ON POWER SUMS OF MATRICES OVER A FINITE COMMUTATIVE RING

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Abstract. In this paper we deal with the problem of computing the sum of the k-th powers of all the elements of the matrix ring $\mathbb{M}_d(R)$ with $d>1$ and R a finite commutative ring. We completely solve the problem in the case $R=\mathbb{Z}/n\mathbb{Z}$ and give some results that compute the value of this sum if R is an arbitrary finite commutative ring R for many values of k and d. Finally, based on computational evidence and using some technical results proved in the paper we conjecture that the sum of the k-th powers of all the elements of the matrix ring $\mathbb{M}_d(R)$ is always 0 unless $d=2$, $\mathrm{card}(R)\equiv 2 \pmod 4$, $1<k\equiv -1,0,1 \pmod 6$ and the only element $e\in R\setminus\{0\}$ such that $2e=0$ is idempotent, in which case the sum is $\mathrm{diag}(e,e)$.

1. introduction

For a ring R we denote by $\mathbb{M}_d(R)$ the ring of $d\times d$ matrices over R. Now, given an integer $k\geq 1$ we define the sum

$$S_k^d(R):=\sum_{M\in\mathbb{M}_d(R)} M^k.$$

This paper deals with the computation of $S_k^d(R)$ in the case when R is finite and commutative.

When $d=1$, the problem of computing $S_k^1(R)$ is completely solved only for some particular families of finite commutative rings. If R is a finite field $\mathbb{F}_q$, the value of $S_k^1(\mathbb{F}_q)$ is well-known. If $R=\mathbb{Z}/n\mathbb{Z}$ the study of $S_k^1(\mathbb{Z}/n\mathbb{Z})$ dates back to 1840 [9] and has been completed in various works [2, 5, 7]. Finally, the case $R=\mathbb{Z}/n\mathbb{Z}[i]$ has been recently solved in [3]. For those rings, we have the following result.

Theorem 1. *Let $k\geq 1$ be an integer.*

i) *Finite fields:*

$$S_k^1(\mathbb{F}_q)=\begin{cases}-1, & \textit{if } (q-1)\mid k\ ;\\ 0, & \textit{otherwise.}\end{cases}$$

ii) *Integers modulo n:*

$$S_k^1(\mathbb{Z}/n\mathbb{Z})=\begin{cases}-\displaystyle\sum_{p\mid n,\, p-1\mid k}\frac{n}{p}, & \textit{if } k \textit{ is even or } k=1 \textit{ or } n\not\equiv 0 \pmod 4;\\ 0, & \textit{otherwise.}\end{cases}$$

iii) *Gaussian integers modulo n:*

$$S_k^1(\mathbb{Z}/n\mathbb{Z}[i]) = \begin{cases} \frac{n}{2}(1+i), & \text{if } k>1 \text{ is odd and } n\equiv 2 \pmod 4; \\ -\displaystyle\sum_{p\in\mathcal{P}(k,n)} \frac{n^2}{p^2}, & \text{otherwise.} \end{cases}$$

where

$$\mathcal{P}(k,n) := \{\text{prime } p : p \mid\mid n, p^2-1 \mid k, p\equiv 3 \pmod 4\}$$

and $p \mid\mid n$ means that $p \mid n$, but $p^2 \nmid n$.

On the other hand, if $d>1$ the problem has been only solved when R is a finite field [1]. In particular, the following result holds.

Theorem 2. *Let $k, d \geq 1$ be integers. Then $S_k^d(\mathbb{F}_q) = 0$ unless $q = 2 = d$ and $1 < k \equiv -1, 0, 1 \pmod 6$ in which case $S_k^d(\mathbb{F}_q) = I_2$.*

In this paper we deal with the computation of $S_k^d(R)$ with $d>1$ and R a finite commutative ring. In particular Section 2 is devoted to completely determine the value of $S_k^d(R)$ in the case $R = \mathbb{Z}/n\mathbb{Z}$ (that we usually write as $\mathbb{Z}_n$). In Section 3 we give some technical results regarding sums of non-commutative monomials over $\mathbb{Z}/n\mathbb{Z}$ which will be used in Section 4 to compute $S_k^d(R)$ for an arbitrary finite commutative ring R in many cases. Finally, we close the paper in Section 5 with the following conjecture based on strong computational evidence

Conjecture 1. *Let $d>1$ and let R be a finite commutative ring. Then $S_k^d(R) = 0$ unless the following conditions hold:*

