $a_1, \dots, a_i$. Consequently, if $a_i = 0$, then $f_k(x, 0) = 0$ for all $k \geqslant i$.

Under the same condition $a_i = 0$, if we delete the $(k+1)$-st row of M^y_λ for $k < i$, then we obtain a lower block-triangular matrix of the form

$$\begin{pmatrix} M_1^{(k)} & 0 \\ * & M_2 \end{pmatrix},$$

where M_2 is the square submatrix of M^y_λ corresponding to the rows $i+1, \dots t+1$ and to the columns $i, \dots t$:

$$M_2 = \begin{pmatrix} a_{i+1,i} - x & a_{i+1} & \cdots & 0 & 0 \\ a_{i+2,i} & a_{i+2,i+1} - x & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{t,i} & \cdots & \cdots & a_{t,t-1} - x & a_t \\ a_{t+1,i} & a_{t+1,i+1} & \cdots & a_{t+1,t-1} & a_{t+1,t} - x \end{pmatrix}.$$

Consequently, the polynomials $f_k(x, 0)$ for $k < i$ are all divisible by the determinant of M_2. Thus, I_x is contained in the ideal generated by $\det(M_2)$, which is a characteristic polynomial of degree $t - i + 1$. Since $t - i + 1 \geqslant 1$ for all $i = 1, \dots, t$, we have $I_x \neq k[x]$. □

As an immediate consequence of Section 3.2 and of Proposition 4.1 we obtain the following.

Corollary 4.2. *The set of n-codimensional ideals of $k[x, y, y^{-1}]$ is the disjoint union*

$$\coprod_{\lambda \vdash n} C^y_\lambda,$$

where C^y_λ is the complement in the affine Gröbner cell C_λ of the union of the hyperplanes $a_i = 0$ where i runs over all integers $i = 1, \dots, t$ such that $d_i \geqslant 1$.

4.2. **On the number of finite-codimensional ideals of $\mathbb{F}_q[x, y, y^{-1}]$.** Recall the positive integer $\nu(\lambda)$ defined by (3.4).

Proposition 4.3. *Let $k = \mathbb{F}_q$. For each partition λ of n, the set C^y_λ is finite and its cardinality is given by*

$$\operatorname{card} C^y_\lambda = (q-1)^{\nu(\lambda)}\, q^{n + \ell(\lambda) - \nu(\lambda)}.$$

Proof. By Corollary 4.2 the set C^y_λ is parametrized by $n + \ell(\lambda)$ parameters subject to the sole condition that $\nu(\lambda)$ of them are not zero. □

Corollary 4.4. *For each integer $n \geqslant 1$, the number $B_n(q)$ of n-codimensional ideals of $\mathbb{F}_q[x, y, y^{-1}]$ is equal to $(q-1)\, q^n\, B^\circ_n(q)$, where*

$$B^\circ_n(q) = \sum_{\lambda \vdash n} (q-1)^{\nu(\lambda)-1}\, q^{\ell(\lambda) - \nu(\lambda)}.$$

Note that $B_n^\circ(q)$ is a polynomial in q since $\nu(\lambda) \geqslant 1$ and $\ell(\lambda) \geqslant \nu(\lambda)$ for all partitions. It is of degree $n-1$ and has integer coefficients. The coefficients of $B_n^\circ(q)$ may be negative, as one can see in Table 3 at the end of the paper.

Remark 4.5. Let v_n be the valuation of the polynomial $B_n^\circ(q)$, i.e. the maximal integer r such that q^r divides $B_n^\circ(q)$. We conjecture that $v_n = 0$, 1, or 2, and that the infinite word $v_1 v_2 v_3 \ldots$ is equal to $0 \prod_{n=1}^{\infty} 01^{2n}02^n$.

Let us now give a product formula for the generating function of the sequence of polynomials $B_n(q)$ and an arithmetical interpretation for two values of $B_n^\circ(q)$.

Theorem 4.6. *(a) Let $B_n(q)$ be the number of ideals of $\mathbb{F}_q[x, y, y^{-1}]$ of codimension n. We have*

$$1 + \sum_{n\geqslant 1} \frac{B_n(q)}{q^n} t^n = \prod_{i\geqslant 1} \frac{1-t^i}{1-qt^i}.$$

(b) Let $B_n^\circ(q)$ be the polynomial $B_n^\circ(q) = (q-1)^{-1} q^{-n} B_n(q)$. It has integer coefficients and satisfies

$$B_n^\circ(1) = \sigma_0(n),$$

where $\sigma_0(n)$ is the number of divisors of n, and

$$B_n^\circ(-1) = \begin{cases} (-1)^{k-1} & \text{if } n = k^2 \text{ for some integer } k,\\ 0 & \text{otherwise.}\end{cases}$$

Proof. (a) Since an analogous proof will be used in Remark 4.7 and in Section 6.2, we give here a detailed proof. Let X be a set and M be the free abelian monoid on X (X is called a basis of M). We say that a function $\varphi : M \to R$ from M to a ring R is *multiplicative* if $\varphi(uv) = \varphi(u)\varphi(v)$ for all couples $(u, v) \in M^2$ of words having no common basis element. Under this condition, it is easy to check the following identity:

$$\sum_{w\in M} \varphi(w) = \prod_{x\in X} \left(1 + \sum_{e\geqslant 1} \varphi(x^e)\right). \tag{4.1}$$

Now, identifying each partition with its planar diagram, we consider a partition λ as a union of rectangular partitions i^{e_i}, with e_i parts of length i, for $e_i \geqslant 1$ and distinct $i \geqslant 1$, which we denote by the formal product $\lambda = \prod_{i\geqslant 1} i^{e_i}$. Thus the set of partitions is equal to the free abelian monoid on $X = \mathbb{N}\setminus\{0\}$. Before we apply (4.1), let us remark that $|\lambda| = \sum_i ie_i$ and $\ell(\lambda) = \sum_i e_i$. Moreover, the multisets $\{e_i \mid i \geqslant 1\}$ and $\{d_i \mid i \geqslant 1\}$ are equal (recall that the integers d_i are those associated with λ in (3.3)); therefore, $\nu(\lambda) = \sum_i 1 = \mathrm{card}\{i \mid e_i \geqslant 1\}$.

