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SOME PROPERTIES OF EVEN MOMENTS OF UNIFORM RANDOM WALKS

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ABSTRACT. We build upon previous work on the densities of uniform random walks in higher dimensions, exploring some properties of the even moments of these densities and extending a result about their modularity.

1. INTRODUCTION

Consider a short random walk of n steps in d dimensions where each step is of unit length and whose direction is chosen uniformly. Following [2], we let $\nu = \frac{d}{2} - 1$ and denote by $p_n(\nu; x)$ the probability density function of the distance x to the origin of this random walk. This paper will be concerned with the even moments of these random walks.

Definition 1.1. *Define*

$$W_n(\nu; s) = \int_0^\infty x^s p_n(\nu; x) \mathrm{dx}$$

as the s^{th} moment of the probability density function.

We know that

Theorem 1.2 (Borwein, Staub, Vignot, Theorem 2.18, [2])**.** *For non-negative integers k, $W_n(\nu; 2k)$ is given by*

$$W_n(\nu; 2k) = \frac{(k+\nu)!\nu!^{n-1}}{(k+n\nu)!} \sum_{k_1+\cdots+k_n=k} \binom{k}{k_1, \ldots, k_n} \binom{k+n\nu}{k_1+\nu, \ldots, k_n+\nu}$$

Theorem 1.3 (Borwein, Staub, Vignot Example 2.23, [2])**.** *For given ν, let $A(\nu)$ be the infinite lower triangular matrix with entries*

$$A_{k,j}(\nu) = \binom{k}{j} \frac{(k+\nu)!\nu!}{(k-j+\nu)!(j+\nu)!}$$

Research of K. G. Hare was supported by NSERC Grant RGPIN-2014-03154.

Research of Ghislain McKay was supported by NSERC Grant RGPIN-2014-03154, the Department of Pure Mathematics, University of Waterloo, and CARMA, University of Newcastle.

for row indices $k = 0, 1, 2, \ldots$ and columns entries $j = 0, 1, 2, \ldots$. Then the moments $W_{n+1}(\nu; 2k)$ are given by the row sums of $A(\nu)^n$.

For a good history of these moments, and random walks in general, see [1, 2, 3, 4].

Example 1.4. *For example, the upper corner of $A(0), A(1)$ and $A(2)$ are given below.*

$$A(0) := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 1 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & \\ 1 & 9 & 9 & 1 & 0 & 0 & 0 & 0 & \\ 1 & 16 & 36 & 16 & 1 & 0 & 0 & 0 & \\ 1 & 25 & 100 & 100 & 25 & 1 & 0 & 0 & \\ 1 & 36 & 225 & 400 & 225 & 36 & 1 & 0 & \\ 1 & 49 & 441 & 1225 & 1225 & 441 & 49 & 1 & \\ \vdots & & & & & & & & \ddots \end{bmatrix}$$

$$A(1) := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 1 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & \\ 1 & 6 & 6 & 1 & 0 & 0 & 0 & 0 & \\ 1 & 10 & 20 & 10 & 1 & 0 & 0 & 0 & \\ 1 & 15 & 50 & 50 & 15 & 1 & 0 & 0 & \\ 1 & 21 & 105 & 175 & 105 & 21 & 1 & 0 & \\ 1 & 28 & 196 & 490 & 490 & 196 & 28 & 1 & \\ \vdots & & & & & & & & \ddots \end{bmatrix}$$

$$A(2) := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 1 & 8/3 & 1 & 0 & 0 & 0 & 0 & 0 & \\ 1 & 5 & 5 & 1 & 0 & 0 & 0 & 0 & \\ 1 & 8 & 15 & 8 & 1 & 0 & 0 & 0 & \\ 1 & 35/3 & 35 & 35 & 35/3 & 1 & 0 & 0 & \\ 1 & 16 & 70 & 112 & 70 & 16 & 1 & 0 & \\ 1 & 21 & 126 & 294 & 294 & 126 & 21 & 1 & \\ \vdots & & & & & & & & \ddots \end{bmatrix}$$

The lower triangular entries of $A(0)$ are the squares of the binomial coefficients $\binom{k}{j}$ and those in $A(1)$ are known as the Naryana numbers [7, A001263]. Using these observations about $A(0)$ and $A(1)$, it is easy to observe that all of the coefficients of $A(0)$ and $A(1)$ are integers. A quick glance at $A(2)$ shows that this is not always true. It was stated that $A_{k,j}(2) \in \frac{1}{3}\mathbb{Z}$ in [2].