1. *$d = 2$,*
2. *$\operatorname{card}(R) \equiv 2 \pmod 4$ and $1 < k \equiv -1, 0, 1 \pmod 6$,*
3. *The unique element $e \in R \setminus \{0\}$ such that $2e = 0$ is idempotent.*

Moreover, in this case

$$S_k^d(R) = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}.$$

2. Power sums of matrices over $\mathbb{Z}_n$

In what follows we will consider integers $n, d > 1$. For the sake of simplicity, M_n^d will denote the set of integer matrices with entries in the range $\{0, \dots, n-1\}$. Furthermore, for an integer $k \geq 1$, let $S_k^d(n) = \sum_{M\in M_n^d} M^k$. Our main goal in this section will be to compute the value of $S_k^d(n)$ modulo n. This is exactly the sum $S_k^d(\mathbb{Z}/n\mathbb{Z})$.

We start with the prime case. If $n = p$ is a prime, we have the following result [1, Corollary 3.2]

Proposition 1. *Let p be a prime. Then, $S_k^d(p) \equiv 0 \pmod p$ unless $d = p = 2$.*

Thus, the case $n = 2$ must be studied separately. In fact, we have

Proposition 2.

$$S_k^2(2) \equiv \begin{cases} 0_2 \pmod 2, & \text{if } k = 1 \text{ or } k \equiv 2,3,4 \pmod 6; \\ I_2 \pmod 2, & \text{if } 1 < k \equiv 0,1,5 \pmod 6. \end{cases}$$

Proof. For every $M \in M_n^2$ it holds that $M^2 \equiv M^8 \pmod 2$. As a consequence $S_k^2(2) \equiv S_{k+6}^2(2) \pmod 2$ for every $k > 1$. Thus, the result follows just computing $S_k^2(2)$ for $1 \le k \le 7$. □

Now, we turn to the prime power case. The following lemma is straightforward

Lemma 1. *Let p be a prime. Then, any element M in $M_{p^{s+1}}^d$ can be uniquely written in the form $A + p^s B$, where $A \in M_{p^s}^d, B \in M_p^d$.*

Using this lemma we can prove the following useful result.

Proposition 3. *Let p be a prime. Then, $S_k^d(p^{s+1}) \equiv p^{d^2} S_k^d(p^s) \pmod{p^{s+1}}$.*

Proof. By the previous lemma we have

$$S_k^d(p^{s+1}) = \sum_{M \in M_{p^{s+1}}^d} M^k = \sum_{A \in M_{p^s}^d} \sum_{B \in M_p^d} (A + p^s B)^k. \tag{1}$$

Using the non-commutative version of the binomial theorem we have that

$$(A + p^s B)^k \equiv A^k + p^s \sum_{t=1}^{k} A^{k-t} B A^{t-1} \pmod{p^{s+1}}.$$

Thus, combining this with (1) we obtain

$$\begin{aligned} S_k^d(p^{s+1}) &\equiv \sum_{B \in M_p^d} \left(\sum_{A \in M_{p^s}^d} A^k \right) + \sum_{t=1}^{k} \sum_{A \in M_{p^s}^d} A^{k-t} \left(p^s \sum_{B \in M_p^d} B \right) A^{t-1} \\ &\equiv p^{d^2} S_k^d(p^s) + \sum_{t=1}^{k} \sum_{A \in M_{p^s}^d} A^{k-t} \left(p^s S_1^d(p) \right) A^{t-1} \\ &\equiv p^{d^2} S_k^d(p^s) \pmod{p^{s+1}} \end{aligned}$$

because $S_1^d(p) \equiv 0 \pmod p$ by Propositions 1 and 2 (depending on whether p is odd or not). □

Remark. Note that Proposition 3 implies that if $S_k^d(p^s) \equiv 0 \pmod{p^s}$, then also $S_k^d(p^{s+1}) \equiv 0 \pmod{p^{s+1}}$.

As a consequence we get the following result which extends Proposition 1.

Corollary 1. $S_k^d(p^s) \equiv 0 \pmod{p^s}$ *unless $d = p = 2$ and $s = 1$.*

Proof. If $p = d = 2$, then Proposition 1 implies that $S_k^2(4) \equiv 2^4 S_k^2(2) \equiv 0 \pmod 4$, so the previous remark leads to $S_k^2(2^s) \equiv 0 \pmod{2^s}$, for every $s > 1$. On the other hand, if d or p is odd, then we know by Proposition 1 that $S_k^d(p) \equiv 0 \pmod p$. Again, the remark gives us $S_k^d(p^s) \equiv 0$, by induction for all $s \ge 1$. □

In order to study the general case the following lemma will be useful. It is an analogue of [6, Lemma 3 i)]

Lemma 2. *If $m \mid n$, then* $S_k^d(n) \equiv \left(\frac{n}{m}\right)^{d^2} S_k^d(m) \pmod m$.