The function $\lambda \mapsto \operatorname{card} C^y_\lambda \, s^{|\lambda|}$ computed in Proposition 4.3 is clearly multiplicative. Applying (4.1), we obtain

$$\begin{aligned}
1 + \sum_{n\geqslant 1} B_n(q) s^n &= 1 + \sum_{|\lambda|\geqslant 1} \operatorname{card} C^y_\lambda \, s^{|\lambda|} \\
&= \prod_{i\geqslant 1} \left(1 + \sum_{e\geqslant 1} \operatorname{card} C^y_{i^e} \, s^{ie} \right) \\
&= \prod_{i\geqslant 1} \left(1 + \sum_{e\geqslant 1} (q-1) q^{ie+e-1} s^{ie} \right) \\
&= \prod_{i\geqslant 1} \left(1 + (q-1)q^{-1} \sum_{e\geqslant 1} (q^{i+1}s^i)^e \right) \\
&= \prod_{i\geqslant 1} \left(1 + (q-1)q^{-1} \frac{q^{i+1}s^i}{1-q^{i+1}s^i} \right) \\
&= \prod_{i\geqslant 1} \frac{(1-q^{i+1}s^i) + (q-1)q^i s^i}{1-q^{i+1}s^i} \\
&= \prod_{i\geqslant 1} \frac{1-q^i s^i}{1-q^{i+1}s^i} \, .
\end{aligned}$$

Finally replace s by $q^{-1}t$.

(b) To compute $B^\circ_n(1)$ we use the formula of Corollary 4.4. Since the values at $q = 1$ of $(q-1)^{\nu(\lambda)-1}$ is 1 if $\nu(\lambda) = 1$ and 0 otherwise and since $\nu(\lambda) = 1$ if and only if $m_1 = \cdots = m_t = d$, in which case $dt = n$, we have

$$B^\circ_n(1) = \sum_{dt=n} 1 = \sum_{d|n} 1 = \sigma_0(n).$$

For $B^\circ_n(-1)$ we use the infinite product expansion of Item (a): replacing $B_n(q)$ by $(q-1)q^n B^\circ_n(q)$, we obtain

$$1 + \sum_{n\geqslant 1} (q-1) B^\circ_n(q) t^n = \prod_{i\geqslant 1} \frac{1-t^i}{1-qt^i} \, .$$

Setting $q = -1$ yields

$$1 - 2\sum_{n\geqslant 1} B^\circ_n(-1) t^n = \prod_{i\geqslant 1} \frac{1-t^i}{1+t^i} \, .$$

Now recall the following identity of Gauss (see [9, (7.324)] or [17, 19.9 (i)]):

$$\prod_{i\geqslant 1} \frac{1-t^i}{1+t^i} = \sum_{k\in\mathbb{Z}} (-1)^k t^{k^2}. \tag{4.2}$$

It follows that

$$1 - 2\sum_{n\geqslant 1} B^\circ_n(-1) t^n = 1 + 2\sum_{k\geqslant 1} (-1)^k t^{k^2},$$

which allows us to conclude. □

Remark 4.7. The results of Theorem 4.6 should be compared to the following ones concerning the number $A_n(q)$ of ideals of $\mathbb{F}_q[x,y]$ of codimension n. Proceeding as in the proof of Theorem 4.6, we deduce from (3.1) that

$$1 + \sum_{n\geqslant 1} A_n(q) s^n = \prod_{i\geqslant 1} \frac{1}{1 - q^{i+1} s^i}\,.$$

Setting $q = -1$, we have

$$1 + \sum_{n\geqslant 1} A_n(-1) s^n = \prod_{i\geqslant 1} \frac{1}{1-(-1)^{i+1} s^i} = \prod_{m\geqslant 1} \frac{1}{(1-s^{2m-1})(1+s^{2m})}. \tag{4.3}$$

Multiplying by $\prod_{m\geqslant 1}(1+s^{2m})^{-1}$ both sides of the Euler identity

$$\prod_{m\geqslant 1} \frac{1}{1-s^{2m-1}} = \prod_{i\geqslant 1} (1+s^i)$$

(see [17, (19.4.7)]), we deduce that the right-hand side of (4.3) is equal to the infinite product

$$\prod_{m\geqslant 1} (1+s^{2m-1}).$$

Thus by [1, Table 14.1, p. 310] or [17, (19.4.4)], the value $A_n(-1)$ is equal to the number[2] of partitions of n with unequal odd parts. Note that $A_n(1)$ is equal to the number[3] of partitions of n. See Table 4 at the end for a list of the polynomials $A_n(q)$ $(1 \leqslant n \leqslant 12)$.

5. Invertible Gröbner cells

Let $\mathrm{Hilb}^n((\mathbb{A}^1_k \backslash \{0\})^2)$ be the Hilbert scheme parametrizing finite subschemes of colength n of the two-dimensional torus, i.e. of the complement of two distinct intersecting lines in the affine plane. Its k-points are in bijection with the set of ideals of $k[x,y,x^{-1},y^{-1}]$ of codimension n. By Section 3.2 this set of ideals is the disjoint union over the partitions λ of n of the sets $C_\lambda^{x,y}$, where $C_\lambda^{x,y}$ consists of the ideals $I \in C_\lambda$ such that both x and y are invertible in $k[x,y]/I$. We call $C_\lambda^{x,y}$ the *invertible Gröbner cell* associated to the partition λ.