We define

$$r_\nu := \min\left\{r > 0 : A_{k,j}(\nu) \in \frac{1}{r}\mathbb{Z}, j, k \geq 0\right\}.$$

Using this notation we see that $r_0 = r_1 = 1$ and $r_2 = 3$. It is not immediately that r_ν is well defined and finite for all ν, (although we will show that this is the case).

In Section 2 we show that

Theorem 1.5. *For $\nu \geq 1$ we have $r_\nu \mid \frac{(2\nu-1)!}{\nu!}$.*

This is not best possible. In Section 3 we prove the opposite direction

Theorem 1.6. *For $\nu \geq 1$ we have $\binom{2\nu-1}{\nu} \mid r_\nu$.*

We conjecture that this is in fact best possible. That is, we conjecture

Conjecture 1.7. *For $\nu \geq 1$ we have $r_\nu = \binom{2\nu-1}{\nu}$.*

We present evidence for this conjecture in Section 4 and 5.

Next we consider a result by Borwein, Nuyens, Straub and Wan in [1] about the modularity of moments. They showed that

Theorem 1.8. *For primes p, we have*

$$W_n(0; 2p) \equiv n \mod p.$$

We extend this in Section 6 to get

Theorem 1.9. *Let*

- *$p = k$ be prime with $2\nu < p$, or*
- *$p = k + \nu$ be prime with $\nu < p$.*

Then

$$W_n(\nu; 2k) \equiv n \mod p.$$

If $p^2 = k$ with p prime then

$$W_n(0; 2k) \equiv n \mod p^2.$$

It is worth remarking that if both $p_1 := k$ and $p_2 := k + \nu$ are prime with $2\nu < p_1$ (and hence $\nu < 2\nu < p_1 < p_2$), then clearly $W_n(\nu; 2k) \equiv n \mod p_1 p_2$ by the Chinese Remainder Theorem.

In Section 7 we discuss some of the open problems related to this research.

2. A proof of Theorem 1.5: $r_\nu | (2\nu-1)!/\nu!$

To prove Theorem 1.5, we make use of the following remark and lemma:

Remark 2.1. *There are multiple equivalent ways of representing $A_{k,j}(\nu)$. The three most common that we will use are:*

$$\begin{aligned} A_{k,j}(\nu) &= \binom{k}{j}\frac{(k+\nu)!\nu!}{(k-j+\nu)!(j+\nu)!} \\ &= \binom{k}{j}\binom{k+\nu}{j}\binom{j+\nu}{j}^{-1} \\ &= \binom{k+\nu}{j}\binom{k+\nu}{j+\nu}\binom{k+\nu}{\nu}^{-1} \end{aligned}$$

Lemma 2.2. *For integers $1 \le \nu \le j$ we have*

$$\gcd((j-\nu+1)(j-\nu+2)\cdots j,\ (j+1)(j+2)\cdots(j+\nu)) \;\Big|\; (2\nu-1)!$$

Proof. Let $A_{j,\nu} = j-\nu+1,\dots,j$ and $B_{j,\nu} = j+1,\dots,j+\nu$. Let $\pi(A_{j,\nu})$ and $\pi(B_{j,\nu})$ be the products of these sequences. Let p be a prime number and $v_p(x)$ be the p-adic valuation of x. We see that for $p^\alpha > 2\nu$ that there is at most one term in $A_{j,\nu} \cup B_{j,\nu}$ that is divisible by p^α. Without loss of generality we may assume that such a term, if it exists, is in $A_{j,\nu}$. We see that $v_p(B_{j,\nu}) = v_p(B_{j+p^\alpha k,\nu})$ for all k by translation. Further, if there exists a term in $A_{j,\nu}$ that is divisible by p^α, then, by translations we can assume that this term is divisible by an arbitrarily high power of p. Hence we can assume that, if such a term exists, then we can find a translate of this sequence so that

$$v_p(\gcd(\pi(A_{j+p^\alpha k,\nu}),\pi(B_{j+p^\alpha k,\nu}))) = v_p(\pi(B_{j+p^\alpha k,\nu})).$$

We see that if $p^\beta \le \nu$ then there are at most $\left\lceil \frac{\nu}{p^\beta}\right\rceil$ terms in $B_{j+p^\alpha k,\nu}$ are are divisible by p^β. We see that if $\nu < p^\beta \le 2\nu$ then there are at most $\left\lceil \frac{2\nu}{p^\beta}\right\rceil - 1$ terms in $B_{j+p^\alpha k,\nu}$ are are divisible by p^β. By Chinese remainder theorem we can find such a j so that both the inequalities are exact. This gives us that

$$v_p(\gcd(\pi(A_{j+p^\alpha k,\nu}),\pi(B_{j+p^\alpha k,\nu}))) \le \sum_{p^\beta\le\nu}\left\lceil\frac{\nu}{p^\beta}\right\rceil + \sum_{\nu<p^\beta\le 2\nu}\left\lceil\frac{2\nu}{p^\beta}\right\rceil - 1 \tag{1}$$

and moreover there exists a j so that this is exact.