Proof. Given a matrix $M \in M_n^d$, let $M = \left(m_{i,j}\right)$ with $1 \leq i, j \leq d$. Then,

$$S_k^d(n) = \sum_{M \in M_n^d} M^k = \sum_{0 \leq m_{i,j} \leq n-1} \left(m_{i,j}\right)^k$$
$$\equiv \left(\frac{n}{m}\right)^{d^2} \sum_{0 \leq m_{i,j} \leq m-1} \left(m_{i,j}\right)^k = S_k^d(m) \pmod{m}$$

□

Now, we can prove the main result of this section.

Theorem 3. *The following congruence modulo n holds:*

$$S_k^d(n) \equiv \begin{cases} \frac{n}{2} \cdot I_2, & \textit{if } d = 2,\ n \equiv 2 \pmod{4} \textit{ and } 1 < k \equiv 0, 1, 5 \pmod{6}; \\ 0_2, & \textit{otherwise.} \end{cases}$$

Proof. Let $n = 2^s p_1^{r_1} \cdots p_t^{r_t}$ be the prime power decomposition of n.

If $1 \leq i \leq t$, we have by Lemma 2 and Corollary 1 that

$$S_k^d(n) \equiv \left(\frac{n}{p_i^{r_i}}\right)^{d^2} S_k^d(p_i^{r_i}) \equiv 0 \pmod{p_i^{r_i}}.$$

On the other hand, using again Lemma 2 we have that

$$S_k^d(n) \equiv \left(\frac{n}{2^s}\right)^{d^2} S_k^d(2^s) \pmod{2^s}.$$

Hence, Corollary 1 implies that $S_k^d(n) \equiv 0 \pmod{2^s}$ unless $d = p = 2$ and $s = 1$.

To conclude, it is enough to apply Proposition 2 together with the Chinese Remainder Theorem. □

The following corollary easily follows from Theorem 3 and it confirms the conjecture stated in the sequence A017593 from the OEIS [8].

Corollary 2. $S_n^2(n) \not\equiv 0 \pmod{n}$ *if and only if* $n \equiv 6 \pmod{12}$.

As a further application of Theorem 3 application we are going to compute the sum of the powers of the Hamilton quaternions over $\mathbb{Z}/n\mathbb{Z}$.

Proposition 4. *For every $n \in \mathbb{N}$ and $l > 0$, it holds that*

$$\sum_{z \in \mathbb{Z}_n[i,j,k]} z^l = 0.$$

Proof. Since for all $z \in \mathbb{Z}_2[i, j, k]$ we have that $z^2 \in \mathbb{Z}_2$, we deduce that $z^4 = z^2$, and so it can be straightforwardly checked that

$$\sum_{z \in \mathbb{Z}_2[i,j,k]} z^l = 0.$$

Now, if $s > 1$, observing that

$$\mathbb{Z}_{2^s}[i, j, k] \cong \left\{ \begin{pmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{pmatrix} : a, b, c, d \in \mathbb{Z}_{2^s} \right\}$$

we can adapt Lemma 1, Proposition 3 and Corollary 1 to inductively obtain that

$$\sum_{z\in\mathbb{Z}_{2^s}[i,j,k]} z^l = 0.$$

Finally, if $n = 2^s m$ with m odd we know [4, Theorem 4] that

$$\mathbb{Z}_n[i,j,k] \cong \mathbb{Z}_{2^s}[i,j,k] \times \mathbb{Z}_m[i,j,k] \cong \mathbb{Z}_{2^s}[i,j,k] \times \mathbb{M}_2(\mathbb{Z}_m)$$

and the result follows from Theorem 3. □

3. Sums of non-commutative monomials over $\mathbb{Z}_n$

We will now consider a more general setting. Let $r \geq 1$ be an integer and consider $w(x_1,\dots,x_r)$ a monomial in the non-commuting variables $\{x_1,\dots,x_r\}$ of total degree k. In this situation, we define the sum

$$S_w^d(n) := \sum_{A_1,\dots,A_r\in M_n^d} w(A_1,\dots,A_r).$$

Note that if $r = 1$, then $w(x_1) = x_1^k$ and $S_w^d(n) = S_k^d(n)$ so we recover the situation from Section 2. Thus, in what follows we assume $r > 1$.

We want to study the value of $S_w^d(n)$ modulo n. To do so we first introduce two technical lemmas that extend [1, Lemma 2.3].