When the ground field is finite, so is $C_\lambda^{x,y}$. The aim of this section is to compute the cardinality of $C_\lambda^{x,y}$ when $k = \mathbb{F}_q$.

5.1. **The cardinality of an invertible Gröbner cell.** Recall the non-negative integers $d_1, \ldots, d_t$ defined by (3.3) and the positive integer $v(\lambda)$ defined by (3.4). We now give a formula for card $C_\lambda^{x,y}$.

Theorem 5.1. *Let $k = \mathbb{F}_q$, n an integer $\geqslant 1$ and λ be a partition of n. Then*

$$\operatorname{card} C_\lambda^{x,y} = (q-1)^{2v(\lambda)}\, q^{n-\ell(\lambda)} \prod_{\substack{i=1,\ldots,t\\ d_i\geqslant 1}} \frac{q^{2d_i}-1}{q^2-1}\,.$$

The theorem will be proved in Section 5.3. It has the following straightforward consequences.

[2] See Sequence A000700 in [19].

[3] See Sequence A000041 in [19].

Corollary 5.2. *Let $k = \mathbb{F}_q$ and λ be a partition of n.*

(a) card $C_\lambda^{x,y}$ *is a monic polynomial in q with integer coefficients; it is of degree $n + \ell(\lambda)$.*

(b) The polynomial card $C_\lambda^{x,y}$ *is divisible by $(q-1)^2$. The quotient*

$$P_\lambda(q) = \frac{\operatorname{card} C_\lambda^{x,y}}{(q-1)^2}$$

is a monic polynomial in q with integer coefficients and of degree $n + \ell(\lambda) - 2$.

(c) If the partition λ is rectangular, i.e., if $v(\lambda) = 1$, in which case $d_2 = \cdots = d_t = 0$ and $d = d_1$ is a divisor of n, then

$$P_\lambda(q) = q^{n-d}\,\frac{q^{2d}-1}{q^2-1} = q^{n-d}\,\left(1 + q^2 + \cdots + q^{2d-2}\right).$$

In this case, $P_\lambda(1) = d$.

(d) If $v(\lambda) \geqslant 2$, then $P_\lambda(q)$ is divisible by $(q-1)^2$, and $P_\lambda(1) = 0$.

Remark 5.3. The polynomials $P_\lambda(q)$ may have negative coefficients. For instance, if λ is the partition of 4 corresponding to $t = 2$, $d_1 = 1$, $d_2 = 2$, then

$$P_\lambda(q) = q^5 - 2q^4 + 2q^3 - 2q^2 + q.$$

The rest of the section is devoted to the proof of Theorem 5.1.

5.2. **A criterion for the invertibility of** x**.** In Section 4 we introduced the algebra map $p_y : k[x,y] \to k[x]$ sending x to itself and y to 0. Similarly, let $p_x : k[x,y] \to k[x]$ be the algebra map sending x to 0 and y to itself. Then by Lemma 2.2, the set $C_\lambda^{x,y}$ consists of the ideals $I \in C_\lambda$ such that $p_x(I) = k[y]$ and $p_y(I) = k[x]$. We already have a criterion for $p_y(I) = k[x]$ (see Proposition 4.1). We shall now give a necessary and sufficient condition for $p_x(I)$ to be equal to $k[y]$.

Resuming the notation of Section 4, we see that $p_x(I)$ can be identified with the ideal of $k[y]$ generated by the polynomials $f_0(0,y), \ldots, f_t(0,y) \in k[y]$ obtained from the polynomials $f_0(x,y), \ldots, f_t(x,y)$ by setting $x = 0$. The polynomials $f_0(0,y), \ldots, f_t(0,y)$ are the maximal minors of the matrix

$$M_\lambda^x = \begin{pmatrix} y^{d_1}+p_1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ p_{2,1} & y^{d_2}+p_2 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ p_{3,1} & p_{3,2} & y^{d_3}+p_3 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ p_{i-1,1} & p_{i-1,2} & p_{i-1,3} & \cdots & y^{d_{i-1}}+p_{i-1} & 0 & 0 & \cdots & 0 \\ p_{i,1} & p_{i,2} & p_{i,3} & \cdots & p_{i,i-1} & y^{d_i}+p_i & 0 & \cdots & 0 \\ p_{i+1,1} & p_{i+1,2} & p_{i+1,3} & \cdots & p_{i+1,i-1} & p_{i+1,i} & y^{d_{i+1}}+p_{i+1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ p_{t,1} & p_{t,2} & p_{t,3} & \cdots & p_{t,i-1} & p_{t,i} & p_{t,i+1} & \cdots & y^{d_t}+p_t \\ p_{t+1,1} & p_{t+1,2} & p_{t+1,3} & \cdots & p_{t+1,i-1} & p_{t+1,i} & p_{t+1,i+1} & \cdots & p_{t+1,t} \end{pmatrix}$$

obtained from the matrix M_λ of (3.5) by setting $x = 0$.