We observe that

$$v_p((2\nu-1)!) = \sum_{p^\beta\le 2\nu-1}\left\lfloor\frac{2\nu-1}{p^\beta}\right\rfloor.$$

ν	Equation (1)	$(2\nu-1)!$
1	1	1
2	$2\cdot 3$	$2\cdot 3$
3	$2^3\cdot 3\cdot 5$	$2^3\cdot 3\cdot 5$
4	$2^3\cdot 3^2\cdot 5\cdot 7$	$2^4\cdot 3^2\cdot 5\cdot 7$
5	$2^6\cdot 3^3\cdot 5\cdot 7$	$2^7\cdot 3^4\cdot 5\cdot 7$
6	$2^6\cdot 3^3\cdot 5^2\cdot 7\cdot 11$	$2^8\cdot 3^4\cdot 5^2\cdot 7\cdot 11$
7	$2^7\cdot 3^4\cdot 5^2\cdot 7\cdot 11\cdot 13$	$2^{10}\cdot 3^5\cdot 5^2\cdot 7\cdot 11\cdot 13$
8	$2^7\cdot 3^4\cdot 5^2\cdot 7^2\cdot 11\cdot 13$	$2^{11}\cdot 3^6\cdot 5^3\cdot 7^2\cdot 11\cdot 13$
9	$2^{11}\cdot 3^4\cdot 5^2\cdot 7^2\cdot 11\cdot 13\cdot 17$	$2^{15}\cdot 3^6\cdot 5^3\cdot 7^2\cdot 11\cdot 13\cdot 17$
10	$2^{11}\cdot 3^6\cdot 5^2\cdot 7^2\cdot 11\cdot 13\cdot 17\cdot 19$	$2^{16}\cdot 3^8\cdot 5^3\cdot 7^2\cdot 11\cdot 13\cdot 17\cdot 19$

TABLE 1. Prime factorization of Eq (1) and $(2\nu-1)!$.

Observe that if $p^\beta < \nu$ then

$$\left\lfloor \frac{2\nu-1}{p^\beta} \right\rfloor \geq \left\lceil \frac{\nu}{p^\beta} \right\rceil .$$

If $p^\beta = \nu$ then

$$\left\lfloor \frac{2\nu-1}{p^\beta} \right\rfloor = \left\lceil \frac{\nu}{p^\beta} \right\rceil = 1.$$

If $\nu < p^\beta \leq 2\nu - 1$ then

$$\left\lfloor \frac{2\nu-1}{p^\beta} \right\rfloor = 1 \geq \left\lceil \frac{2\nu}{p^\beta} \right\rceil - 1.$$

Lastly if $p^\beta = 2\nu$ then

$$\left\lfloor \frac{2\nu-1}{p^\beta} \right\rfloor = 0 \geq \left\lceil \frac{2\nu}{p^\beta} \right\rceil - 1.$$

Hence $v_p(\gcd(\pi(A_{j,\nu}), \pi(B_{j,\nu})) \leq v_p((2\nu-1)!)$ which gives that

$$\gcd(\pi(A_{j,\nu}), \pi(B_{j,\nu})) \;\big|\; (2\nu-1)!$$

as required. □

It is worth remarking that for any fixed $\nu \geq 4$, we can find tighter lower bounds for the gcd by using (1) directly. This can be used to tighten the results of Theorem 1.5 for specific ν. Unfortunately even when tightened in this way, we cannot achieve the conjectured bound. See Table 1

We are now ready to prove Theorem 1.5.