Lemma 3. *Let $\tau \geq 1$ be an integer and let $\beta_i > 0$ for every $1 \leq i \leq \tau$. If p is an odd prime,*

$$\sum_{x_1,\dots,x_\tau} x_1^{\beta_1}\cdots x_\tau^{\beta_\tau} \equiv \begin{cases} (-p^{s-1})^\tau, & \textit{if } p-1 \mid \beta_i \textit{ for every } i;\\ 0, & \textit{otherwise.}\end{cases} \pmod{p^s}$$

where the sum is extended over $x_1,\dots,x_\tau$ in the range $\{0,\dots,p^s-1\}$. Also, if some $\beta_i = 0$, then $\sum_{x_1,\dots,x_\tau} x_1^{\beta_1}\cdots x_\tau^{\beta_\tau} \equiv 0 \pmod{p^s}$.

Proof. It is enough to apply [6, Lemma 3 ii)] which states that

$$\sum_{x_i=0}^{p^s-1} x_i^{\beta_i} \equiv \begin{cases} -p^{s-1}, & \textit{if } p-1 \mid \beta_i;\\ 0, & \textit{otherwise.}\end{cases} \pmod{p^s}$$

for every $1 \leq i \leq \tau$. Observe that, if $\beta_i = 0$, then:

$$\sum_{x_1,\dots,x_\tau} x_1^{\beta_1}\cdots x_\tau^{\beta_\tau} = \sum_{x_i}\sum_{x_j, j\neq i} x_1^{\beta_1}\cdots x_{i-1}^{\beta_{i-1}} x_{i+1}^{\beta_{i+1}}\cdots x_\tau^{\beta_\tau} \equiv 0 \pmod{p^s}$$

□

Remark. Observe that in the previous situation, if $\tau \geq 2$ and $s > 1$, it easily follows that $\sum_{x_1,\dots,x_\tau} x_1^{\beta_1}\cdots x_\tau^{\beta_\tau} \equiv 0 \pmod{p^s}$ regardless the values of $\beta_i \geq 0$.

Lemma 4. *Let $\tau \geq 1$ be an integer and let $\beta_i > 0$ for every $1 \leq i \leq \tau$. Then,*

$$\sum_{x_1,\dots,x_\tau} x_1^{\beta_1}\cdots x_\tau^{\beta_\tau} \equiv \begin{cases} 1, & \textit{if } s = 1;\\ 0, & \textit{if } s > 1 \textit{ and } \beta_i > 1 \textit{ and odd for some } i;\\ (-1)^A (2^{s-1})^B, & \textit{if } s > 1 \textit{ and } \beta_i = 1 \textit{ or even for every } i\end{cases} \pmod{2^s}$$

where the sum is extended over $x_1, \dots, x_\tau$ in the range $\{0, \dots, 2^s-1\}$, $A = card\{\beta_i : \beta_i = 1\}$ and $B = card\{\beta_i : \beta_i \text{ is even}\}$. Also, if some $\beta_i = 0$, then $\sum_{x_1,\dots,x_\tau} x_1^{\beta_1} \cdots x_\tau^{\beta_\tau} \equiv 0 \pmod{2^s}$.

Proof. It is enough to apply [6, Lemma 3 iii)] which states that

$$\sum_{x_i=0}^{2^s-1} x_i^{\beta_i} \equiv \begin{cases} 2^{s-1}, & \text{if } s=1 \text{ or } s>1 \text{ and } \beta_1>1 \text{ is even;} \\ -1, & \text{if } s>1 \text{ and } \beta_i = 1; \\ 0, & \text{if } s>1 \text{ and } \beta_1 > 1 \text{ is odd.} \end{cases} \pmod{p^s}$$

for every $1 \leq i \leq \tau$. The proof of the case when some $\beta_i = 0$ is identical to that of the previous lemma. □

As a consequence, we get the following results.

Proposition 5. *Let p be an odd prime and let $s > 1$ be an integer. Then,*

$$S_w^d(p^s) \equiv 0 \pmod{p^s}.$$

Proof. Let $A_l = \left(a_{i,j}^l\right)_{1\leq i,j\leq d}$ for every $1 \leq l \leq r$. Note that each entry in the matrix $S_w^d(p^s)$ is a homogeneous polynomial in the variables $a_{i,j}^l$. Observe also that these variables are summation indexes in the range $\{0, \dots, p^s-1\}$. Hence, the number of variables is $rd^2 > 2$ and, since $s > 1$, the Remark 3 can be applied to the sum of its monomials, and the result follows. □

Proposition 6. *Let $s > 1$ be an integer. Assume that one of the following conditions holds:*

i) $k \leq rd^2$,
ii) $k > rd^2$ *and* $k + rd^2$ *is even.*

Then, $S_w^d(2^s) \equiv 0 \pmod{2^s}$.