Let μ_i be the determinant of the submatrix M_i of M_λ^x corresponding to the rows $(i+1), \ldots, (t+1)$ and to the columns $i, \ldots, t$. We have $\mu_t = p_{t+1,t}$ and

$$\mu_i = \begin{vmatrix} p_{i+1,i} & y^{d_{i+1}}+p_{i+1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p_{t,i} & p_{t,i+1} & \cdots & y^{d_t}+p_t \\ p_{t+1,i} & p_{t+1,i+1} & \cdots & p_{t+1,t} \end{vmatrix}$$

if $1 \leqslant i < t$. Expanding μ_i along its first column, we obtain

$$\mu_i = \sum_{j=1}^{t-i+1} p_{i+j,i}\, q_{i+j,i}\,, \tag{5.1}$$

where

(5.2)

$$q_{i+j,i} = \begin{cases} \mu_{i+1} & \text{if } j = 1, \\ (-1)^{j-1}\,(y^{d_{i+1}} + p_{i+1}) \cdots (y^{d_{i+j-1}} + p_{i+j-1})\, \mu_{i+j} & \text{if } 1 < j < t-i+1, \\ (-1)^{t-i}\,(y^{d_{i+1}} + p_{i+1}) \cdots (y^{d_{t-1}} + p_{t-1})(y^{d_t} + p_t) & \text{if } j = t-i+1. \end{cases}$$

Observe also that

$$f_i(0,y) = \begin{cases} \mu_1 & \text{if } i = 0, \\ (y^{d_1} + p_1) \cdots (y^{d_i} + p_i)\, \mu_{i+1} & \text{if } 1 \leqslant i < t, \\ (y^{d_1} + p_1) \cdots (y^{d_t} + p_t) & \text{if } i = t. \end{cases} \tag{5.3}$$

Lemma 5.4. *If $1 \leqslant i \leqslant j \leqslant t$, then μ_i belongs to the ideal $(\mu_j, y^{d_j} + p_j)$ generated by μ_j and $(y^{d_j} + p_j)$.*

Proof. The case $i = j$ is obvious. Otherwise, consider the matrix M_i whose determinant is μ_i; the column of M_i containing the entry $y^{d_j} + p_j$ can be written as the sum of a column containing only the entry $y^{d_j} + p_j$, the other entries being zero, and of a column whose top entry is zero and the bottom ones form the first column of the matrix M_j whose determinant is μ_j. Therefore by the multilinearity property of determinants, μ_i is the sum of a determinant which is a multiple of $y^{d_j} + p_j$ and of another determinant which is a multiple of μ_j; indeed, this second determinant is block-triangular with one diagonal block equal to μ_j. □

Here is our criterion for the invertibility of x.

Proposition 5.5. *We have $p_x(I_\lambda) = k[y]$ if and only if $y^{d_i} + p_i$ and μ_i are coprime for all $i = 1, \ldots, t$.*

Proof. (a) Let us first check that the above condition is sufficient. The fact that $y^{d_t} + p_t$ and μ_t are coprime implies that by (5.3) the gcd of $f_t(0,y)$ and of $f_{t-1}(0,y)$ is $(y^{d_1} + p_1) \cdots (y^{d_{t-1}} + p_{t-1})$. Now the gcd of the latter and of $f_{t-2}(0,y)$ is $(y^{d_1} + p_1) \cdots (y^{d_{t-2}} + p_{t-2})$ in view of the fact that $y^{d_{t-1}} + p_{t-1}$ and μ_{t-1} are coprime. Repeating this argument, we find that the gcd of $f_0(0,y), \ldots, f_t(0,y)$ is 1, which implies that $p_x(I_\lambda) = k[y]$.

(b) Conversely, suppose that $y^{d_j} + p_j$ and μ_j are not coprime for some j, i.e., $(\mu_j, y^{d_j} + p_j) \neq k[y]$. By (5.3) and Lemma 5.4, $f_0(0,y), \ldots, f_{j-1}(0,y)$ belong to the ideal $(\mu_j, y^{d_j} + p_j)$. On the other hand, again by (5.3), the remaining polynomials $f_j(0,y), \ldots, f_t(0,y)$ are divisible by $y^{d_j} + p_j$, hence belong to $(\mu_j, y^{d_j} + p_j)$. Therefore, $p_x(I_\lambda) \subseteq (\mu_j, y^{d_j} + p_j) \neq k[y]$. □

For the proof of Theorem 5.1, we shall also need the following result.

Lemma 5.6. *If $y^{d_j} + p_j$ and μ_j are coprime for all $j > i$, then the polynomials $q_{i+1,i}, \ldots, q_{t+1,i}$ of (5.2) are coprime.*

Proof. Proceeding as in Part (a) of the proof of Proposition 5.5 and using (5.2), one shows by descending induction on j that the gcd of $q_{j+1,i}, \dots, q_{t+1,i}$ is

$$(y^{d_{i+1}} + p_{i+1}) \cdots (y^{d_j} + p_j).$$

In particular, for $j = i+1$, the gcd of $q_{i+2,i}, \dots, q_{t+1,i}$ is $(y^{d_{i+1}} + p_{i+1})$. The conclusion follows from this fact together with the coprimality of $(y^{d_{i+1}} + p_{i+1})$ and of $q_{i+1,i} = \mu_{i+1}$. □

5.3. **Proof of Theorem 5.1.** By Propositions 4.1 and 5.5, it is enough to count the entries of the matrix M_λ over $\mathbb{F}_q[y]$ such that $p_i(0) \neq 0$ and $y^{d_i} + p_i$ and μ_i are coprime for all $i = 1, \dots t$. We consider these conditions successively for $i = t, t-1, \dots, 1$.

Assume first that all integers $d_1, \dots, d_t$ are non-zero. For $i = t$, $y^{d_t} + p_t$ is a monic polynomial of degree d_t with non-zero constant term, $\mu_t = p_{t+1,t}$ is of degree $< d_t$, and both polynomials are coprime. It follows from Lemma 2.5 (or from Proposition 2.3 with $d = d_t$ and $h = 1$) that we have $(q-1)^2(q^{2d_t}-1)/(q^2-1)$ possible choices for the last column of M_λ.