Proof of Theorem 1.5. We fix integers $\nu \geq 0$ and $0 \leq j \leq k$. We consider 2 cases:

If $0 \le j \le \nu - 1$ then we have

$$A_{k,j}(\nu) = \binom{k}{j}\binom{k+\nu}{j}\binom{j+\nu}{j}^{-1} = \binom{k}{j}\binom{k+\nu}{j} j! \frac{\nu!}{(j+\nu)!}$$

by our assumption on j we know $j+\nu \le 2\nu-1$, hence $(j+\nu)! \mid (2\nu-1)!$, and therefore

$$A_{k,j}(\nu) \in \tfrac{\nu!}{(j+\nu)!}\mathbb{Z} \subseteq \tfrac{\nu!}{(2\nu-1)!}\mathbb{Z}.$$

Otherwise we may assume that $j \ge \nu$. Then we have

$$\begin{aligned}
A_{k,j}(\nu) &= \binom{k+\nu}{j}\binom{k+\nu}{j+\nu}\binom{k+\nu}{\nu}^{-1} \\
&= \frac{(k+\nu)\cdots(k+1)\cdot k\cdots(k+\nu-j+1)}{j!}\cdot \\
&\quad \frac{(k+\nu)\cdots(k-j+1)}{(j+\nu)!}\cdot\frac{\nu!}{(k+\nu)\cdots(k+1)} \\
&= \frac{k\cdots(k+\nu-j+1)}{j!}\cdot\frac{(k+\nu)\cdots(k-j+1)}{(j+\nu)!}\cdot\nu! \\
&= \frac{(k+\nu)\cdots(k+1)}{(j+\nu)\cdots(j+1)\cdot j\cdots(j-\nu+1)}\binom{k}{j-\nu}\binom{k}{j}\nu!
\end{aligned}$$

Next observe that

$$\begin{aligned}
\frac{(k+\nu)\cdots(k+1)}{(j+\nu)\cdots(j+1)}\binom{k}{j-\nu}\binom{k}{j} &= \binom{k}{j-\nu}\binom{k+\nu}{j+\nu} \\
\frac{(k+\nu)\cdots(k+1)}{(j)\cdots(j-\nu+1)}\binom{k}{j-\nu}\binom{k}{j} &= \binom{k+\nu}{j}\binom{k}{j}
\end{aligned}$$

are both integers, hence there exists $p, q \in \mathbb{Z}$ such that

$$A_{k,j}(\nu) = \frac{(k+\nu)\cdots(k+1)}{(j+\nu)\cdots(j+1)\cdot j\cdots(j-\nu+1)}\binom{k}{j-\nu}\binom{k}{j}\nu! = \frac{p}{q}\nu!$$

and where $q \mid \gcd((j+\nu)\cdots(j+1),\ j\cdots(j-\nu+1))$.

By Lemma 2.2 and transitivity of divisibility, $q \mid (2\nu-1)!$ hence there exists p' such that

$$A_{k,j}(\nu) = p' \cdot \frac{\nu!}{(2\nu-1)!}.$$

Thus, for all integers $\nu \ge 0$ we have $r_\nu \mid \frac{\nu!}{(2\nu-1)!}$ as desired. □

3. A proof of Theorem 1.6: $\binom{2\nu-1}{\nu} \mid r_\nu$

Theorem 1.6 is an immediate corollary of:

Lemma 3.1. *Let $p^\alpha|\binom{2\nu-1}{\nu}$. Let $p^r \geq p^\alpha$ and $p^r > \nu$. Then the denominator of $A_{p^r-1,\nu-1}(\nu)$ is divisible by p^α.*

Proof. Let $p^\alpha|\binom{2\nu-1}{\nu}$. Let $p^r \geq p^\alpha$ and $p^r > \nu$. Notice that

$$A_{p^r-1,\nu-1}(\nu) = \binom{p^r+\nu-1}{\nu-1}\binom{p^r-1}{\nu-1}\binom{2\nu-1}{\nu-1}^{-1}.$$

Consider the first term.

$$\binom{p^r+\nu-1}{\nu-1} = \frac{(p^r+\nu-1)\cdots(p^r+1)}{(\nu-1)\cdots 1}.$$

Observe that each factor of the top is equivalent mod p^r to the matching factor in the bottom. Hence $\binom{p^r+\nu-1}{\nu-1} \equiv 1 \mod p$.

The second term is similar, with each term on the top equivalent mod p^r to the additive inverse of the associated factor on the bottom. Hence $\binom{p^r-1}{\nu-1} \equiv (-1)^\nu \mod p$.

Hence

$$A_{p^r-1,\nu-1}(\nu) = \frac{1}{p^\alpha}\cdot\frac{a}{b}$$

with p co-prime to a. □

4. The case $\nu = 3$ and $\nu = 4$

We see that $r_1 = 1 = \binom{1}{1}$ and $r_2 = 3 = \binom{3}{2}$. In this section we show the next two cases of Conjecture 1.7 hold, namely that $r_3 = 10 = \binom{5}{3}$ and $r_4 = 35 = \binom{7}{4}$.