Proof. Just like in the previous proposition each entry in the matrix $S_w^d(2^s)$ is a homogeneous polynomial in the rd^2 variables $a_{i,j}^l$. Hence, it is a sum of elements of the form

$$\sum_{a_{i,j}^l \in \mathbb{Z}_{2^s}} \prod (a_{i,j}^l)^{\beta_{i,j,l}}.$$

Observe that $\sum_{i,j,l} \beta_{i,j,l} = k$ so, if $k < rd^2$ it follows that some $\beta_{i,j,l} = 0$, and so each monomial sum is 0 mod 2^s (because of Lemma 3). Therefore, each entry in the matrix $S_w^d(p)$ is $0 \pmod{2^s}$ in this case, as claimed.

Now, assume that $k \geq rd^2$ and $k + rd^2$ is even (in particular if $k = rd^2$). Due to Lemma 4 an element $\sum_{a_{i,j}^l \in \mathbb{Z}_{2^s}} \prod (a_{i,j}^l)^{\beta_{i,j,l}}$ is $0 \pmod{2^s}$ unless in one of its monomials the set of rd^2 exponents $\beta_{i,j,l}$ is formed by exactly $rd^2 - 1$ ones and 1 even value. But in this case $k = (rd^2 - 1) + 2\alpha$ so $k + rd^2$ is odd, a contradiction. Consequently, each entry in the matrix $S_w^d(p)$ is also $0 \pmod{2^s}$ in this case and the result follows. □

As Remark 3 and Lemma 4 point out, the case $s = 1$ must be considered separately. In this case, we have the following result.

Proposition 7. *Let p be a prime. Assume that one of the following conditions holds:*

i) $k < rd^2(p-1)$,
ii) k *is not a multiple of* $p-1$.

Then, $S_w^d(p) \equiv 0 \pmod{p}$.

Proof. If $p=2$ condition ii) cannot hold and if condition i) holds, we can apply the same argument of the proof of the first part of Proposition 6 to get the result.

Now, if p is odd, again each entry in the matrix $S_w^d(p)$ is a homogeneous polynomial in the rd^2 variables $a_{i,j}^l$. Hence, it is a sum of elements of the form

$$\sum_{a_{i,j}^l \in \mathbb{Z}_p} \prod (a_{i,j}^l)^{\beta_{i,j,l}}.$$

We have that $\sum_{i,j,l} \beta_{i,j,l} = k$ so, if $k < rd^2(p-1)$ or if it is not a multiple of $p-1$ it follows that some $\beta_{i,j,l}$ is either 0 or not a multiple of $p-1$. In either case the corresponding element is $0 \pmod{p}$ due to Lemma 3 and, consequently, each entry in the matrix $S_w^d(p)$ is also $0 \pmod{p}$ as claimed. □

Observe that in the previous results we have considered sums of the form

$$S_w^d(p^s) = \sum_{A_1,\dots,A_r \in M_{p^s}^d} w(A_1,\dots,A_r),$$

where all the matrices A_i belong to the same matrix ring $M_{p^s}^d$. The following proposition will be useful in the next section and deals with the case when the matrices A_i belong to different matrix rings. First, we introduce some notation. Given a prime p, let

$$S_w^d(p^{s_1},\dots,p^{s_r}) := \sum_{A_i \in M_{p^{s_i}}^d} w(A_1,\dots,A_r).$$

If $s_1 = \cdots = s_r = s$, then $S_w^d(p^{s_1},\dots,p^{s_r}) = S_w^d(p^s)$ and we are in the previous situation.

Proposition 8. *With the previous notation, if* $s_1 > 1$*, then*

$$S_w^d(p^{s_1+1}, p^{s_2},\dots,p^{s_r}) \equiv p^{d^2} S_w^d(p^{s_1}, p^{s_2},\dots,p^{s_r}) \pmod{p^{s_1+1}}.$$