For $i = t-1$, it follows from (5.1) that $\mu_{t-1} = P_1Q_1 + P_2Q_2$, where $Q_1 = q_{t,t-1}$ and $Q_2 = -q_{t+1,t-1}$, which are coprime by Lemma 5.6, $P_1 = p_{t,t-1}$ and $P_2 = p_{t+1,t-1}$, which are both polynomials of degree $< d_{t-1}$. The polynomial $P = y^{d_{t-1}} + p_{t-1}$ is monic of degree d_{t-1} with non-zero constant term, and $Q = \mu_{t-1} = P_1Q_1 + P_2Q_2$ is coprime to P by the coprimality condition. It then follows from Proposition 2.3 applied to the case $d = d_{t-1}$ and $h = 2$ that there are

$$(q-1)^2 q^{d_{t-1}} \frac{q^{2d_{t-1}} - 1}{q^2 - 1}$$

possible choices for the $(t-1)$-st column of M_λ.

In general, the polynomial $P = y^{d_i} + p_i$ is monic of degree d_{t-1} with non-zero constant term, and is assumed to be coprime to $Q = \mu_i = \sum_{j=1}^{t-i+1} p_{i+j,i}\, q_{i+j,i}$. By Lemma 5.6 the polynomials $q_{i+1,i}, \dots, q_{t+1,i}$ are coprime. Applying Proposition 2.3 to the case $d = d_i$ and $h = t+1-i$, we see that there are

$$(q-1)^2 q^{(t-i)d_i} \frac{q^{2d_i} - 1}{q^2 - 1}$$

possible choices for the i-th column of M_λ.

In the end we obtain a number of possible entries for M_λ equal to

$$\prod_{i=1}^{t} (q-1)^2 q^{(t-i)d_i} \frac{q^{2d_i} - 1}{q^2 - 1} = q^{n-\ell(\lambda)} \prod_{i=1}^{t} (q-1)^2 \frac{q^{2d_i} - 1}{q^2 - 1}$$

since $\ell(\lambda) = \sum_{i=1}^{t} d_i$ and $n = |\lambda| = \sum_{i=1}^{t} (t-i+1)\, d_i$. We have thus proved the theorem when all $d_1, \dots, d_t$ are non-zero.

Let E be the subset of $\{1, \dots, t\}$ consisting of those subscripts i for which $d_i = 0$. (Note that 1 does not belong to E since $d_1 > 0$.) Assume now that E is non-empty and set $t' = t - \operatorname{card} E$. By assumption $t' < t$. For any positive integer $i \leqslant t'$, let d'_i be equal to the i-th non-zero d_i. The integers $d'_1 = d_1, d'_2, \dots d'_{t'}$ are positive.

Recall that if $i \in E$, then the i-th column of the matrix M_λ is zero except for the (i,i)-entry which is 1. Permuting rows and columns, we may rearrange M_λ into a matrix M'_λ of the form

$$M'_\lambda = \begin{pmatrix} M_\nu & 0 \\ N & I_{t-t'} \end{pmatrix},$$

where $I_{t-t'}$ is an identity matrix of size $(t - t')$. The $(t' + 1) \times t'$-matrix M_ν is of the form (3.5) with t replaced by t', the sequence $d_1, \ldots, d_t$ by the shorter sequence $d'_1, \ldots, d'_{t'}$, and the partition λ by the partition ν associated to the sequence $d'_1, \ldots, d'_{t'}$.

Let f'_i be the determinant of the square matrix obtained from M'_λ by deleting its $(i + 1)$-st row. It is clear that up to sign and to reordering the maximal minors $f'_0, \ldots, f'_t$ of M'_λ are the same as those of M_λ. In view of the special form of M'_λ, observe that

$$f'_i = \begin{cases} f_i^{(\nu)} & \text{if } 0 \leqslant i \leqslant t', \\ 0 & \text{if } t' < i \leqslant t. \end{cases}$$

where $f_i^{(\nu)}$ is the determinant of the $t' \times t'$-matrix obtained from M_ν by deleting its $(i + 1)$-st row.

The number of possible entries of M_λ, which is the same as the number of possible entries of M'_λ, is then the product of the number of possible entries of N, which is a power of q, and of the number of possible entries of M_ν. Since $d'_1, \ldots, d'_{t'}$ are positive, by the first part of the proof, we know that the number of possible entries of M_ν is the product of a power of q by

$$\prod_{i=1}^{t'} (q-1)^2 \, \frac{q^{2d'_i} - 1}{q^2 - 1}.$$

In other words, the number of possible entries of M_λ is

$$q^c \prod_{\substack{i=1,\ldots,t \\ d_i \geqslant 1}} (q-1)^2 \, \frac{q^{2d_i} - 1}{q^2 - 1}$$

for some non-negative integer c. Now since the invertible Gröbner cell $C_\lambda^{x,y}$ is a Zarisky open subset of the affine Gröbner cell C_λ, the degree of the previous polynomial in q must be the same as the degree of the cardinal of C_λ, which is $q^{n+\ell(\lambda)}$ by Section 3.1. This suffices to establish that $c = n - \ell(\lambda)$ and to complete the proof of the theorem.

5.4. **Proof of Theorem 1.1.** By our remark at the beginning of Section 5, the number $C_n(q)$ of ideals of $\mathbb{F}_q[x, y, x^{-1}, y^{-1}]$ of codimension n is given by

$$C_n(q) = \sum_{\lambda \vdash n} \operatorname{card} C_\lambda^{x,y}, \tag{5.4}$$

where $C_\lambda^{x,y}$ is the *invertible Gröbner cell* associated to the partition λ. The equality in Theorem 1.1 follows then from the formula for card $C_\lambda^{x,y}$ given in Theorem 5.1.

By Corollary 5.2 (a) card $C_\lambda^{x,y}$ is a monic polynomial which has integer coefficients and whose degree is $n + \ell(\lambda)$. Therefore, $C_n(q)$ has integer coefficients and its degree is $\max\{n + \ell(\lambda) \mid \lambda \vdash n\}$. Now $\ell(\lambda)$ is maximal if and only if $\lambda = 1^n$, in which case $\ell(\lambda) = n$. Therefore $C_n(q)$ is monic and its degree is $2n$.