We first need the Lemma

Lemma 4.1. *Let n and k be non-negative integers. If n is even and k is odd then $\binom{n}{k}$ is even.*

Proof. By Kummer's theorem [5], 2 divides $\binom{n}{k}$ when there is at least one carry when k and $n-k$ are added in base 2. Since n is even and k is odd, $n-k$ is odd. The least significant bit of an odd integer represented in base 2 is always 1. Hence both k and $n-k$ have a 1 in the least significant place. Thus when they are added, this will result in a carry. So 2 divides $\binom{n}{k}$. □

We now follow the proof of Theorem 1.5 using $\nu = 3$ to show:

Theorem 4.2. *Conjecture 1.7 holds for $\nu = 3$. That is $r_3 = \binom{5}{3} = 10$.*

Proof. We have that $10|r_3$ by Theorem 1.6.

As in the proof of Theorem 1.5, we first consider the case where $0 \leq j \leq 2$. A quick calculation shows that

$$A_{k,0}(3)\binom{5}{3} = 10$$
$$A_{k,1}(3)\binom{5}{3} = \frac{5(k+3)k}{2}$$
$$A_{k,2}(3)\binom{5}{3} = \frac{(k-1)(k+2)(k+3)k}{4}$$

By considering the cases of k even or odd, we see that all of these values are always integers, and hence $A_{k,0}(3), A_{k,1}(3), A_{k,2}(3) \in \frac{1}{10}\mathbb{Z}$.

If $j \geq 3$ then, as in the proof of Theorem 1.5, we have

$$\begin{aligned} A_{k,j}(3) &= \frac{3!}{(j+3)(j+2)(j+1)}\binom{k+3}{j}\binom{k}{j} \\ &= \frac{3!}{j(j-1)(j-2)}\binom{k}{j-3}\binom{k+3}{j+3}. \end{aligned}$$

We see that if $8 \nmid \gcd((j+3)(j+2)(j+1), j(j-1)(j-2))$ then

$$A_{k,j}(3) \in \frac{2!3!}{5!}\mathbb{Z}$$

as required. Hence we may assume that $8 \mid \gcd((j+3)(j+2)(j+1), j(j-1)(j-2))$. If j is even then $8 \mid (j+3)(j+2)(j+1)$ implies that $j \equiv 6 \mod 8$. We observe that $8 \mid j(j-1)(j-2)$ and $16 \nmid j(j-1)(j-2)$. In this case we observe that one of $\binom{k}{j-3}$ and $\binom{k+3}{j+3}$ is also even by Lemma 4.1. Hence we may write

$$A_{k,j}(3) = \frac{2}{8} \cdot \frac{p}{q}$$

where q is odd. This implies that

$$A_{k,j}(3) \in \frac{2!3!}{5!}\mathbb{Z}$$

as required.

Similarly if j is odd, then $j \equiv 1 \mod 8$, and $8 \mid (j+1)(j+2)(j+3)$ and $16 \nmid (j+1)(j+2)(j+3)$. Further one of $\binom{k+3}{j}$ and $\binom{k}{j}$ is even, and hence

$$A_{k,j}(3) = \frac{2}{8} \cdot \frac{p}{q}$$

where q is odd. Again this implies that

$$A_{k,j}(3) \in \frac{2!3!}{5!}\mathbb{Z}$$

as required. □

Theorem 4.3. *Conjecture 1.7 holds for $\nu = 4$. That is $r_4 = \binom{7}{4} = 35$.*

Proof. We have that $35|r_4$ by Theorem 1.6.

As in the proof of the previous theorem, we first consider the case where $0 \le j \le 3$. A quick calculation shows that

$$\begin{aligned} A_{k,0}(4)\binom{7}{4} &= 35 \\ A_{k,1}(4)\binom{7}{4} &= 7k(k+4) \\ A_{k,2}(4)\binom{7}{4} &= \frac{7(k-1)k(k+3)(k+4)}{12} \\ A_{k,3}(4)\binom{7}{4} &= \frac{(k-2)(k-1)k(k+2)(k+3)(k+4)}{36} \end{aligned}$$

By considering the various cases for k mod 12 (resp. 36), we see that these expressions are always integers, and hence $A_{k,0}(4), A_{k,1}(4), A_{k,2}(4), A_{k,3}(4) \in \frac{1}{35}\mathbb{Z}$.