Proof. Since $s_1 > 1$ we have that $2s_1 > s_1 + 1$ so, due to Lemma 1

$$\begin{aligned} S_w^d(p^{s_1+1}, p^{s_2}, \dots, p^{s_t}) &= \sum_{\substack{A_1 \in M^d_{p^{s_1+1}} \\ A_i \in M^d_{p^{s_i}}}} w(A_1, \dots, A_t) = \\ &= \sum_{\substack{B \in M^d_{p^{s_1}}, C \in M^d_p \\ A_i \in M^d_{p^{s_i}}}} w(B + p^{s_1}C, A_2, \dots, A_r) \equiv \\ &\equiv \sum_{\substack{B \in M^d_{p^{s_1}}, C \in M^d_p \\ A_i \in M^d_{p^{s_i}}}} \left(w(B, A_2, \dots, A_r) + p^{s_1} \sum_l w_l(B, C, A_2, \dots, A_r) \right) = \\ &= p^{d^2} S_w^d(p^{s_1}, \dots, p^{s_r}) + p^{s_1} \sum_l \sum_{\substack{B \in M^d_{p^{s_1}}, C \in M^d_p \\ A_i \in M^d_{p^{s_i}}}} w_l(B, C, A_2, \dots, A_r) \\ &\quad \pmod{p^{s_1+1}}. \end{aligned}$$

Where $w_l(x, y, x_2, \dots, x_r)$ denotes the monomial $w(x_1, x_2, \dots, x_r)$ where the $l-th$ ocurrence of the term x_1 is substituted by y and the remaining ones by x (for instance, $w(x_1, x_2) = x_1^2 x_2 x_1$ gives us $w_1(x, y, x_2) = yxx_2x, w_2(x, y, x_2) = xyx_2x, w_3(x, y, x_2) = x^2x_2y$).

But, for every l, the monomial $w_l(B, C, A_2, \dots, A_r)$ contains C only once and with exponent 1. Hence,

$$\sum_{\substack{B \in M^d_{p^{s_1}}, C \in M^d_p \\ A_i \in M^d_{p^{s_i}}}} w_l(B, C, A_2, \dots, A_r) \equiv 0 \pmod{p}$$

because $S_1^d(p) \equiv 0 \pmod{p}$ and the result follows. □

The following corollary in now straightforward.

Corollary 3. *Assume that $S_w^d(p^s) \equiv 0 \pmod{p^s}$. Let us consider $s_1 \geq s_2 \geq \dots \geq s_r = s$. Then,*

$$S_w^d(p^{s_1}, \dots, p^{s_r}) \equiv 0 \pmod{p^{s_1}}.$$

Proof. Just apply the previous proposition repeatedly. □

4. Power sums of matrices over a finite commutative ring

In this section we will use the results from Section 3 to compute $S_k^d(R)$ for an arbitrary finite commutative ring R in many cases.

First of all, note that if $\text{char}(R) = n = p_1^{s_1} \cdots p_t^{s_t}$, then $R \cong R_1 \times \dots \times R_t$, where $\text{char}(R_i) = p_i^{s_i}$ and each R_i is a subring of characteristic $p_i^{s_i}$ and, in particular, a $Z_{p_i^{s_i}}-$module. This allows us to restrict ourselves to the case when $\text{char}(R)$ is a prime power.

The simplest case arises when R is a free $\mathbb{Z}_{p^s}-$module for an odd prime p.

Proposition 9. *Let p be an odd prime and let R be a finite commutative ring of characteristic p^s, such that R is a free $\mathbb{Z}_{p^s}-$module of rank r. Then,*

i) *If $s > 1$, $S_k^d(R) = 0$ for every $k \geq 1$ and $d \geq 2$.*

ii) *If* $s = 1$, $S_k^d(R) = 0$ *for every* $d \geq 2$ *and* k *such that either* $k < rd^2(p-1)$ *or* k *is not a multiple of* $p - 1$.

Proof. Note that under the previous assumptions and using Proposition 5 or Proposition 7 (depending on whether $s > 1$ or $s = 1$), it follows that

$$\sum_{A_1,\dots,A_r \in M_{p^s}^d} (x_1 A_1 + \cdots + x_r A_r)^k \equiv 0 \pmod{p^s}$$

because each entry of such a matrix is a polynomial in $x_1, \dots, x_r$ whose coefficients are 0 modulo p^s.