Since $\nu(\lambda) \geqslant 1$, it follows from the formula in Theorem 5.1 that card $C_\lambda^{x,y}$ is divisible by $(q - 1)^2$ for each invertible Gröbner cell. Therefore, the polynomial $C_n(q)$ is divisible by $(q - 1)^2$.

6. Proofs of the corollaries

We now start the proofs of Corollary 1.2 and of Corollary 1.4.

6.1. **Proof of Corollary 1.2.** Since $C_n(q)$ and $(q-1)^2$ are both monic with integer coefficients, so is $P_n(q)$. The latter is the sum over all partitions of n of the polynomials $P_\lambda(q)$ (introduced in Corollary 5.2 (b)). By Corollary 5.2 (c)–(d), we have $P_\lambda(1) = 0$ if $\nu(\lambda) \geqslant 2$ and, if $\nu(\lambda) = 1$, then λ is of the form t^d, where $dt = n$, in which case $P_\lambda(1) = d$. The desired formula for $P_n(1)$ follows.

6.2. **Proof of Corollary 1.4.** As in the proof of Theorem 4.6 we consider each partition λ as a union of rectangular partitions i^{e_i}, with e_i parts of length i, for $e_i \geqslant 1$ and distinct $i \geqslant 1$. Recall that $|\lambda| = \sum_i ie_i$, $\ell(\lambda) = \sum_i e_i$, and $\nu(\lambda) = \sum_i 1$. To indicate the dependance of e_i on λ, we write $e_i = e_i(\lambda)$. We then obtain the following statement.

Proposition 6.1. *We have the infinite product expansion*

$$1 + \sum_\lambda \operatorname{card} C_\lambda^{x,y}\, s_1^{e_1(\lambda)}\, s_2^{e_2(\lambda)} \cdots = \prod_{i\geqslant 1} \frac{(1-q^i s_i)^2}{(1-q^{i+1}s_i)(1-q^{i-1}s_i)} .$$

Proof. Proceeding as in the proof of Theorem 4.6 and using Theorem 5.1, we deduce that the left-hand side is equal to

$$1 + \sum_\lambda \prod_{i\geqslant 1} (q-1)^2\, \frac{q^{2e_i}-1}{q^2-1} q^{ie_i - e_i}\, s_i^{e_i},$$

which in turn is equal to

$$\begin{aligned}
&\prod_{i\geqslant 1} \left(1 + \frac{(q-1)^2}{q^2-1} \sum_{e_i \geqslant 1} \big((q^{i+1}s_i)^{e_i} - (q^{i-1}s_i)^{e_i}\big)\right)\\
&\quad = \prod_{i\geqslant 1} \left(1 + \frac{(q-1)^2}{q^2-1} \left(\frac{q^{i+1}s_i}{1-q^{i+1}s_i} - \frac{q^{i-1}s_i}{1-q^{i-1}s_i}\right)\right)\\
&\quad = \prod_{i\geqslant 1} \left(1 + \frac{(q-1)^2}{q^2-1}\, \frac{(q^2-1)q^{i-1}s_i}{(1-q^{i+1}s_i)(1-q^{i-1}s_i)}\right)\\
&\quad = \prod_{i\geqslant 1} \left(1 + \frac{(q-1)^2 q^{i-1}s_i}{(1-q^{i+1}s_i)(1-q^{i-1}s_i)}\right)\\
&\quad = \prod_{i\geqslant 1} \frac{(1-q^i s_i)^2}{(1-q^{i+1}s_i)(1-q^{i-1}s_i)} .
\end{aligned}$$

□

Proof of Corollary 1.4. (a) Replace s_i by $(t/q)^i$ in Proposition 6.1, use (5.4) and Theorem 1.1, and observe that $(1-qt^i)(1-q^{-1}t^i) = 1-(q+q^{-1})t^i + t^{2i}$.

(b) The infinite product is clearly invariant under the transformation $q \leftrightarrow q^{-1}$; thus, $C_n(q^{-1}) = q^{-2n}\, C_n(q)$. Together with $\deg\, C_n(q) = 2n$, this implies that $C_n(q)$ is palindromic. The polynomial $P_n(q)$ is palindromic as a quotient of two palindromic polynomials. □

6.3. **An alternative proof of Corollary 1.4 (a).** After we made public a first version of this article, we learnt of an alternative geometric approach to the polynomials $C_n(q)$. Indeed, Göttsche and Soergel determined the mixed Hodge structure of the punctual Hilbert schemes of any smooth complex algebraic surface (see [11, Th. 2]). Applying their result to the Hilbert scheme $H^n_{\mathbb{C}} = \mathrm{Hilb}^n(\mathbb{C}^\times \times \mathbb{C}^\times)$ of n points of the complex two-dimensional torus, Hausel, Letellier and Rodriguez-Villegas observed in [16, Th. 4.1.3] that the compactly supported mixed Hodge polynomial $H_c(H^n_{\mathbb{C}}; q, u)$ of $H^n_{\mathbb{C}}$ fits into the equality of formal power series

$$1 + \sum_{n\geqslant 1} H_c(H^n_{\mathbb{C}}; q, u)\, \frac{t^n}{q^n} = \prod_{i\geqslant 1} \frac{(1 + u^{2i+1}t^i)^2}{(1 - u^{2i+2}qt^i)(1 - u^{2i}q^{-1}t^i)}\,. \tag{6.1}$$