If $j \ge 4$ then, as in the previous proof, we have

$$\begin{aligned} A_{k,j}(4) &= \frac{4!}{(j+4)(j+3)(j+2)(j+1)}\binom{k+4}{j}\binom{k}{j} \\ &= \frac{4!}{j(j-1)(j-2)(j-3)}\binom{k}{j-4}\binom{k+4}{j+4}. \end{aligned}$$

From equation (1) or Table 1 we have that

$$\gcd((j+4)(j+3)(j+2)(j+1), j(j-1)(j-2)(j-3)) \mid 7!/2$$

Hence we have that $A_{k,j}(4) \in \frac{2\cdot 4!}{7!}\mathbb{Z}$. We still need to show that there is an additional factor of 3 in the numerator.

To prove the result, we need to show that one of three things occurs

- $9 \nmid \gcd((j+4)(j+3)(j+2)(j+1), j(j-1)(j-2)(j-3))$
- $3 \mid \binom{k+4}{j}\binom{k}{j}$, or
- $3 \mid \binom{k}{j-4}\binom{k+4}{j+4}$.

If $(j+4)(j+3)(j+2)(j+1) \equiv j(j-1)(j-2)(j-3) \equiv 0 \mod 9$ then $j \equiv 2 \mod 9$ or $j \equiv 6 \mod 9$. Hence if $j \equiv 0, 1, 3, 4, 5, 7, 8 \mod 9$ then $A_{k,j}(4) \in \frac{3!\cdot 4!}{7!}\mathbb{Z}$ as required.

k	j	a	b		
$k \equiv 0 \mod 3$	$j \equiv 2 \mod 3$	2	2	$f \equiv 0 \mod 3$	$g \equiv 1 \mod 3$
$k \equiv 1 \mod 3$	$j \equiv 2 \mod 3$	4	1	$f \equiv 0 \mod 3$	$g \equiv 2 \mod 3$
$k \equiv 2 \mod 3$	$j \equiv 2 \mod 3$	0	3	$f \equiv 0 \mod 3$	$g \equiv 2 \mod 3$

TABLE 2. Cases when $j \equiv 2 \mod 9$

k	j	a	b		
$k \equiv 0 \mod 3$	$j \equiv 0 \mod 3$	2	4	$f \equiv 0 \mod 3$	$g \equiv 2 \mod 3$
$k \equiv 1 \mod 3$	$j \equiv 0 \mod 3$	1	3	$f \equiv 0 \mod 3$	$g \equiv 1 \mod 3$
$k \equiv 2 \mod 3$	$j \equiv 0 \mod 3$	0	2	$f \equiv 0 \mod 3$	$g \equiv 2 \mod 3$

TABLE 3. Cases when $j \equiv 6 \mod 9$

If $j \equiv 2 \mod 9$. then $27 \nmid (j+1)(j+2)(j+3)(j+4)$ so we have that 9 divides the gcd exactly.

Consider

$$\binom{k+4}{j}\binom{k}{j} = \frac{f_{a,b}(k,j)}{g_{a,b}(k,j)}\binom{k+a}{j}\binom{k+b}{j} \tag{2}$$

where $f_{a,b}(k,j)$ and $g_{a,b}(k,j)$ are polynomials. With careful choices of a and b we can construct $f_{a,b}$ and $g_{a,b}$ such that $f_{a,b}(k,j)$ will have more factors of 3 than $g_{a,b}$.

For example, if $a = b = 2$ then

$$f_{2,2}(k,j) = (k+4)(k+3)(k+2-j)(k-j+1)$$
$$g_{2,2}(k,j) = (k-j+4)(k-j+3)(k+2)(k+1)$$

Using the fact that $j \equiv 2 \mod 3$, we see that for $k \equiv 0 \mod 3$ that $f_{2,2}(k,j) \equiv 0 \mod 3$ and $g_{2,2}(k,j) \equiv 1 \mod 3$ and hence $\binom{k+4}{j}\binom{k}{j} \equiv 0 \mod 3$. A similar argument is given for $k \equiv 1 \mod 3$ and $k \equiv 2 \mod 3$, summarized in Table 2. Hence if $j \equiv 2 \mod 9$ then $A_{k,j}(4) \in \frac{3!\cdot 4!}{7!}\mathbb{Z}$ as required.

If $j \equiv 6 \mod 9$ then $27 \nmid j(j-1)(j-2)(j-3)$ so we have that 9 divides the gcd exactly.