Consequently, for every $g_1, \dots, g_r \in R$ we have that

$$\sum_{A_1,\dots,A_r \in M_{p^s}^d} (g_1 A_1 + \cdots + g_r A_r)^k = 0.$$

Now, since R is free of rank r we can take a basis $g_1, \dots, g_r$ of R so that $M_{p^s}^d = \{g_1 A_1 + \cdots + g_r A_r | A_i \in M_{p^s}^d\}$. Therefore

$$S_k^d(R) = \sum_{A_1,\dots,A_r \in M_{p^s}^d} (g_1 A_1 + \cdots + g_r A_r)^k.$$

This concludes the proof. □

If $p = 2$, we have the following version of Proposition 9

Proposition 10. *Let* R *be a finite commutative ring of characteristic* 2^s, *such that* R *is a free* $\mathbb{Z}_{2^s}-$*module of rank* r. *Then,*

i) *If* $s > 1$, $S_k^d(R) = 0$ *for every* $d \geq 2$ *and* k *such that* $k \leq rd^2$ *or* $k > rd^2$ *with* $k + rd^2$ *even.*

ii) *If* $s = 1$, $S_k^d(R) = 0$ *for every* $d \geq 2$ *and* k *such that either* $k < rd^2$.

Proof. The proof is similar to that of Proposition 9, using Proposition 6 or Proposition 7 depending on whether $s > 1$ or $s = 1$. □

Remark. Note that if R is a finite commutative ring of characteristic p^s and $s = 1$, then R is necessarily free. Consequently, to study the non-free case we may assume that $s > 1$.

Assume that elements $g_1, \dots, g_r$ form a minimal set of generators of a non-free $\mathbb{Z}_{p^s}-$module R. Since R is non-free and $\mathrm{char}(R) = p^s$, it follows that $r > 1$ and also $s > 1$. For every $i \in \{1, \dots, r\}$ let $1 \leq s_i \leq s$ be minimal such that $p^{s_i} g_i = 0$. Note that it must be $s_i = s$ for some i and $s_j < s$ for some j. There is no loss of generality in assuming that $s = s_1 \geq \cdots \geq s_r$ and at least one of the inequalities is strict. Note that $p^{s_1}, \dots, p^{s_r}$ are the invariant factors of the $\mathbb{Z}-$module R. With this notation we have the following result extending Proposition 9.

Proposition 11. *Let* p *be an odd prime and let* R *be a finite commutative ring of characteristic* p^s, *such that* R *is a non-free* $\mathbb{Z}_{p^s}-$*module. Then,*

i) *If* $s_r > 1$, $S_k^d(R) = 0$ *for every* $k \geq 1$ *and* $d \geq 2$.

ii) *If* $s_r = 1$, $S_k^d(R) = 0$ *for every* $d \geq 2$ *and* k *such that either* $k < rd^2(p-1)$ *or* k *is not a multiple of* $p - 1$.

Proof. First of all, observe that

$$S_k^d(R) = \sum_{A_i \in M^d_{p^{s_i}}} (g_1 A_1 + \cdots + g_r A_r)^k.$$

In both situations i) and ii) it follows that $S_w^d(p^{s_r}) \equiv 0 \pmod{p^{s_r}}$. Moreover, we are in the conditions of Corollary 3, so it follows that $S_w^d(p^s, p^{s_2}, \dots, p^{s_r}) \equiv 0 \pmod{p^s}$. Consequently all the coefficients of the above sum are 0 modulo p^s and the result follows. □

The corresponding result for $p = 2$ is as follows.

Proposition 12. *Let R be a finite commutative ring of characteristic 2^s, such that R is a non-free $\mathbb{Z}_{p^s}-$module. Then,*

i) *If $s_r > 1$, $S_k^d(R) = 0$ for every $d \geq 2$ and k such that $k \leq rd^2$ or $k > rd^2$ with $k + rd^2$ even.*
ii) *If $s_r = 1$, $S_k^d(R) = 0$ for every $d \geq 2$ and k such that either $k < rd^2$.*

Proof. It is identical to the proof of Proposition 11. □

5. Conjectures and further work

Given a finite commutative ring R of characteristic n, we have seen in the last section that $S_k^d(R) = 0$ for many values of k, d and n. In this section we present two conjectures based on strong computational evidence which, being true, would let us to give a general result about $S_k^d(R)$.

With the notation from the previous section, given an r-tuple of integers $\kappa = (k_1, \dots, k_r)$, we consider the set of monomials in the non-commuting variables $\{x_1, \dots, x_r\}$

$$\Omega_\kappa := \{w : \deg_{x_i}(w) = k_i, \text{ for every } i\}.$$

The following conjectures are based on computational evidence.

Conjecture 2. *With the previous notation, let $s_1 \geq s_2 \geq \cdots \geq s_r$. Then*

$$S_w^d(p^{s_1}, p^{s_2}, \dots, p^{s_r}) \equiv 0 \pmod{p^{s_1}},$$

unless $d = p = 2$ and $s_i = 1$ for all i.