Setting $u = -1$ in (6.1), we obtain an infinite product expansion for the generating function of the E-polynomial $E(H^n_{\mathbb{C}}; q) = H_c(H^n_{\mathbb{C}}; q, -1)$ of $H^n_{\mathbb{C}}$, namely

$$1 + \sum_{n\geqslant 1} E(H^n_{\mathbb{C}}; q)\, \frac{t^n}{q^n} = \prod_{i\geqslant 1} \frac{(1 - t^i)^2}{1 - (q + q^{-1})t^i + t^{2i}}\,. \tag{6.2}$$

Now, $H^n_{\mathbb{C}}$ is strongly polynomial-count in the sense of Nick Katz (see [13, Appendix]), probably a well-known fact (which also follows from the computations in the present paper). Therefore, by [13, Th. 6.1.2] the E-polynomial counts the number of elements of H^n over the finite field $\mathbb{F}_q$, which is also the number $C_n(q)$ of ideals of codimension n of $\mathbb{F}_q[x, y, x^{-1}, y^{-1}]$. Thus (6.2) implies the equality of Corollary 1.4 (a).

Remark 6.2. In the same vein as above, there is a geometric explanation of the palindromicity of the polynomials $C_n(q)$. In [5] de Cataldo, Hausel, Migliorini observed that any diffeomorphism between $\mathbb{C}^\times \times \mathbb{C}^\times$ and the cotangent bundle $E \times \mathbb{C}$ of the elliptic curve $E = \mathbb{C}/\mathbb{Z}[i]$ induces a linear isomorphism of graded vector spaces between the cohomology groups of the corresponding Hilbert schemes: $H^*(H^n_{\mathbb{C}}, \mathbb{Q}) \cong H^*(\mathrm{Hilb}^n(E \times \mathbb{C}), \mathbb{Q})$. This isomorphism does not preserve the mixed Hodge structures, as the one on the right-hand side is pure whereas the one on the left-hand side is not. Nevertheless, such an isomorphism identifies the weight filtration on $H^*(H^n_{\mathbb{C}}, \mathbb{Q})$ with the perverse Leray filtration on $H^*(\mathrm{Hilb}^n(E \times \mathbb{C}), \mathbb{Q})$ associated to the natural projective map from $\mathrm{Hilb}^n(E \times \mathbb{C})$ to the n-th symmetric product of $\mathbb{C}$ induced by the projection of $E \times \mathbb{C}$ on the second factor. The perverse Leray filtration is "palindromic" as a consequence of the relative hard Lefschetz theorem for the map above (see [5, Th. 4.1.1 and Th. 4.3.2]).

Note that Hausel, Letellier and Rodriguez-Villegas observed a similar palindromicity for the E-polynomial of certain character varieties and termed it "curious Poincaré duality" in [15, Cor. 5.2.4] (see also [13, Cor. 3.5.3], [14, Cor. 1.4]).

Remark 6.3. The natural action of the group $\mathbb{C}^\times \times \mathbb{C}^\times$ on itself induces an action on the Hilbert scheme $H^n_{\mathbb{C}}$. Consider the GIT quotient $\widetilde{H}^n_{\mathbb{C}} = H^n_{\mathbb{C}} /\!/ (\mathbb{C}^\times \times \mathbb{C}^\times)$. Using [13, Th. 2.2.12] and [15, Sect. 5.3], we see that the E-polynomial of $\widetilde{H}^n_{\mathbb{C}}$ is given by

$$E(\widetilde{H}^n_{\mathbb{C}}; q) = E(H^n_{\mathbb{C}}; q)/(q-1)^2 = C_n(q)/(q-1)^2 = P_n(q).$$

Recall from the introduction (see also the appendix below) that the coefficients of $P_n(q)$ are all non-negative. Therefore, $\widetilde{H}^n_{\mathbb{C}}$ provides an example of a polynomial-count variety with odd cohomology and a counting polynomial with non-negative coefficients. This implies non-trivial cancellation for the mixed Hodge numbers of $\widetilde{H}^n_{\mathbb{C}}$. No similar positivity phenomenon was observed for the character varieties investigated by Hausel, Letellier and Rodriguez-Villegas.

Appendix A. The coefficients of the polynomials $C_n(q)$ and $P_n(q)$

We now state the results of the companion paper [18] on the coefficients of the polynomials $C_n(q)$ and $P_n(q)$.

Since $C_n(q)$ and $P_n(q)$ are palindromic of respective degrees $2n$ and $2n-2$, we may expand $C_n(q)$ and $P_n(q)$ as follows:

$$C_n(q) = c_{n,0}\, q^n + \sum_{i=1}^{n} c_{n,i}\, \left(q^{n+i} + q^{n-i}\right),$$

where $c_{n,0}, c_{n,1}, c_{n,2} \ldots$ are integers, and

$$P_n(q) = a_{n,0}\, q^{n-1} + \sum_{i=1}^{n-1} a_{n,i}\, \left(q^{n+i-1} + q^{n-i+1}\right),$$

where $a_{n,0}, a_{n,1}, a_{n,2} \ldots$ are integers.

By Theorem 1.1 of [18] the coefficients $c_{n,i}$ of $C_n(q)$ are given by the following formulas: (a) For the central coefficients $c_{n,0}$ we have

$$c_{n,0} = \begin{cases} 2\,(-1)^k & \text{if } n = k(k+1)/2 \text{ for some integer } k \geqslant 1, \\ 0 & \text{otherwise.} \end{cases}$$

(b) For the non-central coefficients ($i \geqslant 1$) we have

$$c_{n,i} = \begin{cases} (-1)^k & \text{if } n = k(k+2i+1)/2 \text{ for some integer } k \geqslant 1, \\ (-1)^{k-1} & \text{if } n = k(k+2i-1)/2 \text{ for some integer } k \geqslant 1, \\ 0 & \text{otherwise.} \end{cases}$$

Note that in Item (b) the first two conditions are mutually exclusive.