Consider

$$\binom{k+4}{j+4}\binom{k}{j-4} = \frac{f_{a,b}(k,j)}{g_{a,b}(k,j)}\binom{k+a}{j-4}\binom{k+b}{j+4} \tag{3}$$

As before, we can break this into cases, as described in Table 3 □

5. ADDITIONAL SUPPORT FOR CONJECTURE 1.7

We have computationally checked that for all $k, j, \nu \leq 200$ that Conjecture 1.7 holds, Further, using the techniques of Theorems 4.2 and

4.3 we have computationally verified that for all $j, \nu \leq 15$ and all k that Conjecture 1.7 holds. It is not unreasonable to think that Conjecture 1.7 can hold in general. Indeed, if we plot the non-integer entries in the lower triangular part of $A(\nu)$ and colour them based on the prime factorization of their denominators in reduced form we obtain the fractal pattern seen in Figure (1). This suggests that there is far more structure to the matrix $A(\nu)$ that we are currently exploiting. We note that from equation (1) combined with Theorem 1.5 we would be able to prove that $r_5 | 2^3 \cdot 3^2 \cdot 7$. We conjecture that $r_5 = \binom{7}{4} = 2 \cdot 3^2 \cdot 7$. In this image of $A(5)$, denominators are coloured red for 2, blue for 3, green for 7 and orange for 3^2. If the denominator had contained any additional factors of 2, 3 or 5 then we would have coloured this value black. None occurred. Assuming that primes always give rise to the associated fractals early on, as seen in Figure 1, we would be led to believe that $4 \nmid r_5$.

6. Proof of Theorem 1.9: $W_n(v; 2k) \equiv n$

Proof of Theorem 1.9. We rewrite (1.2) as

$$W_n(\nu; 2k) = \sum_{k_1 + \cdots + k_n = k} \frac{k! \cdot (k+\nu)! \cdot \nu!^{n-1}}{k_1! \cdots k_n! \cdot (k_1+\nu)! \cdots (k_n+\nu)!}$$

Let $p = k$ be prime with $2\nu < p$ or let $p = k + \nu$ be prime with $\nu < p$. We claim that there does not exist indices $1 \leq i < j \leq n$ such that $k_i + \nu \geq p$ and $k_j + \nu \geq p$. Indeed, this would lead to

$$2p \leq k_i + k_j + 2\nu \leq (k_1 + \cdots + k_n) + 2\nu = k + 2\nu.$$

If $p = k$ then $2\nu < p$ by assumption and hence $2p \leq k + 2\nu < 2p$, a contradiction. If $p = k + \nu$ then $\nu < p$ by assumption and hence $2p \leq (k + \nu) + \nu < 2p$, a contradiction.

If instead $k = p^2$ and $\nu = 0$ it is easy to see that there does not exist indices $1 \leq i < j \leq n$ such that $k_i + \nu \geq p^2$ and $k_j + \nu \geq p^2$.

We consider 2 cases:

If there exists $1 \leq i \leq n$ such that $k_i = k$ then clearly $k_j = 0$ for $j \neq i$ and hence

$$\frac{k! \cdot (k+\nu)! \cdot \nu!^{n-1}}{k_1! \cdots k_n \cdot (k_1+\nu)! \cdots (k_n+\nu)!} = \frac{k! \cdot (k+\nu)! \cdot \nu!^{n-1}}{k! \cdot 0! \cdots 0! \cdot (k+\nu)! \cdot \nu! \cdots \nu!} = 1$$

Assume that $k_i < k$ for all $1 \leq i \leq n$.