Conjecture 3. *If $p = 2 = d$ and $r > 1$ then for every $\kappa \in \mathbb{N}^r$*

$$\sum_{w \in \Omega_\kappa} \sum_{A_i \in M_2^d} w(A_1, \dots, A_r) \equiv 0 \pmod{2}.$$

The next lemma extends Lemma 2 in some sense. Its proof is straightforward.

Lemma 5. *Let R_1 and R_2 be finite commutative rings, and let $R = R_1 \times R_2$ be its direct product. Then*

$$S_k^d(R) = (card(R_2)^{d^2} \cdot S_k^d(R_1), card(R_1)^{d^2} \cdot S_k^d(R_2)) \in \mathbb{M}_d(R_1) \times \mathbb{M}_d(R_2)$$

Now, the following proposition would follow from Conjectures 2 and 3.

Proposition 13. *Let R be a finite commutative ring of characteristisc p^s for some prime p. Then $S_k^d(R) = 0$ unless $d = 2$, $R = \mathbb{Z}/2\mathbb{Z}$ and $1 < k \equiv -1, 0, 1 \pmod{6}$. Moreover, in this case $S_k^d(R) = I_2$.*

Proof. Assume that $\langle g_1 \dots, g_r\rangle$ is a minimal set of generators of R as $\mathbb{Z}_{p^s}$-module. Let $s = s_1 \geq s_2 \geq \cdots \geq s_r$ be integers such that the order of g_i is p^{s_i}; i.e., $s_1, \dots, s_r$ are minimal such that $p^{s_i} g_i = 0$.

In this situation,

$$S_k^d(R) = \sum_{A_i \in M^d_{p^{s_i}}} (g_1 A_1 + ... + g_r A_r)^k = 0,$$

unless $d = p = 2$, $s = r = 1$ and $1 < k \equiv -1, 0, 1 \pmod 6$ due to Conjecture 2.

On the other hand, if $d = p = 2$, $s = r = 1$ and $1 < k \equiv -1, 0, 1 \pmod 6$ it follows that

$$S_k^2(R) = \sum_{A \in M_2^2} (g_1 A)^k = \begin{pmatrix} g_1^k & 0 \\ 0 & g_1^k \end{pmatrix}.$$

But since in this case $R = \{0, g_1\}$, there are only two possibilities: $g_1^2 = g_1$ (and hence $R = \mathbb{Z}/2\mathbb{Z}$) or $g_1^2 = 0$ and the result follows. □

Finally, the next general result holds provided Conjectures 2 and 3 are correct. It is Conjecture 1, as stated in the introduction to the paper.

Theorem 4. *Let $d > 1$ and let R be a finite commutative ring. Then $S_k^d(R) = 0$ unless the following conditions hold:*

(1) $d = 2$,
(2) $\mathrm{card}(R) \equiv 2 \pmod 4$ *and* $1 < k \equiv -1, 0, 1 \pmod 6$,
(3) *The unique element $e \in R \setminus \{0\}$ such that $2e = 0$ is idempotent.*

Moreover, in this case

$$S_k^d(R) = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}.$$

Proof. First, observe that if $\mathrm{card}(R) \equiv 2 \pmod 4$, then R has $2m$ elements, where m is odd. Therefore, the $2-$primary component of the additive group R has only two elements, and so there is a unique element $e \in R$ of additive order 2.

Now, if R is of characteristic p^s for some prime, the result follows from the above proposition. Hence, we assume that R has composite characteristic. Let $R = R_1 \times R_2$ with R_1 the zero ring or $\mathrm{char}(R_1) = 2^s$ and $\mathrm{char}(R_2)$ odd. Due to Lemma 5 and Proposition 13 it follows that $S_k^d(R) = (\mathrm{card}(R_2)^{d^2} \cdot S_k^d(R_1), 0)$.

Now, $S_k^d(R_1) = 0$ unless $d = 2 = p$, $R_1 = \mathbb{Z}/2\mathbb{Z}$ and $1 < k \equiv -1, 0, 1 \pmod 6$ in which case

$$S_k^d(R) = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix},$$

where $e = (1, 0) \in R_1 \times R_2$ is the only idempotent of R such that $2e = 0$. □

Remark. Note that if, in addition, R is unital then the element e from the previous theorem is just $e = \frac{\mathrm{card}(R)}{2} \cdot 1_R$. Also note that if $S_k^d(R) \neq 0$, then $R \cong \mathbb{Z}/2\mathbb{Z} \times R_2$ with $\mathrm{card}(R_2)$ odd or $R_2 = \{0\}$.

We close the paper with a final conjecture.

Conjecture 4. *Theorem 4 remains true if R is non-commutative.*

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