As for the coefficients of $P_n(q)$, the coefficient $a_{n,i}$ is by [18, Th. 1.2] equal to the number of divisors d of n such that

$$\frac{i + \sqrt{2n+i^2}}{2} < d \leqslant i + \sqrt{2n+i^2}.$$

It follows that all coefficients $a_{n,i}$ of $P_n(q)$ are non-negative integers.

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TABLE 1. *The polynomials $C_n(q)$*

n	$C_n(q)$
1	$q^2 - 2q + 1$
2	$q^4 - q^3 - q + 1$
3	$q^6 - q^5 - q^4 + 2q^3 - q^2 - q + 1$
4	$q^8 - q^7 - q + 1$
5	$q^{10} - q^9 - q^7 + q^6 + q^4 - q^3 - q + 1$
6	$q^{12} - q^{11} + q^7 - 2q^6 + q^5 - q + 1$
7	$q^{14} - q^{13} - q^{10} + q^9 + q^5 - q^4 - q + 1$
8	$q^{16} - q^{15} - q + 1$
9	$q^{18} - q^{17} - q^{13} + q^{12} + q^{11} - q^{10} - q^8 + q^7 + q^6 - q^5 - q + 1$
10	$q^{20} - q^{19} - q^{11} + 2q^{10} - q^9 - q + 1$
11	$q^{22} - q^{21} - q^{16} + q^{15} + q^7 - q^6 - q + 1$
12	$q^{24} - q^{23} + q^{15} - q^{14} - q^{10} + q^9 - q + 1$

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TABLE 2. *The polynomials $P_n(q)$*

n	$P_n(q)$	$P_n(1)$
1	1	1
2	$q^2 + q + 1$	3
3	$q^4 + q^3 + q + 1$	4
4	$q^6 + q^5 + q^4 + q^3 + q^2 + q + 1$	7
5	$q^8 + q^7 + q^6 + q^2 + q + 1$	6
6	$q^{10} + q^9 + q^8 + q^7 + q^6$ $+2q^5 + q^4 + q^3 + q^2 + q + 1$	12
7	$q^{12} + q^{11} + q^{10} + q^9 + q^3 + q^2 + q + 1$	8
8	$q^{14} + q^{13} + q^{12} + q^{11} + q^{10} + q^9 + q^8$ $+q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1$	15
9	$q^{16} + q^{15} + q^{14} + q^{13} + q^{12} + q^9$ $+q^8 + q^7 + q^4 + q^3 + q^2 + q + 1$	13
10	$q^{18} + q^{17} + q^{16} + q^{15} + q^{14} + q^{13}$ $+q^{12} + q^{11} + q^{10} + q^8 + q^7 + q^6$ $+q^5 + q^4 + q^3 + q^2 + q + 1$	18
11	$q^{20} + q^{19} + q^{18} + q^{17} + q^{16} + q^{15}$ $+q^5 + q^4 + q^3 + q^2 + q + 1$	12
12	$q^{22} + q^{21} + q^{20} + q^{19} + q^{18} + q^{17} + q^{16} + q^{15}$ $+q^{14} + 2q^{13} + 2q^{12} + 2q^{11} + 2q^{10} + 2q^9 + q^8$ $+q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1$	28

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TABLE 3. *The polynomials* $B_n^\circ(q)$

n	$B_n^\circ(q)$	$B_n^\circ(1)$	$B_n^\circ(-1)$
1	1	1	1
2	$q+1$	2	0
3	q^2+q	2	0
4	q^3+q^2+q	3	-1
5	$q^4+q^3+q^2-1$	2	0
6	$q^5+q^4+q^3+q^2$	4	0
7	$q^6+q^5+q^4+q^3-q-1$	2	0
8	$q^7+q^6+q^5+q^4+q^3-q$	4	0
9	$q^8+q^7+q^6+q^5+q^4-q^2-q$	3	1
10	$q^9+q^8+q^7+q^6+q^5+q^4-q^2-q$	4	0
11	$q^{10}+q^9+q^8+q^7+q^6+q^5-q^3-2q^2-q$	2	0
12	$q^{11}+q^{10}+q^9+q^8+q^7+q^6+q^5-q^3-q^2+1$	6	0

TABLE 4. *The polynomials* $A_n(q)$

n	$A_n(q)$	$A_n(1)$	$A_n(-1)$
1	q^2	1	1
2	q^4+q^3	2	0
3	$q^6+q^5+q^4$	3	1
4	$q^8+q^7+2q^6+q^5$	5	1
5	$q^{10}+q^9+2q^8+2q^7+q^6$	7	1
6	$q^{12}+q^{11}+2q^{10}+3q^9+3q^8+q^7$	11	1
7	$q^{14}+q^{13}+2q^{12}+3q^{11}+4q^{10}+3q^9+q^8$	15	1
8	$q^{16}+q^{15}+2q^{14}+3q^{13}+5q^{12}+5q^{11}+4q^{10}+q^9$	22	2
9	$q^{18}+q^{17}+2q^{16}+3q^{15}+$ $+5q^{14}+6q^{13}+7q^{12}+4q^{11}+q^{10}$	30	2
10	$q^{20}+q^{19}+2q^{18}+3q^{17}+5q^{16}+$ $+7q^{15}+9q^{14}+8q^{13}+5q^{12}+q^{11}$	42	2
11	$q^{22}+q^{21}+2q^{20}+3q^{19}+5q^{18}+$ $+7q^{17}+10q^{16}+11q^{15}+10q^{14}+5q^{13}+q^{12}$	56	2
12	$q^{24}+q^{23}+2q^{22}+3q^{21}+5q^{20}+7q^{19}+$ $+11q^{18}+13q^{17}+15q^{16}+12q^{15}+6q^{14}+q^{13}$	77	3