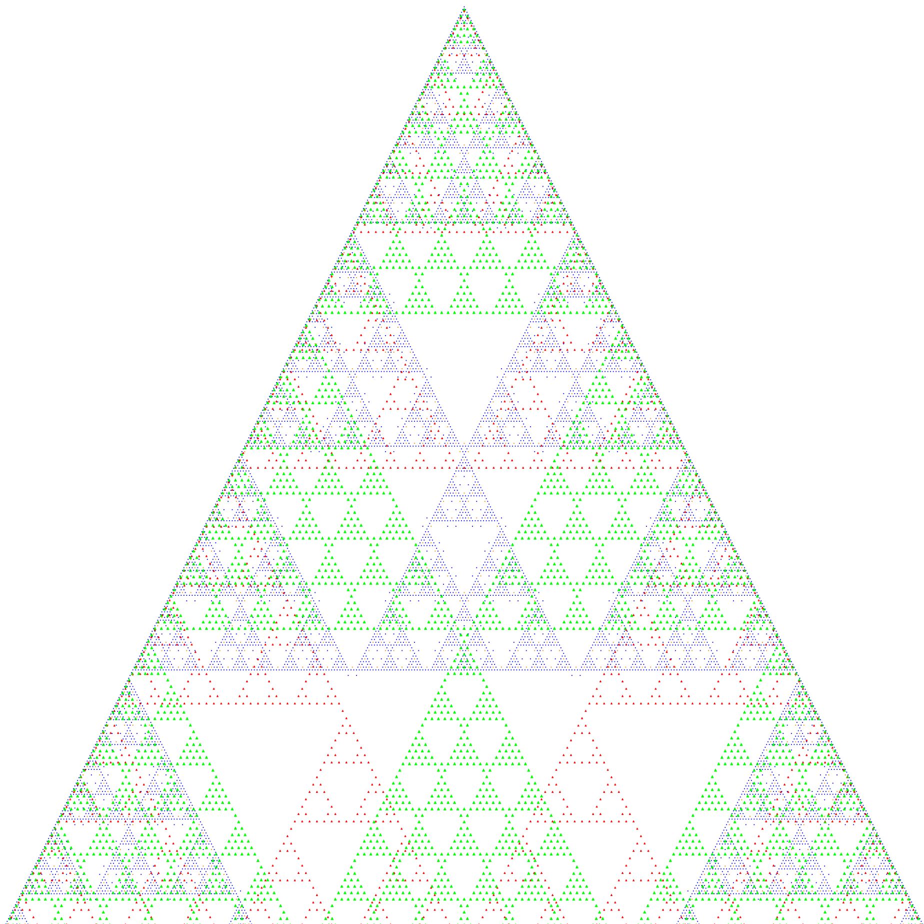

FIGURE 1. Non-integer entries of the first 1000 rows of $A(5)$

If $p = k$ we see that $p|k!$ and $p|(k+\nu)!$. We further see that at most one term in the denominator is divisible by p. Hence

$$\frac{k! \cdot (k+\nu)! \cdot \nu!^{n-1}}{k_1! \cdots k_n! \cdot (k_1+\nu)! \cdots (k_n+\nu)!}$$

can be written as $p\frac{a}{b}$ where $p \nmid b$, and thus is equivalent to 0 mod p.

If $p = k + \nu$ we see that $p|(k+\nu)!$. We further see that no term in the denominator is divisible by p. Hence

$$\frac{k! \cdot (k+\nu)! \cdot \nu!^{n-1}}{k_1! \cdots k_n! \cdot (k_1+\nu)! \cdots (k_n+\nu)!}$$

can be written as $p\frac{a}{b}$ where $p \nmid b$, and thus is equivalent to 0 mod p.

If $p^2 = k$ and $\nu = 0$ we see that $p^{p+1}|k!$ and $p^{p+1}|(k+\nu)!$. We further see that we have at most $2p$ factors of p in the denominator, with equality only if $p|k_i$ for all i. Hence

$$\frac{k! \cdot (k+\nu)! \cdot \nu!^{n-1}}{k_1! \cdots k_n! \cdot (k_1+\nu)! \cdots (k_n+\nu)!}$$

can be written as $p^2 \frac{a}{b}$ where $p \nmid b$, and thus is equivalent to 0 mod p^2.

Thus there are only n terms in the sum for $W_n(\nu; 2k)$ which are not $0 \mod p$ (resp $0 \mod p^2$), namely when $k_i = k$ for some k. In this case the term is $1 \mod p$ (resp $1 \mod p^2$) hence

$$W_n(\nu; 2k) \equiv n \mod p \quad (\text{resp. } W_n(0; 2k) \equiv n \mod p^2)$$

□

7. Comments

We showed in Section 4 that Conjecture 1.7 held for the case $\nu = 3$ and $\nu = 4$. It is probably that this technique could be extended computationally for any fixed ν, although this is not clear. It is not clear that this technique would be extendable to arbitrary ν without additional ideas.

In Section 6 we showed how the ideas of modularity of $W_n(\nu; k)$ could be extended to $k = p^2$ or $\nu > 0$. It appears that something is also happening in the case when $k = p^2 \neq 4$ and $\nu = 1$, although it is unclear how one would prove this. There are most likely many other relations that can be found when considering W_n modulo a well chosen prime power.

8. Acknowledgements

The authors would like to thank Jon Borwein for many useful discussions and suggestions, without which this paper would not have been possible.

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